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The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: http://www.tandfonline.com/loi/uamm20

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To cite this article: Ying Chai, Liping Yuan & Tudor Zamfirescu (2018) Rupert Property of Archimedean Solids, The American Mathematical Monthly, 125:6, 497-504, DOI: 10.1080/00029890.2018.1449505

To link to this article: https://doi.org/10.1080/00029890.2018.1449505

Rupert Property of Archimedean Solids

Ying Chai, Liping Yuan, and Tudor Zamfirescu

Abstract. We say that a polytope $\mathcal{P}$ has the Rupert property if we can make a hole large enough in $\mathcal{P}$ to permit another copy of $\mathcal{P}$ to pass through. In this article, we show that among the 13 Archimedean solids, 8 have this property, namely, the cuboctahedron, the truncated octahedron, the truncated cube, the rhombicuboctahedron, the icosidodecahedron, the truncated cuboctahedron, the truncated icosahedron, and the truncated dodecahedron.

1. INTRODUCTION. An Archimedean solid is a highly symmetric, semi-regular convex polyhedron with two or more types of regular polygons as faces and locally congruent at vertices. There are 13 types in all; see Figure 1. Archimedean solids, by virtue of their high degree of symmetry, are widely applied in educational toys, architecture and art, and so forth. And they also have close connections with astronomy, biology, and chemistry. Recently, other properties of Archimedean solids have been investigated, for example dense packings of Archimedean solids [**7**, **8**], and acute triangulations of their surfaces [**1**, **2**].

More than three hundred years ago, Prince Rupert (Prinz Ruprecht von der Pfalz) won a wager whether a hole large enough can be cut in one of two congruent cubes to permit the second to pass through the first. About one hundred years later, Pieter Nieuwland proved that, taking the first cube to have edge-length 1, the largest second cube that can pass through the first has edge-length $\frac{3\sqrt{2}}{4}$. In 1950, Schreck [**5**] gave a detailed review of Rupert's problem and Nieuwland's proof. In 1968, Scriba [**6**] found out that the tetrahedron and the octahedron have the same property. In 2016, Jerrard, Wetzel, and Yuan [**4**] added that the dodecahedron and the icosahedron also have that property, i.e., we can find through any Platonic solid a hole large enough to permit a congruent copy to pass through; what "passing through" exactly means will be revealed in the next section. We call this property the *Rupert property*. So, all five Platonic solids have the Rupert property. Suppose that a polytope $\mathcal{P}$ has the Rupert property. It is natural to ask how large a polytope $\mathcal{P}'$ similar to $\mathcal{P}$ can be to pass through a hole in $\mathcal{P}$, i.e., how large can a positive scalar ν be, such that the polytope $\nu\mathcal{P}$ passes through a suitable hole in $\mathcal{P}$? We call this *Nieuwland's question* after P. Nieuwland (1764–1794), who asked and answered this question for the cube. Define the *Nieuwland constant* $\nu(\mathcal{P})$ of the polytope $\mathcal{P}$ by

$$\nu(\mathcal{P}) = \sup\{\nu > 0 :\ \nu\mathcal{P} \text{can pass through a suitable hole in } \mathcal{P}\}.$$

Many convex bodies, such as all universal stoppers (see [**3**]), enjoy the Rupert property, but it is easy to see that the unit ball in $\mathbb{R}^3$ does not.

In this article, we discuss the Rupert property of Archimedean solids, claim that the cuboctahedron, the truncated octahedron, the truncated cube, the rhombicuboctahedron, the icosidodecahedron, the truncated cuboctahedron, the truncated icosahedron, and the truncated dodecahedron have the Rupert property, and provide a lower bound

doi.org/10.1080/00029890.2018.1449505

MSC: Primary 52B10

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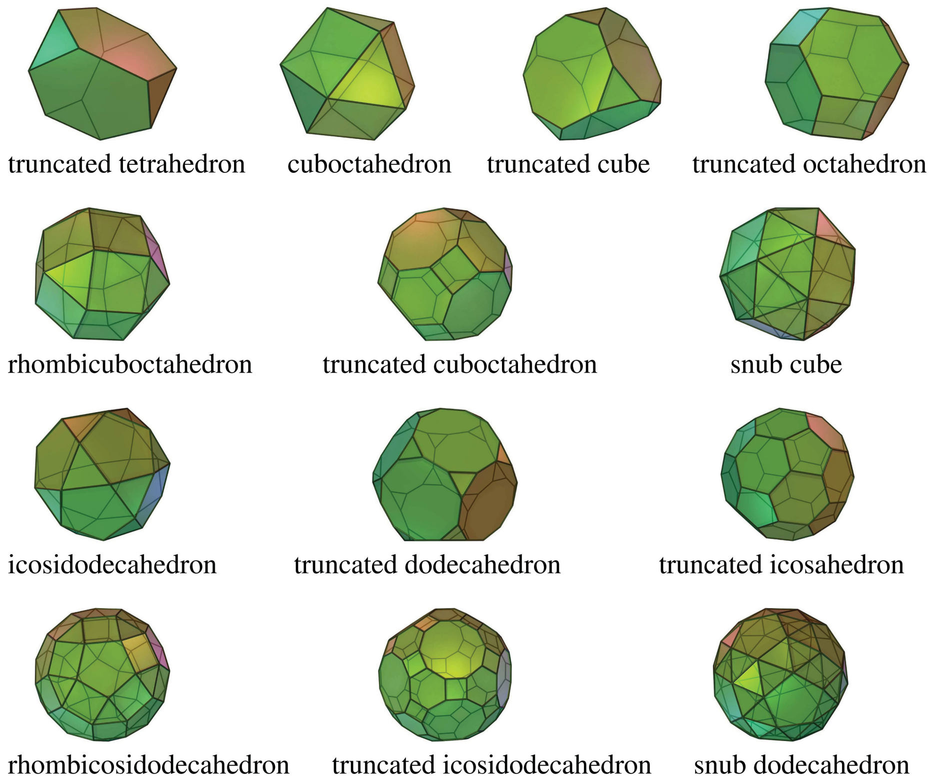

Figure 1. Archimedean solids.

of the Nieuwland constant for each of them. In Section 3, we will prove the case of the cuboctahedron in detail. The results for the remaining seven Archimedean solids will be listed in Section 4, and the details of the proofs can be seen in the online supplement.

2. PRELIMINARIES. The set $C \subset \mathbb{R}^d$ is called a *convex set* if for all $x_1, x_2 \in C$, $\lambda_1 x_1 + \lambda_2 x_2 \in C$ for any $\lambda_1,\ \lambda_2 \in [0,\ 1]$ with $\lambda_1 + \lambda_2 = 1$. If V is a subset of $\mathbb{R}^d$, the *convex hull* conv$V$ of V is the intersection of all convex sets that contain V, and int$V$, bd$V$ denote its relative interior and boundary, respectively. If V is a finite set of points, then conv$V$ is called a *polytope*. Let π_n be a plane in $\mathbb{R}^3$ with normal vector $\boldsymbol{n}$, and P_n the orthogonal projection of $\mathbb{R}^3$ onto π_n. Let τ be a simple closed curve that lies in the plane π_n, and I_τ be the domain in π_n interior to τ. A *hole* [4] H_τ with directrix τ and direction $\boldsymbol{n}$, see Figure 2, is the set

$$\{\boldsymbol{y} + t\boldsymbol{n} \in \mathbb{R}^3 : \boldsymbol{y} \in I_\tau,\ t \in \mathbb{R}\}.$$

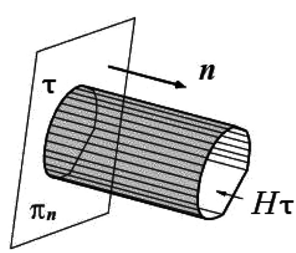

Figure 2. H_τ.

That a polytope $\mathcal{P}$ has the Rupert property means that there are vectors $\boldsymbol{n}, \boldsymbol{m}$ and an isometry μ of π_n onto π_m such that

$$\mu(P_n(\mathcal{P})) \subset \mathrm{int} P_m(\mathcal{P}).$$

We say that $\mathcal{P}$ passes through the hole H_τ with directrix $\tau = \mathrm{bd} P_m(\mathcal{P})$ and direction $\boldsymbol{m}$. $P_n(\mathcal{P})$ is the inner projection of $\mathcal{P}$, denoted by P_i, and $P_m(\mathcal{P})$ is the outer projection of $\mathcal{P}$, denoted by P_o.

For distinct $a, b \in \mathbb{R}^d$, let $\overline{ab}$ denote the line segment from a to b and l_{ab} the line through a, b. The vector $\overrightarrow{ab}$ is the direction vector of l_{ab} from a to b. $\|\cdot\|$ is the Euclidean norm.

Now let $\boldsymbol{e_x} = (1,\ 0,\ 0), \boldsymbol{e_y} = (0,\ 1,\ 0), \boldsymbol{e_z} = (0,\ 0,\ 1)$ be the standard basis for $\mathbb{R}^3$. And let Π_{xy} be the plane spanned by $\boldsymbol{e_x},\ \boldsymbol{e_y}$, the original x-axis be the new x-axis, and the original y-axis be the new y-axis. Thus, $P_{\boldsymbol{e_z}}$ denotes the orthogonal projection of $\mathbb{R}^3$ onto Π_{xy}.

Suppose a polytope $\mathcal{P}$ in $\mathbb{R}^3$ has vertex set $\{a_1,\ a_2, \ldots,\ a_k\}$ $(k \in \mathbb{Z}^+)$, where $a_i = (x_i,\ y_i,\ z_i)$ $(i = 1,\ 2, \ldots,\ k)$. Denote $P_{e_z}(a_i)$ by i_z. For the sake of convenience, we express i_z in the form of $(x_i,\ y_i)$. And then $P_{e_z}(\mathcal{P}) = \mathrm{conv}\{i_z :\ i = 1,\ 2, \ldots,\ k\}$. Let $T_x,\ T_y,\ T_z$ denote the rotational transformations of $\mathbb{R}^3$ around the $x,\ y,\ z$-axis by an angle α, β, γ, respectively. The rotation angle is positive if and only if the rotation obeys the right-hand rule. Then for all $p = (x,\ y,\ z) \in \mathbb{R}^3$,

$$T_x(p) = (x \quad y \quad z)\mathbf{A}_{x(\alpha)}, \qquad T_y(p) = (x \quad y \quad z)\mathbf{A}_{y(\beta)},$$

$$T_z(p) = (x \quad y \quad z)\mathbf{A}_{z(\gamma)},$$

where

$$\mathbf{A}_{x(\alpha)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad \mathbf{A}_{y(\beta)} = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix},$$

$$\mathbf{A}_{z(\gamma)} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathcal{P}(x(\alpha),\ y(\beta),\ z(\gamma))$ means that $\mathcal{P}$ is rotated about the x-axis by an angle α, then about the y-axis by an angle β, and then about the z-axis by an angle γ. Whence the vertex coordinates of $\mathcal{P}(x(\alpha),\ y(\beta),\ z(\gamma))$ are

$$(x_i \quad y_i \quad z_i)\mathbf{A}_{x(\alpha)}\mathbf{A}_{y(\beta)}\mathbf{A}_{z(\gamma)} \quad (i = 1,\ 2, \ldots,\ k).$$

By the definition of the Rupert property, we only need to find two $\mathcal{P}_j = \mathcal{P}(x(\alpha_j), y(\beta_j), z(\gamma_j))$ $(j = 1,\ 2)$, that satisfy $P_{e_z}(\mathcal{P}_1) \subset P_{e_z}(\mathcal{P}_2)$. We have $P_{e_z}(\mathcal{P}_1) = P_i$ and $P_{e_z}(\mathcal{P}_2) = P_o$.

3. THE CUBOCTAHEDRON. We treat here in detail the case of the cuboctahedron.

Theorem 1. *The cuboctahedron $\mathcal{C}$ has the Rupert property.*

Proof. The cuboctahedron $\mathcal{C}$ of edge length $\sqrt{2}$ is shown in Figure 3, and the coordinates of the vertices are given in Table 1.

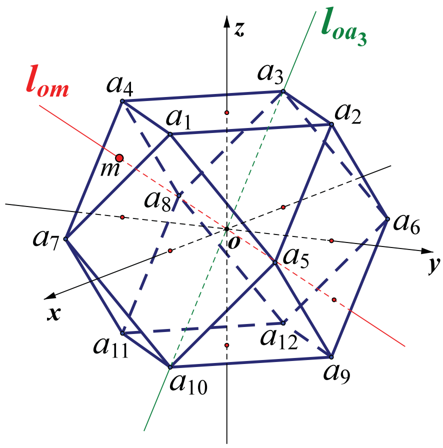

Figure 3. Cuboctahedron $\mathcal{C}$.

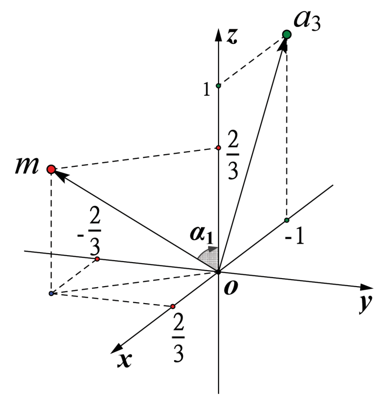

Figure 4. Position of m.

Table 1. Vertex coordinates of $\mathcal{C}$.

Vertex	Coordinates
$a_1 = -a_{12}$	$(1,\ 0,\ 1)$
$a_2 = -a_{11}$	$(0,\ 1,\ 1)$
$a_3 = -a_{10}$	$(-1,\ 0,\ 1)$
$a_4 = -a_9$	$(0,\ -1,\ 1)$
$a_5 = -a_8$	$(1,\ 1,\ 0)$
$a_6 = -a_7$	$(-1,\ 1,\ 0)$

In Figure 3, $m = (\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3})$ is the center of the triangular face $a_1a_4a_7$. Obviously, the angle α_1 between $\overrightarrow{om}$ and the positive z-axis equals $\arcsin \frac{\sqrt{6}}{3}$ and the angle between the projection of $\overrightarrow{om}$ on Π_{xy} and the positive x-axis is $\frac{\pi}{4}$. See Figure 4.

First, we consider the projection of $\mathcal{C}$ along l_{om}.

Now, we rotate $\mathcal{C}$ by an angle $-\frac{\pi}{4}$ about the z-axis, and then by an angle $-\alpha_1$ about the x-axis. The cuboctahedron obtained is denoted by $\mathcal{C}(z(-\frac{\pi}{4}), x(-\alpha_1))$, and the vertex a_i gets the new label $a_{\underline{i}}$ $(i = 1, 2, \ldots, 12)$. Then

$$a_{\underline{i}} = \begin{pmatrix} x_{\underline{i}} & y_{\underline{i}} & z_{\underline{i}} \end{pmatrix} = \begin{pmatrix} x_i & y_i & z_i \end{pmatrix} \mathbf{A}_{z(-\frac{\pi}{4})} \mathbf{A}_{x(-\alpha_1)}.$$

After rotation, the vector $\overrightarrow{om}$ coincides with the z-axis. $\overrightarrow{om}$ is the direction vector of l_{om}, so the projection of $\mathcal{C}(z(-\frac{\pi}{4}), x(-\alpha_1))$ onto Π_{xy} is the same as the projection of $\mathcal{C}$ along l_{om}. Take P_o to be this projection, shown in Figure 5 by the solid line segments. The coordinates of the vertices are given in Table 2.

To find the inner projection P_i, we consider the projection of $\mathcal{C}$ along l_{oa_3}.

Rotate $\mathcal{C}$ about y-axis by $\frac{\pi}{4}$; the new cuboctahedron is denoted by $\mathcal{C}(y(\frac{\pi}{4}))$. The vertices of $\mathcal{C}(y(\frac{\pi}{4}))$ have the same names as those of $\mathcal{C}$, i.e., a_i $(i = 1, 2, \ldots, 12)$. The new coordinates are

$$\begin{pmatrix} x_i & y_i & z_i \end{pmatrix} \mathbf{A}_{y(\frac{\pi}{4})}.$$

After rotation, the direction vector of l_{oa_3}, $\overrightarrow{oa_3}$, coincides with the z-axis. Therefore the projection of $\mathcal{C}(y(\frac{\pi}{4}))$ onto Π_{xy} is the same as the projection of $\mathcal{C}$ along l_{oa_3}, shown

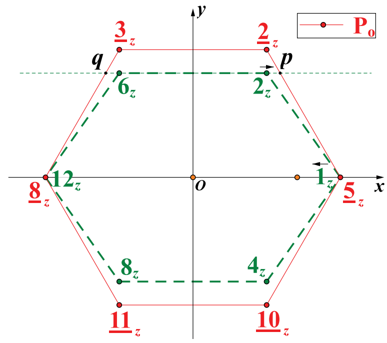

Figure 5. P_o and $P_{e_z}(\mathcal{C}(y(\frac{\pi}{4})))$.

Table 2. Vertex coordinates in Figure 5.

vertex	coordinates	vertex	coordinates
$\underline{2}_z = -\underline{11}_z$	$(\frac{\sqrt{2}}{2},\ \frac{\sqrt{6}}{2})$	$1_z = -12_z$	$(\sqrt{2},\ 0)$
$\underline{3}_z = -\underline{10}_z$	$(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{6}}{2})$	$2_z = -8_z$	$(\frac{\sqrt{2}}{2},\ 1)$
$\underline{5}_z = -\underline{8}_z$	$(\sqrt{2},\ 0)$	$6_z = -4_z$	$(-\frac{\sqrt{2}}{2},\ 1)$

in Figure 5 by the dashed lines. The coordinates are given in Table 2.

If $\mathcal{C}(y(\frac{\pi}{4}))$ is rotated by an angle β (> 0) about the y-axis, the vertex 2_z (see Figure 5) will move closer to the point p along the line l_{pq}, and the vertex 1_z will move closer to the point o along the x-axis. Denote the cuboctahedron at the new place by $\mathcal{C}(y(\frac{\pi}{4}), y(\beta))$; the vertex a_i gets the new label $a_{i'}$ ($i = 1, 2, \ldots, 12$),

$$a_{i'} = \begin{pmatrix} x_{i'} & y_{i'} & z_{i'} \end{pmatrix} = \begin{pmatrix} x_i & y_i & z_i \end{pmatrix} \mathbf{A}_{y(\frac{\pi}{4})} \mathbf{A}_{y(\beta)}.$$

Then we choose a suitable β, such that $1'_z$, $2'_z$ move into the interior of P_o and $\overline{1'_z 2'_z}$ is parallel to $\overline{2_z \underline{5}_z}$. Because $\overline{1'_z 2'_z}$ is parallel to $\overline{2_z \underline{5}_z}$ if and only if $\|2'_z - p\| = \|1'_z - \underline{5}_z\|$, we only need to choose a β satisfying $\|2'_z - p\| = \|1'_z - \underline{5}_z\|$.

First of all, we consider the change of edge $\overline{2_z 6_z}$ when $\mathcal{C}(y(\frac{\pi}{4}))$ is rotated to $\mathcal{C}(y(\frac{\pi}{4}), y(\beta))$. In Figure 6, o' is the intersection of the positive y-axis and the face $a_2 a_5 a_9 a_6$. Clearly, $p = (\sqrt{2} - \frac{\sqrt{3}}{3},\ 1)$. So

$$\|2_z - p\| = \sqrt{2} - \frac{\sqrt{3}}{3} - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3}.$$

The edge-length of $\mathcal{C}$ is $\sqrt{2}$; therefore $\|a_2 - o'\| = 1$. We have

$$\|2'_z - 2_z\| = \cos\left(\frac{\pi}{4} - \beta\right) - \cos\left(\frac{\pi}{4}\right);$$

see Figure 6. Thus,

$$\|2'_z - p\| = \|2_z - p\| - \|2'_z - 2_z\| = \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3} - \cos\left(\frac{\pi}{4} - \beta\right) + \frac{\sqrt{2}}{2}$$

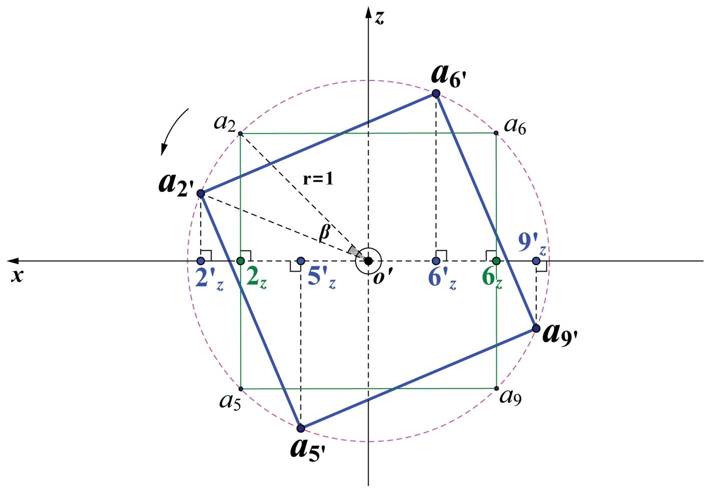

Figure 6. Change of edge $\overline{2_z 6_z}$ when $\mathcal{C}(y(\frac{\pi}{4}))$ is rotated to $\mathcal{C}(y(\frac{\pi}{4}), y(\beta))$.

$$= \sqrt{2} - \frac{\sqrt{3}}{3} - \cos\left(\frac{\pi}{4} - \beta\right).$$

In order to calculate $\|1'_z - \underline{5}_z\|$, we have to think about the changes of the projections of $a_1,\ a_{12}$; see Figure 7.

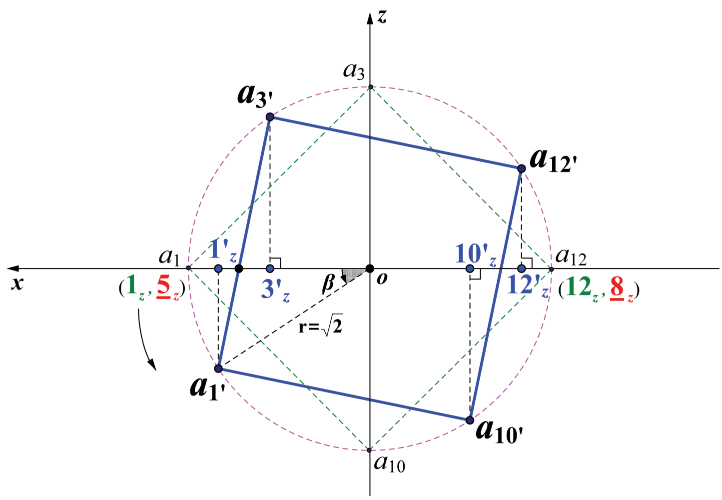

Figure 7. Changes of the projections of $a_1,\ a_{12}$ when $\mathcal{C}(y(\frac{\pi}{4}))$ is rotated to $\mathcal{C}(y(\frac{\pi}{4}), y(\beta))$.

Obviously, $\|a_1\| = \sqrt{2}$, which implies that

$$\|1'_z - \underline{5}_z\| = \sqrt{2} - \sqrt{2}\cos\beta.$$

From $\|2'_z - p\| = \|1'_z - \underline{5}_z\|$, we get

$$\beta = \arccos\frac{\sqrt{6} + 2\sqrt{3}}{6} \approx 9.73561^\circ.$$

In this way, we then find the projection P_i of $\mathcal{C}(y(\frac{\pi}{4}), y(\arccos\frac{\sqrt{6}+2\sqrt{3}}{6}))$ onto Π_{xy}. Thus, $\mathcal{C}$ has the Rupert property. ■

Theorem 2. *The Nieuwland constant of the cuboctahedron* $\mathcal{C}$ *satisfies the inequality* $\nu(\mathcal{C}) > 1.01461$.

Proof. Inspect Figure 5. The intersection of the lines

$$l_{o2_z}: \ y = \sqrt{2}x,$$

$$l_{\underline{2}_z\underline{5}_z}: \ y = -\sqrt{3}x + \sqrt{6}$$

is $p = (3\sqrt{2} - 2\sqrt{3},\ 6 - 2\sqrt{6})$. Because $\frac{\sqrt{2}}{2} < 3\sqrt{2} - 2\sqrt{3} < \sqrt{2}$, we have $p \in \overline{\underline{2}_z\underline{5}_z}$, whence the intersection c_2 of $l_{o2'_z}$ and $l_{\underline{2}_z\underline{5}_z}$ belongs to $\overline{\underline{2}_z\underline{5}_z}$. Thus,

$$\nu(\mathcal{C}) \geqslant \frac{\|c_2\|}{\|2'_z\|} = \frac{\|\underline{5}_z\|}{\|1'_z\|} = \frac{\sqrt{2}}{\sqrt{2}\cos\beta} = \frac{\sqrt{2}}{\frac{2\sqrt{3}+2\sqrt{6}}{6}} = 2\sqrt{3} - \sqrt{6} > 1.01461. \quad ■$$

4. FURTHER RESULTS. Using methods similar to those in Section 3 we can prove that seven other Archimedean solids also enjoy the Rupert property. We only list the results here; for details of the proofs, please see the online supplement.

Theorem 3. *The truncated octahedron, the truncated cube, the rhombicuboctahedron, the icosidodecahedron, the truncated cuboctahedron, the truncated icosahedron, and the truncated dodecahedron have the Rupert property.*

Theorem 4. *For the Niewland constant of the*

(i) truncated octahedron $\mathcal{O}$*, we have* $\nu(\mathcal{O}) > 1.00815$.

(ii) truncated cube $\mathcal{T}$*, we have* $\nu(\mathcal{T}) > 1.02036$.

(iii) rhombicuboctahedron $\mathcal{R}$*, we have* $\nu(\mathcal{R}) > 1.00609$.

(iv) icosidodecahedron $\mathcal{I}$*, we have* $\nu(\mathcal{I}) > 1.00015$.

(v) truncated cuboctahedron $\mathcal{U}$*, we have* $\nu(\mathcal{U}) > 1.00370$.

(vi) truncated icosahedron $\mathcal{J}$*, we have* $\nu(\mathcal{J}) > 1.00004$.

(vii) truncated dodecahedron $\mathcal{D}$*, we have* $\nu(\mathcal{D}) > 1.00014$.

The treatment of the remaining five cases appears to be quite hard, harder than those solved by us in this article.

Open problems. Prove that the truncated tetrahedron, the snub cube, the rhombicosidodecahedron, the truncated icosidodecahedron and the snub dodecahedron, all enjoy the Rupert property. Also, provide estimates of their Nieuwland constants.

Conjecture. Every convex polytope has the Rupert property.

ACKNOWLEDGMENT. The second author gratefully acknowledges financial support by National Natural Science Foundation of China (11471095), Outstanding Youth Science Foundation of Hebei Province (A2013205189), and Program for Excellent Talents in University, Hebei Province (GCC2014043).

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