Acta Math. Hungar.
https://doi.org/10.1007/s10474-018-0823-x

THE KATCHALSKI–LEWIS TRANSVERSAL PROBLEM FOR REGULAR POLYGONS

Q. DU[1,2], L. YUAN[1,2] and T. ZAMFIRESCU[1,3,4,*]

[1]College of Mathematics and Information Science, Hebei Normal University, 050024 Shijiazhuang, P.R. China

[2]Hebei Key Laboratory of Computational Mathematics and Applications, 050024 Shijiazhuang, P.R. China
e-mails: duqingdan@126.com, lpyuan@hebtu.edu.cn

[3]Fachbereich Mathematik, Technische Universität Dortmund, 44221 Dortmund, Germany

[4]Institute of Mathematics "Simion Stoilow", Roumanian Academy, Bucharest, Romania
e-mail: tuzamfirescu@gmail.com

(Received October 8, 2017; accepted March 7, 2018)

Dedicated to Tibor Bisztriczky, Gábor Fejes Tóth, and Endre Makai on the occasion of their 70th birthday

Abstract. If every k-membered subfamily of a family of plane convex bodies has a line transversal, then we say that this family has property $T(k)$. We say that a family $\mathcal{F}$ has property $T-m$, if there exists a subfamily $\mathcal{G} \subset \mathcal{F}$ with $|\mathcal{F}-\mathcal{G}| \leq m$ admitting a line transversal. Heppes [7] posed the problem whether there exists a convex body K in the plane such that if $\mathcal{F}$ is a finite $T(3)$-family of disjoint translates of K, then $m=3$ is the smallest value for which $\mathcal{F}$ has property $T-m$. In this paper, we study this open problem in terms of finite $T(3)$-families of pairwise disjoint translates of a regular $2n$-gon ($n \geq 5$). We find out that, for $5 \leq n \leq 34$, the family has property $T-3$; for $n \geq 35$, the family has property $T-2$.

1. Introduction

This paper deals with a problem of Heppes on transversal properties of disjoint translates which was motivated by an old (disproved) conjecture of

* Corresponding author.

The authors gratefully acknowledge financial support by National Natural Science Foundation of China (No. 11471095), Outstanding Youth Science Foundation of Hebei Province (No. 2013205189), Program for Excellent Talents in University, Hebei Province (No. GCC2014043) and Key Project of the Education Department of Hebei province (No. zd2017043).

Key words and phrases: $2n$-gon, line transversal, translate.

Mathematics Subject Classification: 52A35.

0236-5294/$20.00

Katchalski and Lewis. The problem asks about the existence of a planar convex body K with the following properties.

(1) If $\mathcal{F}$ is a family of disjoint translates of K in which every three members admit a line transversal, then there is a line that intersects all but at most 3 members of $\mathcal{F}$.

(2) There exists a family $\mathcal{F}$ of disjoint translates of K in which any three members admit a line transversal, but any line misses at least 3 members of $\mathcal{F}$.

If K exists, we call such a body a *Heppes body*.

Heppes' problem is natural for the following reasons: On the one hand, in a series of papers, Heppes showed that the unit disk misses Property (2), while Holmsen showed that the parallelogram fails enjoying Property (1).

In this paper we make some progress on Heppes' problem. We consider the case when the convex set K is the regular $2n$-gon and show that

(A) If $n \geq 35$, then K does not satisfy Property (2).

(B) If $5 \leq n \leq 34$, then K does satisfy Property (1).

The result (A) definitely rules out the regular $2n$-gon K as a Heppes body, for sufficiently large n, while the result (B) states that for n between 5 and 34, K might possibly be a Heppes body, it is a Heppes body candidate. Also, (A) can be considered as a strengthening of the result of Heppes, as the regular $2n$-gon tends to the disk as n tends to infinity.

Our general approach is very reminiscent of the work by Heppes, who showed that the unit disk fails to enjoy Property (2).

A *line transversal* to a family of convex bodies is a (straight) line having a non-empty intersection with every member of the family. We also say that this family has property T. A family $\mathcal{F}$ of at least k convex bodies is a $T(k)$-family if any subfamily of $\mathcal{F}$ with k members has property T. Alternatively, we also say that this family has property $T(k)$.

We say that a family $\mathcal{F}$ has property $T - m$ if there exists a subfamily $\mathcal{G} \subset \mathcal{F}$ with $|\mathcal{F} - \mathcal{G}| \leq m$ enjoying property T.

Over the years, considerable effort has been devoted to finding some conditions on the finite family $\mathcal{F}$ of disjoint translates of a convex body such that, for some integer n, $T(n)$ would imply T.

Danzer [2] proved that every finite $T(5)$-family of disjoint congruent discs in the plane has property T. Grünbaum [4] proved that every finite $T(5)$-family of disjoint translates of a rectangle has property T. Based on the above results, Grünbaum [4] formulated the conjecture below concerning families of pairwise disjoint translates of a convex body.

Conjecture 1. *Let $\mathcal{F}$ be a finite family of pairwise disjoint translates of a convex body K. If $\mathcal{F}$ has property $T(5)$, then it has property T.*

Katchalski [10] proved that when the above family has property $T(128)$, this family has property T. Tverberg [12] gave the proof for Conjecture 1.

Tverberg's result is sharp in the sense that 5 cannot be replaced by 4. However, if the family has only property $T(3)$, then there exist interesting consequences. Katchalski and Lewis [11] proved the following.

Theorem 1. *There exists a constant $m \in \mathbb{Z}^+$ such that every finite $T(3)$-family of disjoint translates of a convex body in the plane has property $T - m$.*

Katchalski and Lewis [11] proved that $m < 192\pi$, and conjectured that $m = 2$. Afterwards many authors worked on improving the upper bound for m or confirming the smallest value of m for a given convex body, that is, dealing with the so-called *Katchalski–Lewis transversal problem*.

Due to the standard reductions described by Tverberg [12], we may assume that K is a centrally symmetric convex body. For families of discs, Heppes [6] proved the following.

Theorem 2. *If $\mathcal{F}$ is a $T(3)$-family of $n > 5$ disjoint congruent discs, then $\mathcal{F}$ has property $T - 2$.*

On the other hand, by a construction of Bezdek [1], we have the following lower bound.

Theorem 3. *To every $n > 5$, there exists a $T(3)$-family of n disjoint congruent discs without property $T - 1$.*

Theorems 2 and 3 imply that $m = 2$ is the smallest value such that every finite $T(3)$-family of disjoint congruent discs has property $T - m$.

For families of squares, Holmsen [7,8] found out the following.

Theorem 4. *The smallest value of m, for which every finite $T(3)$-family of disjoint translates of a square has property $T - m$, is 4.*

Theorems 2–4 together show that, while the smallest possible number m does not depend on the size of the finite family, it does depend on the shape of the convex body. Since in a sense the discs are the roundest and the squares the least round centrally symmetric convex bodies, Heppes [6] posed the problem whether there exists a centrally symmetric convex body K in the plane such that every finite $T(3)$-family of disjoint translates of K has property $T - 3$, and moreover, $m = 3$ is the smallest value for which the family has property $T - m$.

In the present paper, we probe into the above open problem and determine an upper bound for the Katchalski–Lewis transversal problem about finite $T(3)$-families of disjoint translates of a regular $2n$-gon.

Theorem 5. *Let $\mathcal{F}$ be a finite $T(3)$-family of disjoint translates of a regular $2n$-gon, where $n \geq 5$. The following statements hold:*

(1) *$\mathcal{F}$ has property $T - 3$ for $5 \leq n \leq 34$;*

(2) *$\mathcal{F}$ has property $T - 2$ for $n \geq 35$.*

2. Notation

Throughout this paper the convex body $R_n(a)$ will be a regular $2n$-gon in $\mathbb{R}^2$ centered at a, the circumcircle of which has diameter 1. Put $U_n(a) = 2R_n(\mathbf{0}) + a$, where $\mathbf{0}$ is the coordinate origin. Denote by $C(a)$ the circumcircle of $R_n(a)$, then put $D(a) = \operatorname{conv} C(a)$, $U(a) = 2D(\mathbf{0}) + a$. Denote by $C^\circ(a)$ the circle inscribed in $R_n(a)$, then put $D^\circ(a) = \operatorname{conv} C^\circ(a)$, $U^\circ(a) = 2D^\circ(\mathbf{0}) + a$.

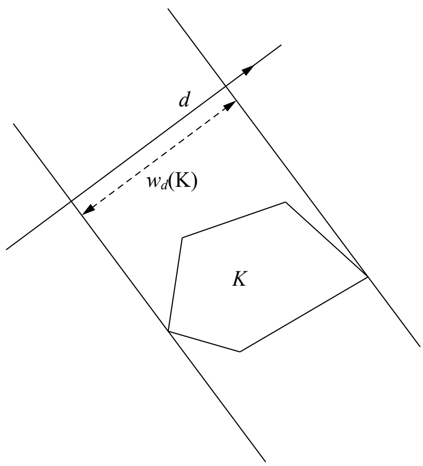

Fig. 1: Width of K in the direction d

The *width* $w_d(K)$ of a convex body K in the direction d is the minimum distance between the two parallel supporting lines of K that are perpendicular to d and contain K between them (see Fig. 1). The minimum of $w_d(K)$ taken over all possible directions d is the *width* of the convex body K. The R_n-*width* of K in the direction d is $w_d^{R_n}(K) = w_d(K)/w_d(R_n)$, where $R_n = R_n(\mathbf{0})$.

A *transversal strip* of a family $\mathcal{F}$ of discs with diameter 1 is a closed parallel strip intersecting all members of $\mathcal{F}$. Let $w(\mathcal{F}) > 0$ denote the width of a narrowest transversal strip S of the family $\mathcal{F}$. Then $w(\mathcal{F}) + 1, w(\mathcal{F}) + 2$ are the smallest widths of a strip parallel to S covering all centers, respectively all discs.

Let $K(\mathbf{0})$ be a centrally symmetric convex body centered at the origin $\mathbf{0}$, and $K(a) = K(\mathbf{0}) + a, K(b) = K(\mathbf{0}) + b$ be two disjoint translates of $K(\mathbf{0})$. The *sheaf* of $K(a)$ and $K(b)$ is the union of all common transversal lines of $\{K(a), K(b)\}$. This is a simply connected unbounded domain, the boundary of which lies in the union of the four common supporting lines of $K(a)$ and $K(b)$. It is denoted by $\Sigma_K(a, b)$ (see Fig. 2). The locus

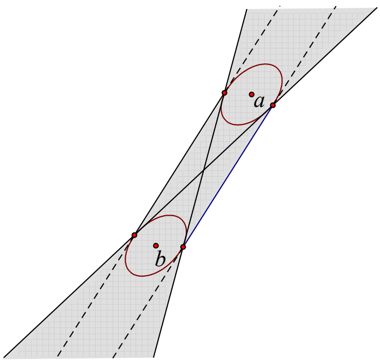

Fig. 2: $\Sigma_K(a,b)$

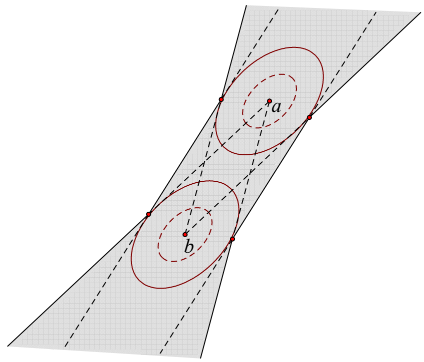

Fig. 3: $\Sigma^c_K(a,b)$

of the centers of the translates of $K(\mathbf{0})$ which have non-empty intersection with the sheaf $\Sigma_K(a,b)$ is called the *center sheaf* belonging to a and b, which is denoted by $\Sigma^c_K(a,b)$. Two of the (at most) six lines generating the boundary of the center sheaf are non-separating supporting lines to both convex bodies $K'(a)$ and $K'(b)$, where $K'(a)=2K(\mathbf{0})+a, K'(b)=2K(\mathbf{0})+b$; the other four are supporting lines of one of the two enlarged convex bodies, passing through the center of the other one (see Fig. 3). Denote by $\Sigma^c_K(p_1p_2,q_1q_2)=\bigcup_{a\in p_1p_2, b\in q_1q_2}\Sigma^c_K(a,b)$ the *generalized center sheaf* belonging to p_1p_2 and q_1q_2, where p_1p_2 and q_1q_2 denote the line-segments on the x- and y-axis connecting p_1 and p_2, q_1 and q_2, respectively.

For $a,b\in\mathbb{R}^2$, ab denotes the line-segment from a to b, $\overline{ab}$ is used for the line through the points a and b, and $d(a,b)=\|a-b\|$ for the Euclidean distance between a and b. For $p\in\mathbb{R}^2$, we write $p=(x_p,y_p)$.

3. Proof of Theorem 5

In order to prove Theorem 5, we shall make use of the following lemma in [3].

LEMMA 1. *Every finite $T(3)$-family of congruent discs with diameter 1 admits a transversal strip of width less than 0.67, if the distance between the centers of any two members of the family is greater than 0.95.*

With every member of the finite family $\mathcal{F}$ of pairwise disjoint translates of the regular $2n$-gon $R_n(\mathbf{0})$, we can associate a circumscribed disc of diameter 1. In this way, we obtain a family $\mathcal{F}'$ of discs, and $\mathcal{F}'$ inherits property $T(3)$ from $\mathcal{F}$. Of course, the centers of the discs are the same as the centers of the $2n$-gons.

Due to the disjointness condition of $\mathcal{F}$, the distance between the centers of any two members of $\mathcal{F}$ is at least $\cos\frac{\pi}{2n}>0.95$ for $n\geq 5$. By Lemma 1, the width of the narrowest transversal strip of $\mathcal{F}'$ is less than 0.67, hence there exists a narrowest strip S^c covering all centers of discs in $\mathcal{F}'$, of width $w < 1.67$. We suppose that S^c is horizontal.

We can assume that $\mathcal{F}$ satisfies the following conditions (see [5], [6]).

(i) no three $2n$-gons have a common supporting line;

(ii) no three centers are the vertices of a right triangle;

(iii) no pair of $2n$-gons have a common supporting line at angle τ or $-\tau$ to the x-axis, where

$$(*) \qquad \tau = \begin{cases} \arccos(\frac{1}{2.4} \cdot \cos\frac{\pi}{10}) & \text{for } 5 \le n \le 34, \\ \arccos(\frac{1}{2.34} \cdot \cos\frac{\pi}{70}) & \text{for } n \ge 35. \end{cases}$$

Because S^c is narrowest, there exist three basic centers of $\mathcal{F}'$(i.e. $\mathcal{F}$) on the boundary of the strip S^c such that two centers b and c lie on one of the boundary lines and are strictly separated from each other by the vertical line through the third center a, lying on the other boundary line of S^c. In a canonically chosen coordinate system, these three points are $a(0,\alpha)$, $b(\beta,0)$ and $c(\gamma,0)$, $\alpha = -w < 0$, $\beta > 0$, $\gamma < 0$.

In the following, let the regular $2n$-gon $R_n(\mathbf{0})$ centered at the origin $\mathbf{0}$ be a member of the family $\mathcal{F}$. The vertices of $R_n(\mathbf{0})$ are labeled o_0, o_1, $\ldots$, o_{2n-1} in clockwise order from the positive y-axis (see Fig. 4). Let θ denote the angle of the positive y-axis and $\overline{\mathbf{0}o_0}$, thus $\theta \in [0, \frac{\pi}{n})$. For any $2n$-gon $R_n(m) = R_n(\mathbf{0}) + m \in \mathcal{F}$, the vertices of $R_n(m)$ corresponding to the vertices of $R_n(\mathbf{0})$ are labeled m_0, m_1, $\ldots$, m_{2n-1}, respectively.

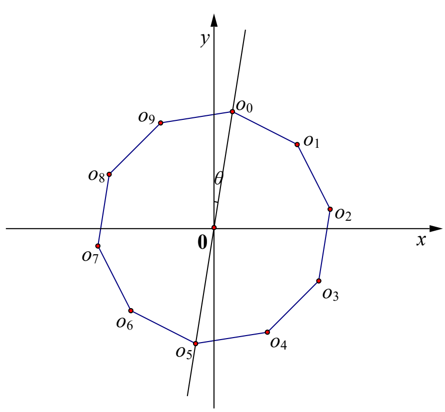

Fig. 4: A regular 10*-gon* $R_5(\mathbf{0})$

The width of $R_n(\mathbf{0})$ in the vertical direction v is:

$$w_v(R_n(\mathbf{0})) = \begin{cases} \cos\theta, & \theta \in [0, \frac{\pi}{2n}], \\ \cos(\frac{\pi}{n} - \theta), & \theta \in (\frac{\pi}{2n}, \frac{\pi}{n}). \end{cases}$$

We only need to prove that Theorem 5 holds for $w > w_v(R_n(\mathbf{0}))$. In fact, for $w \le w_v(R_n(\mathbf{0}))$, the family $\mathcal{F}$ has a common horizontal transversal line, so Theorem 5 holds. By computer simulation in Maple, the interval $(w_v(R_n(\mathbf{0})), 1.67)$ is divided, to prove Theorem 5, as $(w_v(R_n(\mathbf{0})), \varpi)$ $\cup\, [\varpi, 1.67)$, where $\varpi = 1.2$ if $5 \le n \le 34$, $\varpi = 1.17$ if $n \ge 35$.

A transversal line of all but 3 (or 2) members of $\mathcal{F}$ for $5 \le n \le 34$ ($n \ge 35$) is called a *candidate transversal line.* The $2n$-gons not met by the candidate transversal line are called *exceptional* $2n$*-gons* and their centers *exceptional centers.*

3.1. Proof of Theorem 5 for $w \in [\varpi, 1.67)$. Let ζ be the length of the intersection of $R_n(\mathbf{0})$ with the x-axis. We have

(1) if $n \equiv 0 \pmod 2$, then

$$\zeta = \frac{\cos\frac{\pi}{2n}}{\cos(\frac{\pi}{2n} - \theta)}, \quad \text{where } \theta \in \left[0, \frac{\pi}{n}\right);$$

(2) if $n \equiv 1 \pmod 2$, then

$$\zeta = \begin{cases} \frac{\cos\frac{\pi}{2n}}{\cos\theta}, & \theta \in [0, \frac{\pi}{2n}], \\ \frac{\cos\frac{\pi}{2n}}{\cos(\frac{\pi}{n} - \theta)}, & \theta \in (\frac{\pi}{2n}, \frac{\pi}{n}). \end{cases}$$

The diameter of the inscribed disc $D^\circ(\mathbf{0})$ of $R_n(\mathbf{0})$ is $\iota = \cos\frac{\pi}{2n}$.

Assume $\beta \le |\gamma|$. The disjointness hypothesis on the $2n$-gons implies $\beta - \gamma > \zeta \ge \iota$. Since $\beta \le -\gamma$, we obtain that $\gamma \le -\frac{\zeta}{2} \le -\frac{\iota}{2}$.

In the following, we show that, for $n = 5, 6$, the line $y = \alpha + \frac{1}{2} w_v(R_n(\mathbf{0}))$, which is the upper horizontal supporting line of $R_n(a)$, will be our candidate transversal line for $\mathcal{F}$; the strip bounded by the x-axis and $y = \alpha + w_v(R_n(\mathbf{0}))$ will be denoted by S^*; for $n \ge 7$, the line $y = \alpha + \frac{\iota}{2}$, which is the upper horizontal supporting line of $D^\circ(a)$, will be our candidate transversal line for $\mathcal{F}$; the strip bounded by the x-axis and $y = \alpha + \iota$ will also be denoted by S^*. Clearly, all exceptional centers are in the strip S^*; w_0 denotes the width of S^*.

Let $\rho = \sqrt{\frac{\alpha^2}{4\alpha^2 - 1}}$. Then $p = (\rho, 0)$ and $-p = (-\rho, 0)$ are two points such that the line parallel to $\overline{ap}$ and passing through the origin $\mathbf{0}$ is tangent to $D(p), D(-p)$ and $D(a)$. Hence, it can be assumed that $\beta \le \rho$. Otherwise, $\beta > \rho$ would imply $\gamma < -\rho$, so $D(p)$, $D(-p)$ and $D(a)$, as well as $R_n(p)$,

$R_n(-p)$ and $R_n(a)$, would have no common transversal line. This, together with the assumption $\beta \le |\gamma|$, implies that $R_n(a)$, $R_n(b)$ and $R_n(c)$ have no common transversal line, in contradiction to property $T(3)$.

Clearly, ρ is an increasing function of α which attains its maximum value in each interval at the largest value of α. Thus,

$$\beta \le \rho \le \rho_0 = \begin{cases} \frac{6}{\sqrt{119}}, & 5 \le n \le 34, \\ \frac{1.17}{\sqrt{4\cdot 1.17^2-1}}, & n \ge 35. \end{cases}$$

holds in the whole interval $\alpha \in (-1.67, -\varpi]$.

Now, we introduce the following notation. Let $\lambda^l(K_1, K_2)$ and $\lambda^r(K_1, K_2)$ denote the common non-horizontal non-separating supporting lines of the convex bodies K_1 and K_2 on the left and on the right, respectively.

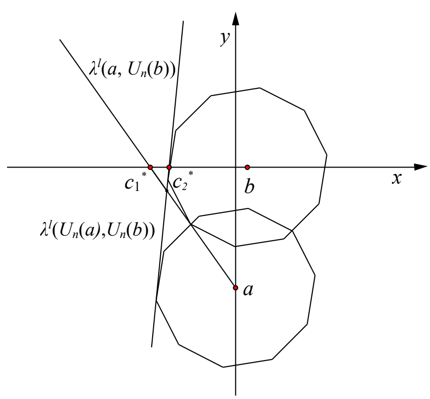

Fig. 5: c_1^* *and* c_2^*

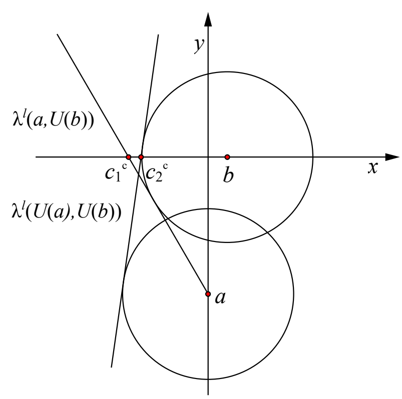

Fig. 6: c_1^c *and* c_2^c

For given a and b, let $c^*(\gamma^*, 0)$ denote the leftmost point of the center sheaf $\Sigma^c_{R_n}(a, b)$ on the x-axis. This point is determined by $\lambda^l(a, U_n(b))$ or $\lambda^l(U_n(a), U_n(b))$. Let $c_1^*(\gamma_1^*(\alpha, \beta, \theta), 0)$ be the intersection point of $\lambda^l(a, U_n(b))$ with the x-axis, and $c_2^*(\gamma_2^*(\alpha, \beta, \theta), 0)$ the intersection point of $\lambda^l(U_n(a), U_n(b))$ with the x-axis (see Fig. 5).

Let

$$i_{\lambda^l(a,U_n(b))} = \frac{\frac{3\pi}{2} - \arctan\frac{\alpha+\beta\sqrt{\alpha^2+\beta^2-1}}{\alpha\sqrt{\alpha^2+\beta^2-1}-\beta} - \theta}{\frac{\pi}{n}}.$$

If

$$\frac{\beta + \sin\left(\theta + \frac{\pi\lfloor i_{\lambda^l(a,U_n(b))}\rfloor}{n}\right)}{\cos\left(\theta + \frac{\pi\lfloor i_{\lambda^l(a,U_n(b))}\rfloor}{n}\right) - \alpha} - \frac{\beta + \sin\left(\theta + \frac{\pi\lceil i_{\lambda^l(a,U_n(b))}\rceil}{n}\right)}{\cos\left(\theta + \frac{\pi\lceil i_{\lambda^l(a,U_n(b))}\rceil}{n}\right) - \alpha} \le 0,$$

then

$$\gamma_1^*(\alpha,\beta,\theta) = \frac{-\alpha\big(\beta+\sin\big(\theta+\frac{\pi\lfloor i_{\lambda^l(a,U_n(b))}\rfloor}{n}\big)\big)}{\cos\big(\theta+\frac{\pi\lfloor i_{\lambda^l(a,U_n(b))}\rfloor}{n}\big)-\alpha};$$

otherwise,

$$\gamma_1^*(\alpha,\beta,\theta) = \frac{-\alpha(\beta+\sin(\theta+\frac{\pi\lceil i_{\lambda^l(a,U_n(b))}\rceil}{n}))}{\cos\big(\theta+\frac{\pi\lceil i_{\lambda^l(a,U_n(b))}\rceil}{n}\big)-\alpha}.$$

Set

$$i_{\lambda^l(U_n(a),U_n(b))} = \frac{\frac{3\pi}{2}-\arctan\frac{\beta}{\alpha}-\theta}{\frac{\pi}{n}}.$$

If $i_{\lambda^l(U_n(a),U_n(b))} - \lfloor i_{\lambda^l(U_n(a),U_n(b))}\rfloor - 0.5 \le 0$, then

$$\gamma_2^*(\alpha,\beta,\theta) = \frac{\beta}{\alpha}\cos\Big(\theta+\frac{\pi}{n}\big\lfloor i_{\lambda^l(U_n(a),U_n(b))}\big\rfloor\Big)+\beta+\sin\Big(\theta+\frac{\pi}{n}\big\lfloor i_{\lambda^l(U_n(a),U_n(b))}\big\rfloor\Big);$$

otherwise,

$$\gamma_2^*(\alpha,\beta,\theta) = \frac{\beta}{\alpha}\cos\Big(\theta+\frac{\pi}{n}\big\lceil i_{\lambda^l(U_n(a),U_n(b))}\big\rceil\Big)+\beta+\sin\Big(\theta+\frac{\pi}{n}\big\lceil i_{\lambda^l(U_n(a),U_n(b))}\big\rceil\Big).$$

Now we consider the circumscribed discs $U(a)$, $U(b)$ of $U_n(a)$, $U_n(b)$. Let $c^{\mathrm{c}}(\gamma^{\mathrm{c}},0)$ denote the leftmost point of the center sheaf $\Sigma_D^{\mathrm{c}}(a,b)$ on the x-axis. This point is determined by $\lambda^l(a,U(b))$ or $\lambda^l(U(a),U(b))$. Let $c_1^{\mathrm{c}}(\gamma_1^{\mathrm{c}}(\alpha,\beta),0)$ ($c_2^{\mathrm{c}}(\gamma_2^{\mathrm{c}}(\alpha,\beta),0)$) be the point of intersection of $\lambda^l(a,U(b))$ ($\lambda^l(U(a),U(b))$) with the x-axis (see Fig. 6). So we have $\gamma_1^{\mathrm{c}} \le \gamma_1^*$, $\gamma_2^{\mathrm{c}} \le \gamma_2^*$, and

$$\gamma_1^{\mathrm{c}}(\alpha,\beta) = \frac{-\alpha(\sqrt{\alpha^2+\beta^2-1}+\alpha\beta)}{1-\alpha^2}, \quad \gamma_2^{\mathrm{c}}(\alpha,\beta) = \frac{\sqrt{\alpha^2+\beta^2}}{\alpha}+\beta.$$

As the distance between any two centers of $\mathcal{F}$ is at least $\cos\frac{\pi}{2n}$, the length of the projection on the x-axis of the line-segment connecting any two centers in S^* is at least $\xi = \sqrt{\cos^2\frac{\pi}{2n}-w_0^2}$. Furthermore, we have $d(b,c_1^*) = \beta-\gamma_1^*$, $d(b,c_2^*) = \beta-\gamma_2^*$, $d(b,c_1^{\mathrm{c}}) = \beta-\gamma_1^{\mathrm{c}}$, and $d(b,c_2^{\mathrm{c}}) = \beta-\gamma_2^{\mathrm{c}}$. So $d(b,c_1^{\mathrm{c}}) \ge d(b,c_1^*)$, $d(b,c_2^{\mathrm{c}}) \ge d(b,c_2^*)$, and the length of bc is at most $\max\{d(b,c_1^*),d(b,c_2^*)\} \le \max\{d(b,c_1^{\mathrm{c}}),d(b,c_2^{\mathrm{c}})\}$.

Proposition 6. *For any* $\alpha\in(-1.67,-\varpi]$, $\beta\in(0,\rho_0]$, *we have the following conclusions*:
(1) *if* $n=5$, *then* $d(b,c_1^*)<3\xi$, $d(b,c_2^*)<3\xi$;
(2) *if* $6\le n\le 34$, *then* $d(b,c_1^{\mathrm{c}})<3\xi$, $d(b,c_2^{\mathrm{c}})<3\xi$;
(3) *if* $n\ge 35$, *then* $d(b,c_1^{\mathrm{c}})<2\xi$, $d(b,c_2^{\mathrm{c}})<2\xi$.

Proof. (1) If $n=5$, then for all vertices of $U_5(a), U_5(b)$ which might be touching points of $\lambda^l(a, U_5(b))$ and $\lambda^l(U_5(a), U_5(b))$, we have $d(b, c_1^*) < 3\xi$, $d(b, c_2^*) < 3\xi$.

(2) If $n=6$, then $d(b, c_1^c) - 3\xi$, $d(b, c_2^c) - 3\xi$ are increasing functions of θ for $\theta \in [0, \frac{\pi}{12})$, and they will reach the maximum values at $\theta = \frac{\pi}{12}$; $d(b, c_1^c) - 3\xi$, $d(b, c_2^c) - 3\xi$ are decreasing functions of θ for $\theta \in (\frac{\pi}{12}, \frac{\pi}{6})$, and they will attain the maximum values at $\theta = \frac{\pi}{12}$. Routine calculation shows that these maximum values are less than 0 for $\alpha \in (-1.67, -\varpi]$ and $\beta \in (0, \rho_0]$.

If $n \geq 7$, then $d(b, c_1^c) - 3\xi$, $d(b, c_1^c) - 2\xi$ are decreasing functions of β and n, and $d(b, c_2^c) - 3\xi$, $d(b, c_2^c) - 2\xi$ are decreasing functions of n, increasing functions of β. Hence, we have the following results:

If $7 \leq n \leq 34$, then $d(b, c_1^c) - 3\xi$ will attain the maximum value at $\beta = 0$, $n = 7$, and $d(b, c_2^c) - 3\xi$ will attain the maximum value at $\beta = \rho_0$, $n = 7$. One can verify that these maximum values are less than 0 for $\alpha \in (-1.67, -\varpi]$;

If $n \geq 35$, then $d(b, c_1^c) - 2\xi$ will attain the maximum value at $\beta = 0$, $n = 35$, and $d(b, c_2^c) - 2\xi$ will attain the maximum value at $\beta = \rho_0$, $n = 35$. Again, one can prove that these maximum values are negative for $\alpha \in (-1.67, -\varpi]$. □

Proposition 6 implies that $d(b, c) < 3\xi$ for $5 \leq n \leq 34$, and $d(b, c) < 2\xi$ for $n \geq 35$.

Corollary 1. *For $w \in [\varpi, 1.67)$, there exists at most one center for $5 \leq n \leq 34$ and no center for $n \geq 35$ in S^* between the vertical lines $x = \gamma$ and $x = \beta$.*

Proof. For $5 \leq n \leq 34$, we assume that there exist two centers m_1, m_2 in S^* between the vertical lines $x = \gamma$ and $x = \beta$, and the x-coordinate of m_2 is not less than that of m_1. The projections $m_1'c$, $m_2'm_1'$, bm_2' on the x-axis of the line-segments m_1c, m_2m_1, bm_2 have length at least ξ, so $d(b, c) = d(b, m_2') + d(m_2', m_1') + d(m_1', c) \geq 3\xi$, in contradiction with Proposition 6. Therefore, the first conclusion holds as claimed.

The proof for $n \geq 35$ is completely analogous. □

To complete the rest of the proof, the original intervals $\alpha \in (-1.67, -\varpi]$ and $\beta \in (0, \rho_0]$ will be cut into a few smaller pieces. It is assumed in the following that $\alpha \in [\alpha_2, \alpha_1]$ and $\beta \in [\beta_1, \beta_2]$ hold, where $[\alpha_2, \alpha_1]$ is one of the nine subintervals

$$[-1.67, -1.61],\ [-1.61, -1.5],\ [-1.5, -1.4],\ [-1.4, -1.3],\ [-1.3, -1.25],$$
$$[-1.25, -1.2],\ [-1.2, -1.18],\ [-1.18, -1.175],\ [-1.175, -1.17];$$

and $[\beta_1, \beta_2]$ is, independently, one of the three subintervals

$$[0, 0.04],\ [0.04, 0.2],\ [0.2, \rho_0].$$

Let $[\gamma_2, \gamma_1]$ denote the feasible interval of γ for $\alpha \in [\alpha_2, \alpha_1]$ and $\beta \in [\beta_1, \beta_2]$. The endpoints of the above mentioned subintervals will be denoted by a_2, a_1, b_1, b_2, c_2, c_1, respectively. This subdivision defines 18 ($5 \leq n \leq 34$) and 27 ($n \geq 35$) analogous subproblems.

The generalized center sheaf $\Sigma^c_{R_n}(a_2a_1, b_1b_2)$ defined by the segments a_2a_1 and b_1b_2 lies between two polygonal lines, where the left one consists of parts of the lines

$$\lambda^l(a_1, U_n(b_1)), \quad \lambda^l(U_n(a_1), U_n(b_1)), \quad \lambda^l(b_2, U_n(a_1)),$$

and the right one is consisting of parts of the lines

$$\lambda^r(a_1, U_n(b_2)), \quad \lambda^r(U_n(a_2), U_n(b_2)), \quad \lambda^r(b_1, U_n(a_1)).$$

Analogously, the boundaries of $\Sigma^c_{R_n}(a_2a_1, c_2c_1)$ are included in the union of

$$\lambda^l(a_1, U_n(c_2)), \quad \lambda^l(U_n(a_2), U_n(c_2)), \quad \lambda^l(c_1, U_n(a_1)),$$

and

$$\lambda^r(a_1, U_n(c_1)), \quad \lambda^r(U_n(a_1), U_n(c_1)), \quad \lambda^r(c_2, U_n(a_1)).$$

In the same way, we can get the boundaries of $\Sigma^c_D(a_2a_1, b_1b_2)$ and $\Sigma^c_D(a_2a_1, c_2c_1)$. In addition, $\Sigma^c_{R_n}(a_2a_1, b_1b_2) \subset \Sigma^c_D(a_2a_1, b_1b_2)$, $\Sigma^c_{R_n}(a_2a_1, c_2c_1) \subset \Sigma^c_D(a_2a_1, c_2c_1)$.

To every box $[\alpha_2, \alpha_1] \times [\beta_1, \beta_2]$, with $\alpha \in [\alpha_2, \alpha_1]$, $\beta \in [\beta_1, \beta_2]$, a feasible interval $[\gamma_2, \gamma_1]$ can be calculated for the x-coordinate of $c = (\gamma, 0)$ based on the condition $\beta \leq -\gamma$, the disjointness hypothesis and the $T(3)$-property of the family $\mathcal{F}$. Therefore, we have

(1) for $n = 5$, 6, $\gamma \leq \gamma_1 = \min\{\beta_2 - \zeta, -\frac{\zeta}{2}\}$, $\gamma \geq \gamma_2 = \min\{\gamma_1^*(\alpha_1, \beta_1, \theta), \gamma_2^*(\alpha_1, \beta_1, \theta)\}$;

(2) for $n \geq 7$, $\gamma \leq \gamma_1 = \min\{\beta_2 - \iota, -\frac{\iota}{2}\}$, $\gamma \geq \gamma_2 = \min\{\gamma_1^c(\alpha_1, \beta_1), \gamma_2^c(\alpha_1, \beta_1)\}$.

We consider (for any arbitrary subproblem) the intersection Q of the generalized center sheaves $\Sigma^c_{R_n}(a_2a_1, b_1b_2)$, $\Sigma^c_{R_n}(a_2a_1, c_2c_1)$ and the strip S^* of $\alpha = \alpha_2$. Clearly, the domain

$$Q = S^* \cap \Sigma^c_{R_n}(a_2a_1, b_1b_2) \cap \Sigma^c_{R_n}(a_2a_1, c_2c_1)$$

contains all exceptional centers, and

$$Q \subset S^* \cap \Sigma^c_D(a_2a_1, b_1b_2) \cap \Sigma^c_D(a_2a_1, c_2c_1) =: Q_1.$$

All centers but a, b, c must lie, by disjointness, outside of $U_n(a)$, $U_n(b)$ and $U_n(c)$, thus for any choice in the parameter boxes, these centers cannot lie in $U_n(a_2) \cap U_n(a_1)$, $U_n(b_1) \cap U_n(b_2)$ and $U_n(c_2) \cap U_n(c_1)$.

Proposition 7. (1) *For $n = 5, 6$, if $m \in Q$ is to the right of the line $x = \beta_1$, then $m \in U_n(b_1) \cap U_n(b_2)$; if $m \in Q$ is to the left of the line $x = \gamma_1$, then $m \in U_n(c_2) \cap U_n(c_1)$.*

(2) *For $n \geq 7$, if $m \in Q_1$ is to the right of the line $x = \beta_1$, then $m \in U^\circ(b_1) \cap U^\circ(b_2)$; if $m \in Q_1$ is to the left of the line $x = \gamma_1$, then $m \in U^\circ(c_2) \cap U^\circ(c_1)$.*

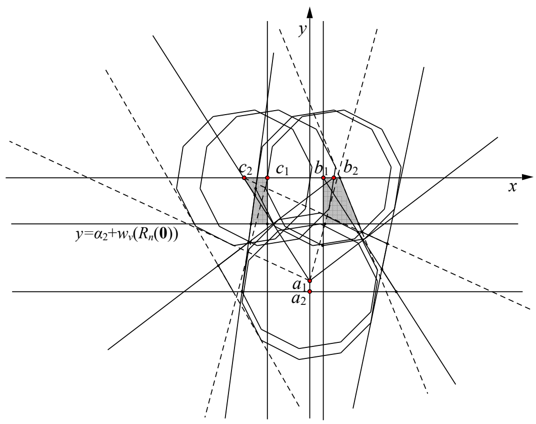

Fig. 7: Q^l and Q^r

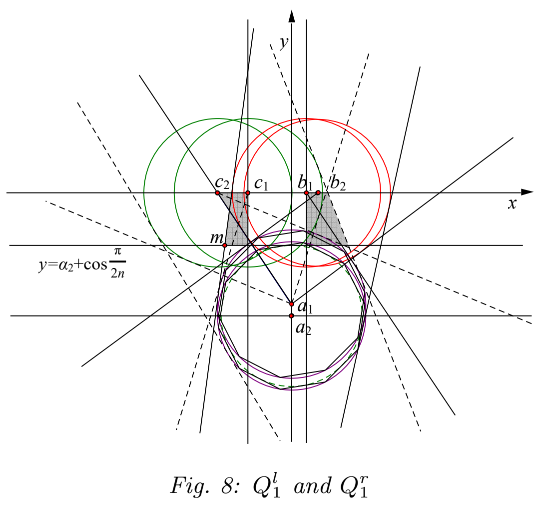

Fig. 8: Q_1^l and Q_1^r

Proof. Denote by Q^l the part of polygon Q to the left of the vertical line $x = \gamma_1$, and by Q^r the part of polygon Q to the right of $x = \beta_1$ (in Fig. 7 the domains Q^l and Q^r are shaded). Similarly, denote by Q_1^l and Q_1^r the part of polygon Q_1 to the left of $x=\gamma_1$ and to the right of $x=\beta_1$, respectively (in Fig. 8 the domains Q_1^l and Q_1^r are shaded). We can check that for $n=5,6$, all possible vertices of Q^l lie in $U_n(c_2)\cap U_n(c_1)$, and all possible vertices of Q^r lie in $U_n(b_1)\cap U_n(b_2)$. Hence, $Q^l\subset U_n(c_2)\cap U_n(c_1)$, $Q^r\subset U_n(b_1)\cap U_n(b_2)$. For $n\geq 7$, all possible vertices of Q_1^l lie in $U^\circ(c_2)\cap U^\circ(c_1)$, and all possible vertices of Q_1^r lie in $U^\circ(b_1)\cap U^\circ(b_2)$. Hence, $Q_1^l\subset U^\circ(c_2)\cap U^\circ(c_1)$, $Q_1^r\subset U^\circ(b_1)\cap U^\circ(b_2)$. Therefore, the claim of the proposition holds.

We illustrate our method of proving this proposition with the example of showing that the intersection point m of $\lambda^l(U(a_1),U(b_1))$ with the line $y=\alpha_2+\cos\frac{\pi}{2n}$ lies in $U^\circ(c_2)\cap U^\circ(c_1)$ for $n\geq 7$. We only prove the case of $\alpha\in[-1.67,-1.61]$, $\beta\in[0.04,0.2]$; the same conclusion can be obtained in a similar way for other boxes. At present, $\gamma_1=0.2-\iota$, $\gamma_2=\gamma_1^c(\alpha_1,\beta_1)$. The coordinates of m are

$$\Big(\beta_1+\frac{\sqrt{\alpha_1^2+\beta_1^2}}{\alpha_1}-\frac{\alpha_2\beta_1}{\alpha_1}-\frac{\beta_1}{\alpha_1}\cos\frac{\pi}{2n}\,,\ \alpha_2+\cos\frac{\pi}{2n}\Big).$$

Moreover, both $d^2(c_1,m)-\cos^2\frac{\pi}{2n}$ and $d^2(c_2,m)-\cos^2\frac{\pi}{2n}$ are decreasing functions of n, so they attain the maximum values at $n=7$, and these maximum values are negative. Hence, $d(c_1,m)<\cos\frac{\pi}{2n}$ and $d(c_2,m)<\cos\frac{\pi}{2n}$. So $m\in U^\circ(c_2)\cap U^\circ(c_1)$, as desired. □

COROLLARY 2. *For $w\in[\varpi,1.67)$, $n\geq 5$, there is no center in S^* to the left of the line $x=\gamma$ and no center in S^* to the right of the line $x=\beta$.*

Corollaries 1 and 2 imply Theorem 5 for $w\in[\varpi,1.67)$.

3.2. Proof of Theorem 5 for $w\in(w_v(R_n(0)),\varpi)$. In this part, the inequality $\beta\leq|\gamma|$ is not assumed. Rotate the x-axis around $\mathbf{0}$ by angles τ and $-\tau$, and obtain the lines λ^l and λ^r, respectively, where τ is the same as in $(*)$.

Let H be the convex hull of all centers of $\mathcal{F}$, and the vertices of H labelled by a_0, a_1,a_2, … in anticlockwise order from the center a. Clearly, $a_0=(0,\alpha)$. Then we have $y_0=-w$ and $y_i\geq -w$ for $i\geq 1$. For each a_i, the lines λ_i^l, λ_i^{l-}, λ_i^{sl}, λ_i^{cl} and λ_i^{dl} parallel to λ^l support $U_n(a_i),R_n(a_i)$, $U^\circ(a_i)$, $U(a_i)$ and $D^\circ(a_i)$ from above, respectively. Similarly, the lines λ_i^r, λ_i^{r-}, λ_i^{sr}, λ_i^{cr} and λ_i^{dr} parallel to λ^r support $U_n(a_i)$, $R_n(a_i)$, $U^\circ(a_i)$, $U(a_i)$ and $D^\circ(a_i)$ from above, respectively.

PROPOSITION 8. *For any $i\geq 0$, the intersection point of λ_i^{dl} and λ_i^{dr} lies on or above the x-axis.*

PROOF. For $i>0$, the intersection point of λ_i^{dl} and λ_i^{dr} lies above the intersection point of λ_0^{dl} and λ_0^{dr}. So we only need to show that the conclusion holds for $i=0$.

Let p be the intersection of the line

$$\lambda_0^{dl}: \ y-\alpha-\frac{1}{2}\cos\frac{\pi}{2n}\cos\tau-\tan\tau\Big(x+\frac{1}{2}\sin\tau\cos\frac{\pi}{2n}\Big)=0$$

and the line

$$\lambda_0^{dr}: \ y-\alpha-\frac{1}{2}\cos\frac{\pi}{2n}\cos\tau+\tan\tau\Big(x-\frac{1}{2}\sin\tau\cos\frac{\pi}{2n}\Big)=0.$$

If $5\le n\le 34$, we have $y_p=-1.2+\frac{1}{2}\tan\tau\sin\tau\cos\frac{\pi}{2n}+\frac{1}{2}\cos\tau\cos\frac{\pi}{2n}\ge 0$; if $n\ge 35$, then $y_p=-1.17+\frac{1}{2}\tan\tau\sin\tau\cos\frac{\pi}{2n}+\frac{1}{2}\cos\tau\cos\frac{\pi}{2n}\ge 0$. $\square$

By Proposition 8, the intersection point of λ_i^l and λ_i^r also lies above the x-axis for any $i\ge 0$. Consider a lower supporting line $\lambda(\delta)$ of H passing through vertices a_i and a_j ($j=i$ or $j=i+1$), where δ denotes the angle from the x-axis to $\lambda(\delta)$. (By our assumption (i), there are at most two centers on one supporting line.) Then the lines $\lambda^+(\delta)$, $\lambda^{++}(\delta)$, $\lambda^-(\delta)$ parallel to $\lambda(\delta)$ are the upper supporting lines of $R_n(a_i), U_n(a_i)$, and $U^\circ(a_i)$, respectively.

Now we consider the intersection of the closed half-plane $y\le 0$ and the above open half-plane bounded by $\lambda^{++}(\delta)$. It is cut into (at most) three parts by the lines λ_i^l and λ_j^r (see Fig. 9). The part strictly to the left of λ_i^l, the part strictly to the right of λ_j^r are denoted by $L(\delta)$, $R(\delta)$, respectively; the remaining part is denoted by $Q(\delta)$, as shown in Fig. 9. Denote by $n_\ell(\delta)$ and $n_r(\delta)$ the number of centers lying in $L(\delta)$ and $R(\delta)$, respectively.

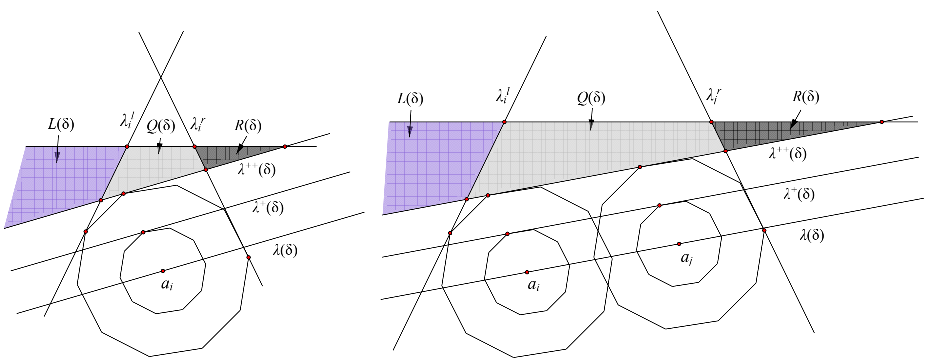

Fig. 9: $Q(\delta)$ for $j=i$ and for $j=i+1$

Here, we need the following simple (and well-known) fact.

LEMMA 2. *The disjoint sets K_1, K_2 and K_3 have no common transversal if and only if each of them can be strictly separated from the union of the other two sets.*

The next result and its corollary are essentially due to Kaiser [9].

Proposition 9. *For every* δ, *either* $n_\ell(\delta) = 0$ *or* $n_r(\delta) = 0$.

Proof. Suppose, on the contrary, that there exists δ such that a center g belongs to $L(\delta)$ and a center g' belongs to $R(\delta)$. Then $R_n(a_i)$ is weakly separated from $R_n(g) \cup R_n(g')$ by $\lambda^+(\delta)$, $R_n(g)$ is weakly separated from $R_n(a_i) \cup R_n(g')$ by λ_i^{l-}, and $R_n(g')$ is weakly separated from $R_n(a_i) \cup R_n(g)$ by λ_j^{r-}. Evidently, a sufficiently small upward translate of $\lambda^+(\delta)$ provides strict separation of $R_n(a_i)$ from the other two $2n$-gons. By assumption (iii), no $2n$-gon apart from $R_n(a_i)$ touches λ_i^{l-}, thus a sufficiently small translation of λ_i^{l-} to the left results in a line strictly separating $R_n(g)$ from the other two $2n$-gons. The case of $R_n(g')$ is analogous. Then, by Lemma 2, the $2n$-gons $R_n(a_i), R_n(g)$ and $R_n(g')$ do not enjoy the property $T(3)$, a contradiction. □

Clearly, $n_\ell(-\tau) = 0$ and $n_\ell(\delta)$ is a left-continuous increasing function of δ; $n_r(\tau) = 0$ and $n_r(\delta)$ is right-continuous and decreasing.

Corollary 3. *There exists* $\delta^* \in [-\tau, \tau]$ *such that*

$$n_\ell(\delta^*) = n_r(\delta^*) = 0.$$

From now on, $\lambda^+(\delta^*)$ will be the candidate transversal line, and the number of exceptional centers will be denoted again by n_{ex}. By Corollary 3, all exceptional centers lie in the domain $Q(\delta^*)$. The supporting line $\lambda(\delta^*)$ is called *a balanced supporting line* of H.

Proposition 10. *There exists a balanced supporting line of* H *passing through* a.

Proof. Suppose that none of the supporting lines through a is balanced. Then the angle δ^* between a balanced supporting line and the x-axis must be greater than $\delta_{0,1}$, the angle between $\overline{a_0 a_1}$ and x-axis. This balanced supporting line supports H at a_i for some $i \geq 1$. So, $n_\ell(\delta^*) = 0$. The domain lying above $\lambda^{++}(\delta^*)$ and to the left of λ_1^l has no center, and $n_\ell(\delta_{0,1}) = 0$ also holds. Since $\lambda(\delta_{0,1})$ is not a balanced supporting line, $n_r(\delta_{0,1}) \geq 1$.

Let p be the intersection point of λ_0^{sr} with the line $y = \varpi + \alpha$, δ_1 the angle formed by the x-axis and the other supporting line of $U^\circ(a_0)$ at the point m passing through p (see Fig. 10).

The coordinates of the point p are

$$(x_p, y_p) = \Big(\frac{\cos\frac{\pi}{2n} \cos\tau - \varpi}{\tan\tau} + \sin\tau \cos\frac{\pi}{2n} \,,\, \varpi + \alpha \Big).$$

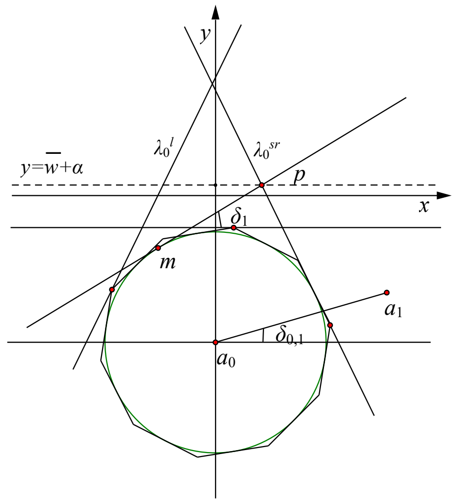

Fig. 10: Balance supporting line through a_0

The coordinates of the point m are

$$x_m = \frac{x_p \cos^2 \frac{\pi}{2n} - \cos \frac{\pi}{2n} \sqrt{(y_p - \alpha)^4 + x_p^2 (y_p - \alpha)^2 - (y_p - \alpha)^2 \cos^2 \frac{\pi}{2n}}}{(y_p - \alpha)^2 + x_p^2},$$

$$y_m = \alpha + \frac{\cos^2 \frac{\pi}{2n} - x_m x_p}{y_p - \alpha}.$$

Hence, $\delta_1 = \arctan \frac{y_m - y_p}{x_m - x_p}$. (Observe that δ_1 does not depend on α and is a decreasing function of n.)

Thus, $\delta_1 = 0.348\ldots < 0.35$ for $5 \le n \le 34$; $\delta_1 = 0.247\ldots < 0.25$ for $n \ge 35$.

Furthermore, if $\delta_{0,1} > \delta_1$, then the domain $R(\delta_{0,1})$ would be empty, which contradicts $n_r(\delta_{0,1}) \ge 1$. Hence $\delta_{0,1} \le \delta_1$. We have $x_1 > \cos \frac{\pi}{2n} \cos \delta_1$. Some calculation shows that the origin $\mathbf{0}$ lies strictly above the line λ_1^{cl}, and therefore above the line λ_1^l. Consequently, the domain lying above $\lambda^{++}(\delta^*)$ and to the left of λ_1^l contains a center c, which contradicts $n_\ell(\delta^*) = 0$. $\square$

By Proposition 10, the balance supporting line $\lambda(\delta^*)$ of H through a_0 either only passes through a_0 or passes through a_0 and a_1. In the following, we will prove that $\lambda^+(\delta^*)$ is a candidate transversal line for both cases.

Case 1: *The line $\lambda(\delta^*)$ only passes through a_0.* First we present the following lemma.

LEMMA 3. (1) *For $5 \le n \le 34$, the domain $Q(\delta^*)$ contains at most three centers of $\mathcal{F}$, i.e. $n_{\rm ex} \le 3$;*

(2) *For $n \geq 35$, the domain $Q(\delta^*)$ contains at most two centers of $\mathcal{F}$, i.e. $n_{\text{ex}} \leq 2$.*

PROOF. Let $k_{\lambda_0^l} = \frac{n(2\pi-\tau-\theta)}{\pi}$. If $k_{\lambda_0^l} - \lfloor k_{\lambda_0^l} \rfloor \geq 0.5$, then $k = \lceil k_{\lambda_0^l} \rceil$; otherwise $k = \lfloor k_{\lambda_0^l} \rfloor$.

Let $j_{\lambda_0^r} = \frac{n(\tau-\theta)}{\pi}$. If $j_{\lambda_0^r} - \lfloor j_{\lambda_0^r} \rfloor \geq 0.5$, then $j = \lceil j_{\lambda_0^r} \rceil$; otherwise $j = \lfloor j_{\lambda_0^r} \rfloor$.

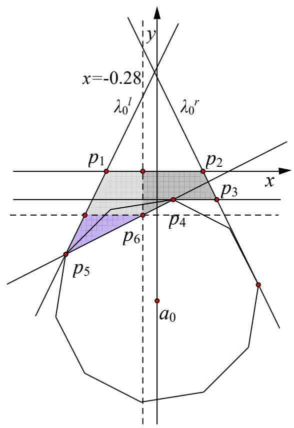

Fig. 11: $Q(\delta^*)$ contains at most three centers

(1) For $5 \leq n \leq 74$ and $w = \varpi$, let $p_1 = (x_{p_1}, 0)$, $p_2 = (x_{p_2}, 0)$ be the intersection points of the x-axis with λ_0^l, λ_0^r respectively; p_3 the intersection point of the line $y = y_0 + w_v(R_n(\mathbf{0}))$ and λ_0^r; p_4 the point where the line $y = y_0 + w_v(R_n(\mathbf{0}))$ touches $U_n(a_0)$; p_5 the intersection point of $U_n(a_0)$ and λ_0^l (see Fig. 11). We have

$$x_{p_1} = \frac{-y_0 - \cos(\theta + \frac{\pi k}{n})}{\tan\tau} + \sin\Big(\theta + \frac{\pi k}{n}\Big),$$

$$x_{p_2} = \frac{y_0 + \cos(\theta + \frac{\pi j}{n})}{\tan\tau} + \sin\Big(\theta + \frac{\pi j}{n}\Big),$$

$$x_{p_3} = \frac{\cos(\theta + \frac{\pi j}{n}) - w_v(R_n(\mathbf{0}))}{\tan\tau} + \sin\Big(\theta + \frac{\pi j}{n}\Big),$$

$$y_{p_3} = y_{p_4} = y_0 + w_v(R_n(\mathbf{0})),$$

$$x_{p_4} = \sin\theta, \text{ for } \theta \in \Big[0, \frac{\pi}{2n}\Big], \quad x_{p_4} = \sin\Big(\theta - \frac{\pi}{n}\Big), \text{ for } \theta \in \Big(\frac{\pi}{2n}, \frac{\pi}{n}\Big),$$

$$x_{p_5} = \sin\Big(\theta + \frac{\pi k}{n}\Big), \quad y_{p_5} = y_0 + \cos\Big(\theta + \frac{\pi k}{n}\Big).$$

Clearly, $Q(\delta^*) \subset p_1p_2p_3p_4p_5$.

For $5 \le n \le 34$, let p_6 be the intersection of $\overline{p_4p_5}$ and the line $x = -0.25$, p_7 be the orthogonal projection of p_1 on $\overline{p_4p_5}$. Elementary computations show that, for $n = 5, 6, 12$, the lines $\overline{p_1p_7}$ and $x = -0.25$ cut the domain $Q(\alpha^*)$ into three parts with diameter less than $\cos\frac{\pi}{2n}$; for $n \ge 7$ and $n \ne 12$, the lines $\overline{p_1p_7}$ and $x = -0.25$ cut the pentagon $p_1p_2p_3p_4p_5$ into three parts with diameter less than $\cos\frac{\pi}{2n}$, too.

Similarly, for $35 \le n \le 74$, the line $x = -0.31$ cuts this pentagon into two parts with diameter less than $\cos\frac{\pi}{2n}$.

If $w < \varpi$, then $Q(\delta^*)$ is included in $Q(\delta^*)$ obtained for $w = \varpi$. Thus, the claim of the lemma holds in this case as well.

(2) For $n \ge 75$ and $w = \varpi$, let $p_1 = (x_{p_1}, 0)$, $p_2 = (x_{p_2}, 0)$ be intersection points of the x-axis and λ_0^{cl}, λ_0^{cr} respectively; p_3 the intersection point of the line $y = y_0 + w_v(R_n(\mathbf{0}))$ and λ_0^{cr}; p_4 the point where the line $y = y_0 + w_v(R_n(\mathbf{0}))$ touches $U_n(a_0)$; p_5 the intersection point of $\overline{pp_4}$ and λ_0^{cl}. Let $p = (\sin(2\pi - \tau - \frac{\pi}{n}), y_0 + \cos(2\pi - \tau - \frac{\pi}{n}))$, see Fig. 12.

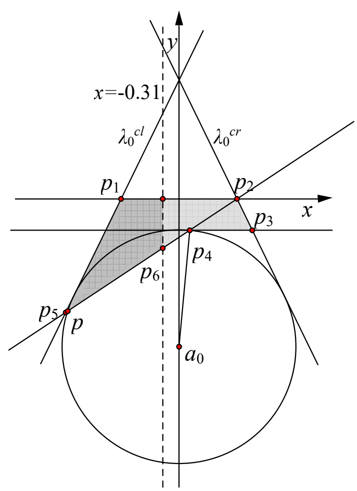

Fig. 12: $Q(\delta^*)$ contains at most two centers

We have

$$x_{p_1} = \frac{-y_0 - \cos\tau}{\tan\tau} - \sin\tau, \quad x_{p_2} = \frac{y_0 + \cos\tau}{\tan\tau} + \sin\tau,$$

$$x_{p_3} = \frac{-w_v(R_n(\mathbf{0})) + \cos\tau}{\tan\tau} + \sin\tau, \quad y_{p_3} = y_{p_4} = y_0 + w_v(R_n(\mathbf{0})),$$

$$x_{p_4} = \sin\theta, \text{ for } \theta \in \Big[0, \frac{\pi}{2n}\Big]; \quad x_{p_4} = \sin\Big(\theta - \frac{\pi}{n}\Big), \text{ for } \theta \in \Big(\frac{\pi}{2n}, \frac{\pi}{n}\Big).$$

It is obvious that $Q(\delta^*) \subset p_1p_2p_3p_4p_5$. Let p_6 be the intersection point of $\overline{p_4p_5}$ and the line $x = -0.31$. This pentagon $p_1p_2p_3p_4p_5$ is divided by $x = -0.31$ into two parts with diameter less than $\cos\frac{\pi}{2n}$. If $w < \varpi$, then $Q(\delta^*) \subset p_1p_2p_3p_4p_5$. Thus, the conclusion also holds. $\square$

Case 2: *The line* $\lambda(\delta^*)$ *passes through* a_0 *and* a_1. So $\delta^* = \delta_{0,1}$.

PROPOSITION 11. *If* $\delta_{0,1} \geq 0.35$, *then*
(1) $n_{\mathrm{ex}} \leq 3$ *for* $5 \leq n \leq 34$;
(2) $n_{\mathrm{ex}} \leq 2$ *for* $n \geq 35$.

PROOF. For $\delta_{0,1} \geq 0.35$, there is no center in the domain above the line $\lambda^{++}(\delta_{0,1})$, below the x-axis and to the right of λ_0^r. Hence, by Lemma 3, the statements hold. $\square$

Consequently, it will be assumed that $\delta_{0,1} < 0.35$. In the following, we show that Theorem 5 holds for all regular $2n$-gons ($n \geq 5$) when $w < 1.2$. Firstly, we prove the following proposition.

PROPOSITION 12. *The point* c *is the single exceptional center in the half plane* $x \leq 0$.

PROOF. First, we show that the diameter of the intersection of the half plane $x \leq 0$ and $Q(\delta_{0,1})$ is less than $\cos\frac{\pi}{2n}$ for any $\delta_{0,1} < 0.35 =: \eta$, $w \in (w_v(R_n(\mathbf{0})), 1.2)$ and $n \geq 5$.

Let l be the upper tangent line of $U^\circ(a_0)$, where the angle between the x-axis and l is η. So the equation of l is

$$y - \alpha - \cos\frac{\pi}{2n}\cos\eta - \tan\eta \cdot \left(x + \sin\eta\cos\frac{\pi}{2n}\right) = 0.$$

For $w = 1.2$, let p_1, p_2 be the intersection points of λ_0^{cl} with the x-axis, l, respectively, p_3 the intersection point of the line $y = y_0 + \cos\frac{\pi}{2n}$ and the y-axis (see Fig. 13). Therefore, for any $\delta_{0,1} < \eta$, $w \in (w_v(R_n(\mathbf{0})), 1.2)$, the part of $Q(\delta_{0,1})$ lying on or below the x-axis is included in the quadrilateral $\mathbf{0}p_1p_2p_3$.

The coordinates of p_2 are

$$x_{p_2} = \frac{\cos\frac{\pi}{2n}\cos\eta - \cos\tau + \tan\eta\sin\eta\cos\frac{\pi}{2n} - \tan\tau\sin\tau}{\tan\tau - \tan\eta},$$

$$y_{p_2} = \cos\eta - 1.2 + \tan\eta \cdot (x_{p_2} + \sin\eta).$$

By computing, we establish that the diameter of the quadrilateral $\mathbf{0}p_1p_2p_3$ is less than $\cos\frac{\pi}{2n}$. $\square$

Next, we show that the $x > 0$ part of $Q(\delta_{0,1})$ contains at most two exceptional centers for $5 \leq n \leq 34$, and at most one exceptional center for $n \geq 35$.

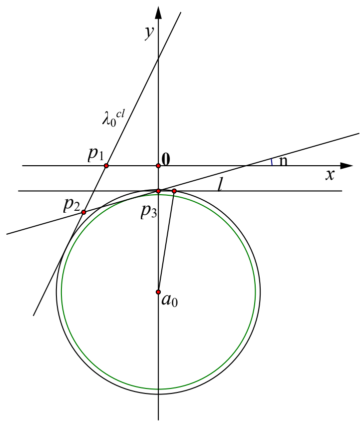

Fig. 13: Single exceptional center c

Suppose, on the contrary, that, for $5 \leq n \leq 34$, $Q(\delta_{0,1})$ contains three exceptional centers in the half plane $x > 0$, namely, $f_1 = (\mu_1, \nu_1)$, $f_2 = (\mu_2, \nu_2)$, $f_3 = (\mu_3, \nu_3)$, where $0 < \mu_1 < \mu_2 < \mu_3$; for $n \geq 35$, $Q(\delta_{0,1})$ contains two exceptional centers in the half plane $x > 0$, say, $g_1 = (\mu'_1, \nu'_1)$, $g_2 = (\mu'_2, \nu'_2)$, where $0 < \mu'_1 < \mu'_2$.

For $\theta \in [0, \frac{\pi}{2n}]$, $\mu_1 \leq \sin\theta$ and $\mu'_1 \leq \sin\theta$, we have $\nu_1 > y_0 + \frac{\cos\frac{\pi}{2n}}{\cos(\frac{\pi}{2n}-\theta)} =: -\kappa$, $\nu'_1 > y_0 + \frac{\cos\frac{\pi}{2n}}{\cos(\frac{\pi}{2n}-\theta)} = -\kappa$; for $\theta \in [0, \frac{\pi}{2n}]$, $\mu_1 > \sin\theta$ and $\mu'_1 > \sin\theta$, or $\theta \in (\frac{\pi}{2n}, \frac{\pi}{n})$, let $\kappa = -y_0 - w_v(R_n(\mathbf{0}))$. Hence, the disjointness hypothesis on $2n$-gons and Proposition 12 imply that

$$\mu_1 - \gamma \geq \xi_{\min}, \quad \mu_2 - \mu_1 \geq \xi_{\min}, \quad \mu_3 - \mu_2 \geq \xi_{\min},$$

where $\xi_{\min} = \min_{w\in(w_v(R_n(\mathbf{0})),1.2),n\in[5,34]} \sqrt{\cos^2\frac{\pi}{2n} - \kappa^2} > 0.9178$,

$$\mu'_1 - \gamma \geq \xi'_{\min}, \quad \mu'_2 - \mu'_1 \geq \xi'_{\min},$$

where $\xi'_{\min} = \min_{w\in(w_v(R_n(\mathbf{0})),1.2),n\geq 35} \sqrt{\cos^2\frac{\pi}{2n} - \kappa^2} > 0.9785$.

Proposition 13. *The inequalities* $\gamma > -0.31$, $\mu_1 > 0.6078$, $\mu_2 > 1.5256$, $\mu_3 > 2.4434$, $\mu'_1 > 0.6685$, $\mu'_2 > 1.647$ *hold.*

Proof. Since the family $\mathcal{F}$ has property $T(3)$, f_1, f_2, f_3, g_1, g_2 must lie in $\Sigma^c_{R_n}(a, c) \subset \Sigma^c_D(a, c)$. As a result, they must lie on or to the left of the lines $\lambda^r(a, U_n(c))$, $\lambda^r(U_n(a), U_n(c))$, $\lambda^r(U_n(a), c)$. Let m_1, m'_1 be the intersection points of the x-axis and $\lambda^r(a, U_n(c))$, $\lambda^r(a, U(c))$ respectively; let m_2, m'_2 be the intersection points of $\lambda^r(U_n(a), U_n(c))$, $\lambda^r(U(a), U(c))$ and the line $y = y_0 + w_v(R_n(\mathbf{0}))$, respectively.

We have

$$x_{m_1'} = \frac{-\alpha\big(\alpha - \gamma\sqrt{\alpha^2+\gamma^2-1}\big)}{\alpha\sqrt{\alpha^2+\gamma^2-1}+\gamma},$$

$$x_{m_2'} = -\frac{\gamma}{\alpha}\big(\alpha + w_v(R_n(\mathbf{0}))\big) + \gamma - \frac{\sqrt{\alpha^2+\gamma^2}}{\alpha}\,.$$

Let

$$i_{\lambda^r(a,R_n(c))} = \frac{\frac{\pi}{2} - \arctan\frac{\alpha-\gamma\sqrt{\alpha^2+\gamma^2-1}}{-\gamma-\alpha\sqrt{\alpha^2+\gamma^2-1}} - \theta}{\frac{\pi}{n}}\,.$$

If

$$\frac{\gamma+\sin\big(\theta+\frac{\pi}{n}\lfloor i_{\lambda^r(a,R_n(c))}\rfloor\big)}{\cos\big(\theta+\frac{\pi}{n}\lfloor i_{\lambda^r(a,R_n(c))}\rfloor\big)-\alpha} - \frac{\gamma+\sin\big(\theta+\frac{\pi}{n}\lceil i_{\lambda^r(a,R_n(c))}\rceil\big)}{\cos\big(\theta+\frac{\pi}{n}\lceil i_{\lambda^r(a,R_n(c))}\rceil\big)-\alpha} \ge 0,$$

then $i = \lfloor i_{\lambda^r(a,R_n(c))}\rfloor$; otherwise $i = \lceil i_{\lambda^r(a,R_n(c))}\rceil$. Hence,

$$x_{m_1} = \frac{-\alpha\big(\gamma+\sin(\theta+\frac{\pi}{n}i)\big)}{\cos(\theta+\frac{\pi}{n}i)-\alpha}\,.$$

Let

$$j_{\lambda^r(R_n(a),R_n(c))} = \frac{\frac{\pi}{2} - \arctan\frac{\gamma}{\alpha} - \theta}{\frac{\pi}{n}}\,.$$

If $j_{\lambda^r(R_n(a),R_n(c))} - \lfloor j_{\lambda^r(R_n(a),R_n(c))}\rfloor \ge 0.5$, then $j = \lceil j_{\lambda^r(R_n(a),R_n(c))}\rceil$; otherwise $j = \lfloor j_{\lambda^r(R_n(a),R_n(c))}\rfloor$. So,

$$x_{m_2} = -\frac{\gamma}{\alpha}\Big(w_v(R_n(\mathbf{0})) - \cos\Big(\theta+\frac{\pi}{n}j\Big)\Big) + \sin\Big(\theta+\frac{\pi}{n}j\Big).$$

For $-1.2 < \alpha < -w_v(R_n(\mathbf{0}))$, if $\gamma \le -0.31$, we have the following results.

(a) For $n = 5$, we have $x_{m_1}, x_{m_2} < 2\xi_{\min}$. Then $\mu_3 < 2\xi_{\min}$, a contradiction.

(b) For $6 \le n \le 34$, we have $x_{m_2'} < 0.9$, $x_{m_1'} - \gamma < 2.7$. Then $\mu_3 - \gamma < 2.7 < 3\xi_{\min}$, a contradiction.

(c) For $n \ge 35$, we have $x_{m_2'} < 0.9$, $x_{m_1'} - \gamma < 1.83$. Then $\mu_2' - \gamma < 1.83 < 2\xi'_{\min}$, a contradiction.

Hence, $\gamma > -0.31$, and the proof is complete. □

Since $\mu_1 > \sin\theta$, $\mu_1' > \sin\theta$, the centers f_1, f_2, f_3, g_1, g_2 are all above the line $y = y_0 + w_v(R_n(\mathbf{0}))$. Of course, they are all above the line $y = y_0 + \iota$, the upper horizontal supporting line of $U^\circ(a_0)$. On the other hand, the lower boundary line $\lambda^{++}(\delta_{0,1})$ of $Q(\delta_{0,1})$ runs below f_3, g_2, and so does the line

$\lambda^-(\delta_{0,1})$. Hence, the inequality $\arctan\frac{-y_0-\iota}{\mu_3} < 0.102$ holds for $5 \le n \le 34$; the inequality $\arctan\frac{-y_0-\iota}{\mu_2'} < 0.122$ holds for $n \ge 35$.

COROLLARY 4. *Let $\delta_{0,1}$ be the angle formed by the x-axis and the candidate transversal line $\lambda^+(\delta_{0,1})$.*

(1) *If* $5 \le n \le 34$, *then* $\delta_{0,1} \in [0, 0.102)$, *and the angle ϑ formed by the x-axis and $\overline{cf_3}$ satisfies $\vartheta \in (-0.102, 0]$;*

(2) *If $n \ge 35$, then $\delta_{0,1} \in [0, 0.122)$, and the angle ϑ' formed by the x-axis and $\overline{cg_2}$ satisfies $\vartheta' \in (-0.122, 0]$.*

PROOF. We only prove the first part of this corollary; the second part can be obtained in a similar way.

For $5 \le n \le 34$, evidently, the angle of the x-axis and $\lambda^-(\delta_{0,1})$ is also $\delta_{0,1}$. Denote by $p_1 = (x_{p_1}, 0)$, $p_2 = (0, y_{p_2})$ the intersection points of $\lambda^-(\delta_{0,1})$ and the x-axis, y-axis, respectively. We have $x_{p_1} \ge \mu_3$, $y_{p_2} \ge y_0 + \iota$, so $\delta_{0,1} = \arctan\frac{|y_{p_2}|}{x_{p_1}} \le \arctan\frac{-y_0-\iota}{\mu_3} < 0.102$.

Because f_3, g_2 are above the line $y = y_0 + \iota$,

$$\vartheta = -\arctan\frac{|\nu_3|}{\mu_3 - \gamma} > -\arctan\frac{-y_0 - \iota}{\mu_3} > -0.102. \qquad \square$$

In order to prove Proposition 14, we need some preparation first.

Let v_1, v_2 be the intersection points of $\lambda^l(U^\circ(c), U^\circ(f_3))$ with $\lambda^l(U(c), f_3)$, $\lambda^l(c, U(f_3))$, respectively (see Fig. 14). Let u_1 be the touching point of $\lambda^l(U_n(c), f_3)$ with $U_n(c)$; u_2 the touching point of $\lambda^l(c, U_n(f_3))$ with $U_n(f_3)$. The lines $\lambda^l(U_n(c), f_3)$ and $\lambda^l(c, U_n(f_3))$ intersect $\lambda^l(U_n(c), U_n(f_3))$ at s_1, s_2, respectively. Let s_3, s_4 be the orthogonal projections of c, f_3 on $\lambda^l(U_n(c), U_n(f_3))$ (see Fig. 15). Denote by m_1, m_2 the touching points of $\lambda^l(U^\circ(c), U^\circ(f_3))$ with $U^\circ(c)$ and $U^\circ(f_3)$, respectively.

For $5 \le n \le 34$, we have the following statements.

When the point f_3 moves along any line l_1, the label i of u_1 on $U_n(c)$ is determined by

$$i = \frac{\frac{\pi}{2} - \arctan\dfrac{\frac{\nu_3}{\mu_3-\gamma} - (\mu_3-\gamma)\sqrt{(\frac{\nu_3}{\mu_3-\gamma})^2 + 1 - \frac{1}{(\mu_3-\gamma)^2}}}{1 + \frac{\nu_3}{\mu_3-\gamma}(\mu_3-\gamma)\sqrt{(\frac{\nu_3}{\mu_3-\gamma})^2 + 1 - \frac{1}{(\mu_3-\gamma)^2}}} - \theta}{\frac{\pi}{n}}.$$

By Corollary 4, i is an increasing function of $(\mu_3 - \gamma)$. Therefore, it remains constant or increases, while f_3 moves to the right along the line l_1.

The label j of u_2 on $U_n(f_3)$ is

$$j = \frac{\frac{3\pi}{2} - \arctan\dfrac{-\frac{\nu_3}{\mu_3-\gamma} - (\mu_3-\gamma)\sqrt{(\frac{\nu_3}{\mu_3-\gamma})^2 + 1 - \frac{1}{(\mu_3-\gamma)^2}}}{-1 + \frac{\nu_3}{\mu_3-\gamma}(\mu_3-\gamma)\sqrt{(\frac{\nu_3}{\mu_3-\gamma})^2 + 1 - \frac{1}{(\mu_3-\gamma)^2}}} - \theta}{\frac{\pi}{n}}.$$

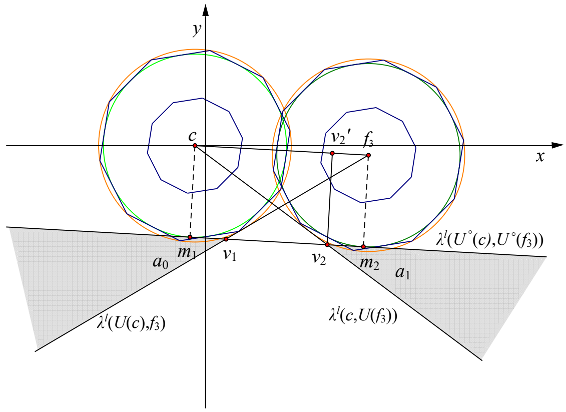

Fig. 14: Regular 2n-gons replaced by circumcircles and incircles

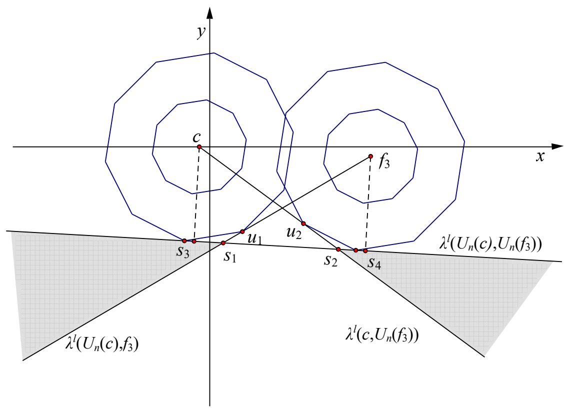

Fig. 15: Regular 2n-gons

By Corollary 4, j is a decreasing function of $\mu_3 - \gamma$. Therefore, it remains constant or decreases while f_3 moves to the right along the line l_1.

Moreover, the labels of the touching points of $\lambda^l(U_n(c), U_n(f_3))$ with $U_n(c)$, $U_n(f_3)$ remain invariant. Hence, when f_3 moves to the right along the line l_1, $d(s_1, s_3)$ and $d(s_2, s_4)$ are decreasing, while $d(s_1, s_2)$ is increasing. So when $\mu_3 - \gamma$ attains the minimum value, $d(s_1, s_3)$, $d(s_2, s_4)$ reach the maximum values and $d(s_1, s_2)$ attains the minimum value. In addition, $d(s_1, s_2) > d(v_1, v_2)$, $d(s_1, s_3) < d(v_1, m_1)$, $d(s_2, s_4) < d(v_2, m_2)$,

$$d(v_1, v_2) = 2\phi(t) = 2\Big(\cos\frac{\pi}{2n}\sqrt{4t^2 - 1} - t\Big),$$

$$d(v_1, m_1) = d(v_2, m_2) = \psi(t) = 2t - \cos\frac{\pi}{2n}\sqrt{4t^2 - 1}\,,$$

where $d(c, f_3) = 2t$.

For $n \geq 35$, we can obtain analogous conclusions if we replace f_3 by g_2. Based on the above discussion, we have the following.

PROPOSITION 14. (1) *For* $5 \leq n \leq 34$, $d(a_0, a_1) > 2.124$, $\mu_3 - x_1 < 0.416$; (2) *For* $n \geq 35$, $d(a_0, a_1) > 1.404$, $\mu_2' - x_1 < 0.399$.

PROOF. We only prove the first part of this proposition; the second can be proved similarly.

For $5 \leq n \leq 34$, let $d(c, f_3) = 2t$, and v_2' be the orthogonal projection of v_2 on $\overline{cf_3}$. Due to the above discussion, when $t = \frac{3}{2}\xi_{\min}$, $d(a_0, a_1)$ attains the minimum value and $\mu_3 - x_1$ reaches the maximum value. Since the family $\mathcal{F}$ has property $T(3)$, a_0, a_1 must lie in the two components which contain the points v_1 and v_2. On the other hand, because $d(a_0, c) < d(a_0, f_3)$, both a_0 and v_1 are in the same connected component.

We have $x_{v_1} < \psi(\frac{3}{2}\xi_{\min}) < 0.314$. For $x_1 > \cos\frac{\pi}{2n} \cdot \cos 0.102 > 0.946$, a_1 and v_2 are in the same connected component. Moreover, the angle of $\overline{v_2 v_2'}$ and y-axis is less than 0.102, so $x_{v_2'} - x_{v_2} < \sin(0.102) < 0.102$.

Hence, $d(a_0, a_1) \geq d(v_1, v_2) = 2\phi(t) > 2.124$, $\mu_3 - x_1 < \psi(t) + 0.102 < 0.416$. $\square$

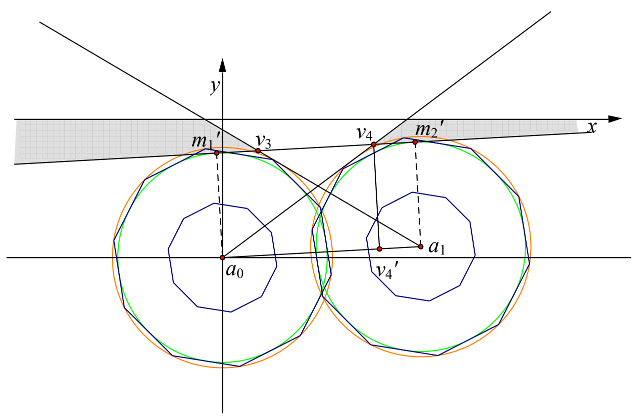

Fig. 16: Regular 2n-gons replaced by circumcircles and incircles

To obtain Proposition 15, we first present the following facts.

For $n \geq 5$, let v_3, v_4 be the intersection points of $\lambda^l(U^\circ(a_0), U^\circ(a_1))$ with $\lambda^l(U(a_0), a_1)$, $\lambda^l(a_0, U(a_1))$, respectively (see Fig. 16). Denote the touching point of $\lambda^l(U_n(a_0), a_1)$ with $U_n(a_0)$ by u_1', and the touching point of $\lambda^l(a_0, U_n(a_1))$ with $U_n(a_1)$ by u_2'. The lines $\lambda^l(U_n(a_0), a_1)$ and $\lambda^l(a_0, U_n(a_1))$ intersect $\lambda^l(U_n(a_0), U_n(a_1))$ at s_1', s_2'. Let s_3', s_4' be the orthogonal projections of a_0, a_1 on $\lambda^l(U_n(a_0), U_n(a_1))$ (see Fig. 17). Let the touching points of $\lambda^l(U^\circ(a_0), U^\circ(a_1))$ with $U^\circ(a_0)$, $U^\circ(a_1)$ be m_1', m_2', respectively.

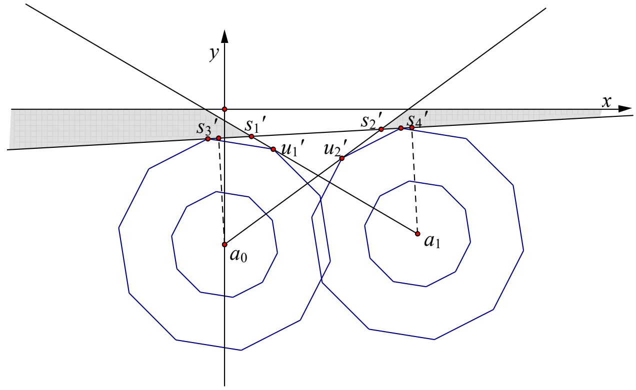

Fig. 17: Regular 2n-gons

When the point a_1 moves along any line l_2, the label i' of u_1' on $U_n(a_0)$ is determined by

$$i' = \frac{\frac{\pi}{2} - \arctan \frac{\frac{y_1-y_0}{x_1}+x_1\sqrt{(\frac{y_1-y_0}{x_1})^2+1-\frac{1}{x_1^2}}}{1-\frac{y_1-y_0}{x_1}x_1\sqrt{(\frac{y_1-y_0}{x_1})^2+1-\frac{1}{x_1^2}}} - \theta}{\frac{\pi}{n}}.$$

By Corollary 4, i' is a decreasing function of x_1. Therefore, it remains constant or decreases, while a_1 moves to the right along the line l_2.

The label j' of u_2' on $U_n(a_1)$ is

$$j' = \frac{\frac{3\pi}{2} - \arctan \frac{-\frac{y_1-y_0}{x_1}+x_1\sqrt{(\frac{y_1-y_0}{x_1})^2+1-\frac{1}{x_1^2}}}{-1-\frac{y_1-y_0}{x_1}x_1\sqrt{(\frac{y_1-y_0}{x_1})^2+1-\frac{1}{x_1^2}}} - \theta}{\frac{\pi}{n}}.$$

By Corollary 4, j' is an increasing function of x_1. Therefore, it does not decrease while a_1 moves to the right along the line l_2.

Moreover, the labels of the touching points of $\lambda^l(U_n(a_0), U_n(a_1))$ with $U_n(a_0)$, $U_n(a_1)$ remain invariant. Hence, when a_1 moves on the line l_2 to the right, $d(s_1', s_3')$ and $d(s_2', s_4')$ are decreasing, and $d(s_1', s_2')$ is increasing. So when x_1 attains the minimum value, $d(s_1', s_3')$, $d(s_2', s_4')$ reach the maximum values, and $d(s_1', s_2')$ attains the minimum value. In addition, $d(s_1', s_2') > d(v_3, v_4)$, $d(s_1', s_3') < d(v_3, m_1')$, $d(s_2', s_4') < d(v_4, m_2')$,

$$d(v_3, v_4) = 2\phi(t) = 2\Big(\cos\frac{\pi}{2n}\sqrt{4t^2-1} - t\Big),$$

$$d(v_3, m_1') = d(v_4, m_2') = \psi(t) = 2t - \cos\frac{\pi}{2n}\sqrt{4t^2-1}\,,$$

where $d(a_0, a_1) = 2t$.

Now, we have the following proposition.

Proposition 15. (1) *For* $5 \le n \le 34$, $x_1 - \mu_1 < 0.444$.
(2) *For* $n \ge 35$, $x_1 - \mu_1' < 0.542$.

Proof. In this proposition we also only prove the first part.

For $5 \le n \le 34$, c, f_1, f_2, f_3 must lie above the line $\lambda^l(U^\circ(a_0), U^\circ(a_1))$ and in the connected components which contain v_3, v_4. As $d(a_0, a_1) > 2.124$, $x_{v_3} < \psi(1.062) < 0.342$, by Proposition 13, f_1, f_2, f_3 are in the same connected component as v_4. Let v_4' be the orthogonal projection of v_4 on $\overline{a_0a_1}$. The angle of $\overline{v_4v_4'}$ and the y-axis is less than 0.102, so $x_{v_4'} - x_{v_4} < \sin(0.102) < 0.102$. Because of the previous knowledge, $x_1 - \mu_1 < \psi(1.062) + 0.102 < 0.444$. □

Proposition 14 and Proposition 15 imply that
(a) $\mu_3 - \mu_1 < 0.86$ for $5 \le n \le 34$;
(b) $\mu_2' - \mu_1' < 0.941$ for $n \ge 35$.

But (a) and (b) contradict the previous fact. Therefore, in the $x > 0$ part of $Q(\delta_{0,1})$, there exist at most two exceptional centers for $5 \le n \le 34$, and at most one exceptional center for $n \ge 35$. Hence, for $w \in (w_v(R_n(\mathbf{0})), 1.2)$, Theorem 5 holds.

References

[1] A. Bezdek, On the transversal-conjecture of Katchalski and Lewis, in: *Intuitive Geometry* (Szeged, 1991), Colloq. Math. Soc. János Bolyai, vol. 63, North-Holland (Amsterdam, 1994), pp. 23–25.

[2] L. Danzer, Über ein Problem aus der Kombinatorischen Geometrie, *Arch. Math.*, **8** (1957), 347–351.

[3] Qingdan Du and Liping Yuan, An upper bound on the transversal width of smaller overlapping $T(3)$-families of discs, manuscript.

[4] B. Grünbaum, On common transversals, *Arch. Math.*, **9** (1958), 465–469.

[5] A. Heppes, New upper bound on the transversal width of $T(3)$-families of discs, *Discrete Comput. Geom.*, **34** (2005), 463–474.

[6] A. Heppes, Proof of the Katchalski–Lewis transversal conjecture for $T(3)$-families of congruent discs, *Discrete Comput. Geom.*, **38** (2007), 289–304.

[7] A. Holmsen, A transversal theorem in the plane, unpublished manuscript, 2001.

[8] A. Holmsen, New bounds on the Katchalski–Lewis transversal problem, *Discrete Comput. Geom.*, **29** (2003), 395–408.

[9] T. Kaiser, Line transversals to unit disks, *Discrete Comput. Geom.*, **28** (2002), 379–387.

[10] M. Katchalski, A conjecture of Grünbaum on common transversals, *Math. Scand.*, **59** (1986), 192–198.

[11] M. Katchalski and T. Lewis, Cutting families of convex sets, *Proc. Amer. Math. Soc.*, **79** (1980), 457–461.

[12] H. Tverberg, Proof of Grünbaum's conjecture on common transversals for translates, *Discrete Comput. Geom.*, **4** (1989), 191–203.