Results Math (2018) 73:19

1422-6383/18/010001-8
published online February 8, 2018
https://doi.org/10.1007/s00025-018-0777-x

Results in Mathematics

CrossMark

Critical Points on Convex Surfaces

Tudor Zamfirescu

Abstract. The cut locus $C(x)$ of some point x on an open convex surface is a forest and has measure zero. However, we show here that topologically it can be quite large, namely residual. All critical points with respect to x belong to $C(x)$. We also show that, irrespective of how large $C(x)$ might be, there is a Jordan arc in $C(x)$ containing all critical points.

Mathematics Subject Classification. 52A15, 53C45, 53A05.

Keywords. Open convex surface, critical point, cut locus.

1. Introduction

Let S be a *convex surface*, which is by definition the boundary of a convex set with non-empty interior in the 3-dimensional Euclidean space, but is neither a plane, nor the union of two planes. If S is compact, then it is said to be a *closed convex surface*. If S is unbounded, then it is called an *open convex surface*.

We denote by ρ the intrinsic metric of S and, for $x \in S$, let $\rho_x : S \to \mathbb{R}$ be defined by $\rho_x(y) = \rho(x, y)$. In this paper we investigate the critical points of x on S with respect to this function ρ_x.

An important role will be played by the *cut locus* $C(x)$ of x, defined as the set of all $y \in S$ such that no *segment*, i.e. shortest path, from x to y can be extended as a segment beyond y. The cut locus was introduced by Poincaré [5] in 1905. Among other things, it is known that $C(x)$ is a *forest*, i.e. a union of pairwise disjoint trees, including the cases of a single tree, a point, and the empty set. For an introduction to the cut locus, see for example [4].

For any tree, a point of the tree is called an *extremity* if its deletion does not disconnect the tree. Let $E(x)$ denote the set of all extremities of trees which are components of $C(x)$.

A point y is called *critical* with respect to x (and ρ_x) if for any tangent direction τ at y there exists a segment from y to x with direction σ at y such that $\angle\tau,\sigma \leq \pi/2$ (see, for instance, [3], p. 2). For example, all relative maxima of ρ_x and all relative minima of $\rho_x|_{C(x)\setminus E(x)}$ are critical points. Let $Q(x)$ be the set of all critical points with respect to x. It is easily seen that $Q(x) \subset C(x)$.

Let $\mathcal{S}$ be the space of all convex surfaces in $\mathbb{R}^3$. Its subspace $\mathcal{S}^c$ of all closed convex surfaces, endowed with the well-known Pompeiu–Hausdorff metric, is a Baire space.

Contrary to the case of Riemannian surfaces, on an arbitrary convex surface the cut locus may be residual in the surface. In fact, we proved in 1982 that this holds for most closed convex surfaces, in the sense of Baire categories, that is for all surfaces in $\mathcal{S}^c$ except those in a first category subset [8].

With possibly so large cut loci, it is surprising that the distribution of the critical points with respect to x is always very nice: they all lie on a single *Y-tree*, i.e. a tree with at most 3 extremities, included in $C(x)$, except for the case of boundaries of tetrahedra of a very special type. This result, proven in [10] for closed surfaces, will be extended here to all convex surfaces; it even takes a stronger form in the case of open surfaces: the Y-tree is a Jordan arc, or a point, or empty.

For related work on farthest points in the case of closed convex surfaces, see [9]. Further research on the location of critical points can be found in [2,7,11].

An arc $J \subset S$ from a to b will be called *obtuse* (*non-acute*) if, for any point $c \in J\setminus\{a, b\}$, $\angle acb > \pi/2$ ($\angle acb \geq \pi/2$).

In the following, T_x denotes the space of all unit tangent vectors at $x \in S$.

For $M \subset S$, cl$M$ means the topological closure of M. If σ is a segment, $\lambda\sigma$ denotes its length. For any Borel set $M \subset S$, the *Gauß image* of M is the set of all outer normal unit vectors at points of M, and the curvature $\omega(M)$ of M is the measure on the unit sphere of that Gauß image (see [1], p. 207).

For $x, y \in \mathbb{R}^3$, xy denotes the line-segment from x to y, and $\overline{xy}$ the line containing x, y.

2. Residual Cut Loci

A point in a convex surface S which is not interior to any segment is called an *endpoint* of S. Let $E(S)$ be the set of endpoints of S.

Since $E(S) \subset C(x)$ for every $x \in S$, to show that there are open convex surfaces all cut loci of which are residual, it suffices to show that there exist such surfaces S with $E(S)$ residual in S. Exceptionally, we present the result of this section in arbitrary dimension.

We shall use the following mentioned result.

Lemma I [8]. *On most closed convex hypersurfaces, most points are endpoints.*

Theorem 0. *There exist open convex hypersurfaces $S \subset \mathbb{R}^d$ with $E(S)$ residual in S.*

Proof. Let $P \subset \mathbb{R}^d$ be the hyperparaboloid

$$x_d = x_1^2 + x_2^2 + \cdots + x_{d-1}^2.$$

For $i = 1, 2, 3, \ldots$, take the hyperplane H_i of equation $x_d = i$, the halfspace H_i^- defined by $x_d \leq i$, and the halfspace H_i^+ defined by $x_d \geq i$.

Let C_i be the intersection of all halfspaces containing the origin, bounded by hyperplanes tangent to P at the points of $P \cap H_i$ ($i = 1, 2, 3, \ldots$). Consider the convex body

$$D_i = C_i \cap H_{i-3}^+ \cap H_{i+3}^- \ (i = 1, 2, 3, \ldots),$$

and put $D_0 = \mathbb{R}^d$. The intersection $D_{i-1} \cap D_i \cap D_{i+1}$ is another convex body, and $F_i = D_{i-1} \cap (\mathrm{bd}D_i) \cap D_{i+1}$ is a piece of a cone ($i = 1, 2, 3, \ldots$).

By Lemma I, we can choose D_i^* to be a convex body with $E(\mathrm{bd}D_i^*)$ residual in $\mathrm{bd}D_i^*$, close to D_i (in the sense of Pompeiu–Hausdorff distance). Then

$$F_i^* = D_{i-1}^* \cap (\mathrm{bd}D_i^*) \cap D_{i+1}^*$$

also has residually many endpoints and is close to F_i; so, the open convex surface $\bigcup_{i=1}^{\infty} F_i^*$ has the desired property. □

3. Auxiliary Material

We recall Alexandrov's comparison theorem, the convex version of the Pizzeti–Toponogov comparison theorem.

Lemma C [1]. *If a triangle on a convex surface with segments as sides has angles α, β, γ, and a triangle in $\mathbb{R}^2$ with the same side-lengths has respective angles $\alpha_E, \beta_E, \gamma_E$, then $\alpha \geq \alpha_E$, $\beta \geq \beta_E$, $\gamma \geq \gamma_E$.*

Lemma O. *Let S be a convex surface, $x \in S$, and $J \subset C(x)$ a Jordan arc joining critical points a, b. If for some segments from x to a and b their angle at x is at most (less than) $\pi/2$, then J is non-acute (obtuse).*

Proof. Choose the two (possibly coinciding) segments from x to a determining the domains A, A^* (with A possibly void) such that $A^* \supset J \backslash \{a\}$ and the angle α at a between the segments towards A^* is minimal.

From the definition of a critical point it follows that $\alpha \leq \pi$. Similarly, we obtain segments from x to b and domains B, B^* with analogous properties. The hypothesis implies that one of these segments from x to a and one of those from x to b make an angle $\gamma \leq \pi/2$. (The case of strict inequality is treated analogously.)

Let y be an arbitrary point of J different from a and b. Suppose, on the contrary, that for some segments σ_{ay}, σ_{by}, from a and b to y, one of the angles δ_1, δ_2 at y, say δ_2, is less than $\pi/2$.

Consider the points $a_E, b_E, x_E, y_E \in \mathbb{R}^2$ such that the line $\overline{x_E y_E}$ separates a_E from b_E, the 3-point set $\{a_E, x_E, y_E\}$ is isometric to $\{a, x, y\} \subset S$ and $\{b_E, x_E, y_E\}$ is isometric to $\{b, x, y\} \subset S$.

Let $\alpha_E, \gamma_E, \beta_E, \delta_E$ be the respective angles of the quadrilateral $a_E x_E b_E y_E$.

The arc $\sigma_{ay} \cup \sigma_{by}$ divides $S \backslash (A \cup B)$ into two quadrilateral domains Q_1, Q_2 with angles $\alpha_1, \beta_1, \gamma_1, \delta_1$ and $\alpha_2, \beta_2, \gamma_2, \delta_2$ at a, b, x, y respectively. We have $\alpha_i \geq \alpha_E$, $\beta_i \geq \beta_E$, $\gamma_i \geq \gamma_E$, $\delta_i \geq \delta_E$ $(i = 1, 2)$. To see this, it suffices to join y with x by two segments, one through each quadrilateral domain (this is known to be possible [1]), and apply Lemma C to the four geodesic triangles. Since $\delta_2 < \pi/2$, we must have $\delta_E < \pi/2$, whence $\alpha_E + \beta_E + \gamma_E > 3\pi/2$. Since $\alpha_1 + \alpha_2 = \alpha \leq \pi$ and $\beta_1 + \beta_2 \leq \pi$, we must also have $\alpha_E \leq \pi/2$ and $\beta_E \leq \pi/2$, hence $\gamma_E > \pi/2$. This implies $\gamma_1 > \pi/2$ and $\gamma_2 > \pi/2$. Since γ equals γ_1 or γ_2, a contradiction is obtained. □

Lemma B. *For any convex surface S and point $x \in S$, the set $Q(x)$ is bounded.*

Proof. Of course, the lemma is meaningful only for open surfaces, so assume S is an open convex surface. Now, suppose $Q(x)$ contains the unbounded sequence $\{q_n\}_{n=1}^{\infty}$. With at most one exception, each of these points q_n has curvature less than π and is therefore joined to x by at least two segments. We choose σ_n, σ'_n among them such that the bounded domain determined by $\sigma_n \cup \sigma'_n$ be maximal. Let τ_n, τ'_n be the directions of σ_n, σ'_n at x. By selecting a subsequence if necessary, we may assume that the sequences of directions $\{\tau_n\}_{n=1}^{\infty}$ and $\{\tau'_n\}_{n=1}^{\infty}$ converge to, say, $\tau, \tau' \in T_x$, respectively.

Let $\varepsilon \in (0, \pi/10)$. For n large enough, $\angle \tau_n, \tau < \varepsilon/2$.

Let σ_{mn} be a segment from q_n to q_m $(n \neq m)$.

Consider the points $\overline{x}, \overline{q_n}, \overline{q_m} \in \mathbb{R}^2$ with $\|\overline{x} - \overline{q_n}\| = \lambda\sigma_n$, $\|\overline{x} - \overline{q_m}\| = \lambda\sigma_m$, $\|\overline{q_n} - \overline{q_m}\| = \lambda\sigma_{nm}$. For $m > n$ large enough, $\lambda\sigma_m$ becomes as large as necessary for $\angle \overline{x}\, \overline{q_n}\, \overline{q_m} > \pi - \varepsilon$ to hold. This implies that $\angle \sigma_n, \sigma_{nm} > \pi - \varepsilon$ and $\angle \sigma'_n, \sigma_{nm} > \pi - \varepsilon$. Thus, no segment from x to q_n makes an angle at most $\pi/2$ with σ_{nm}, which contradicts $q_n \in Q(x)$. □

4. The Arc Containing $Q(x)$ on Open Convex Surfaces

Let S be an open convex surface, and $x \in S$. In the forest $C(x)$ with its possibly infinitely many trees and uncountably many endpoints, we shall single out the antipodal arc of x, an arc in $C(x)$ entirely containing $Q(x)$.

Theorem 1. *For any open convex set S and point $x \in S$, if $Q(x)$ contains more than one point, then there is a single Jordan arc J_x joining critical points, lying in $C(x)$ and containing $Q(x)$.*

Proof. Every pair of critical points belonging to the same tree in $C(x)$ can be uniquely joined by a Jordan arc. We do this for all such pairs of points. The union of all these arcs is a subforest F of $C(x)$, which, by Lemma B, is bounded, because the points of F farthest from x lie in $Q(x)$.

Suppose some component T of F has at least three extremities a, b, c. Then these are critical points; from the definition it follows that there are (possibly coinciding) segments from x to a determining two domains A, A^*, one of which, say A^*, contains $T\backslash\{a\}$ and has at a an angle $\alpha \leq \pi$, and the other having at x an angle $\alpha_x \geq 0$, but being possibly empty (the case $\alpha_x = 0$). Similarly, we obtain segments from x to b and c, and domains B, B^* and C, C^* with analogous properties. Of course, at most one of the domains A, B, C, say C, is unbounded. By respecting elementary intersection properties of segments (see [1]), either $A \subset B$, or $B \subset A$, or $A \cap B = \emptyset$. But, clearly, $A \neq B$. Since $T \cap A = T \cap B = \emptyset$, but $a, b \in T$, the first two possibilities are excluded for non-empty A, respectively B. So, $A \cap B = \emptyset$.

We have

$$\omega(A \cup \{a\}) = 2\pi - \alpha + \alpha_x \geq \pi + \alpha_x, \qquad \omega(B \cup \{b\}) \geq \pi + \beta_x.$$

Therefore

$$\omega(S) \geq \omega(A \cup \{a\}) + \omega(B \cup \{b\}) \geq 2\pi.$$

Since S is open, we must have the equality case, which implies $\alpha_x = \beta_x = 0$, whence $A = B = \emptyset$ and $\omega(S\backslash\{a, b\}) = 0$. Hence S has vanishing curvature except at two points, which is impossible for non-degenerate surfaces.

Hence, every component of F has at most two extremities. Suppose F is disconnected, choose as above the bounded domain A corresponding to an extremity of one component of F, and choose the bounded domain $A^\diamond$ corresponding to an extremity of another component of F. Then $A \cap A^\diamond = \emptyset$, and the same contradiction as before is obtained. In conclusion, F is a single tree with at most two extremities a, b. This means that F is a Jordan arc J_x joining a to b. □

We call J_x the *antipodal arc* of x on S.

Lemma G. *Let uv be a line-segment included in the convex surface $S \subset \mathbb{R}^3$. Let σ be a segment of S from u to w orthogonal at u to $\overline{uv}$, and assume that, for every $t \in \sigma$, the line-segment $uv + t - u$ lies in S. Then σ is a line-segment or $\omega(\{u\}) < \pi$.*

Proof. Let σ' be a segment of S from v to w. Consider the surface $S' = \mathrm{bdconv}(\sigma \cup \sigma')$. At u, the Gauß image of S is included in the Gauß image of S'. Assume σ is not a line-segment. Since S is not degenerate, σ and uw make a non-vanishing angle ν at u. Then the curvature of $\{u\}$ in S' equals $2\pi - (\frac{\pi}{2} + \frac{\pi}{2} + \nu) < \pi$, which a fortiori yields $\omega(\{u\}) < \pi$ on S. □

Theorem 2. *On any open convex surface S, if $x \in S$ and* $\mathrm{card}Q(x) > 1$, *then J_x is obtuse and has at least one extremity b joined to x by segments opposite at b. The latter assertion remains true if $Q(x)$ reduces to a single point $\{b\}$.*

Proof. We use the notation of Lemma O; in particular, let $J = J_x$ join a to b.

One of the domains A, B, say B, is unbounded.

Consider the unbounded domain $B' \subset B$ with the union of two segments from x to b as boundary, such that its angle β' at b be minimal.

If $\beta' < \pi$, then $\rho_x(z) < \rho_x(b)$ for points $z \in C(x) \cap B'$ close to B'. This yields that the absolute minimum of ρ_x on $C(x) \cap \mathrm{cl}B'$ lies in B', which is absurd, being a critical point. Hence, $\beta' \geq \pi$. Since $b \in Q(x)$, $\beta' \leq \pi$. Thus, $\beta' = \pi$. This guarantees that J_x admits two segments from x to b opposite at b.

We now prove that J_x is obtuse. Suppose, on the contrary, that for some point y of J_x different from a and b and some segments σ_a, σ_b from a and b to y, one of the angles δ_1, δ_2 between them at y, say δ_2, is at most $\pi/2$.

The arc $\sigma_a \cup \sigma_b$ divides $S \backslash (A \cup B)$ into two quadrilateral domains Q_1, Q_2 with angles $\alpha_1, \beta_1, \gamma_1, \delta_1$ and $\alpha_2, \beta_2, \gamma_2, \delta_2$ at a, b, x, y respectively. By Lemma O, $\gamma_1 \geq \pi/2$ and $\gamma_2 \geq \pi/2$.

Similarly, $\sigma_a \cup \sigma_b$ divides $S \backslash (A \cup B')$ into two quadrilateral domains Q_1', Q_2' with angles $\alpha_1, \beta_1', \gamma_1', \delta_1$ and $\alpha_2, \beta_2', \gamma_2', \delta_2$ at a, b, x, y respectively. Clearly,

$$\gamma_1' + \gamma_2' \geq \gamma_1 + \gamma_2 \geq \pi.$$

Let γ^* be the angle of $S \backslash B'$ at x. Then

$$\omega(S \backslash \mathrm{cl}B') = \pi + \gamma^*,$$

whence $\gamma^* \leq \pi$. This can happen only if $A = \emptyset$ and $\gamma_1' + \gamma_2' = \gamma_1 + \gamma_2 = \pi$. Then $\gamma^* = \pi$, $\omega(S \backslash \mathrm{cl}B') = 2\pi$, $\omega(\mathrm{cl}B') = 0$, and $\gamma_E = \pi/2$. (Recall the notation in the proof of Lemma O.) It follows further that $\alpha_E = \beta_E = \pi/2$, and $\delta_E = \pi/2$. Hence,

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \pi/2$$

and $\delta_2 = \pi/2$. Consequently, $\omega(\{a\}) = \pi$ and $\omega(Q_2') = 0$.

Because $\omega(S) = 2\pi$ and $\omega(\mathrm{cl}B') = 0$, $\mathrm{cl}B'$ must be cylindrical, and if the segments joining x to b were line-segments then S would be degenerate. Hence, $\mathrm{cl}B'$ is a non-planar piece of a cylinder. Since $\omega(Q_2' \cup \mathrm{cl}B') = 0$ too, $Q_2' \cap \mathrm{cl}B'$ contains no vertex, and $Q_2' \cup \mathrm{cl}B'$ is a larger non-planar piece of a cylinder, a half-line (generator) of which contains the segment (there is only one because $A = \emptyset$) from x to a. Also, σ_a is congruent to one of the segments from x to b; hence, σ_a is not a line-segment. Since $\alpha_1 = \pi/2$, Lemma G implies $\omega(\{a\}) < \pi$, and a contradiction is obtained. □

5. Non-acute Arcs in the Y-tree Containing $Q(x)$

On any closed convex surface $S \subset \mathbb{R}^3$, each cut locus is a tree. We have shown in [10] that in this case all critical points belong to a Y-tree, or else S is a tetrahedral surface of a special type. The latter case will be tacitly excluded in the rest of this paper.

Theorem 3. *Let S be a convex surface and $J \subset C(x)$ be an arc joining two points critical with respect to x. If $a \in J$ is an additional critical point, then at least one of the two arcs into which a decomposes J is non-acute.*

Proof. If b, c are the extremities of J, let σ_b be a segment from x to b, σ_c a segment from x to c, and σ_a, σ_a' two segments from x to a separating together b from c (see [5] or [6]). Let γ_b be the angle between σ_a and σ_b, and γ_b' the angle between σ_a' and σ_b. Also, consider the analogous angles γ_c ad γ_c'.

Since

$$\gamma_b + \gamma_b' + \gamma_c + \gamma_c' \leq 2\pi,$$

one of these angles, say γ_b, is at most $\pi/2$. Now, by Lemma O, the subarc of J joining a to b is non-acute. □

On closed convex surfaces the antipodal tree Y_x of x, being a Y-tree, has at most one ramification point.

Theorem 4. *If S is a closed convex surface, $x \in S$, and Y_x has a critical ramification point y, then at least two of the three arcs into which y decomposes Y_x are non-acute.*

Proof. Let a, b, c be the endpoints of Y_x. One of the Jordan arcs in Y_x from y to a and from y to b, say the second, is non-acute, by Theorem 3. Similarly, one of the arcs from y to a and from y to c is non-acute. □

Theorem 4 assumes that the Y-tree Y_x has a critical ramification point. Is this possible?

Remark. For the ramification point y of Y_x to be critical with respect to x, it is sufficient that the degree of y in $C(x)$ is 3 and no angle formed by the branches of Y_x at y is less than $\pi/2$.

This can be seen in the following way. There must exist precisely three segments from x to y forming at y the angles $\alpha_1, \alpha_1', \alpha_2, \alpha_2', \alpha_3, \alpha_3'$ with the neighbouring branches of Y_x, respectively, where $\alpha_1' = \alpha_2$, $\alpha_2' = \alpha_3$, $\alpha_3' = \alpha_1$. These equalities, as well as the existence of directions of the branches of T_x at y have been established in [5,6].

Thus, $\alpha_1 + \alpha_1' \geq \pi/2$ implies $\alpha_3' + \alpha_1 + \alpha_1' + \alpha_2 \geq \pi$, whence $\alpha_2' + \alpha_3 \leq \pi$. This and the other two analogous inequalities guarantee that $y \in Q(x)$.

References

[1] Alexandrov, A.D.: Die innere Geometrie der konvexen Flächen. Akademie-Verlag, Berlin (1955)

[2] Bárány, I., Itoh, J., Vîlcu, C., Zamfirescu, T.: Every point is critical. Adv. Math. **235**, 390–397 (2013)

[3] Cheeger, J., Gromov, M., Okonek, C., Pansu, P.: Geometric Topology: Recent Developments. Lecture Notes in Mathematics, vol. 1504. Springer, Berlin (1991)

[4] Kobayashi, S.: On conjugate and cut loci. Glob. Differ. Geom. **27**, 140–169 (1989)

[5] Poincaré, H.: Sur les lignes géodésiques des surfaces convexes. Trans. Am. Math. Soc. **6**, 237–274 (1905)

[6] Shiohama, K., Tanaka M.: Cut loci and distance spheres on Alexandrov surfaces. In: Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992). Séminaires et Congres, vol. 1, pp. 531–559. Societe Mathematique de France, Paris (1996)

[7] Vîlcu, C., Zamfirescu, T.: Multiple farthest points on Alexandrov surfaces. Adv. Geom. **7**, 83–100 (2007)

[8] Zamfirescu, T.: Many endpoints and few interior points of geodesics. Invent. Math. **69**, 253–257 (1982)

[9] Zamfirescu, T.: Farthest points on convex surfaces. Math. Z. **226**, 623–630 (1997)

[10] Zamfirescu, T.: Extreme points of the distance function on convex surfaces. Trans. Am. Math. Soc. **350**, 1395–1406 (1998)

[11] Zamfirescu, T.: On the critical points of a Riemannian surface. Adv. Geom. **6**, 493–500 (2006)

Tudor Zamfirescu
Fachbereich Mathematik
Universität Dortmund
44221 Dortmund
Germany

and

Institute of Mathematics “Simion Stoilow”
Roumanian Academy
Bucharest
Roumania

and

College of Mathematics and Information Science
Hebei Normal University
Shijiazhuang 050024
People’s Republic of China
e-mail: tudor.zamfirescu@mathematik.uni-dortmund.de

Received: January 16, 2017.

Accepted: December 11, 2017.