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Results in Mathematics

Locating Diametral Points

Jin-ichi Itoh, Costin Vîlcu, Liping Yuan, and Tudor Zamfirescu

Abstract. Let K be a convex body in $\mathbb{R}^d$, with $d = 2, 3$. We determine sharp sufficient conditions for a set E composed of 1, 2, or 3 points of bd$K$, to contain at least one endpoint of a diameter of K. We extend this also to convex surfaces, with their intrinsic metric. Our conditions are upper bounds on the sum of the complete angles at the points in E. We also show that such criteria do not exist for $n \geq 4$ points.

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1. Introduction

The tangent cone at a point x in the boundary bd$K$ of a convex body K can be defined using only neighborhoods of x in bd$K$. So, one doesn't normally expect to get global information about K from the size of the tangent cones at one, two or three points. Nevertheless, in some cases this is what happens!

A *convex body* K in $\mathbb{R}^d$ is a compact convex set with interior points; we shall consider only the cases $d = 2, 3$. A *convex surface* in $\mathbb{R}^3$ is the boundary of a convex body in $\mathbb{R}^3$.

Let S be a convex surface and x a point in S. Consider homothetic dilations of S with the centre at x and coefficients of homothety tending to infinity. The limit surface is called the *tangent cone at* x (see [1]), and is denoted by T_x.

If K is a planar convex body, then the tangent cone at a boundary point is an angle.

If K is a convex body in $\mathbb{R}^3$, then the tangent cone at $x \in$ bd$K$ can be unfolded in the plane, producing an angle the measure of which is *the complete angle* at x, denoted by θ_x.

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Denote by ρ the intrinsic metric on the convex surface S (which is derived from the ambient Euclidean distance).

We shall call *diameter* each line-segment in K, or arc in S, of length equal to the extrinsic, respectively intrinsic, maximal distance between pairs of points in K or in S.

An endpoint of some (intrinsic or extrinsic) diameter is called an (intrinsic, respectively extrinsic) *diametral point*.

In this paper, we provide criteria for finding extrinsic diametral points in convex bodies $K \subset \mathbb{R}^d$, $d = 2, 3$, and criteria for finding intrinsic diametral points in convex surfaces $S = \mathrm{bd}K \subset \mathbb{R}^3$. Our criteria consist of upper bounds on the sum of the complete angles at 1, 2, or 3 points.

We also show that such criteria do not exist for $n \geq 4$ points.

Related to our results in Sect. 3 is the following one, obtained by Itoh and Vîlcu [4]. Each point y in a convex surface S with complete angle $\theta_y \leq \pi$ is a farthest point on S, i.e., y is at maximal intrinsic distance from some point in S.

Passing from planar convex bodies to convex surfaces is not always obvious. For example, while the diameter of a convex polygon P (in the plane, diameter means extrinsic diameter) with n vertices can be computed in time $O(n)$ [8], the intrinsic diameter of a convex polyhedral surface in $\mathbb{R}^3$ with n vertices can be computed in time $O(n^8 \log n)$ [2].

Also, it is well-known that diameters of convex polygons must join vertices, but this is not always true for geodesic diameters of convex polyhedral surfaces [7]. The result in [7] that the diameter is generically realized by five geodesic segments (see the definition below) was later proved by Zalgaller [10].

There is a nice connection between the lengths of intrinsic and extrinsic diameters of a convex surface, considered by several authors, see [5,6,11]: for any convex surface S, the former is less than or equal to $\pi/2$ times the latter, and equals it if and only if S is a surface of revolution of constant width.

Our results provide another connection. The endpoints of extrinsic and intrinsic diameters of convex bodies and surfaces are in general distinct; yet, in some cases, they can be found in the same set, see the remarks at the end of the paper.

A pair of points *sees a line-segment under the angle* α if the sum of the two angles under which they see the line-segment equals α.

Let σ be an extrinsic diameter of the convex body K. A pair of points $u, v \in K \backslash \sigma$ is said to be a *σ-separated pair* if the line-segment uv meets σ [12].

A *geodesic segment* on the convex surface S is an arc (path) on S realizing the intrinsic distance between its endpoints. If σ is an intrinsic diameter, i.e. a longest geodesic segment, of S, then a pair of points $u, v \in S \backslash \sigma$ is said to be *σ-separated* if some geodesic segment from u to v meets σ [12].

For $M \subset \mathbb{R}^d$, we denote by $\overline{M}$ its affine hull, by $\mathrm{int}M$ the relative interior of M (i.e., in the topology of $\overline{M}$) and by $\mathrm{bd}M$ the relative boundary of M.

For distinct $x, y \in \mathbb{R}^d$, let xy be the line-segment from x to y; thus, $\overline{xy}$ is the line through x, y. We put $x_1 \dots x_n = \mathrm{conv}\{x_1, \dots, x_n\}$.

2. Planar Convex Bodies

Let K be a planar convex body and x a boundary point of K.

We denote by X the angle of bd$K$ at x towards K (so $X \le \pi$), and keep this habit for any boundary point; so, Y is the angle at y, and so on.

We shall repeatedly use the next result.

Lemma 2.1. (Zamfirescu [12]) *For any diameter uv of a planar convex body, every uv-separated pair sees uv under an angle not less than $5\pi/6$.*

Lemma 2.2. *Assume in the convex quadrilateral $Q = xyzw$ we have $X+Y \le \pi$. Then at least one of the vertices x, y is a diametral point of Q.*

Proof. Assume x, y are not diametral points of Q. Then the side zw is longer than the diagonals xz and yw, whence $W < \angle wxz < X$ and $Z < \angle wyz < Y$. It follows that

$$2\pi = X + Y + Z + W < 2(X + Y) \le 2\pi,$$

absurd. □

Theorem 2.3. *Let K be a planar convex body.*

(i) *Any point $x \in$ bd$K$ with $X \le \pi/3$ is a diametral point of K. If K has two such points, they determine a diameter of K.*

(ii) *Among any two points $x, y \in$ bd$K$ with $X + Y \le 5\pi/6$ there exists a diametral point of K.*

(iii) *Among any three points $x, y, z \in$ bd$K$ with $X+Y+Z \le 4\pi/3$ there exists a diametral point of K.*

Proof. (i) Assume the existence of a diameter yz of K, with y, z different from x. It follows that in the triangle xyz the angle at x is not smaller than the other two, whence the triangle is equilateral. Hence, xy and xz are diameters, too. Thus, (i) is proven.

For the rest of the proof [parts (ii) and (iii)], assume the conclusion does not hold, and let uv be a diameter of K.

(ii) If x and y are not uv-separated, then we have the quadrilateral $xyvu$. By Lemma 2.2, $X+Y \ge \angle uxy + \angle xyv > \pi$, which contradicts our hypothesis.

So x and y are uv-separated; then, by Lemma 2.1, $X + Y \ge 5\pi/6$. This and the hypothesis imply $X + Y = 5\pi/6$.

Assume that K is not the quadrilateral $xuyv$. Then the sum of the angles of $xuyv$ at x and y is less than $5\pi/6$, in contradiction with Lemma 2.1, applied to $xuyv$.

So K is the quadrilateral $xuyv$. Slightly moving x out of K along the line $\overline{xy}$ would provide quadrilaterals $K' = x'uyv$ with x', y uv-separated and

$X' + Y < 5\pi/6$. By Lemma 2.1, uv is no longer a diameter of K', so x' is a diametral point of K'. Now, let x' converge back towards x. Then $K' \to K$, which implies that x is a diametral point of K, contradicting our assumption.

(iii) Assume first that x, y, z are all on one side of $\overline{uv}$. Lemma 2.2 gives

$$X + Y > \pi, \qquad X + Z > \pi, \qquad Y + Z > \pi,$$

so $X + Y + Z > 3\pi/2$, contradicting $X + Y + Z \leq 4\pi/3$.

Hence, we can assume that x, y are on one side of $\overline{uv}$ and z on the other side. The previous case (ii) and Lemma 2.2 imply

$$X + Y > \pi, \qquad X + Z > 5\pi/6, \qquad Y + Z > 5\pi/6.$$

Summing up, we get $X+Y+Z > 4\pi/3$, which contradicts the hypothesis. □

All bounds in Theorem 2.3 are sharp, as one can see from the following examples.

(i) Consider an isosceles triangle $\Delta = xyz$ with $\|x - y\| = \|x - z\|$ and $X = \pi/3 + \varepsilon$, with ε arbitrarily small. Clearly, x is not a diametral point of Δ.

(ii) Consider a convex quadrilateral $Q' = x'uy'v$ with $\|x' - v\| = \|x' - y'\| = \|y' - v\| = \|u - v\|$ and $\|u - x'\| = \|u - y'\|$, see Fig. 1a. Then, in Q', $X' = Y' = U/2 = 5\pi/12$.

Let x and y be interior points of Q', on the line $\overline{x'y'}$, arbitrarily close to x' and y', respectively. Then, in $Q = xuyv$, we have $X + Y = 5\pi/6 + \varepsilon$, with ε arbitrarily small; moreover, uv is the unique diameter of Q.

(iii) Consider an equilateral triangle $\Delta = uvz'$ and let m be the midpoint of uv, see Fig. 1b. On the circle of diameter uv, take points x, y separated

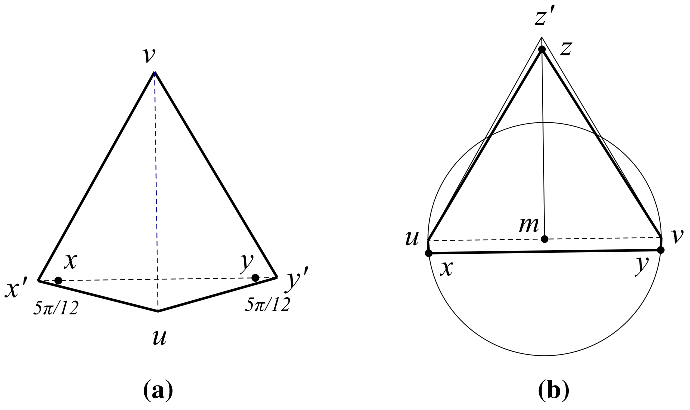

FIGURE 1. (ii) and (iii)

from z' by $\overline{uv}$, such that $xy \| uv$ and x is arbitrarily close to u. Then, in $xuz'vy$, $X = Y = \pi/2 + \varepsilon$, with ε arbitrarily small.

Take a point z on $z'm$, such that $\|z - z'\|$ equals the distance between the parallel lines $\overline{xy}$ and $\overline{uv}$. Of course, in $xuzvy$, $Z = \angle uzv = \pi/3 + \varepsilon'$, with $\varepsilon' > 0$.

Then $X + Y + Z = 4\pi/3 + 2\varepsilon + \varepsilon'$, and $2\varepsilon + \varepsilon'$ converges to 0 as $x \to u$. Moreover, uv is the unique diameter of $xyvzu$.

Corollary 2.4. *If the planar convex body K, symmetric about $\mathbf{0}$, has a boundary point x with $X \leq 5\pi/12$, then $x(-x)$ is a diameter of K.*

Proof. The sum of the angles at x and $-x$ is less than or equal to $5\pi/6$. Now, by Theorem 2.3 (ii), x or x' is a diametral point of K. But, by Theorem 4 in [12], the endpoints of each diameter of K are symmetric with respect to $\mathbf{0}$. So, $x(-x)$ is a diameter of K. □

The above approach cannot be extended to $n \geq 4$ points.

Remark 2.5. There is no non-trivial constant $c(n)$ depending only on $n \geq 4$, to guarantee that, for any planar convex body K, among any n points $x_1, \ldots, x_n$ in bd$K$ with $\sum_{i=1}^n X_i \leq c(n)$ there exists a diametral point of K.

Proof. Suppose that such a constant $c(n)$ does exist.

Notice that any points $x_1, \ldots, x_n$ in the boundary of any planar convex body K, form a convex n-gon; hence $\sum_{i=1}^n X_i \geq (n-2)\pi$, and therefore $c(n) \geq (n-2)\pi$.

Next we show that, for any $\varepsilon > 0$, there exist a planar polygon P and n vertices of P with $\sum_{i=1}^n X_i < (n-2)\pi + \varepsilon$, none of which is a diametral point of P. This implies $c(n) \leq (n-2)\pi$, hence necessarily $c(n) = (n-2)\pi$. In this case, K is precisely the convex n-gon with vertices $x_1, \ldots, x_n$ and, trivially, at least two of them are diametral points.

Let uv be a diameter of a circle C. Consider $n \geq 4$ points $x_1, \ldots, x_n$ on C, at least two of them on each side of the line $\overline{uv}$, such that no two of them are diametrally opposite. Let $x_1, \ldots, x_k$ be on one side and $x_{k+1}, \ldots, x_n$ on the other side of $\overline{uv}$.

Of course, in the n-gon $x_1 \ldots x_n$, we have $\sum_{i=1}^n \angle x_i x_{i+1} x_{i+2} = (n-2)\pi$, where indices i are taken modulo n. Let P denote the $(n+2)$-gon $uvx_1 \ldots x_n$. By taking x_1, x_n close to u, and x_k, x_{k+1} close to v, we get $\sum_{i=1}^n X_i < (n-2)\pi + \varepsilon$, with ε arbitrarily small. But P has the unique diameter uv. □

3. Convex Bodies in $\mathbb{R}^3$

We obtain here results similar to Theorem 2.3, locating diametral points of convex bodies in $\mathbb{R}^3$.

We denote by θ_x the complete angle at the point x in bd$K$, and by ω_x the curvature at x; hence, $\omega_x = 2\pi - \theta_x$.

Theorem 3.1. *Let K be a convex body in $\mathbb{R}^3$.*

(i) *Any point $x \in \mathrm{bd}K$ with $\theta_x \le 2\pi/3$ is a diametral point of K. If K has two such points, they determine a diameter of K.*

(ii) *Among any two points $x, y \in \mathrm{bd}K$ with $\theta_x + \theta_y \le 3\pi/2$ there exists a diametral point of K.*

(iii) *Among any three points $x, y, z \in \mathrm{bd}K$ with $\theta_x + \theta_y + \theta_z \le 9\pi/4$ there exists a diametral point of K.*

Proof. (i) Assume there exists $K \subset \mathbb{R}^3$ and a point $x \in \mathrm{bd}K$ with $\theta_x \le 2\pi/3$, which is not diametral.

Let uv be a diameter of K. In the planar convex body $K \cap \overline{xuv}$, the angle X at x must be at most $\theta_x/2 \le \pi/3$. By Theorem 2.4 (i), $X > \pi/3$, and a contradiction is obtained.

(ii) Assume there exists $K \subset \mathbb{R}^3$ and points x, y on $\mathrm{bd}K$ with $\theta_x + \theta_y \le 3\pi/2$, none of which is a diametral point of K.

Let uv be a diameter of K. We consider the (possibly degenerate) tetrahedron $T = uvxy$.

We unfold $xuv \cup yuv$ on a plane, with x, y coming on different sides of $\overline{uv}$. The resulting quadrilateral Q has angles X, Y, U, V at the points corresponding to x, y, u, v, respectively. Now, unfold $uxy \cup vxy$ on a plane, with u, v coming on different sides of $\overline{xy}$. The resulting quadrilateral Q' has angles X', Y', U', V' at the points corresponding to x, y, u, v. In Q', the length of the diagonal corresponding to uv equals at least $\|u - v\| > \|x - y\|$. By Theorem 2.3 (ii), $X' + Y' > 5\pi/6$, whence $U' + V' < 2\pi - (5\pi/6) = 7\pi/6$.

We have

$$X + X' + Y + Y' + U + U' + V + V' = 4\pi$$

in $\mathrm{bd}T$.

Since $X + X' + Y + Y' \le \theta_x + \theta_y \le 3\pi/2$, we have $U + U' + V + V' \ge 5\pi/2$. This, together with the inequality $U' + V' < 7\pi/6$ obtained above, yields $U + V > 4\pi/3$. This implies $X + Y < 2\pi/3$. Hence, $X < \pi/3$ or $Y < \pi/3$. Thus, uv cannot be a longest side, in xuv or in yuv, and a contradiction is obtained.

(iii) Suppose $\theta_x + \theta_y + \theta_z \le 9\pi/4$, but there is no diametral point among x, y, z. Then, by (ii), $\theta_x + \theta_y > 3\pi/2$, $\theta_y + \theta_z > 3\pi/2$, $\theta_z + \theta_x > 3\pi/2$. It follows that $2\theta_x + 2\theta_y + 2\theta_z > 9\pi/2$, in contradiction with our hypothesis. □

Theorem 3.2. *If the convex body K, symmetric with respect to $\mathbf{0}$, has a boundary point x with $\theta_x \le 5\pi/6$, then $x(-x)$ is a diameter of K.*

Proof. Obviously, $\theta_x = \theta_{-x}$. Assume that $x(-x)$ is not a diameter. Then consider a diameter, which, by Theorem 4 in [12], must join diametrally opposite points. Let $y(-y)$ be that diameter. In the parallelogram $xy(-x)(-y)$, the diagonal $y(-y)$ is the unique diameter of it, so x and $-x$ are not diametral points. By Theorem 2.3 (ii), $\angle yx(-y) + \angle y(-x)(-y) > 5\pi/6$.

But $\angle yx(-y) + \angle y(-x)(-y) \leq (\theta_x/2) + (\theta_{-x}/2) \leq 5\pi/6$, and we got a contradiction. □

4. Convex Surfaces

In this section we investigate intrinsic diameters on convex surfaces. We obtain results similar to those in Sects. 2 and 3. Roughly speaking, as soon as the curvature concentrated at some points is large enough, they become eligible as diametral points.

Lemma 4.1. [The Pizzetti–Alexandrov comparison theorem ([1], p. 132)] *The angles of any geodesic triangle in a convex surface are not smaller than the corresponding angles of the Euclidean triangle with the same side-lengths.*

Lemma 4.2 follows from Alexandrov's *Konvexitätsbedingung* ([1], p. 130).

Lemma 4.2. *Consider a convex surface S. Let $abc \subset S$ and $a'b'c' \subset \mathbb{R}^2$ be two triangles as in Lemma 4.1. If $d \in bc$, $d' \in b'c'$ and $\rho(b,d) = \|b' - d'\|$, then $\rho(a,d) \geq \|a' - d'\|$.*

The following statement is well-known. For a thorough introduction to the theory of critical points for distance functions, see [3].

Lemma 4.3. *Each endpoint of a diameter on a convex surface is critical with respect to the other. Consequently, each digon determined by two diameters, with no third diameter passing through its interior, has both endpoint angles at most π.*

Theorem 4.4. *Let S be a convex surface.*

- (i) *Any point $x \in S$ with $\theta_x \leq 2\pi/3$ is a diametral point of S. If S has two such points, they determine a diameter of S.*
- (ii) *Among any two points $x, y \in S$ with $\theta_x + \theta_y \leq 5\pi/3$ there exists a diametral point of S.*
- (iii) *Among any three points $x, y, z \in S$ with $\theta_x + \theta_y + \theta_z \leq 5\pi/2$ there exists a diametral point of S.*

Proof. (i) Assume a point x on S verifies $\theta_x \leq 2\pi/3$ and is not a diametral point of S. Let yz be a diameter of S. There are two geodesic triangles with vertices at x, y, z on S, at least one of which has an angle less than or equal to $\pi/3$ at x. A contradiction now follows from the assumption that x is not a diametral point, Lemma 4.1 and Theorem 2.3 (i).

Assume now that there are $x, y \in S$ with $\theta_x, \theta_y \leq 2\pi/3$, and take $z \in S \setminus \{x, y\}$. Join x, y and z by geodesic segments to form two triangles on S. At least one of them has its angle at x less than or equal to $\pi/3$, so yz is not a diameter of S or xy is a diameter, by the preceding argument. Analogously, xz is not a diameter of S or xy is a diameter.

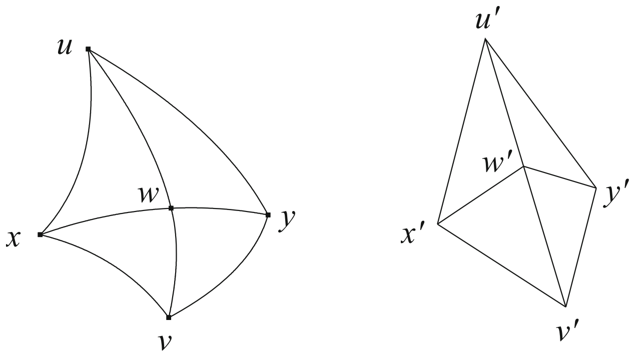

FIGURE 2. The points x, y are not inside one digon

Since this holds for any $z \in S$ and x, y are diametral points of S, xy must be a diameter of S.

For the rest of the proof, assume the conclusions are false and let uv be a diameter of S.

The geodesic segments joining u and v determine on S one or several digons.

(ii) The points x, y are not inside one digon, say D, determined by geodesic segments from u to v, see Fig. 2. Indeed, by Lemma 4.3, the total curvature of the interior of D is at most 2π, hence

$$2\pi \geq \omega_x + \omega_y = 4\pi - (\theta_x + \theta_y) \geq \frac{7}{3}\pi > 2\pi,$$

absurd.

Therefore, the points x, y are in distinct digons, and so x and y are uv-separated, for some diameter uv. Let $\{w\} = uv \cap xy$. Consider the points x', y', u', v', w' in $\mathbb{R}^2$ such that $w' \in u'v'$, $x'y' \cap u'v' \neq \emptyset$, $\|u' - v'\| = \rho(u, v)$, $\|u' - x'\| = \rho(u, x)$, $\|u' - y'\| = \rho(u, y)$, $\|v' - x'\| = \rho(v, x)$, $\|v' - y'\| = \rho(v, y)$, $\|u' - w'\| = \rho(u, w)$. By Lemma 4.2, $\|x' - w'\| \leq \rho(x, w)$ and $\|y' - w'\| \leq \rho(y, w)$.

By Lemma 4.1, $\angle uxv \geq \angle u'x'v'$ and $\angle uyv \geq \angle u'y'v'$.

But

$$\begin{aligned} \|x' - y'\| &\leq \|x' - w'\| + \|w' - y'\| \leq \rho(x, w) + \rho(w, y) \\ &= \rho(x, y) < \rho(u, v) = \|u' - v'\|. \end{aligned}$$

By Theorem 2.3 (ii),

$$\angle uxv + \angle uyv \geq \angle u'x'v' + \angle u'y'v' > 5\pi/6.$$

But

$$\angle uxv + \angle uyv \leq (\theta_x/2) + (\theta_y/2) \leq 5\pi/6,$$

and a contradiction is obtained.

(iii) Notice that the points x, y, z cannot be all in the same digon determined by geodesic segments from u to v. Indeed, for three points x, y, z in the same digon, we have, by Lemma 4.3, $\omega_x+\omega_y+\omega_z \leq 2\pi$, hence $\theta_x+\theta_y+\theta_z \geq 4\pi$, contradicting the hypothesis.

Assume first that x, y are in one digon, and z in another one. Then, by (ii), $\theta_x + \theta_z > 5\pi/3$ and $\theta_y + \theta_z > 5\pi/3$. At (ii) we saw that $\theta_x + \theta_y \geq 2\pi$. Summing up these inequalities, we get $\theta_x+\theta_y+\theta_z > 8\pi/3 > 5\pi/2$, impossible.

Hence, x, y, z are in different digons. By (ii), we have $\theta_x + \theta_y > 5\pi/3$, $\theta_y+\theta_z > 5\pi/3$, and $\theta_x+\theta_z > 5\pi/3$, hence $\theta_x+\theta_y+\theta_z > 5\pi/2$, in contradiction with the hypothesis. □

Corollary 4.5. *If the convex surface S, symmetric about* $\mathbf{0}$*, has a point x with $\theta_x \leq 5\pi/6$, then there exists a diameter of S from x to $-x$.*

Proof. Since $\theta_x = \theta_{-x}$, we have $\theta_x+\theta_{-x} \leq 5\pi/3$. Now, Theorem 4.4 (ii) implies that x or x' is a diametral point of S. By Proposition 6 in [9], each diameter of S is realized between diametrally opposite points. □

The endpoints of extrinsic and intrinsic diameters of convex bodies or surfaces are in general distinct.

The hypotheses of Corollaries 3.2 and 4.5, are the same. Also, those of Theorems 4.4 and 3.1 might be simultaneously verified. In these cases, endpoints of both extrinsic and intrinsic diameters of $S = \text{bd}K$ can be found in the same subset of S composed by 1, 2, or 3 points.

Conjecture. In Theorem 3.1 (ii), the inequality $\theta_x + \theta_y \leq 5\pi/3$ suffices to guarantee the existence of a diametral point in $\{x, y\}$.

Open questions. Are the bounds $9\pi/4$ in Theorem 3.1 (iii) and $5\pi/2$ in Theorem 4.4 (iii) optimal?

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Jin-ichi Itoh
School of Education
Sugiyama Jogakuen University
17-3 Hoshigaoka-motomachi, Chikusa-ku
Nagoya 464-8662
Japan
e-mail: j-itoh@sugiyama-u.ac.jp

Costin Vîlcu
Simion Stoilow Institute of Mathematics of the Roumanian Academy
P.O. Box 1-764
014700 Bucharest
Roumania
e-mail: Costin.Vilcu@imar.ro

Liping Yuan
Hebei Key Laboratory of Computational Mathematics and Applications, School of Mathematical Sciences
Hebei Normal University
Shijiazhuang 050024
People's Republic of China
e-mail: lpyuan@hebtu.edu.cn

Tudor Zamfirescu
Fachbereich Mathematik
Technische Universität Dortmund
44221 Dortmund
Germany
e-mail: tudor.zamfirescu@mathematik.tu-dortmund.de

and

Roumanian Academy
Bucharest
Roumania

and

School of Mathematical Sciences
Hebei Normal University
Shijiazhuang 050024
People's Republic of China

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