Hindawi
Complexity
Volume 2020, Article ID 5608720, 6 pages
https://doi.org/10.1155/2020/5608720

Research Article

The Property of Hamiltonian Connectedness in Toeplitz Graphs

Ayesha Shabbir,[1,2] Muhammad Faisal Nadeem ,[3] and Tudor Zamfirescu[4,5]

[1]*Preparatory Year Deanship, King Faisal University, 31982 Hofuf, Al Ahsa, Saudi Arabia*
[2]*Department of Mathematics, University of Lahore, Gujrat Campus, Gujrat, Pakistan*
[3]*Department of Mathematics, COMSATS, University Islamabad Lahore Campus, Pakistan*
[4]*Faculty of Mathematics, University of Dortmund, 44221 Dortmund, Germany*
[5]*Romanian Academy, Bucharest, Romania*

Correspondence should be addressed to Muhammad Faisal Nadeem; mfaisalnadeem@ymail.com

Received 17 August 2019; Revised 3 January 2020; Accepted 18 January 2020; Published 12 March 2020

Academic Editor: Dimitri Volchenkov

A spanning path in a graph G is called a Hamiltonian path. To determine which graphs possess such paths is an NP-complete problem. A graph G is called Hamiltonian-connected if any two vertices of G are connected by a Hamiltonian path. We consider here the family of Toeplitz graphs. About them, it is known only for $n = 3$ that $T_n\langle p, q\rangle$ is Hamiltonian-connected, while some particular cases of $T_n\langle p, q, r\rangle$ for $p = 1$ and $q = 2, 3, 4$ have also been investigated regarding Hamiltonian connectedness. Here, we prove that the nonbipartite Toeplitz graph $T_n\langle 1, q, r\rangle$ is Hamiltonian-connected for all $1 < q < r < n$ and $n \geq 5r - 2$.

1. Introduction

A path in a finite undirected graph G is called a *Hamiltonian path* if it visits each vertex of G exactly once. We call the graph G *Hamiltonian-connected* if for any pair of distinct vertices x and y of G, there exists a Hamiltonian path from x to y. In 1963, Ore introduced the family of Hamiltonian-connected graphs [13]. The Hamiltonian path problem, is the computational complexity problem of finding Hamiltonian paths in graphs, and related graphs are among the most famous NP-complete problems, see [14]. In this paper, we are investigating this property of Hamiltonian connectedness for some classes of Toeplitz graphs.

Let $n, t_1, t_2, \ldots, t_k \in \mathbb{N}$ such that $1 \leq t_1 < t_2 < \cdots < t_k < n$. An undirected *Toeplitz graph* $T_n\langle t_1, t_2, \ldots, t_k\rangle$ is a symmetric graph with the vertex set $\{1, 2, \ldots, n\}$ and with an edge (i, j) between the vertices i and j if and only if $|j - i| = t_l$ for some $l \in \{1, 2, \ldots, k\}$. The integers $t_1, t_2, \ldots, t_k$ are called *entries* or *jumps*. The adjacency matrix of any such graph is a symmetric Toeplitz matrix. Toeplitz graphs were introduced by G. Sierksma. The undirected Toeplitz graphs were first investigated by van Dal et al. [14] with respect to hamiltonicity. Heuberger [8] extended this study in 2002, while the directed case was studied in [9–11]. For results regarding different properties of Toeplitz graphs such as connectivity, bipartiteness, planarity, and colourability, see [3–8]. The well-known circulant graphs are particular cases of Toeplitz graphs. In fact, for each Toeplitz graph T, there exists a circulant graph C such that T is a spanning subgraph of C.

For $n, t_1, t_2, \ldots, t_k \in \mathbb{N}$, a *circulant graph* $C_n(t_1, t_2, \ldots, t_k)$ is a regular graph of degree $2k$ or $2k - 1$ with the vertex set $\{0, 1, 2, \ldots, n - 1\}$, in which two vertices i and j are adjacent if and only if $j - i = t_l \pmod n$, for some $l \in \{1, 2, \ldots, k\}$. Circulant graphs are Cayley graphs on the abelian group Z_n, i.e., the circulant graph $C_n(t_1, t_2, \ldots, t_k)$ is the Cayley graph $Cay(Z_n, \{t_1, t_2, \ldots, t_k\})$. Furthermore, note that $C_n(1, 2, \ldots, \lfloor n/2\rfloor) \cong K_n$ and $C_n(1) \cong C_n$ (a cycle or cyclic graph on n vertices).

For results regarding connectivity of the Toeplitz graph, see [14], where it is shown that the graph $T_n\langle t_1, t_2, \ldots, t_k\rangle$ has at least $\gcd(t_1, t_2, \ldots, t_k)$ components. Therefore, for $\gcd(t_1, t_2, \ldots, t_k) > 1$, the corresponding Toeplitz graph is disconnected. But, one may find a disconnected graph even for $\gcd(t_1, t_2, \ldots, t_k) = 1$, e.g., $T_6\langle 3, 5\rangle$.

In [11], it has been proven that $T_n\langle 1, 2\rangle$ is Hamiltonian-connected only for $n = 3$, while $T_n\langle 1, 2, s\rangle$ is Hamiltonian-connected for all values of n and s. The present paper is a sequel of [12], where it is proved that $T_n\langle t_1, t_2\rangle$ is Hamiltonian-connected only for $n = 3$. Thereafter, the case $k = 3$ was considered and it was shown that the graphs $T_n\langle 1,3,t_3\rangle$ and $T_n\langle 1,4,t_3\rangle$ are Hamiltonian-connected. Here, we are presenting a more general result about $T_n\langle 1,t_2,t_3\rangle$ with $1<t_2<t_3<n$ under the assumption that t_2 and t_3 are not both odd because otherwise the corresponding Toeplitz graph becomes bipartite, hence not Hamiltonian-connected [12]. A Toeplitz graph becomes circulant if for each entry t_i, $n-t_i$ also occurs as an entry for all $i=1,2,\ldots,k$, see [1]. In this special class of Toeplitz graphs, we prove here the existence of Hamiltonian-connected graphs.

The following results are needed to prove our first result.

Theorem 1 (see [2]). *A connected Cayley graph on an abelian group is Hamiltonian-connected if and only if it is neither cyclic nor bipartite.*

Theorem 2 (see [15]). *The circulant graph $C_n(t_1,t_2,\ldots,t_r)$ is connected if and only if $gcd(t_1,t_2,\ldots,t_r,n)=1$.*

Theorem 3 (see [7]). *A connected circulant graph $C_n(t_1,t_2,\ldots,t_r)$ is bipartite if and only if $t_1,t_2,\ldots,t_r$ are odd and n is even.*

2. Main Results

We start with the following result which is a consequence of Theorem 1.

Theorem 4. *If n is an odd integer and $k\geq 2$ such that $t_k\leq\lfloor n/2\rfloor$ and $gcd(t_1,\ldots,t_k,n)=1$, then $T_n\langle t_1,\ldots,t_k,n-t_k,\ldots,n-t_1\rangle$ is Hamiltonian-connected.*

Proof. Since $T=T_n\langle t_1,t_2,\ldots,t_k,n-t_k,\ldots,n-t_1\rangle\cong C_n(t_1,t_2,\ldots,t_k)$, under the given conditions and by Theorems 2 and 3, T is a connected noncyclic and nonbipartite Cayley graph $\mathrm{Cay}(Z_n,\{t_1,t_2,\ldots,t_k\})$. Hence, it is Hamiltonian-connected by Theorem 1. □

To prove our next main results, we need the following notation and lemmas.

Let T be a Toeplitz graph and p,q $(p<q)$ be two vertices of T. The symbols $G_{p,q}$ and $\overline{G}_{q,p}$ stand for the paths $(p,p+1)(p+1,p+2)\cdots(q-1,q)$ and $(q,q-1)(q-1,q-2)\ldots(p+1,p)$, respectively. By $P_{p,q}$, we mean a path from p to $p+1$ with the set of vertices $\{p,p+1,p+2,\ldots,q-2,q-1,q\}$, and by $P_{q,p}$, we mean a path from q to $q-1$ with the same vertices. Note that the existence of $P_{p,q}$ or $P_{q,p}$ is not guaranteed. Furthermore, it is easy to observe that if T is a Toeplitz graph of order n and there exists a path $(v_1,v_2)\ldots(v_{k-1},v_k)$ in T, then by the symmetry of Toeplitz graphs, there exists another path $(n+1-v_1,n+1-v_2)\ldots(n+1-v_{k-1},n+1-v_k)$ in T.

Lemma 1. *If t is an even integer with $n\in\{t+1,t+3,\ldots,2t-3\}$ or $n\geq 2t-1$, then $T_n\langle 1,t\rangle$ admits a Hamiltonian path from 1 to 2.*

Proof. For $n\in\{t+1,t+3,\ldots,3t-3\}$, the Toeplitz graph $T_n\langle 1,t\rangle$ with t even admits a unique Hamiltonian path which is starting from 1, passing through the edge $(n-1,n)$, and ending at 2. We use these paths as basic paths to construct our desired path in $T=T_n\langle 1,t\rangle$, which are defined as follows:

When $n=t+1$,

$$P:\ (1,t+1)\overline{G}_{t+1,2}. \tag{1}$$

When $n=t+i$ for some $i\in\{3,5,\ldots,t-1\}$,

$$\begin{aligned}P':\ &(1,t+1)\overline{G}_{t+1,i}(i,t+i)(t+i,t+i-1)(t+i-1,i-1)\\ &\cdot(i-1,i-2)(i-2,t+i-2)\ldots(t+3,t+2)(t+2,2).\end{aligned} \tag{2}$$

When $n=2t+i$ for some $i\in\{1,3,\ldots,t-3\}$,

$$\begin{aligned}P'':\ &(1,t+1)G_{t+1,t+i}(t+i,2t+i)\overline{G}_{2t+i,2t}(2t,t)(t,t-1)\\ &\cdot(t-1,2t-1)(2t-1,2t-2)(2t-2,t-2)\ldots\\ &\cdot(i+2,t+i+2)(t+i+2,t+i+1)(t+i+1,i+1)\overline{G}_{i+1,2}.\end{aligned} \tag{3}$$

See also Figures 1(a)–1(c), respectively, for the illustration of P, P', and P''.

Now, by using P, P', and P'', we construct a Hamiltonian path from 1 to 2 in $T_n\langle 1,t\rangle$ as follows.

When $n\equiv 2(\mathrm{mod}(t-1))$, a desired path obtained by using P is shown in Figure 2(a).

When $n\equiv(1+i)(\mathrm{mod}(t-1))$ for some $i\in\{3,5,\ldots,t-1\}$, we use P and P' to get a suitable path shown in Figure 2(b).

Finally, when $n\equiv(2+i)(\mathrm{mod}(t-1))\geq 2t+1$ for some $i\in\{1,3,5,\ldots,t-3\}$, we consider P and P'' to obtain a path given in Figure 2(c) as desired. □

Lemma 2. *If t is an odd integer and n is an even integer, then $T_n\langle 1,t\rangle$ admits a Hamiltonian path from 1 to 2.*

Proof. Let $n\equiv(1+i)(\mathrm{mod}(t-1))$ be an even integer for some $i\in\{1,3,\ldots,t-2\}$. For $i=1$, consider the path shown in Figure 2(a), while for other values of i, follow the path shown in Figure 2(b). □

Immediate consequences of Lemmas 1 and 2 are as follows.

Corollary 1. *Let t be an even integer and $p<q$ be any two vertices of $T=T_n\langle 1,t\rangle$. Then, paths $P_{p,q}$ and $P_{q,p}$ exist in T if $q-p\in\{1,t,t+2,\ldots,2t-4\}$ or $q-p\geq 2t-2$.*

Corollary 2. *Let t be an odd integer and $p<q$ be any two vertices of $T=T_n\langle 1,t\rangle$. Then, paths $P_{p,q}$ and $P_{q,p}$ exist in T if $q-p=1$ or $q-p\geq t$ is odd.*

Lemma 3. *Let t be an even integer and x be any vertex of $T=T_{2t}\langle 1,t\rangle$. Then, there exists a Hamiltonian path from x to $2t$ in T.*

Proof. For $x=1$, the result is trivial. Paths for other values of x are listed in Table 1. □

Now, by using Corollary 1 and Lemma 3, we prove our next lemma.

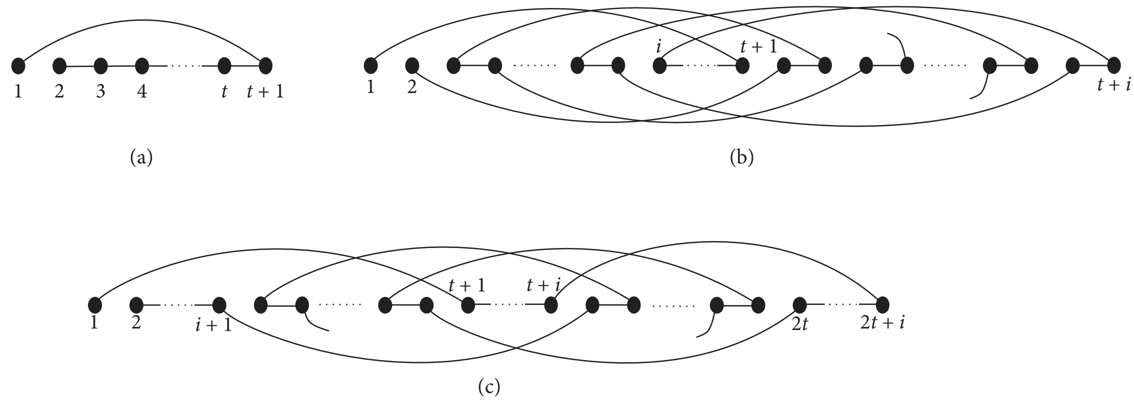

Figure 1: P, P', and P'' basic paths.

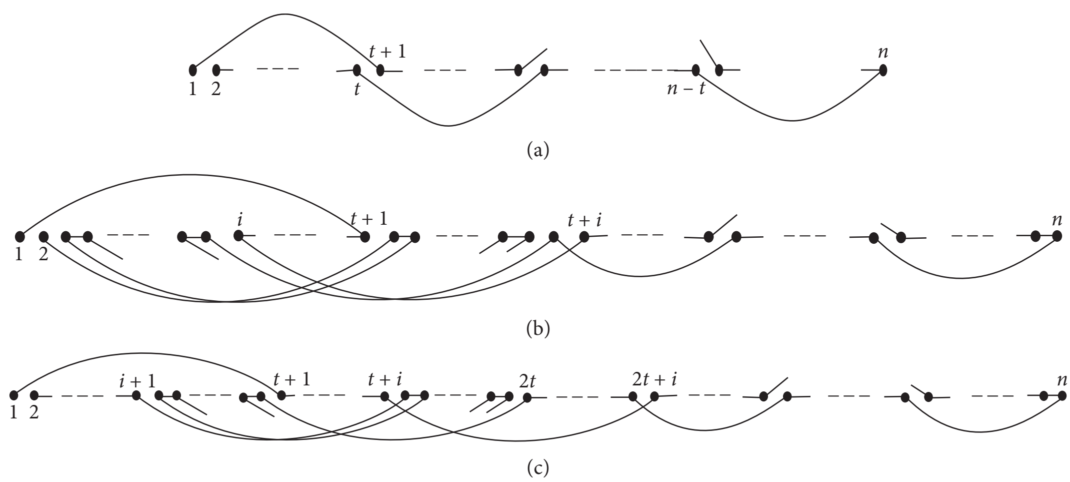

Figure 2: Hamiltonian path from 1 to 2 in T, when t is even. (a) $n \equiv 2(\text{mod } (t-1))$. (b) $n \equiv (1+i)(\text{mod } (t-1))$; $i \in \{3, 5, \ldots, t-1\}$. (c) $n \equiv (2+i)(\text{mod } (t-1)) \geq 2t+1$; $i \in \{1, 3, \ldots, t-3\}$.

Table 1: Hamiltonian paths from x to $2t$ in $\langle T_{2t}\rangle 1, t$ when t is even.

Conditions	Paths
$1 \neq x < t$ is odd	$G_{x,t+1}(t+1,1)(1,2)(2,t+2)(t+2,t+3)(t+3,3)$ $(3,4)\ldots(x-2,x-1)(x-1,t+x-1)G_{t+x-1,2t}$
$x > t$ is odd	$\overline{G}_{x,t+1}(t+1,1)G_{1,x+1-t}(x+1-t,x+1)$ $(x+1,x+2)(x+2,x+2-t)(x+2-t,x+3-t)$ $\ldots(2t-1,t-1)(t-1,t)(t,2t)$
$x \leq t$ is even	$\overline{G}_{x,1}(1,t+1)G_{t+1,t+x+1}(t+x+1,x+1)$ $(x+1,x+2)(x+2,t+x+2)(t+x+2,t+x+3)$ $\ldots(2t-1,t-1)(t-1,t)(t,2t)(t,2t)$
$x > t$ is even	$(x,x-t)(x-t,x-1-t)(x-1-t,x-1)$ $(x-1,x-2)\ldots(t+2,2)(2,1)(1,t+1)\overline{G}_{t+1,x+1-t}$ $(x+1-t,x+1)G_{x+1,2t}$

Lemma 4. *Let $x < y$ be any two vertices of $T_n\langle 1, t\rangle$, where t is an even integer and $n \geq 5t-2$. Then, $T_n\langle 1, t\rangle$ admits a Hamiltonian path from x to y, except from 2 to $t+1$ (by symmetry, another one from $n-t$ to $n-1$).*

Proof. Let $T = T_n\langle 1, t\rangle$ be a Toeplitz graph with t even and $n \geq 5t-2$. Because of symmetry of Toeplitz graphs, it suffices to show that T admits a Hamiltonian path from any vertex $x \in \{1, 2, \ldots, \lceil n/2 \rceil\}$ to each vertex $y \in \{2, 3, \ldots, n\}$ $(x < y)$ of T. Take $x < y$, any two vertices of T, other than the pair $\{2, t+1\}$ of vertices. We split our proof into two cases. ☐

Case 1. $y = x + 1$.

Let $x \in \{1, 2, \ldots, t\}$, then by Corollary 1, we have paths $P_{t,n}$ and $P_{t+2,n}$ in T. By joining $P_{t,n}$ and $P_{t+2,n}$ to the remaining subgraph of T, we obtain desired Hamiltonian paths $\overline{G}_{x,1}(1, t+1)P_{t,n}\overline{G}_{t,x+1}$ and $\overline{G}_{x,3}(3, t+3)P_{t+2,n}(t+2, 2)(2, 1)(1, x+1)$ in T, respectively, for $x \leq t-1$ and $x = t$. For illustration, see Figures 3(a) and 3(b), respectively.

Finally, to obtain Hamiltonian paths for remaining values of x, we assume $a \leq x \leq b$, where $a = t + 1 + i(t-2)$ and $b = t + (i+1)(t-2)$ are integers for some $i \geq 0$ such that $b \leq \lceil n/2 \rceil$. By applying Corollary 1 to T, we get $P_{a,1}$ and $P_{b+1,n}$ in T and construct a desired path $\overline{G}_{x,a}P_{a,1}(a-1, b+2)P_{b+1,n}\overline{G}_{b+1,x+1}$ from x to $x+1$ in T. See also Figure 3(c).

Case 2. $y \neq x + 1$.

Here, we partition the vertex set of T into 5 subsets of vertices, according to Figure 4, and consider the following two subcases.

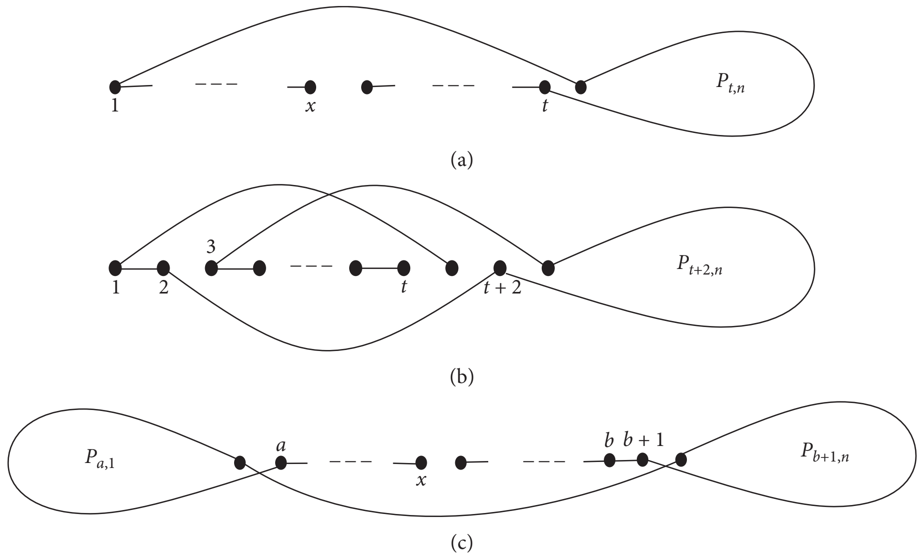

FIGURE 3: Hamiltonian path from x to $x+1$ in T when $x \leq \lceil n/2 \rceil$. (a) $x \leq t-1$. (b) $x = t$. (c) $y = x+1$ for $x > t$.

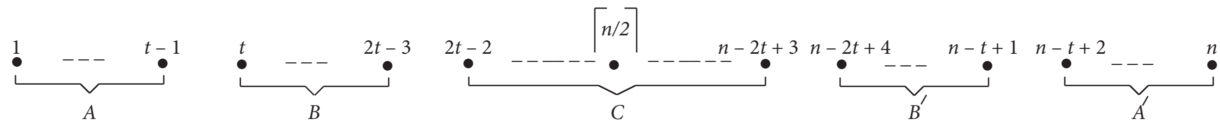

FIGURE 4: Five subsets of vertices.

Case 3. $x, y \in A \cup B \cup C$.

Corollary 1 guarantees the existence of $P_{v,n}$ for any vertex $v < n-2t+3$ in T. Here, by using it, we are getting Hamiltonian paths from x to y in T.

When $x = 1$, we consider the path $G_{1,y-1}P_{y-1,n}$ from 1 to any vertex y in T.

If $x \neq 1$, then possible Hamiltonian paths for different values of x and y are listed in Tables 2–4.

Case 4. $x \in A \cup B \cup C$ and $y \in B' \cup A'$.

The case when $x \in C$ and $y \in B' \cup A'$ is symmetric to the case when $x \in A \cup B$ and $y \in C$. Hence, we remain only with the case when $x \in A \cup B$ and $y \in B' \cup A'$. Here, by Lemma 3, we have a Hamiltonian path in $T_{2t}\langle 1, t\rangle$ from any $x \in A \cup B$ to $2t$. By symmetry, another Hamiltonian path exists from $y \in B' \cup A'$ to $n-2t+1$, of vertex set $\{n-2t+1, n-2t+2, \ldots, n\}$. By joining $2t$ to $n-2t+1$ by the paths $G_{2t,n-2t+1}$, we get a Hamiltonian path from x to y in T.

This completes the proof. □

Now, by using the fact $T_n\langle 1, t, s\rangle \cong T_n\langle 1, t\rangle \cup T_n\langle 1, s\rangle$ and Lemma 4, we prove our next main results.

Theorem 5. *If t is an even integer, then the Toeplitz graph $T_n\langle 1, t, s\rangle$ is hamiltonian-connected for all $n \geq 5t-2$.*

Proof. Let $T = T_n\langle 1, t, s\rangle$ be a Toeplitz graph with t even and $n \geq 5t-2$. Then, because of Lemma 4, we only have to establish the existence of a Hamiltonian path from 2 to $t+1$. For this, we use the entry s along with other two entries 1 and t.

There are five cases to consider. In first four cases, we use $P_{v,n}$ to construct desired paths which exist for any vertex $v \leq n-2t+2$ of T due to Corollary 1.

TABLE 2: Hamiltonian paths from $1 \neq x$ to y in T when t is even.

	$x, y \in A$
Conditions	Paths
y is even	$\overline{G}_{x,1}(1, t+1)G_{t+1,t+x}(t+x, 2t+x)P_{2t+x-1,n}\overline{G}_{2t+x-1,2t}$ $(2t, t)(t, t-1)(t-1, 2t-1)(2t-1, 2t-2)$ $(2t-2, t-2)\ldots(t+x+2, t+x+1)$ $(t+x+1, x+1)G_{x+1,y}$
x is even and y is odd	$\overline{G}_{x,1}(1, t+1)G_{t+1,t+x+1}(t+x+1, x+1)(x+1, x+2)$ $(x+2, t+x+2)\ldots(y-2, y-1)(y-1, t+y-1)$ $G_{t+y-1,2t-1}P_{2t-1,n}(2t, t)\overline{G}_{t,y}$
Both are odd	$\overline{G}_{x,1}(1, t+1)G_{t+1,y+1}(y+1, t+y+1)G_{t+y+1,2t+2}$ $(2t+2, t+2)(t+2, t+3)(t+3, 2t+3)\ldots$ $(t+x-1, t+x)(t+x, 2t+x)\overline{G}_{2t+x,2t+y-1}P_{2t+y-1,n}$ $(2t+y, t+y)\overline{G}_{t+y,t+x+1}(t+x+1, x+1)G_{x+1,y}$

(i) For $s = t+1$, considered path is

$$(2,1)(1, s+1 = t+2)P_{t+2,n}(t+3, 3)G_{3,t+1}. \tag{4}$$

(ii) When $s = t+i$; for some $i \in \{3, 5, \ldots, t-1\}$, a possible path is

$$\begin{gathered}(2,1)(1, s+1 = t+i+1)G_{t+i+1,2t+1}P_{2t+1,n} \\ \cdot (2t+2, t+2)(t+2, t+3)(t+3, 3)(3, 4) \\ \cdot (4, t+4)\ldots(t+i-1, t+i)(t+i, i)G_{i,t+1}.\end{gathered} \tag{5}$$

(iii) $s = t+i$; for some $i \in \{2, 4, \ldots, t-2\}$; in this case, constructed path is

Table 3: Hamiltonian paths from $1 \neq x$ to y in T when t is even.

$1 \neq x \in A$ and $y \in B \cup C$	
Conditions	Paths
$y = t$	$\overline{G}_{x,1}(1, t+1) G_{t+1,t+x} P_{t+x,n}(t+x+1, x+1)\ G_{x+1,t}$
x is odd and $y = t+i$; $i \in \{2, 4, \ldots, x-1\}$	$\overline{G}_{x,i-1}(i-1, t+i-1)(t+i-1, t+i-2)\ (t+i-2, i-2)(i-2, i-3)\ldots(2,1)$ $(1, t+1)\overline{G}_{t+1,x+1}(x+1, t+x+1) P_{t+x,n} \overline{G}_{t+x,y}$
x is odd and $y = t+i$; $i \in \{1, 3, \ldots, x-2\}$	$G_{x,t+1}(t+1, 1)(1,2)(2, t+2)(t+2, t+3)\ (t+3, 3)\ldots(i-2, i-1)(i-1, t+i-1)$ $(t+i-1, 2t+i-1) P_{2t+i-2,n} \overline{G}_{2t+i-2,t+x-1}\ (t+x-1, x-1)(x-1, x-2)$ $(x-2, t+x-2)(t+x-2, t+x-3)\ldots\ (y+2, y+1)(y+1, i+1)(i+1, i)(i, y)$
x is odd and $y \geq t+x$	$G_{x,t+1}(t+1, 1)(1,2)(2, t+2)(t+2, t+3)\ (t+3, 3)\ldots(x-2, x-1)(x-1, t+x-1)$ $G_{t+x-1,y-1} P_{y-1,n}$
x is even and $y = t+1$	$\overline{G}_{x,3}(3, 3+t) G_{t+3,t+x+1}(t+x+1, x+1)\ (x+1, x+2)(x+2, t+x+2)\ldots(t-1, t)$ $(t, 2t)(2t, 2t+1) P_{2t+1,n}(2t+2, t+2)\ (t+2, 2)(2,1)(1, t+1)$
x is even and $y = t+i$; $i \in \{3, 5, \ldots, t-1\}$ such that $i < x+1$	$\overline{G}_{x,1}(1, 1+t) G_{t+1,t+i-1}(t+i-1, 2t+i-1)\ P_{2t+i-2,n} \overline{G}_{2t+i-1,2t}(2t, t)(t, t-1)(t-1, 2t-1)$ $\ldots(t+x+3, t+x+2)(t+x+2, x+2)\ (x+2, x+1)(x+1, t+x+1)\overline{G}_{t+x+1,y}$
x is even and $y = t+x+1$	$\overline{G}_{x,1}(1, t+1) G_{t+1,t+x}(t+x, 2t+x) P_{2t+x-1,n}\ \overline{G}_{2t+x-1,2t}(2t, t)(t, t-1)(t-1, 2t-1)\ldots$ $(t+x+3, t+x+2)(t+x+2, x+2)\ (x+2, x+1)(x+1, y)$
x is even and $y = t+i$; $i \in \{3, 5, \ldots, t-1\}$ such that $i > x+1$	$\overline{G}_{x,1}(1, t+1) G_{t+1,t+x+1}(t+x+1, x+1)\ (x+1, x+2)(x+2, t+x+2)\ldots(i-2, i-1)$ $(i-1, t+i-1)(t+i-1, 2t+i-1) P_{2t+i-2,n}\ \overline{G}_{2t+i-2,2t}(2t, t)$ $(t, t-1)(t-1, 2t-1)\ldots(i+1, i)(i, y)$
x is even and $y = t+i$; $i \in \{2, 4, \ldots, t\}$ such that $i \leq x$	$\overline{G}_{x,i-1}(i-1, t+i-1)(t+i-1, t+i-2)\ (t+i-2, i-2)\ldots(t+2, 2)(2,1)(1, t+1)$ $\overline{G}_{t+1,x+1}(x+1, t+x+1) P_{t+x,n} \overline{G}_{t+x,y}$
x is even and $y = t+i$; $i \in \{2, 4, \ldots, t\}$ such that $i > x$	$(x, x-1)(x-1, t+x-1)(t+x-1, t+x-2)\ (t+x-2, x-2)\ldots(t+2, 2)(2,1)(1, t+1)$ $\overline{G}_{t+1,i-1}(i-1, t+i-1)(t+i-1, t+i-2)\ (t+i-2, i-2)\ldots(x+2, x+1)(x+1, t+x+1)$ $(t+x+1, t+x)(t+x, 2t+x) P_{2t+x-1,n} \overline{G}_{2t+x-1,y}$
x is even and $y \geq 2t+1$	$\overline{G}_{x,1}(1, 1+t) G_{t+1,t+x+1}(t+x+1, x+1)\ (x+1, x+2)(x+2, t+x+2)\ldots(t-1, t)(t, 2t)$ $G_{2t,y-1} P_{y-1,y}$

Table 4: Hamiltonian paths from $1 \neq x$ to y in T when t is even.

$x, y \in B \cup C$	
Conditions	Paths
$x \in \{t+1, t+3, \ldots, 2t-3\}$ and $y = t+i$; $i \in \{2, 4, \ldots, t-2\}$	$\overline{G}_{x,t+1}(t+1, 1) G_{1,t}(t, 2t) G_{2t,t+x+1}\ (t+x+1, x+1)(x+1, x+2)\ (x+2, t+x+2)\ldots(y-2, y-1)\ (y-1, t+y-1) G_{t+y-1,3t-2} P_{3t-2,n}\ (3t-1, 2t-1)\overline{G}_{2t-1,y}$
$x \in \{t+1, t+3, \ldots, 2t-3\}$ and $y = t+i$; $i \in \{1, 3, \ldots, t-1\}$	$\overline{G}_{x,t+1}(t+1, 1) G_{1,i+1}(i+1, t+i+1)\ (t+i+1, t+i+2)(t+i+2, i+2)$ $\ldots(t-1, t)(t, 2t) G_{2t,t+x} P_{t+x+1,n}\ (t+x+1, x+1) G_{x+1,y}$
$x \in \{t+1, t+3, \ldots, 2t-3\}$ and $y = 2t$	$\overline{G}_{x,t+1}(t+1, 1) G_{1,i+1}(i+1, t+i+1)\ (t+i+1, t+i+2)(t+i+2, i+2)\ldots$ $(t-2, 2t-2)(2t-2, 3t-2)\overline{G}_{3t-2,2t+1}\ (2t+1, 3t+1) P_{3t,n}(3t, 3t-1)$ $(3t-1, 2t-1)(2t-1, t-1)(t-1, t)\ (t, 2t = y)$
$x \in \{t+1, t+3, \ldots, 2t-3\}$ and $y > 2t$	$\overline{G}_{x,t+1}(t+1, 1) G_{1,i+1}(i+1, t+i+1)\ (t+i+1, t+i+2)(t+i+2, i+2)$ $\ldots(t-1, t)(t, 2t) G_{2t,y-1} P_{y-1,n}$
$x \in \{t, t+2, \ldots, 2t-4\} \cup C$	$P_{x+1,1} G_{x+1,y-1} P_{y-1,n}$

$$G_{2,i+2}(i+2, t+i+2)(t+i+2, t+i+3) \cdot (t+i+3, i+3)\ldots(t-1, t)(t, 2t)(2t, 2t+1) \cdot P_{2t+1,n}(2t+2, t+2) G_{t+2,t+i+1}(t+i+1, i+1)(1, t+1). \tag{6}$$

(iv) $s \in \{2t, 2t+1, \ldots, n-2t+2\}$; a desired Hamiltonian path is

$$(2,1)(1, s+1) P_{s,n} \overline{G}_{s,2t}(2t, t)(t, t-1)(t-1, 2t-1)\ldots \cdot (t+4, 4)(4,3)(3, t+3) G_{t+3,t+1}. \tag{7}$$

(v) $s \in \{n-2t+3, n-2t+4, \ldots, n-1\}$: here, first, by using Lemma 3, we construct a path $Q_{n-2t+1,s+1}$ in $T_n\langle 1, t\rangle$ from $s+1$ to $n-2t+1$, of vertex set $\{n-2t+1, n-2t+2, \ldots, n\}$. Then, by joining this path to the remaining subgraph of T, we get a Hamiltonian path

$$(2,1)(1, s+1) Q_{n-2t+1,s+1} \overline{G}_{n-2t+1,2t}(2t, t)(t, t-1) \cdot (t-1, 2t-1)\ldots(t+4, 4)(4,3)(3, t+3) G_{t+3,t+1}, \tag{8}$$

from 2 to $t+1$. This concludes the proof. □

Theorem 6. *If t is odd and s is even, then the Toeplitz graph $T_n\langle 1, t, s\rangle$ is Hamiltonian-connected for all $n \geq 5s-2$.*

Proof. Again by virtue of Lemma 4, for s even and $n \geq 5s - 2$, we only need to prove the existence of a Hamiltonian path starting from 2 and ending at $s + 1$. Here, we consider the following four cases:

(i) $s = t + 1$; by Corollary 1, we have $P_{t+2,n}$ in $T_n\langle 1, s\rangle$, which helps us to get a desired path:

$$(2,1)(1,t+1)G_{t+1,3}(3,t+3)P_{t+2,n}. \tag{9}$$

(ii) $s = t + 3$; again by applying Corollary 1 to $T_n\langle 1, s\rangle$, we get $P_{2t+1,n}$ to construct a Hamiltonian path:

$$\begin{aligned}&(2,1)(1,t+1)\overline{G}_{t+1,3}(3,t+3)(t+3,t+2)\\&\quad\cdot(t+2,2t+2)P_{2t+1,n}\overline{G}_{2t+1,t+4}.\end{aligned} \tag{10}$$

(iii) $s \in \{t + 5, t + 7, \ldots, 2t\}$; here,

$$\begin{aligned}&(2,1)(1,t+1)\overline{G}_{t+1,3}(3,t+3)(t+3,t+2)\\&\quad\cdot(t+2,2t+2)G_{2t+2,2t+4}(2t+4,t+4)(t+4,t+5)\\&\quad\cdot(t+5,2t+5)\ldots(s-1,s)(s,s+t)G_{s+t,3t}P_{3t,n}\\&\quad\cdot(3t+1,2t+1)\overline{G}_{2t+1,s+1},\end{aligned} \tag{11}$$

is a desired path, which is constructed by using $P_{3t,n}$, obtained by applying Corollaries 1 and 2 to $T_n\langle 1, s\rangle$.

(iv) $s \geq 2t$; in this case, we use Corollaries 1 and 2 to obtain $P_{s+2,n}$ in $T_n\langle 1, s\rangle$ and $P_{s+2,t+2}$ in $T_n\langle 1, t\rangle$, respectively, which enables us to obtain a Hamiltonian path

$$(2,1)(1,t+1)\overline{G}_{t+1,3}(3,s+3)P_{s+2,n}P_{s+2,t+2}, \tag{12}$$

from 2 to $s + 1$ in T. This completes the proof. □

3. Conclusion

We proved here the existence of a number N such that for $n \geq N$, every nonbipartite Toeplitz graph $T_n\langle 1, t, s\rangle$ is Hamiltonian-connected. Also, the family of Toeplitz graphs, which are also circulant, contains Hamiltonian-connected graphs.

Data Availability

Research data have been provided in the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgements

The first author gratefully acknowledges financial support by the Deanship of Scientific Research, King Faisal University, through the Nasher track under the under grant no. 186236. The third author gratefully acknowledges financial support by NSF of China (no. 11871192) and the Program for Foreign experts of Hebei Province (no. 2019YX002A).

References

[1] R. E. Burkard and W. Sandholzer, "Efficiently solvable special cases of bottleneck travelling salesman problems," *Discrete Applied Mathematics*, vol. 32, no. 1, pp. 61–76, 1991.

[2] C. C. Chen and N. Quimpo, "On strongly Hamiltonian abelian group graphs, Combinatorial Mathematics VIII," in *Lecture Notes in Mathematics*, K. L. McAvaney, Ed., vol. 884, pp. 23–34, Springer, Berlin, Germany, 1981.

[3] R. Euler, "Coloring infinite, planar Toeplitz graphs," Tech. Report, Laboratoire d'Informatique de Brest (LIBr), Brest Cedex, France, 1998.

[4] R. Euler, "Characterizing bipartite Toeplitz graphs," *Theoretical Computer Science*, vol. 263, no. 1-2, pp. 47–58, 2001.

[5] R. Euler, H. LeVerge, and T. Zamfirescu, "A characterization of infinite, bipartite Toeplitz graphs," in *Combinatorics and Graph Theory 95*, Ku Tung-Hsin, Ed., vol. 1, pp. 119–130, Academia Sinica, World Scientific, Singapore, 1995.

[6] R. Euler and T. Zamfirescu, "On planar Toeplitz graphs," *Graphs and Combinatorics*, vol. 29, no. 5, pp. 1311–1327, 2013.

[7] C. Heuberger, "On planarity and colorability of circulant graphs," *Discrete Mathematics*, vol. 268, no. 1–3, pp. 153–169, 2003.

[8] C. Heuberger, "On Hamiltonian Toeplitz graphs," *Discrete Mathematics*, vol. 245, no. 1–3, pp. 107–125, 2002.

[9] S. Malik, "Hamiltonian cycles in directed Toeplitz graphs-2," *Ars Combinatoria*, vol. 116, pp. 303–319, 2014.

[10] S. Malik and A. M. Qureshi, "Hamiltonian cycles in directed Toeplitz graphs," *Ars Combinatoria*, vol. 109, pp. 511–526, 2013.

[11] S. Malik and T. Zamfirescu, "Hamiltonian connectedness in directed Toeplitz graphs," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 53, no. 2, pp. 145–156, 2010.

[12] M. F. Nadeem, A. Shabbir, and T. Zamfirescu, "Hamiltonian connectedness of Toeplitz graphs," *Springer Proceedings in Mathematics & Statistics*, vol. 98, pp. 135–149, 2014.

[13] O. Ore, "Hamiltonian connected graphs," *Journal de Mathématiques Pures et Appliquées*, vol. 42, pp. 21–27, 1963.

[14] R. van Dal, G. Tijssen, Z. Tuza, J. A. A. van der Veen, Ch. Zamfirescu, and T. Zamfirescu, "Hamiltonian properties of Toeplitz graphs," *Discrete Mathematics*, vol. 159, no. 1–3, pp. 69–81, 1996.

[15] E. A. Van Doorn, "Connectivity of circulant digraphs," *Journal of Graph Theory*, vol. 10, no. 1, pp. 9–14, 1986.