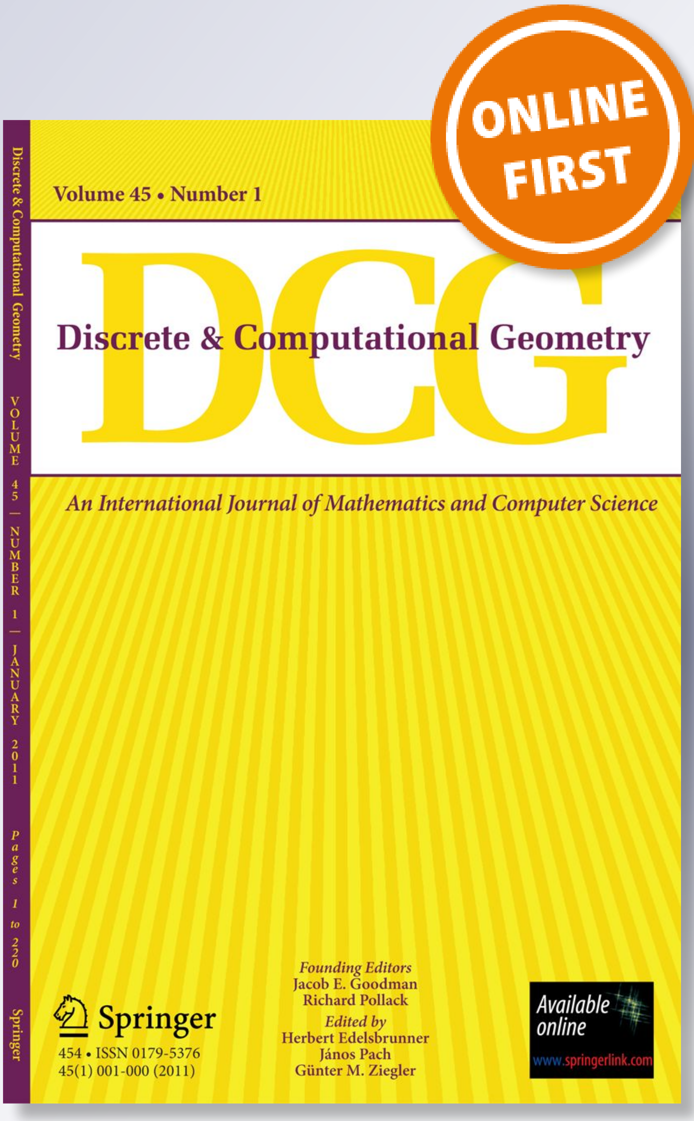
Cages of Small Length Holding Convex Bodies

Augustin Fruchard & Tudor Zamfirescu

Discrete & Computational Geometry

ISSN 0179-5376

Discrete Comput Geom
DOI 10.1007/s00454-019-00144-4



Discrete & Computational Geometry
https://doi.org/10.1007/s00454-019-00144-4

RICKY POLLACK MEMORIAL ISSUE

Cages of Small Length Holding Convex Bodies

Augustin Fruchard[1] · **Tudor Zamfirescu**[2,3,4]

Received: 8 August 2017 / Revised: 22 July 2019 / Accepted: 4 October 2019

Abstract

A cage G, defined as the 1-skeleton of a convex polytope in 3-space, holds a compact set K if G cannot move away without meeting the relative interior of K. The main results of this paper establish the infimum of the lengths of cages holding various compact convex sets. First, planar graphs and Steiner trees are investigated. Then the notion of points *almost fixing* a convex body in the plane is introduced and studied. The last two sections treat cages holding 2-dimensional compact convex sets, respectively the regular tetrahedron.

Keywords Immobilisation · Skeleton · Steiner tree · Convex body

Mathematics Subject Classification 52A15 · 52A40 · 52B10

1 Introduction

A *cage* is the 1-dimensional skeleton of a 3-dimensional convex polytope in $\mathbb{R}^3$. A compact set G is said to *hold* another compact set K if G is disjoint from the relative interior of K and if G cannot be rigidly moved to a position far away without intersecting the relative interior of K on its way. Here, a *move* means a continuous path starting from the identity in the space of isometries of $\mathbb{R}^3$. "Far away" means that

Dedicated to the memory of Ricky Pollack.

Editor in Charge: János Pach

Augustin Fruchard
Augustin.Fruchard@uha.fr

Tudor Zamfirescu
tzamfirescu@yahoo.com

1 Département de Mathématiques, IRIMAS, Université de Haute-Alsace, 68093 Mulhouse, France

2 Fakultät für Mathematik, Technische Universität Dortmund, 44221 Dortmund, Germany

3 Institute of Mathematics "Simion Stoilow", Romanian Academy, 010702 Bucharest, Romania

4 College of Mathematics, Hebei Normal University, 050024 Shijiazhuang, P. R. China

the convex hulls of K and of the moved G are disjoint. Already in 1959, Coxeter [5] raised the problem of finding the infimum of the total lengths of cages holding the ball of radius 1 in $\mathbb{R}^3$. In the following years, Besicovitch [2] and Aberth [1] solved Coxeter's problem. In the present paper, we extend the investigation to other compact convex sets replacing the ball.

The space $\mathbb{R}^3$ is endowed with its Euclidean norm $\|x\| = \sqrt{\langle x|x\rangle}$, where $\langle\cdot|\cdot\rangle$ is the usual scalar product. For distinct $x, y \in \mathbb{R}^3$, let $\overline{xy}$ be the line through x, y and xy the line-segment from x to y. The open line-segment $xy \setminus \{x, y\}$ is denoted by $]xy[$. Given a line $\overline{xy}$ oriented from x to y, $(xy)^+$ denotes the open half-plane on the left of $\overline{xy}$.

The measure of an angle $\widehat{xyz}$ is denoted by $\angle xyz$. The measure of the angle between two lines or planes X and Y is denoted by $\angle(X, Y)$. These angles will be oriented if the context is in the plane, and unoriented if the context is in 3-space.

As usual, for $M \subset \mathbb{R}^d$ with $d \geq 2$, the *convex hull* conv M of M is the intersection of all convex subsets of $\mathbb{R}^d$ containing M, and its *affine hull* aff M is the intersection of all affine subspaces of $\mathbb{R}^d$ containing M. Also, int M and bd M denote its interior and boundary, while rel int M and rel bd M denote its relative interior and relative boundary, that is in the topology of aff M; moreover, $\operatorname{diam} M = \sup_{x,y\in M} \|x - y\| \in \mathbb{R} \cup \{+\infty\}$.

For any closed convex subset M of $\mathbb{R}^3$, let $\pi_M \colon \mathbb{R}^3 \to M$ denote the (metric) *projection*, i.e., $\pi_M(x)$ is the unique point of M such that $\|x - \pi_M(x)\| = \inf_{y\in M} \|x - y\|$. It is known (and easy to prove) that $\langle x - \pi_M(x)| y - \pi_M(x)\rangle \leq 0$ for all $y \in M$ and that π_M is 1-Lipschitz.

Here, a 2- or 3-dimensional compact convex set in $\mathbb{R}^3$ is called a *convex body*. Let $\mathcal{K}$ be the space of all convex bodies in $\mathbb{R}^3$. Equipped with the Pompeiu–Hausdorff distance, $\mathcal{K}$ becomes a metric space. For $K \in \mathcal{K}$, the *width* of K, denoted by wid K, is the smallest distance between two parallel $(\dim K - 1)$-dimensional affine subspaces H, H' of aff K such that $K \subset \operatorname{conv}(H \cup H')$.

The d-dimensional unit ball (centred at $\mathbf{0}$) is denoted by $\mathbb{B}_d$, and $\operatorname{rel\,bd} \mathbb{B}_d = \mathbb{S}_{d-1}$ $(d \geq 2)$. The d-dimensional regular simplex of edge length 1 is denoted by $\mathbb{T}_d$ $(d \geq 2)$. The d-dimensional cube of unit edge length is denoted by $\mathbb{C}_d$ $(d \geq 2)$.

The group of orientation preserving isometries of $\mathbb{R}^d$ is denoted by $\operatorname{Isom}^+\mathbb{R}^d$.

We shall denote by λ the 1-dimensional Hausdorff measure (length).

For $n \geq 2$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$, we put $x_1x_2\ldots x_n = \operatorname{conv}\{x_1, x_2, \ldots, x_n\}$ and, for $d = 2$, $|x_1x_2\ldots x_n| = \lambda \operatorname{bd} x_1x_2\ldots x_n$. In particular, $|xy| = \|x - y\|$.

Given $x_1, \ldots, x_n \in \mathbb{R}^3$, the *perimeter function* is $p(x_1, \ldots, x_n) = \sum_{i=1}^n |x_ix_{i+1}|$, with $x_{n+1} = x_1$. If $n \geq 3$ and $x_1 \ldots x_n$ is a non-degenerate planar convex n-gon with consecutive vertices $x_1, \ldots, x_n$, then $p(x_1, \ldots, x_n) = |x_1 \ldots x_n|$, but $p(x_1, x_2) = 2|x_1x_2|$.

Let $\mathcal{G}(K)$ be the space of all cages in $\mathbb{R}^3$ holding the compact set K and set

$$L(K) = \inf_{G\in\mathcal{G}(K)} \lambda G.$$

We are looking for $L(K)$ for various sets K. Apart from general results, we estimate $L(K)$ for the most common convex bodies. For the unit balls we find $L(\mathbb{B}_2) = 6$, while Besicovitch in [2] proves that $L(\mathbb{B}_3) \leq \frac{8\pi}{3} + 2\sqrt{3}$ and Aberth in [1] proves the equality. For the regular simplices of unit edge length, we obtain $L(\mathbb{T}_2) = \frac{3+\sqrt{3}}{2}$ and $L(\mathbb{T}_3) = 3$. For the cubes of unit edge length, we establish $L(\mathbb{C}_2) = 3\sqrt{2}$ and we conjecture $L(\mathbb{C}_3) = 4 + 3\sqrt{2}$.

The structure of the article is as follows. In Sect. 2 we relate the length of a cage with the perimeter and the length of the Steiner tree joining the projections of the vertices of the cage on some plane; this will yield lower bounds for $L(K)$ for several sets K. Section 3 deals with the notion of almost fixing points, yielding upper bounds for $L(K)$. Planar convex bodies are studied in Sect. 4 and non-planar ones in Sect. 5.

We end this introductory section with the following remark: The function L is not continuous. Indeed, take the square $S = \mathbb{C}_2 = abcd$. For $0 < \eta < 1$, let $x \in ab$ satisfy $|xa| = \eta$ and choose $S_\eta = xbcd$. Then $S_\eta \to S$ if $\eta \to 0$, but we have $L(S_\eta) = 3$ for all η, as Theorem 4.3 shows, while $L(S) = 3\sqrt{2}$, by Theorem 4.7(ii).

2 Geometric Graphs and Steiner Trees

By a *geometric graph*, we mean a pair (G, η), where G is a graph with vertex set $V(G)$ and edge set $E(G)$, and η is an embedding of G in a plane Π, which acts as follows. For any vertex $v \in V(G)$, $\eta(v)$ is a point in Π; for any edge $(v, w) \in E(G)$, $\eta((v, w))$ is the line-segment $\eta(v)\eta(w) \subset \Pi$, reduced to $\{\eta(v)\}$ if $\eta(v) = \eta(w)$. The set $\bigcup_{e\in E(G)} \eta(e)$ will be denoted by $\eta(G)$. Observe that we may have $\eta(v) \in \eta((u, w))$ for distinct $v, u, w \in V(G)$, and $\eta((u, v)) \cap \eta((u', v'))$ may be non-void, even a line-segment, for distinct $u, v, u', v' \in V(G)$. A vertex $v \in V(G)$ is called *external* if $\eta(v) \in \operatorname{bd}\operatorname{conv} \eta(G)$, and an edge $e \in E(G)$ is called *external* if $\eta(e) \subset \operatorname{bd}\operatorname{conv} \eta(G)$.

If

(1) $\eta(G)$ is not a line-segment,
(2) for any point $x \in \Pi$, $\operatorname{card} \left(\eta_{|V(G)}\right)^{-1}(\{x\}) \le 2$,
(3) for any side $s = ab$ of the convex polygon $\operatorname{conv} \eta(G)$, there is at least one path $(v_1, \dots, v_n)$ in G such that $\eta(v_1) = a$, $\eta(v_n) = b$, and $\eta((v_1, \dots, v_n)) = ab$,

then (G, η) is called *convex*.

We denote by P_s one of the paths $(v_1, \dots, v_n)$ satisfying condition (3). Notice that condition (3) implies that the union of $\eta(e)$ for all external edges e equals $\operatorname{bd}\operatorname{conv} \eta(G)$.

Now, for every side s of $\operatorname{conv} \eta(G)$, choose a path P_s and delete from G the edges of P_s, to obtain a graph ∇G.

We call a geometric graph (G, η) *strongly connected* if G has at least three edges and the graph $G \setminus \{e_1, e_2\}$ is connected, for any pair of external edges $e_1, e_2 \in E(G)$.

Lemma 2.1 *If the geometric graph (G, η) is convex and strongly connected, then ∇G is connected.*

Proof Let $v \in V(G)$ be external, let C be the connected component of v in ∇G, and assume that $C \neq \nabla G$.

Case 1. All external vertices belong to C.

Choose $w \in V(G) \setminus C$. Delete any two external edges; the resulting graph contains a path from w to some external vertex, which therefore does not belong to C, contrary to the present assumption.

Case 2. Some external vertex u does not belong to C.

In this case, at least two external edges e_1, e_2 do not belong to C. Since (G, η) is strongly connected, $G \setminus \{e_1, e_2\}$ is connected, which yields $u \in C$, and a contradiction is obtained again. □

For the geometric graph (G, η), we define its length $\mu(G, \eta)$ as follows:

$$\mu(G, \eta) = \lambda\eta(\nabla G) + \lambda \mathrm{bd\,conv}\, \eta(G).$$

Consider n points $v_1, \ldots, v_n \in \mathbb{R}^3$. Let $S(v_1, \ldots, v_n)$ denote the length of the shortest connected rectifiable set containing all n points, called their *Steiner tree*.

Moreover, let

$$f(v_1, \ldots, v_n) = p(v_1, \ldots, v_n) + S(v_1, \ldots, v_n), \tag{1}$$

where p is the perimeter function. Remember that $f(v_1, \ldots, v_n) = |v_1 \ldots v_n| + S(v_1, \ldots, v_n)$ if $n \geq 3$ and $v_1, \ldots, v_n$ are the consecutive vertices of a non-degenerate planar convex polygon, but $f(v_1, v_2) = 3|v_1 v_2|$. The function f will play a central role in this article.

An immediate consequence of Lemma 2.1 is the following.

Corollary 2.2 *Let (G, η) be a convex strongly connected geometric graph, with external vertices $v_1, \ldots, v_n$ such that $\eta(v_1), \ldots, \eta(v_n)$ lie in this order on* bd conv $\eta(G)$. *Then*

$$\mu(G, \eta) \geq f\big(\eta(v_1), \ldots, \eta(v_n)\big).$$

Proof Indeed,

$$\begin{aligned}\mu(G, \eta) = \lambda\eta(\nabla G) + \lambda \mathrm{bd\,conv}\, \eta(G) &\geq S\big(\eta(v_1), \ldots, \eta(v_n)\big) + p\big(\eta(v_1), \ldots, \eta(v_n)\big) \\ &= f\big(\eta(v_1), \ldots, \eta(v_n)\big).\end{aligned}$$

□

We shall use the following obvious fact.

Lemma 2.3 *If $\varphi \colon \mathbb{R}^3 \to \mathbb{R}^3$ is 1-Lipschitz, then*

$$\begin{aligned}&p\big(\varphi(v_1), \ldots, \varphi(v_n)\big) \leq p(v_1, \ldots, v_n),\\ &S\big(\varphi(v_1), \ldots, \varphi(v_n)\big) \leq S(v_1, \ldots, v_n), \ \textit{and therefore}\\ &f\big(\varphi(v_1), \ldots, \varphi(v_n)\big) \leq f(v_1, \ldots, v_n).\end{aligned}$$

Theorem 2.4 *Let G be a cage and P some plane in $\mathbb{R}^3$. If $v_1, \ldots v_n$ are the external vertices of (G, π_P) such that $\pi_P(v_1), \ldots, \pi_P(v_n)$ lie in this order on* bd conv $\eta(G)$, *then*

$$\lambda G \geq f\big(\pi_P(v_1), \ldots, \pi_P(v_n)\big).$$

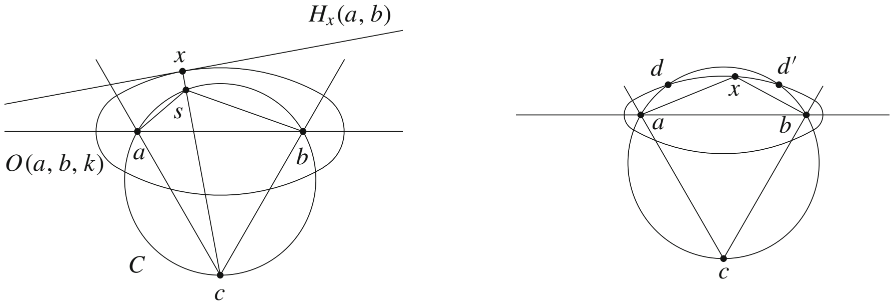


Fig. 1 The locus $\mathcal{O}(a,b,k)$. Left: $k \geq \frac{2}{\sqrt{3}}|ab|$, right: $k < \frac{2}{\sqrt{3}}|ab|$

Proof As a graph, the 1-dimensional skeleton of a polytope is 3-connected (i.e., every pair of vertices can be joined by three paths in the graph having only their endpoints in common), and therefore 3-edge connected (i.e., the graph minus any pair of edges is graph-theoretically connected). Then the geometric graph (G, π_P) is strongly connected. Also, it is easily verified that (G, π_P) is convex. We have $G = \nabla G \cup \big(\bigcup_s P_s\big)$ and $\lambda G \geq \lambda \nabla G + \sum_s \lambda P_s$. By successively using Lemma 2.3 and Corollary 2.2, we obtain

$$\lambda G \geq \lambda \pi_P(\nabla G) + \sum_s \lambda \pi_P(P_s) \geq \mu(G, \pi_P) \geq f\big(\pi_P(v_1), \ldots, \pi_P(v_n)\big). \qquad \square$$

Given $a, b, c \in \mathbb{R}^2$, let $t(a,b,c)$ be the Fermat–Torricelli point, i.e., the unique point $s \in \mathbb{R}^2$ such that $S(a,b,c) = |as| + |bs| + |cs|$. If one of the three angles of the triangle abc is greater than $\frac{2\pi}{3}$, then $t(a,b,c)$ is one of the points a, b, or c.

Now fix $a, b \in \mathbb{R}^2$ and $k > |ab|$, and consider the locus

$$\mathcal{O}(a,b,k) = \big\{x \in \mathbb{R}^2\ ;\ S(a,b,x) = k\big\}.$$

Proposition 2.5 (i) *For any $a, b \in \mathbb{R}^2$ and any $k > |ab|$, $\mathcal{O}(a,b,k)$ is a convex $\mathcal{C}^1$ curve.*

(ii) *For all $x \in \mathbb{R}^2$ there exists a half-plane $H_x(a,b)$ containing x on its boundary such that, for all $d \in H_x(a,b)$ we have $S(a,b,d) \geq S(a,b,x)$.*

Proof (i) Let $c \in \mathbb{R}^2$ be such that abc is an equilateral triangle, labelled clockwise. Let C be the circle circumscribed to abc. Recall that $(ab)^+$ denotes the open half-plane on the left of the line $\overline{ab}$ oriented from a to b, i.e., here not containing c. We describe $\mathcal{O}^+(a,b,k) = \mathcal{O}(a,b,k) \cap (ab)^+$.

Let $x \in \mathcal{O}^+(a,b,k)$. If $x \in (ca)^+$, then $\angle xab > \frac{2\pi}{3}$, hence the Steiner tree connecting a, b, x is the union of the line-segments xa and ab, and $\mathcal{O}^+(a,b,k) \cap (ca)^+$ is an arc of circle of radius $k - |ab|$ centred in a. If $x \in (ac)^+ \cap (cb)^+ \setminus \operatorname{conv} C$, then the Fermat–Torricelli point $s = t(a,b,x)$ is on C, and the Ptolemy theorem shows that $|sa| + |sb| = |sc|$, whence $\mathcal{O}^+(a,b,k) \cap (ac)^+ \cap (cb)^+$ is an arc of circle of centre c and radius k if $k \geq \frac{2}{\sqrt{3}}|ab|$. The curve $\mathcal{O}^+(a,b,k)$ is differentiable at its point on $\overline{ac}$, with a tangent line orthogonal to $\overline{ac}$. See Fig. 1 left.

If $k < \frac{2}{\sqrt{3}}|ab|$, then $\mathcal{O}^+(a,b,k)$ crosses C at some points d, d'. If $x \in (ac)^+ \cap (cb)^+ \cap \operatorname{conv} C$, then the Steiner tree connecting a, b, x is the union of the line-segments ax and xb, hence $\mathcal{O}^+(a,b,k) \cap (ac)^+ \cap (cb)^+ \cap \operatorname{conv} C$ is an arc of ellipse of foci a and b. At d, d' too, $\mathcal{O}^+(a,b,k)$ is differentiable. Indeed, on the one hand, the normal to the ellipse at d must bisect the angle $\widehat{adb}$, on the other hand, denoting by ω the centre of C, we have $\angle adc = \frac{1}{2}\angle a\omega c = \frac{\pi}{3} = \angle cdb$. It follows that both normals to the ellipse and to the circle of centre c coincide at d. See Fig. 1 right.

Now, the convex curve $\mathcal{O}^+(a,b,k)$ is fully described via the symmetry with respect to the bisector of ab, and further $\mathcal{O}(a,b,k)$ via the symmetry with respect to $\overline{ab}$.

(ii) Put $k = S(a,b,x)$, let D_x be the tangent of $\mathcal{O}(a,b,k)$ at x, and choose for $H_x(a,b)$ the half-plane bounded by D_x and not containing ab. □

Remark One can prove that, for all integers $n \geq 3$, all $a_1, \ldots, a_n \in \mathbb{R}^2$, and all $k > S(a_1, \ldots, a_n)$, the locus $\{x \in \mathbb{R}^2 \ ; \ S(a_1, \ldots, a_n, x) = k\}$ is a concatenation of convex $\mathcal{C}^1$ curves, themselves concatenations of arcs of circles and/or ellipses, with possible angular points $b_1, \ldots, b_r$. These points b_j are those for which the Steiner tree of $a_1, \ldots, a_n, b_j$ is not unique, and where this Steiner tree combinatorially changes.

Corollary 2.6 *If $v_1, \ldots, v_n \in \mathbb{R}^2$ and $a, b, c \in v_1 \ldots v_n$, then $S(a,b,c) \leq S(v_1, \ldots, v_n)$ and $f(a,b,c) \leq f(v_1, \ldots, v_n)$.*

Proof Since c (resp. b, a) is in the convex hull of $v_1, \ldots, v_n$, every half-plane bounded by a straight line passing through c (resp. b, a) contains at least one point v_i (resp. v_j, v_k). Choose $c' = v_i$ in the half-plane $H_c(a,b)$ given by Proposition 2.5(ii). Similarly, choose $b' = v_j$ in the half-plane $H_b(a,c')$, then choose $a' = v_k$ in the half-plane $H_a(b',c')$. Then we have

$$\begin{aligned} S(a,b,c) \leq S(a,b,c') &\leq S(a,b',c') \leq S(a',b',c') \\ &= S(v_k, v_j, v_i) \leq S(v_1, \ldots, v_n). \end{aligned}$$

Since $|abc| \leq |v_1 \ldots v_n|$, the inequality for f follows. □

Remark One could expect that, for $a_1, \ldots, a_m \in v_1 \ldots v_n$, we have $S(a_1, \ldots, a_m) \leq S(v_1, \ldots, v_n)$ as soon as $m < n$. This is however false, already for $m = 4$.

Indeed, given an equilateral triangle $\mathbb{T}_2 = abc$ of unit edge length, choose n points $v_1, \ldots, v_n$, some of them close to a, some close to b, and others close to c, such that abc is in their convex hull. Then $S(v_1, \ldots, v_n)$ is close to $S(a,b,c) = \sqrt{3}$, while for $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = \frac{1}{2}(a+b)$, one can check that one of the Steiner trees connecting $a_1, \ldots, a_4$ is obtained by joining a_1 to a_4, then taking the Steiner tree of a_2, a_3 and a_4, cf. Fig. 2; the other Steiner tree is symmetric with respect to

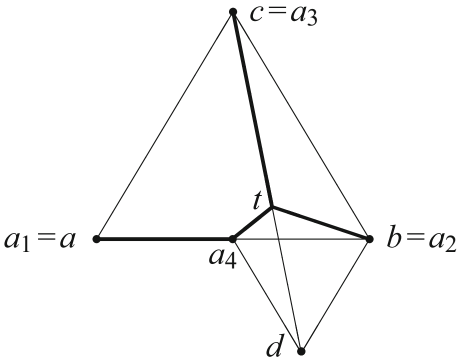


Fig. 2 In bold, a Steiner tree connecting $a_1, \dots, a_4$

the line $\overline{a_3a_4}$. If d is not on a_2a_3 but such that a_2a_4d is equilateral, and if t is the Fermat–Torricelli point associated to a_2, a_3, a_4, from $|ta_4| + |ta_2| = |td|$ one obtains

$$S(a_1, \dots, a_4) = |a_1a_4| + |a_3d| = \frac{1}{2}\left(1 + \sqrt{7}\right) > S(v_1, \dots, v_n).$$

Theorem 2.7 *Let K be a planar convex body, $x, y, z \in \operatorname{bd} K$, and $x', y', z' \in \mathbb{R}^2$, such that $\overline{xx'}$, $\overline{yy'}$, $\overline{zz'}$, are supporting lines of K and the vectors $x' - x$, $y' - y$, $z' - z$ point toward the direct sense on* bd K. *Assume that the order of x, y, z is also in the direct sense on* bd K *and that the Steiner tree determined by x, y, z has a vertex v inside* int K. *Put $\alpha_x = \angle x'xy$, $\beta_x = \angle x'xv$, $\gamma_x = \angle x'xz$, and, analogously, $\alpha_y, \beta_y, \gamma_y$ and $\alpha_z, \beta_z, \gamma_z$.*

If f attains a local minimum at x, y, z, then

$$\begin{aligned}\cos\alpha_x + \cos\beta_x + \cos\gamma_x &= 0,\\ \cos\alpha_y + \cos\beta_y + \cos\gamma_y &= 0,\\ \cos\alpha_z + \cos\beta_z + \cos\gamma_z &= 0.\end{aligned}$$

Moreover, bd K *is differentiable at x, y, z, and the normals at x, y, z are concurrent.*

Proof Consider for a moment $v = (v_1, v_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ fixed, and x variable on $T = \overline{xx'}$. Put $g(x) = |xy| + |xv| + |xz|$. Again for a moment, consider T to be the real line $\mathbb{R} \times \{0\}$, with the real number x corresponding to the point x as its abscissa. We have

$$g(x) = \sqrt{(x - y_1)^2 + y_2^2} + \sqrt{(x - v_1)^2 + v_2^2} + \sqrt{(x - z_1)^2 + z_2^2},$$

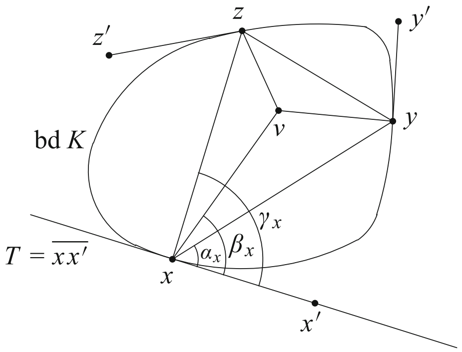


Fig. 3 The law of cosines

whence

$$g'(x) = \frac{x - y_1}{\sqrt{(x-y_1)^2 + y_2^2}} + \frac{x - v_1}{\sqrt{(x-v_1)^2 + v_2^2}} + \frac{x - z_1}{\sqrt{(x-z_1)^2 + z_2^2}}$$
$$= \cos\alpha_x + \cos\beta_x + \cos\gamma_x.$$

Now, assume

$$\cos\alpha_x + \cos\beta_x + \cos\gamma_x \neq 0.$$

Then, for some $\widetilde{x} \in T$ close to x, $g(\widetilde{x}) < g(x)$. This implies $|z\pi_K(\widetilde{x})| + |v\pi_K(\widetilde{x})| + |y\pi_K(\widetilde{x})| < g(x)$, which yields $f(\widetilde{x}, y, z) < f(x, y, z)$, impossible, see Fig. 3.

To prove the second part, suppose that some normals at x, y, z form a non-degenerate triangle. Then suitably and slightly turning the set $\{v, x, y, z\}$ about an interior point of that triangle brings all three points x, y, z in positions x'', y'', z'' outside K. But then

$$f(\pi_K(x''), \pi_K(y''), \pi_K(z'')) < f(x, y, z),$$

which is again impossible. □

In the rest of this section, we compute the infimum of f on some sets; this will be useful in Sect. 4.

Lemma 2.8 *We have* $\inf\{f(x, y, z)\ ;\ x, y, z \in \mathbb{S}_1, \mathbf{0} \in xyz\} = 6$.

Proof Recall the perimeter function p; we have $p(x, y, z) = |xyz|$ if $\operatorname{int} xyz \neq \emptyset$, and $p(x, y, z) = 2|xy|$ if $z \in xy$. It is easily seen that both p and S have no local minimum with distinct x, y, z. It follows that the global minimum of both functions is attained when two of the points x, y, z collapse, i.e., when xyz is a diameter of $\mathbb{S}_1$. □

Lemma 2.9 *If abc is an equilateral triangle of sides $I_1 = bc$, $I_2 = ca$, $I_3 = ab$ and side length δ, then* $\inf\{f(u_1, u_2, u_3)\ ;\ u_k \in I_k\} = \frac{3+\sqrt{3}}{2}\,\delta$. *This infimum is reached precisely at (a', b', c'), where a', b', c' are the midpoints of bc, ca, and ab respectively.*

Proof We prove that both functions p and S have a global minimum on $I_1 \times I_2 \times I_3$ at (a', b', c'), and that this minimum is uniquely reached for p.

The minimality of p implies the so-called incidence/reflection law: $\angle cu_1u_2 = \angle u_3u_1b$, denoted by α, and similarly $\angle au_2u_3 = \angle u_1u_2c = \beta$ and $\angle bu_3u_1 = \angle u_2u_3a = \gamma$. Since the measures of angles in a triangle sum to π, we have $\alpha + \beta = \beta + \gamma = \gamma + \alpha = \frac{2\pi}{3}$, yielding $\alpha = \beta = \gamma = \frac{\pi}{3}$. It follows that the minimum of p is reached only at (a', b', c').

For S, however, there is a 2-dimensional subset of $I_1 \times I_2 \times I_3$ where S attains its global minimum: choose arbitrarily $t \in abc$ and consider $u_1 = \pi_{bc}(t)$, $u_2 = \pi_{ca}(t)$, and $u_3 = \pi_{ab}(t)$. Then t is the Fermat–Torricelli point $t(u_1, u_2, u_3)$ and $S(u_1, u_2, u_3) = \frac{\sqrt{3}}{2}\,\delta$. □

Lemma 2.10 *If abcd is a parallelogram of sides $I_1 = ab$, $I_2 = bc$, $I_3 = cd$, $I_4 = da$, with $\delta = |ac| \le |bd|$, then* $\inf\{f(u_1, \ldots, u_4)\ ;\ u_k \in I_k\} = 3\delta$.

If abcd is not a rectangle, then this infimum is reached precisely at (a, c, c, a). If abcd is a rectangle, then the infimum is reached at both (a, c, c, a) and (b, b, d, d).

Proof As before, we prove that both functions p and S have a global minimum at the aforementioned points. Assume that $abcd$ is not a rectangle, and put $\beta = \angle abc$. We have $\beta < \frac{\pi}{2}$. Firstly, observe that the perimeter function p has no local minimum at a quadruple $(u_1, \ldots, u_4)$ with $u_i \notin \{a, b, c, d\}$ for every i. Indeed, at such four points, the minimality of p implies the incidence/reflection law, which yields $4\beta = 2\pi$, a contradiction. Hence the minimum is attained, say, for $u_1 = a$ or b. Then one easily shows that $u_4 = u_1$ in the first case and $u_2 = u_1$ in the second, and finally that the global minimum of p is attained at $u_1 = u_4 = a$, $u_2 = u_3 = c$.

In the Steiner tree connecting $u_1, \ldots, u_4$, let x_1 be the neighbour of u_1, and x_3 that of u_3, with possibly $x_i = u_i$ for some of the i's. Then

$$\begin{aligned} S(u_1, \ldots, u_4) &= |u_4x_1| + |u_1x_1| + |x_1x_3| + |x_3u_3| + |x_3u_2| \\ &\ge |a\pi_{ab}(x_1)| + |\pi_{ab}(x_1)x_1| + |x_1x_3| + |x_3\pi_{cd}(x_3)| + |\pi_{cd}(x_3)c| \\ &\ge |ac| \end{aligned}$$

if x_1 is the neighbour of u_4 and x_3 the neighbour of u_2 in the Steiner tree, see Fig. 4 left, and

$$\begin{aligned} S(u_1, \ldots, u_4) &= |u_4x_3| + |u_3x_3| + |x_3x_1| + |x_1u_1| + |x_1u_2| \\ &\ge |\pi_{cd}(a)\pi_{cd}(x_3)| + |\pi_{cd}(x_3)x_3| + |x_3x_1| \\ &\quad + |x_1\pi_{ab}(x_1)| + |\pi_{ab}(x_1)\pi_{ab}(c)| \\ &\ge |\pi_{cd}(a)\pi_{ab}(c)| = |ac| \end{aligned}$$

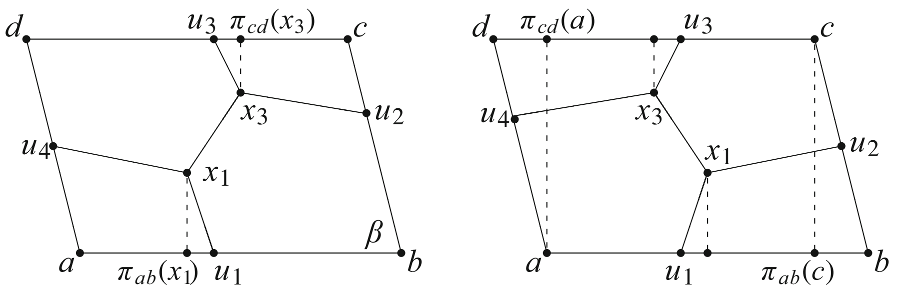


Fig. 4 Left: x_1 is the neighbour of u_4 in the Steiner tree; right: x_1 is the neighbour of u_2

in the case x_1 is the neighbour of u_2 and x_3 that of u_4, see Fig. 4 right. Hence, both functions p and S attain their global minimum at $u_1 = u_4 = a$, $u_2 = u_3 = c$, and this remains true for their sum. The case of a rectangle is clear. $\square$

3 Points Almost Fixing Planar Convex Bodies

In the sense of Baire categories, for most convex bodies $K \subset \mathbb{R}^2$, the inscribed circle C is unique and touches bd K in exactly three points a, b, c, with $o \in \operatorname{int} abc$ [12], where o is the centre of C. Then, obviously, the points a, b, c fix K. However, there are convex bodies which cannot be fixed by any set of three points, e.g. parallelograms. A set of points on the boundary of K is *fixing* K if any small move of K brings some point of the set in int K. More precisely, $a_1, \dots, a_n \in \operatorname{bd} K$ fix K if there is a neighbourhood $\mathcal{V}$ of the identity $\mathbf{id}$ in the group $\operatorname{Isom}^+\mathbb{R}^2$ of planar rotations and translations, such that, for every $f \in \mathcal{V}$ satisfying $f(K) \neq K$, at least one of the a_i belongs to int $f(K)$. This is slightly different from another commonly used definition of fixing, e.g. in [3,4], where a subset H of bd K is said to fix K if $\mathbf{id}$ is isolated in the set $\left\{f \in \operatorname{Isom}^+\mathbb{R}^2\ ;\ H \cap f(\operatorname{int} K) = \emptyset\right\}$. We consider that a rotation of a disc around its centre does not move the disc. Observe that both definitions are equivalent if K is not a disc, since in this case there is a neighbourhood $\mathcal{V}$ of $\mathbf{id}$ in $\operatorname{Isom}^+\mathbb{R}^2$ such that every $f \in \mathcal{V} \setminus \{\mathbf{id}\}$ satisfies $f(K) \neq K$. We now introduce the following related notion.

The points $a_1, \dots, a_n \in \operatorname{bd} K$ *almost fix* the convex body $K \subset \mathbb{R}^2$ if, for any $i \in \{1, \dots, n\}$ and any neighbourhood V_i of a_i, there is a pair of points $a_i', a_i'' \in V_i \cap \operatorname{bd} K$, such that the points $a_1', a_1'', \dots, a_n', a_n''$ fix K.

Theorem 3.1 *For any planar convex body K, there are two or three points almost fixing K.*

Proof If K is a disc, then it is obviously almost fixed by two diametrically opposite boundary points. So, assume from now on that K is not a disc.

Suppose without loss of generality that $C = \mathbb{S}_1$ is an inscribed circle of K.
Case 1. There exists no $a \in C \cap \operatorname{bd} K$ with $-a \in C \cap \operatorname{bd} K$.

Let $a \in C \cap \operatorname{bd} K$ and A be the component of $-a$ in $C \cap \operatorname{int} K$. The set A is an open arc $\widetilde{a'a''} \subset C$ of length less than π. Clearly, the triangle $aa'a''$ is acute and a, a', a'' fix K.

Case 2. There exists $a \in C \cap \operatorname{bd} K$ with $-a \in C \cap \operatorname{bd} K$.

Let $\widetilde{a_1a_2} \subset C$ be the connected component of a in $C \cap \operatorname{bd} K$. Then $\widetilde{a_1a_2} \neq C$ because K is not a disc. If $-a \in \widetilde{a_1a_2}$ and $\lambda\widetilde{a_1a_2} > \pi$ then K is fixed by a_1, a_2 and the midpoint a_3 of $\widetilde{a_1a_2}$. The case $\lambda\widetilde{a_1a_2} = \pi$ (hence $a_1 = \pm a$ and $a_2 = \mp a$) will be treated later on.

If $-a \notin \widetilde{a_1a_2}$, let $\widetilde{a_1^*a_2^*} \subset C$ be the component of $-a$ in $C \cap \operatorname{bd} K$. Then $\widetilde{a_1a_2}$ and $\widetilde{a_1^*a_2^*}$ are disjoint.

Suppose $a_1, a_2 \neq a$ or $a_1^*, a_2^* \neq -a$, say $a_1^*, a_2^* \neq -a$. If $\lambda\widetilde{a_1^*a_2^*} < \pi$, then the triangle $aa_1^*a_2^*$ is acute and a, a_1^*, a_2^* fix K. If $\lambda\widetilde{a_1^*a_2^*} \geq \pi$, then $\lambda\widetilde{a_1a_2} < \pi$. Let m be the midpoint of $\widetilde{a_1a_2}$. Since $m \notin \widetilde{a_1^*a_2^*}$ and $\lambda\widetilde{a_1^*a_2^*} \geq \pi$, we have $-m \in \widetilde{a_1^*a_2^*}$. Then the triangle $(-m)a_1a_2$ is acute, and its vertices fix K.

If $a_1 = a$, $a_1^* = -a$, and $\widetilde{a_1a_2}$, $\widetilde{a_1^*a_2^*}$ are non-degenerate and do not lie both on the same half of C, then we are in the already treated case for some point in the relative interior of $\widetilde{a_1a_2}$ instead of a.

It remains the case that at least one of $\widetilde{a_1a_2}$, $\widetilde{a_1^*a_2^*}$ is degenerate or they lie both on the same half of C.

Let the lines $\Lambda \ni a$, $-\Lambda \ni -a$ be orthogonal to $\overline{-aa}$. Put $bc = \Lambda \cap \operatorname{bd} K$, with b, c perhaps not distinct. Take small arcs $\widetilde{b'b''}$, $\widetilde{c'c''}$, $\widetilde{a'a''} \subset \operatorname{bd} K$ containing $b, c, -a$, respectively, in their relative interior. Then $a', a'', b', b'', c', c''$ fix K, and consequently $-a, b, c$ almost fix K. This remains valid for bc degenerate ($b = c$); in that case $a, -a$ almost fix K. $\square$

Our next statement presents useful criteria for a finite collection of points to fix K or almost fix K. The proof can be found in [7]. We first introduce some notation.

Given a planar convex body K, with $\operatorname{bd} K$ oriented counterclockwise, and $a \in \operatorname{bd} K$, let $T_\ell(a)$, $T_r(a)$ be the left, respectively right, tangent line at $\operatorname{bd} K$ in a. We orient these lines as $\operatorname{bd} K$, i.e., so that $\operatorname{int} K \subset T_\ell(a)^+ \cap T_r(a)^+$. Let $N_\ell(a)$ and $N_r(a)$ be the left and right normals at $\operatorname{bd} K$ in a, oriented in the directions $T_\ell(a)^+$ and $T_r(a)^+$ respectively. Let $L(a)$ be the open sector, union of the left open half-planes bounded by $N_\ell(a)$ and $N_r(a)$:

$$L(a) = N_\ell(a)^+ \cup N_r(a)^+.$$

Let $\overline{L}(a)$ be the corresponding closed sector, and let

$$\vec{L}(a) = \{x \in \mathbb{S}_1\ ;\ a + x \in \overline{L}(a)\}$$

be the *set of directions* of $\overline{L}(a)$; it is a compact subset of $\mathbb{S}_1$. Let $R(a)$ and $\overline{R}(a)$ be the analogous sectors for the right half-planes. Observe that the set of directions of $\overline{R}(a)$ is $-\vec{L}(a)$, hence will not be needed. If $\operatorname{bd} K$ is differentiable at a, then $\overline{L}(a) = \mathbb{R}^2 \setminus R(a)$, otherwise $L(a) \cap R(a)$ is the union of two sectors of vertex a. We will also use the intersection sectors:

$$\ell(a) = N_\ell(a)^+ \cap N_r(a)^+ = \mathbb{R}^2 \setminus \overline{R}(a),$$

$r(a) = \mathbb{R}^2 \setminus \overline{L}(a)$, their corresponding closed sectors $\overline{\ell}(a)$ and $\overline{r}(a)$, and the set of directions of $\overline{\ell}(a)$: $\vec{\ell}(a) = \{x \in \mathbb{S}_1 \ ; \ a + x \in \overline{\ell}(a)\}$.

Now we present the result from [7] that we are going to use.

Theorem 3.2 [7] *Let K be a planar convex body and $a_1 \dots, a_n \in \operatorname{bd} K$.*

(i) *If $a_1, \dots, a_n$ fix K, then both intersections $L(a_1) \cap \dots \cap L(a_n)$ and $R(a_1) \cap \dots \cap R(a_n)$ are empty.*
(ii) *If the three intersections $\overline{L}(a_1) \cap \dots \cap \overline{L}(a_n)$, $\overline{R}(a_1) \cap \dots \cap \overline{R}(a_n)$, and $\vec{L}(a_1) \cap \dots \cap \vec{L}(a_n)$ are empty, then $a_1, \dots, a_n$ fix K.*
(iii) *If $a_1, \dots, a_n$ almost fix K, then both intersections $\ell(a_1) \cap \dots \cap \ell(a_n)$ and $r(a_1) \cap \dots \cap r(a_n)$ are empty.*
(iv) *If the three intersections $\overline{\ell}(a_1) \cap \dots \cap \overline{\ell}(a_n)$, $\overline{r}(a_1) \cap \dots \cap \overline{r}(a_n)$, and $\vec{\ell}(a_1) \cap \dots \cap \vec{\ell}(a_n)$ are empty, then $a_1, \dots, a_n$ almost fix K.*

Remarks The set of directions is needed in items (ii) and (iv): If $K = [-2, 2] \times [0, 1]$ then the three points $a = (-1, 0)$, $b = (1, 0)$, $c = (0, 1)$ do not fix K although the intersections $\overline{L}(a) \cap \overline{L}(b) \cap \overline{L}(c)$ and $\overline{R}(a) \cap \overline{R}(b) \cap \overline{R}(c)$ are empty.

Another commonly used definition of fixing points is the following one [6]: The points $a_1, \dots, a_n \in \operatorname{bd} K$ *weakly fix* K if, for any path $\gamma : [0, 1] \to \operatorname{Isom}^+\mathbb{R}^2$, $t \mapsto \gamma_t$ such that $\gamma_0(K) = K$ and $\gamma_1(K) \neq K$, there exist $i \in \{1, \dots, n\}$ and $t \in [0, 1]$ such that $a_i \in \operatorname{int} \gamma_t(K)$. Obviously, if $a_1, \dots, a_n$ fix K in our first sense, then they weakly fix K. Example 3 of [7] shows that the converse, however, is not true. Nevertheless Theorem 3.2 remains valid with this notion of weakly fixing.

In [4] a condition of second order, i.e., using the curvature of $\operatorname{bd} K$, is given such that three points fix a convex body K with a $\mathcal{C}^2$ boundary.

The example below shows that, although every planar convex body can be almost fixed by at most three points, sometimes it may be more economical to use more points to fix it, in the following sense: it is possible to have inequality (2) below

$$\begin{aligned} &\inf\{f(a, b, c, d) \ ; \ a, b, c, d \text{ almost fix } K\} \\ &\quad < \inf\{f(a, b, c) \ ; \ a, b, c \text{ almost fix } K\}. \end{aligned} \tag{2}$$

Example With $\varepsilon > 0$ small, let K be the parallelogram $a_0b_0c_0d_0$, with $a_0 = (-2, -\varepsilon)$, $b_0 = (1, -\varepsilon)$, $c_0 = (2, \varepsilon)$, and $d_0 = (-1, \varepsilon)$. Its centre is $\omega = (0, 0)$, its ratio of side lengths is almost 3 and its angles are $\delta = \arctan 2\varepsilon$ and $\pi - \delta$, see Fig. 5.

Using Theorem 3.2(i), it is rather tedious but elementary to prove that

$$\inf\{f(a, b, c, d) \ ; \ a, b, c, d \text{ almost fix } K\}$$

is attained for $a' = (-x, -\varepsilon)$, $b' = b_0$, $c' = (x, \varepsilon)$, and $d' = d_0$, where $0 < x < 1$ is such that the angles $\alpha = \angle b_0c'c_0$ and $\beta = \angle a'c'd_0$ satisfy $2 \cos \alpha = 1 + \cos \beta$.

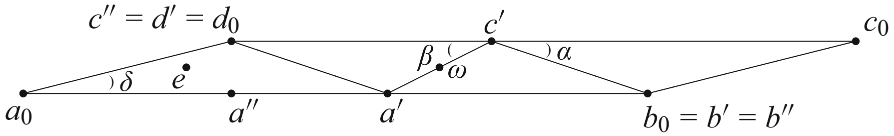


Fig. 5 An example of convex body satisfying (2)

Using $\tan\alpha = \frac{2\varepsilon}{1-x}$ and $\tan\beta = \frac{\varepsilon}{x}$, we obtain $2\alpha^2 = \beta^2 + \mathcal{O}(\varepsilon^4)$, hence $x = (1+2\sqrt{2})^{-1} + \mathcal{O}(\varepsilon)$.

Since the Steiner tree of a', b', c', d' is the polygonal line $d'a' \cup a'c' \cup c'b'$ as soon as $\alpha+\beta < \frac{\pi}{6}$, we have $f(a', b_0, c', d_0) = 2|d_0a'| + |a'b_0| + 2|b_0c'| + |c'd_0| + |a'c'|$, hence

$$\inf\{f(a,b,c,d)\ ;\ a,b,c,d \text{ almost fix } K\} = 6 + \mathcal{O}(\varepsilon^2).$$

Besides, one easily finds that $\inf\{f(a,b,c)\ ;\ a,b,c \text{ almost fix } K\}$ is attained for $a'' = (-1,-\varepsilon)$, $b'' = b_0$, and $c'' = d_0$, yielding

$$\inf\{f(a,b,c)\ ;\ a,b,c \text{ almost fix } K\} = 6 + (\sqrt{3}+2)\varepsilon + \mathcal{O}(\varepsilon^2).$$

We used here that the length $S(a'', b_0, d_0)$ of the Steiner tree joining a'', b_0, and d_0 is $|eb_0| = 2 + \sqrt{3}\varepsilon + \mathcal{O}(\varepsilon^2)$, where $e = (-1-\sqrt{3}\varepsilon, 0)$ is such that $a''d_0e$ is equilateral. As a consequence, (2) is satisfied if ε is small enough.

4 Cages for Planar Convex Bodies in $\mathbb{R}^3$

Let $r(K)$ denote the inradius of the planar convex body K.

Theorem 4.1 *For any planar convex body K, $L(K) \geq 6r(K)$.*

Proof We may suppose $r(K) = 1$, and $\mathbb{B}_2$ to be inscribed in K.

Let G be a cage holding K, and set $G_0 = G \cap \operatorname{aff} K$. Denote by $v_1, \ldots, v_m$ the vertices of G, and write $w_i = \pi_{\operatorname{aff} K}(v_i)$. Take the labelling so that $w_1, \ldots, w_n$ are the external vertices of $\pi_{\operatorname{aff} K}(G)$.

Case 1. $\mathbf{0} \in \operatorname{conv} G_0$.

By the Carathéodory Theorem, there exist $a, b, c \in G_0$ such that $\mathbf{0} \in abc$. Clearly,

$$\mathbf{0} \in \pi_{\mathbb{B}_2}(a)\pi_{\mathbb{B}_2}(b)\pi_{\mathbb{B}_2}(c),$$

too. By successively using Theorem 2.4, Corollary 2.6, and Lemmas 2.3 and 2.8, we obtain

$$\lambda G \geq f(w_1, \ldots, w_n) \geq f(a,b,c) \geq f(\pi_{\mathbb{B}_2}(a), \pi_{\mathbb{B}_2}(b), \pi_{\mathbb{B}_2}(c)) \geq 6.$$

Case 2. $\mathbf{0} \notin \operatorname{conv} G_0$.

We claim that $\operatorname{diam} G > 2$, yielding the result since G is 3-connected.

Consider the shortest arc of $\mathbb{S}_1$ which contains $\mathbb{S}_1 \cap \operatorname{conv}(\{\mathbf{0}\} \cup G_0)$, and denote by α, β its endpoints. Take $a \in \overline{\mathbf{0}\alpha} \cap G_0$, $b \in \overline{\mathbf{0}\beta} \cap G_0$, and consider the diameter $c(-c)$ of $\mathbb{S}_1$, parallel to $\overline{ab}$. We obtain the trapezoid $ab(-c)c$. There are two half-lines starting at a and supporting K, one of which, say L_a, meets $\overline{c(-c)}$ at some point a', with $c \in \overline{\mathbf{0}a'}$. Analogously, some supporting half-line L_b of K from b meets $\overline{c(-c)}$ at b', with $-c \in \overline{\mathbf{0}b'}$. If $\|a-b\| \le 2$ then $\|a'-b'\| \ge \|a-b\|$ and K could escape from the cage via $\operatorname{conv}(L_a \cup L_b \cup K)$, which contradicts the assumption. Hence, indeed, $\operatorname{diam} G \ge \operatorname{diam} G_0 \ge \|a-b\| > 2$. □

Theorems 4.3 and 4.6 below provide a link between points almost fixing a planar convex body and cages holding that body. We will use the following refinement of Kovalyov's theorem.

Lemma 4.2 *Let K be a convex body in $\mathbb{R}^2$ and $\mu \in [0, 1[$. If $f_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f_\mu(x, y) = (x, \mu y)$, then*

(i) *a copy of $f_\mu(K)$ fits into K, i.e., there exists $i \in \operatorname{Isom}^+\mathbb{R}^2$ such that $i(f_\mu(K)) \subset K$, and*

(ii) *moreover, i can be chosen arbitrarily close to* **id** *if μ is close enough to 1, that is:*

For any $\varepsilon > 0$ there exists $\mu_0 < 1$ such that for all $\mu \in [\mu_0, 1[$ there exists $i \in \operatorname{Isom}^+\mathbb{R}^2$ satisfying

$$i(f_\mu(K)) \subset K \quad \text{and} \quad \|i(x) - x\| < \varepsilon \quad \text{for all} \quad x \in K.$$

Proof Item (i) is Kovalyov's theorem, see [8]. Item (ii) follows directly from the proof of Kós and Töröcsik in [9]: As μ tends to 1, the points E' and F' of their proof tend to E and F, hence the isometry i tends to **id** . □

Theorem 4.3 *Let K be a planar convex body in $\mathbb{R}^3$ and $a, b \in \operatorname{rel\,bd} K$. If a, b almost fix K in $\operatorname{aff} K$, then $L(K) \le f(a, b)$.*

Proof The plane $P_0 = \operatorname{aff} K$ is thought to be the horizontal plane.

Case 1. $\overline{ab} \not\subset \operatorname{rel\,bd} K$.

Let a', a'' be close to a and b', b'' close to b such that a', a'', b', b'' fix K. It does not matter whether a' and a'' are on each side of a or not. These points are labelled so that the quadrilateral $a'b'b''a''$ is convex, with $a''b'' \in (a'b')^+$. Let D_0 denote the bisector of $\overline{a'b'}$ and $\overline{a''b''}$.

With $\alpha > 0$ small enough, consider the plane P' passing through $a'b'$ such that $\angle(P_0, P') = \alpha$, with $a''b''$ below P'. Similarly, let P'' be the plane containing $a''b''$, with $\angle(P_0, P') = \alpha$ and $a'b'$ below P'', see Fig. 6 left. Let P be the plane parallel to P_0, below P_0, at a distance α^2. Set $D = P' \cap P''$, $D' = P \cap P'$ and $D'' = P \cap P''$. Observe that $\pi_{P_0}(D) = D_0$, that D' is parallel to $\overline{a'b'}$, and that D'' is parallel to $\overline{a''b''}$.

Let $\delta > 0$ be such that the four measures of angles $\angle(T_r(a), \overline{ab})$, $\angle(\overline{ab}, T_\ell(b))$, $\angle(T_r(b), \overline{ba})$, and $\angle(\overline{ba}, T_\ell(a))$ are at least 5δ. By continuity of the left and right tangent lines, if a', a'' are close enough to a and b', b'' close enough to b, then all measures $\angle(T_r(a'), D_0)$, $\angle(D_0, T_\ell(b'))$, $\angle(T_r(b''), -D_0)$, and $\angle(-D_0, T_\ell(a''))$ are at least 4δ.

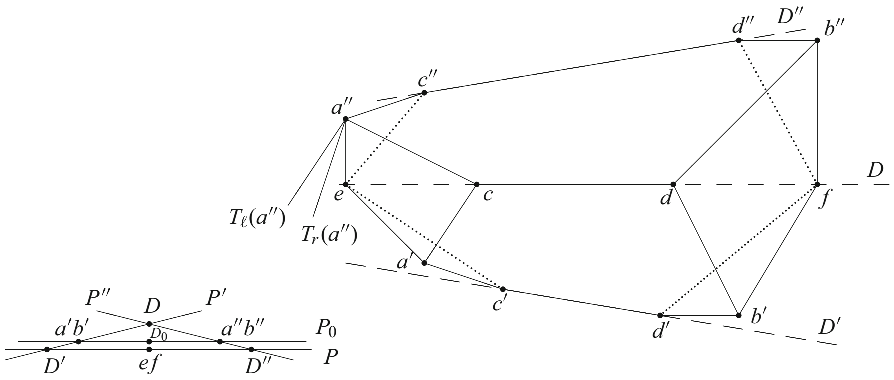


Fig. 6 A cage holding K in the case 1. Left: front view in the case $a'b'$ and $a''b''$ are parallel. Right: view from above

Choose $c, d \in D$, $c', d' \in D'$, and $c'', d'' \in D''$, c, c', c'' close to a and d, d', d'' close to b, such that their projections $c_0 = \pi_{P_0}(c), \ldots, d''_0 = \pi_{P_0}(d'')$ satisfy

$$\angle(\overline{c''_0a''}, T_\ell(a'')), \angle(T_r(a''), \overline{a''c''_0}), \angle(\overline{c'_0a'}, T_\ell(a')) \text{ and } \angle(T_r(a'), \overline{a'c'_0}) \geq 3\delta \quad (3)$$

and similarly at points b', b''. We will see later that the eight line-segments $a'c$, $a'c'$, $a''c$, $a''c''$, $b'd$, $b'd'$, $b''d$, and $b''d''$ hold K. Therefore, as holding cage, we are led to consider the 1-skeleton of the polyhedron $\Pi_0 = a'a''cc'c''b'b''dd'd''$ (the convex hull of $\{a', a'', c, c', c'', b', b'', d, d', d''\}$). Nevertheless we wish to eliminate the edges $a'a''$ and $b'b''$, because they might cross $\operatorname{rel\,int} K$. For this reason, we add to Π_0 two vertices $e \in \operatorname{aff}(a'a''c)$ and $f \in \operatorname{aff}(b'b''d)$, say $e, f \in P$, such that $\pi_{P_0}(e)$ and $\pi_{P_0}(f)$ belong to D_0. The resulting polyhedron $\Pi = a'a''cc'c''b'b''dd'd''ef$ has three hexagonal faces: $a'cdb'd'c'$, $a''cdb''d''c''$, and $ec'd'fd''c''$, two quadrilateral faces: $a'ca''e$ and $b'db''f$, and four triangular faces: $a'ec'$, $a''ec''$, $b'fd'$, and $b''fd''$.

Three edges, namely cd, $c'd'$, and $c''d''$, have a length close to $|ab|$ and the other sixteen have an arbitrarily small length, hence the length of the 1-skeleton of Π, denoted by G in the sequel, is close to $3|ab| = f(a, b)$.

It remains to prove that G holds K if α is small enough. We assume in particular that $\alpha \leq \delta$. By contradiction, suppose that there exists $\varphi\colon [0, 1] \to \operatorname{Isom}^+\mathbb{R}^3$, $t \mapsto \varphi_t$, continuous such that $\varphi_0 = \mathbf{id}$, $\varphi_1(K)$ is far away from K, and $G \cap \operatorname{rel\,int} K_t = \emptyset$ for all $t \in [0, 1]$, where $K_t = \varphi_t(K)$. We assume without loss of generality that $K_t \neq K$ for all $t > 0$.

Let us fix $t > 0$ so small that $\angle(\Delta, \varphi_t(\Delta)) \leq \delta$ for any line Δ. In particular, we have $\angle(P_0, P_t) \leq \delta$, where $P_t = \varphi_t(P_0)$. If t is small enough, then the plane P_t cuts at least four of the eight aforementioned edges of G in four points a_1, a_2, b_1, b_2 close to a', a'', b', b'' respectively. More precisely, we have $a_1 \in P_t \cap (a'c \cup a'c')$, $a_2 \in P_t \cap (a''c \cup a''c'')$, $b_1 \in P_t \cap (b'd \cup b'd')$, and $b_2 \in P_t \cap (b''d \cup b''d'')$.

Let $\pi : P_0 \to P_t$ be the restriction to P_0 of the orthogonal projection onto P_t. The image by π of any point $x \in P_0$ will be denoted by $\widetilde{x}$. Since $\angle a'a_1\widetilde{a'} \leq \alpha \leq \delta$ and the tangent lines at K_t in $\widetilde{a'}$ make an angle at most δ with the corresponding tangent lines at K in a', inequality (3) ensures that, in the plane P_t, the line $\overline{a_1\widetilde{a'}}$ points outwards from $K_t = \varphi_t(K)$, making an angle of measure at least δ with the left and right tangent lines at K_t. The same holds for $\overline{a_2\widetilde{a''}}$, $\overline{b_1\widetilde{b'}}$ and $\overline{b_2\widetilde{b''}}$. By assumption, $a_1, a_2, b_1, b_2 \notin \operatorname{rel\,int} K_t$, whence $\widetilde{a'}, \widetilde{a''}, \widetilde{b'}, \widetilde{b''} \notin \operatorname{rel\,int} K_t$; hence there exist four lines $D(\widetilde{x})$, $\widetilde{x} = \widetilde{a'}, \widetilde{a''}, \widetilde{b'}, \widetilde{b''}$, with $\widetilde{x} \in D(\widetilde{x})$, such that K_t is in each of the four left half-planes $D(\widetilde{x})^+$. Now, by Lemma 4.2, a copy of $\widetilde{K} = \pi(K)$, arbitrarily close to $\widetilde{K}$, fits into K_t. Using the inverse map $\pi^{-1}\colon P_t \to P_0$, we obtain that a copy of K, arbitrarily close to K, fits into the intersection of the half-planes $D(x)^+$, with $x = a', b', a'', b''$, where $D(x) = \pi^{-1}(D(\widetilde{x}))$. Since a', a'', b', b'' fix K, this copy of K close to K must be K itself, and the points a_1, a_2, b_1, b_2 must coincide with a', a'', b', b'' respectively, hence $K_t = K$, a contradiction.

Case 2. $ab \subset \operatorname{rel\,bd} K$.

We orient $P_0 = \operatorname{aff} K$ so that $\operatorname{rel\,int} K \subset (ab)^+$. We first prove that the points a', a'', b', b'' fixing K, a', a'' close to a and b', b'' close to b, can be chosen such that $a', b' \in ab$ and $a'', b'' \in (ab)^+$, i.e., such that a', a'' are on each side of a and b', b'' on each side of b. Observe that $\overline{ab}$ is the right tangent line at rel bd K in a and the left tangent line in b. Denote by A the left tangent line in a and by B the right tangent line in b, both with the same orientation as rel bd K, i.e., with $\operatorname{rel\,int} K \subset A^+ \cap B^+$. Since a, b almost fix K, using Theorem 3.2(iii) we have that both measures $\angle(A, \overline{ab})$ and $\angle(\overline{ab}, B)$ are at least $\frac{\pi}{2}$. Moreover, if $\angle(A, \overline{ab}) = \angle(\overline{ab}, B) = \frac{\pi}{2}$, we also have $A \cap \operatorname{rel\,bd} K = \{a\}$ or $B \cap \operatorname{rel\,bd} K = \{b\}$ (or both).

For short, we say that rel bd K *has a right angle at a* if $\angle(A, \overline{ab}) = \frac{\pi}{2}$ and $A \cap \operatorname{rel\,bd} K \neq \{a\}$.

If rel bd K has no right angle neither at a nor at b, then using Theorem 3.2(ii), we easily verify that for any $a', b' \in \,]ab[\,$, $a'', b'' \in (ab)^+ \cap \operatorname{rel\,bd} K$, a', a'' close to a and b', b'' close to b, with $\operatorname{dist}(a'', ab) = \operatorname{dist}(b'', ab)$, the points a', a'', b', b'' fix K.

If rel bd K has a right angle at a, then we first choose $a', b' \in \,]ab[$ and $b'' \in (ab)^+ \cap \operatorname{rel\,bd} K$, a' close to a and b', b'' close to b. Then we choose $a'' \in (ab)^+ \cap \operatorname{rel\,bd} K$, a'' close to a, such that $\operatorname{dist}(a'', ab) < \operatorname{dist}(b'', ab)$ and $a'' \notin N_\ell(b'')^+$ where $N_\ell(b'')$ is the left normal line at rel bd K in b''. Then Theorem 3.2(ii) implies that the points a', a'', b', b'' fix K, see Fig. 7.

Consider the two planes P, P' containing $a'b'$ and making a small angle of measure α with P_0, with $a''b''$ above P and below P'. Let P'' be the plane containing $a''b''$ and perpendicular to P_0. Choose $c, d \in P \cap P''$ and $c', d' \in P' \cap P''$, c, c' close to a and

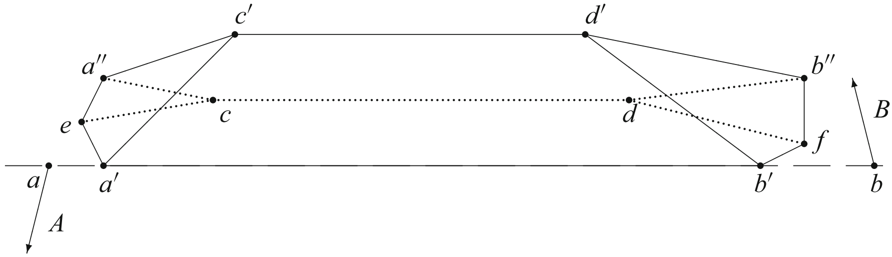


Fig. 7 Perspective view of a cage holding K in the case 2

d, d' close to b, such that the measures of angles $\angle xyz$ are all small enough, where $xyz = b'a'c, b'a'c', a'b'd, a'b'd', b''a''c, b''a''c', a''b''d$, and $a''b''d'$ respectively. As before, we add two vertices $e \in P \cap \operatorname{aff}(a'a''c')$ and $f \in P \cap \operatorname{aff}(b'b''d')$, say with $\|a' - e\| = \|a'' - e\|$ and $\|b' - f\| = \|b'' - f\|$, in order to eliminate the edges $a'a''$ and $b'b''$. The heptahedron $\Pi = a'a''cc'eb'b''dd'f$ has two hexagonal faces $a'ecdfb'$ and $a''cdb''d'c'$, three quadrilateral faces $a'c'd'b'$, $a'c'a''e$ and $b'd'b''f$, and two triangular faces $a''ec$ and $b''fd$. We prove similarly as in case 1 that, for α small enough, the 1-skeleton of Π is a cage holding K. □

We say that the convex body K is *weakly strictly convex* if it possesses two parallel supporting hyperplanes H, H' at distance wid K from each other, such that $H \cap K$ is a single point. Recall that wid K is the width of K.

A consequence of Theorem 4.3 is the following.

Corollary 4.4 *For any planar weakly strictly convex body K in $\mathbb{R}^3$, we have $L(K) \leq$ 3wid K.*

Proof Take the two supporting lines H, H' given by the definition of weakly strict convexity, and the point $\{a\} = H \cap K$. Then the orthogonal projection b of a onto H' belongs to $H' \cap K$.

With $\varepsilon > 0$ small, consider the two lines parallel to $\overline{ab}$ and at distance ε from $\overline{ab}$. These lines cut rel bd K in four points a', a'', b', b'' which fix K. This proves that a, b almost fix K and Theorem 4.3 applies. □

An immediate consequence of Corollary 4.4 and Theorem 4.1 is the following.

Corollary 4.5 *Let K be a planar weakly strictly convex body. Assume that K contains a disc of diameter* wid K. *Then we have $L(K) =$* 3wid K.

In particular, for the unit two-dimensional disc $\mathbb{B}_2$, we have $L(\mathbb{B}_2) = 6$.

Corollary 4.5 also follows from the fact that any cage with $\mathbb{S}_2$ as circumscribed sphere has length more than 6. This was proven by Lillington [10], and generalized to higher dimensions by Linhart [11].

Theorem 4.6 *Let K be a planar convex body in $\mathbb{R}^3$ and $a, b, c \in$* rel bd K *be three points that almost fix K. Then one has $L(K) \leq f(a, b, c)$ in the two following situations:*

(i) rel bd K *contains at most one of the line-segments ab, bc and ca.*
(ii) rel bd K *contains two of the line-segments and K is weakly strictly convex.*

Remarks If the three line-segments ab, ac, and bc are in rel bd K, then K is the triangle abc and $L(K)$ is given by Theorem 4.7 in the sequel.

The example at the end of Sect. 3 shows that the statement of Theorem 4.6 would be false with four points instead of three. Actually, it can be shown that the parallelogram $a'b_0c'd_0$ of Fig. 5 plus its Steiner tree $d_0a' \cup a'c' \cup c'b_0$ cannot be approximated by any cage holding the parallelogram $K = a_0b_0c_0d_0$.

The question remains open whether the inequality is still valid when K is not weakly strictly convex.

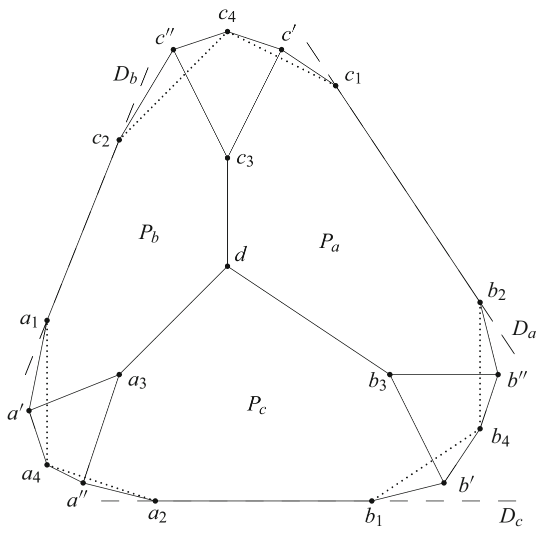


Fig. 8 A cage holding K in the case 1 of Theorem 4.6(i)

Proof If a, b, c are collinear, with $c \in ab$, then we prove, as at the beginning of the proof of case 2 of Theorem 4.3 above, that a, b almost fix K, and Theorem 4.3 applies, since $f(a, b, c) = f(a, b)$. We now assume that a, b, c are not collinear.

(i) *Case 1: None of the line-segments ab, bc or ca is in* rel bd K.

Let $d_0 = t(a, b, c)$ be the Fermat–Torricelli point of a, b, c. We first assume that d_0 is distinct from a, b and c.

For any arbitrarily small $\varepsilon > 0$, let $a', a'', b', b'', c', c'' \in \operatorname{rel\,bd} K$ be six points fixing K, labelled in that cyclic order on rel bd K, with $|aa'|, \dots, |cc''| < \varepsilon^2$.

Let u be the unit vector orthogonal to $P_0 = \operatorname{aff} K$ pointing upward, and put $d = d_0 + \varepsilon u$. Let P be the plane parallel to P_0, below P_0, at a distance ε^3. The plane $P_c = \operatorname{aff}(a''b'd)$ cuts P along a line denoted by D_c, parallel to $a''b'$ and at a distance of order ε^2 from it, see Fig. 8. Take $a_2, b_1 \in D_c$, a_2 at a distance of order ε from a'' and b_1 at a distance of order ε from b', in such a manner that the angles $\widehat{a_2a''b'}$ and $\widehat{b_1b'a''}$ have measures of order ε. Similarly, let $P_a = \operatorname{aff}(b''c'd)$, $D_a = P_a \cap P$, $P_b = \operatorname{aff}(c''a'd)$, $D_b = P_b \cap P$ and choose $b_2, c_1 \in D_a$ and $c_2, a_1 \in D_b$ analogously to a_2 and b_1.

Let $a_3 \in P_b \cap P_c$ be at a distance of order ε from a, so that $\angle(a_3a'd)$ and $\angle(a_3a''d)$ are of order ε, similarly for $b_3 \in P_c \cap P_a$ and $c_3 \in P_a \cap P_b$.

Finally, consider $a_4 \in P \cap \operatorname{aff}(a'a''a_3)$, say with $a_4 \in \pi_P(P_b \cap P_c)$, and similarly b_4, c_4. Then one proves as in the proof of case 1 of Theorem 4.3 that the 1-skeleton of the polyhedron

$$\Pi = da'a''a_1a_2a_3a_4b'b''b_1b_2b_3b_4c'c''c_1c_2c_3c_4$$

holds K. This triskaidecahedron (13 faces) has:

- one enneagonal (9-vertex) face $a_1a_4a_2b_1b_4b_2c_1c_4c_2$,

- three heptagonal faces $da_3a''a_2b_1b'b_3$, $db_3b''b_2c_1c'c_3$, and $dc_3c''c_2a_1a'a_3$,
- three quadrilateral faces $a_3a'a_4a''$, $b_3b'b_4b''$, and $c_3c'c_4c''$,
- and six triangular faces $a_1a'a_4$, $a_2a''a_4$, $b_1b'b_4$, $b_2b''b_4$, $c_1c'c_4$, and $c_2c''c_4$.

Among its 30 edges, six have lengths whose sum is close to $f(a, b, c)$: the edges a_2b_1, b_2c_1, c_2a_1, a_3d, b_3d, and c_3d. The remaining 24 edges have lengths of order ε.

The case where $t(a, b, c)$ is one of the points, say $t(a, b, c) = a$, goes similarly: We choose for d_0 the point on the bisector of $\widehat{bac}$ at the distance $\sqrt{\varepsilon}$ from a. The rest of the proof is the same.

(i) *Case 2: One of the line-segments ab, bc or ca is in* rel bd K.

Assume $ab \subset \operatorname{rel\,bd} K$. Then the plane P is chosen containing ab and making an angle of measure ε^3 with P_0. In this manner $a''b'$ is an edge of the cage and a_1, b_2 are no more needed. The rest of the cage is constructed exactly as above, yielding a hendecahedron (11 faces) $da'a''a_1a_3a_4b'b''b_2b_3b_4c'c''c_1c_2c_3c_4$ with

- one enneagonal face $a_1a_4a''b'b_4b_2c_1c_4c_2$,
- two heptagonal faces $da_3a''a_2b_1b'b_3$, $db_3b''b_2c_1c'c_3$, and $dc_3c''c_2a_1a'a_3$,
- one pentagonal face $da_3a''b'b_3$,
- three quadrilateral faces $a_3a'a_4a''$, $b_3b'b_4b''$, and $c_3c'c_4c''$,
- and four triangular faces $a_1a'a_4$, $b_2b''b_4$, $c_1c'c_4$, and $c_2c''c_4$.

Among its 26 edges, six have lengths whose sum is close to $f(a, b, c)$: the edges $a''b'$, b_2c_1, c_2a_1, a_3d, b_3d, and c_3d. The remaining 20 edges have lengths of order ε.

(ii) Assume ab and ac are in rel bd K. Since a, b, c almost fix K, there exist two parallel supporting lines of K at b and c; this gives wid $K \le |bc|$. Since K is weakly strictly convex, Corollary 4.4 applies, yielding

$$L(K) \le 3 \operatorname{wid} K \le 3|bc| = f(b, c) \le f(a, b, c). \qquad \square$$

Question Is there a planar convex body K in $\mathbb{R}^3$ satisfying

$$L(K) \ne \inf\{f(a, b, c)\ ;\ a, b, c \text{ almost fix } K\}?$$

A parallelogram will be called *acute* if, at each of its vertices, the angles between the diagonal and the sides are acute.

Theorem 4.7 (i) *If $T = abc$ is a triangle of sides $I_1 = bc$, $I_2 = ca$, $I_3 = ab$, then*

$$L(T) = \min\{f(x_1, x_2, x_3)\ ;\ x_k \in I_k\}.$$

In particular, for the equilateral triangle $\mathbb{T}_2$ of unit edge length, we have $L(\mathbb{T}_2) = \frac{3+\sqrt{3}}{2}$.

(ii) *If R is a rectangle of diagonal length δ, then $L(R) = 3\delta$. In particular, for the square $\mathbb{C}_2$, we have $L(\mathbb{C}_2) = 3\sqrt{2}$.*

(iii) *If K is a non-acute parallelogram of diagonal lengths δ and Δ, with $\delta < \Delta$, then $L(K) \ge 3\delta$.*

Proof (i) Since f is continuous and $I_1 \times I_2 \times I_3$ is compact, $\inf\{f(x_1, x_2, x_3)\,;\ x_k \in I_k\}$ is reached at some point (u, v, w). Two cases occur: Either each point is in the relative interior of its side, or two points are at a vertex, say, $u = v = c$, and the third one w is at the foot of the corresponding height. This second case occurs for all non-acute triangles; it can be seen that it occurs also for some acute ones, if the largest angle is close enough to $\frac{\pi}{2}$.

In the first case, the three normals at u, v, w are concurrent by Theorem 2.7, hence the points u, v, w fix the triangle T. In the second case, we choose u', v' arbitrarily close to c such that the three normals at u', v', w are concurrent, and the points u', v', w fix T. This proves that c, w almost fix T. Then Theorem 4.3 or Theorem 4.6 applies, yielding $L(T) \le f(u, v, w)$.

Conversely, let G be a cage holding T. Let $P = \operatorname{aff} T$ and consider the geometric graph $\pi_P(G)$, i.e., the projection of the cage G in the plane P. Let $v_1, \dots, v_n$ denote the external vertices of $\pi_P(G)$.
In the plane P, let S_a denote the half-strip not containing a, bounded by the side bc and the two rays parallel to ab starting from b and from c. Then G intersects S_a at some point x, otherwise T would escape from G by a translation inside S_a.

Similarly, let S_b and S_c denote the analogous half-strips for b and c, see Fig. 9. Then G intersects S_b at some point y and S_c at some point z. Now consider the projections on T: $u = \pi_T(x)$, $y = \pi_T(y)$ and $w = \pi_T(z)$. They satisfy $u \in I_1$, $v \in I_2$, and $w \in I_3$. By using Lemma 2.3, Corollary 2.6, and Theorem 2.4, we obtain

$$f(u, v, w) \le f(x, y, z) \le f(v_1, \dots, v_n) \le \lambda G.$$

Lemma 2.9 proves the equality for the equilateral triangle.

(ii) Let $R = abcd$ be a rectangle and denote its sides by $I_1 = ab$, $I_2 = bc$, $I_3 = cd$, and $I_4 = da$, see Fig. 10. Let G be a cage holding R, let $P = \operatorname{aff} R$, and let $v_1, \dots, v_n$ denote the external points of $\pi_P(G)$. In the plane P, let S be the half-strip not containing R determined by $\overline{da}$, ab, and $\overline{bc}$. Then G intersects S at some point m, otherwise R

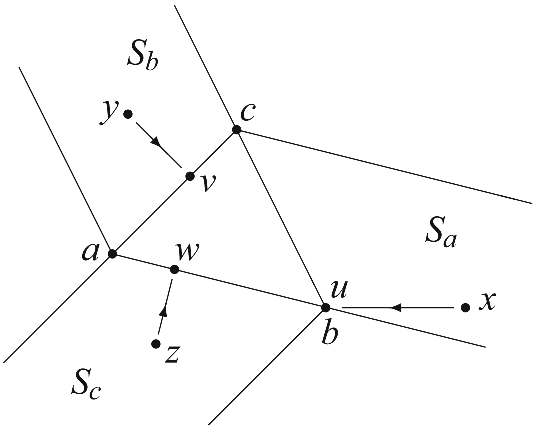


Fig. 9 Proof of Theorem 4.7(i)

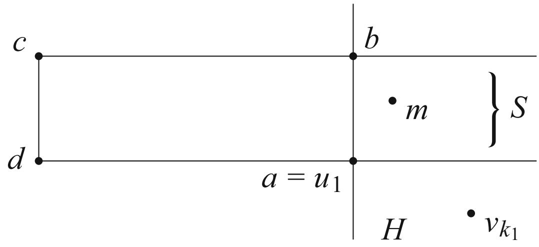


Fig. 10 Proof of Theorem 4.7(ii)

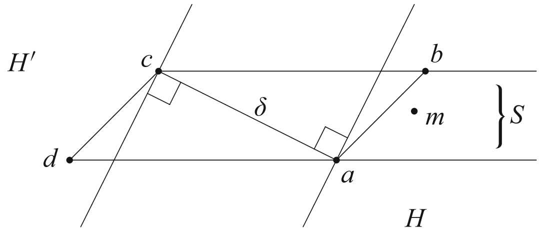


Fig. 11 Proof of Theorem 4.7(iii)

would escape from G by a translation inside S. Let H be the open half-plane not containing R bounded by $\overline{ab}$. Since m is in the convex hull of the points $v_1, \ldots, v_n$, one of these points, v_{k_1}, belongs to H. Therefore the point $u_1 = \pi_R(v_{k_1})$ belongs to I_1. In the same manner, there are $k_2, k_3, k_4 \in \{1, \ldots, n\}$ such that $u_i = \pi_R(v_{k_i})$ belongs to I_i for each $i = 2, 3, 4$. By applying Theorem 2.4, Lemma 2.3, and Lemma 2.10, we then obtain

$$\lambda G \geq f(v_1, \ldots, v_n) \geq f(v_{k_1}, v_{k_2}, v_{k_3}, v_{k_4}) \geq f(u_1, u_2, u_3, u_4) \geq 3\delta.$$

(iii) Let G be a cage holding the parallelogram $K = abcd$, with $|ac| = \delta < |bd| = \Delta$, let $P = \operatorname{aff} K$, and let $v_1, \ldots, v_n$ denote the external points of $\pi_P(G)$, see Fig. 11. In the plane P, let S be the half-strip not containing K determined by $\overline{da}$, ab, and $\overline{bc}$. The cage G intersects S at some point m, otherwise K would escape from G by a translation inside S. Let H be the open half-plane not containing c bounded by the straight line containing a and orthogonal to $\overline{ac}$, i.e., $H = \{x \in P\ ;\ \widehat{cax} \text{ is obtuse}\}$. Since K is non-acute, H contains S and since m is in the convex hull of the points $v_1, \ldots, v_n$, one of these points, say v_k, belongs to H. In the same manner, some point v_l belongs to the symmetric half-plane $H' = \{x \in P\ ;\ \widehat{acx} \text{ is obtuse}\}$. We then obtain

$$\lambda G \geq f(v_1, \ldots, v_n) \geq f(v_k, v_l) = 3|v_k v_l| \geq 3|ac| = 3\delta. \qquad \square$$

Remarks We conjecture that, in the case of an acute parallelogram K of diagonal lengths δ and Δ with $\delta \leq \Delta$, we have $L(K) = 3\delta$.

In the case of a non-acute parallelogram, it is easy to prove that the vertices a and c do not almost fix K any more and that $\inf\{f(x, y, z)\ ;\ x, y, z$ almost fix $K\}$ is attained e.g. for $x = a$, $y = c$ and $z = \pi_{\overline{da}}(c)$, whereas $\inf\{f(x, y)\ ;\ x, y$ almost fix $K\}$ is attained only for $x = b$, $y = d$ and is larger. By Theorem 4.6(i), we then obtain $L(K) \le f\big(a, c, \pi_{\overline{da}}(c)\big)$. We conjecture that this is an equality.

5 Cages for the Regular Tetrahedron

Let G be a cage holding the regular tetrahedron of unit edge length $\mathbb{T}_3 = abcd$. Recall that $\pi_{\mathbb{T}_3}$ is the projection function on $\mathbb{T}_3$.

Lemma 5.1 *We have* $\pi_{\mathbb{T}_3}(G) \cap (ab \cup ac) \neq \emptyset$.

Proof Suppose the intersection in the statement is empty. Imagine $P_0 := \operatorname{aff} abc$ to be horizontal and d above it. Denote by P_0^- the half-space below, bounded by P_0. Let Π_a be the vertical plane (i.e., orthogonal to aff abc) which includes bc. Similarly, let Π_b and Π_c be the vertical planes which include ac and ab respectively, see Fig. 12.

Denote by Π_a^+ the half-space containing a and bounded by Π_a. We have

$$G \cap \Pi_b \cap \Pi_a^+ \cap P_0^- = G \cap \Pi_c \cap \Pi_a^+ \cap P_0^- = G \cap P_0 \cap \Pi_a^+ \setminus abc = \emptyset.$$

Since G holds $\mathbb{T}_3$, G contains a point x below aff abc, in the triangular prism $\Delta = \pi_{\mathbb{T}_3}^{-1}(\operatorname{int} abc)$. We claim that this prism also contains a vertex of G. To see this, take $x \in G \cap \Delta$ to be farthest from Π; this is possible by compactness of G and because the two boundaries of Δ which are parts of Π_b and Π_c do not cross G. If x is not a vertex of G, then the side e of G to which x belongs must have an endpoint v below aff abc. If $v \notin \Pi^+$, then the other endpoint v' of e equals x (otherwise $\pi_{\mathbb{T}_3}(v') \in ab \cup ac$). If $v \in \Pi^+$, then both endpoints v, v' of e lie in Π^+, and $x \in \{v, v'\}$, or e is parallel to $\overline{bc}$ and x can be chosen in $\{v, v'\}$.

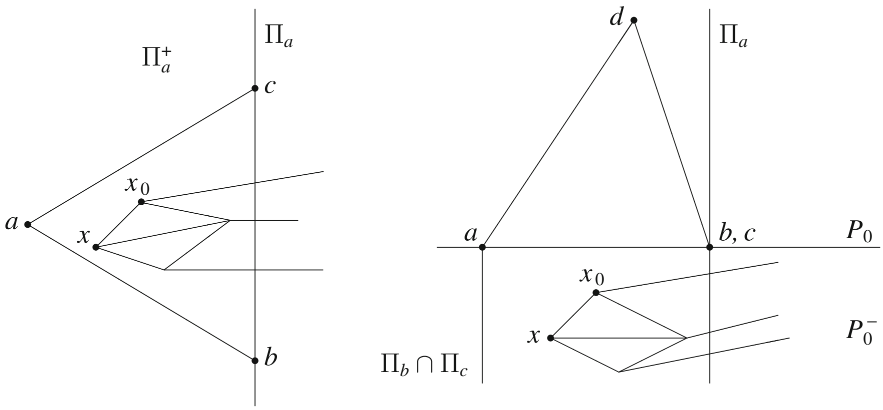


Fig. 12 Proof of Lemma 5.1. Left: top view, right: front view

Now, among all vertices of conv G which are in Δ, choose one x_0 such that the angle between x_0bc and abc is minimal. Then aff x_0bc separates int $\mathbb{T}_3$ from int conv G. Thus, G cannot hold $\mathbb{T}_3$, the seeked contradiction. □

Theorem 5.2 *For the regular tetrahedron $\mathbb{T}_3$ of unit edge length, we have $L(\mathbb{T}_3) = 3$.*

Proof Let $\mathbb{T}_3 = abcd$. A plane parallel to ab and to cd, and close (at distance η) to ab cuts $\mathbb{T}_3$ along a rectangle $Q = pqrs$ with $p \in ac$, $q \in ad$, $r \in bd$, $s \in bc$. Let G be the 1-skeleton of the pentahedron $abpqrs$. We shall prove below that G holds $\mathbb{T}_3$. As $\eta \to 0$, the length of six sides of G tends to 0, while the length of each of the remaining three tends to 1. Hence, $L(\mathbb{T}_3) \leq 3$. In order to prove that G holds $\mathbb{T}_3$, it obviously suffices to show that Q holds $\mathbb{T}_3$.

Suppose Q can go far away without meeting rel int $\mathbb{T}_3$. Then aff Q must contain a vertex of $\mathbb{T}_3$ somewhere on Q's way. Of course, that vertex cannot be c or d; let us assume it is b. Let Q' be that position of Q. Let x, y be the points where aff Q' meets ac, ad, respectively. Put $\Delta = xyb$. Note that $x \in ap$ implies $\|x - b\| > \|p - r\|$. But then, Δ is too long to fit inside Q'. Hence, $p \in ax$. Analogously, $q \in ay$. To fit in Q', Δ must have xb and yb at most as long as pr. This implies $\|p - x\| > \|a - p\|$ and $\|q - y\| > \|a - q\|$, whence $\|x - y\| > 2\|p - q\|$.

Let $p' \in pc$ satisfy $\|p - p'\| = \|a - p\|$, and analogously choose q'. As Q was chosen close to ab, x is close to a, $\angle p'bq'$ is close to 0, and therefore the height h of $p'bq'$ at p' (or q') is larger than $\|p - q\|$. Obviously, both heights of Δ at x and y are larger than h. Thus, Δ does not fit into Q'.

Conversely, let G be a cage holding $\mathbb{T}_3$. If $\pi_{\mathbb{T}_3}(G)$ does not meet some side e of $\mathbb{T}_3$, let e' denote the side opposite to e; if $\pi_{\mathbb{T}_3}(G)$ meets all sides, choose opposite e, e' arbitrarily.

By Lemma 5.1, there exist $w_1, w_2, w_3, w_4 \in G$ respectively projecting via $\pi_{\mathbb{T}_3}$ into the four sides different from e and e', which we denote by $e_1, \ldots, e_4$.

Consider the projection function π_P onto a plane P parallel to e and e'. Then $\pi_P(\mathbb{T}_3)$ is a square C, of side length $\frac{1}{\sqrt{2}}$ and sides $I_1, \ldots, I_4$, with $I_i = \pi_P(e_i)$, see Fig. 13.

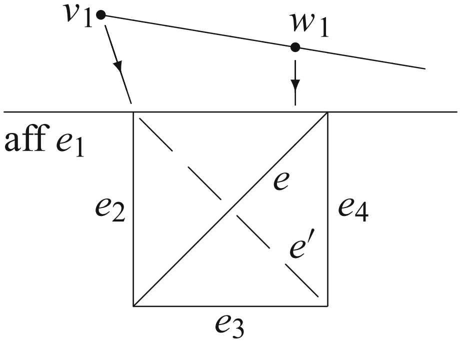


Fig. 13 Proof of Theorem 5.2

Of course, $\pi_P(w_i) \notin \operatorname{int} C$, for $i = 1, \ldots, 4$, otherwise $\pi_{\mathbb{T}_3}(w_i)$ would be on a face of $\mathbb{T}_3$ instead of the side e_i.

We have $\pi_C(\pi_P(w_1)) \in I_1$, hence $\pi_P(w_1)$ is in the closed half-plane bounded by aff e_1 and not containing int C. Therefore there exists an external vertex v_1 of $\pi_P(G)$ in the same half-plane, hence also satisfying $\pi_C(v_1) \in I_1$. We do the same with w_2, w_3 and w_4. This gives four external vertices $v_1 \ldots, v_4$ of $\pi_P(G)$, with possible coincidences, such that $\pi_C(v_i) \in I_i$. By using Theorem 2.4 and Lemmas 2.3 and 2.10, we then obtain

$$\lambda G \geq f(v_1, \ldots, v_4) \geq f(\pi_C(v_1), \ldots, \pi_C(v_4)) \geq \inf_{u_i \in I_i} f(u_1, \ldots, u_4) = 3. \qquad \square$$

Acknowledgements The authors warmly thank the referee for his/her careful reading. The authors gratefully acknowledge partial support from GDRI ECO-Math. The second author also thanks for the financial support by the NSF of China (11871192).

References

1. Aberth, O.: An isoperimetric inequality for polyhedra and its application to an extremal problem. Proc. Lond. Math. Soc. **13**, 322–336 (1963)
2. Besicovitch, A.S.: A cage to hold a unit-sphere. In: Proceedings of Symposia in Pure Mathematics, vol. VII, pp. 19–20. American Mathematical Society, Providence (1963)
3. Bracho, J., Fetter, H., Mayer, D., Montejano, L.: Immobilization of solids and mondriga quadratic forms. J. Lond. Math. Soc. **51**(1), 189–200 (1995)
4. Bracho, J., Montejano, L., Urrutia, J.: Immobilization of smooth convex figures. Geom. Dedicata **53**(2), 119–131 (1994)
5. Coxeter, H.S.M.: Review 1950. Math. Rev. **20**, 322 (1959)
6. Czyzowicz, J., Stojmenovic, I., Urrutia, J.: Immobilizing a shape. Int. J. Comput. Geom. Appl. **9**(2), 181–206 (1999)
7. Fruchard, A.: Fixing and almost fixing a convex figure (2017). hal-01573119
8. Kovalyov, M.D.: Covering a convex figure by its images under dilatation. Ukrainskij Geom. Sbornik **27/84**, 57–68 (1984). (in Russian)
9. Kós, G., Törőcsik, J.: Convex disks can cover their shadow. Discrete Comput. Geom. **5**(6), 529–531 (1990)
10. Lillington, J.N.: A conjecture for polytopes. Proc. Cambr. Philos. Soc. **76**, 407–411 (1974)
11. Linhart, J.: Kantenlängensumme, mittlere Breite und Umkugelradius konvexer Polytope. Arch. Math. **29**, 558–560 (1977)
12. Zamfirescu, T.: Inscribed and circumscribed circles to convex curves. Proc. Am. Math. Soc. **80**(3), 455–457 (1980)