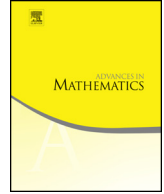Contents lists available at ScienceDirect

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With respect to whom are you critical?

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A R T I C L E   I N F O

A B S T R A C T

For any compact Riemannian surface S and any point y in S, Q_y^{-1} denotes the set of all points in S for which y is a critical point, and $|Q_y^{-1}|$ its cardinality. We proved [2] together with Imre Bárány that $|Q_y^{-1}| \geq 1$, and that equality for all $y \in S$ characterizes the surfaces homeomorphic to the sphere. Here we show, for any orientable surface S and any point $y \in S$, the following two main results. There exists an open and dense set of Riemannian metrics g on S for which y is critical with respect to an odd number of points in S, and this is sharp. If S is the torus then $|Q_y^{-1}| \leq 5$, and if S has genus $g \geq 2$ then $|Q_y^{-1}| \leq 8g - 5$. Properties involving points at globally maximal distance on S are eventually presented.

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1. Introduction

In this paper, by *surface* we always mean a 2-dimensional compact Riemannian manifold, unless explicitly stated otherwise.

For any surface S, denote by ρ its (intrinsic) metric, and by ρ_x the *distance function* from $x \in S$, given by $\rho_x(y) = \rho(x, y)$. A *segment* between x and y in S is a path from x to y of length $\rho(x, y)$. A point $y \in S$ is called *critical* with respect to ρ_x (or to x), if for any tangent direction τ of S at y there exists a segment from y to x whose tangent direction at y makes a non-obtuse angle with τ.

For an excellent survey of critical point theory for distance functions see [8].

For any point x in S, denote by Q_x the set of all critical points with respect to x, and by Q the *critical point mapping* associating to any point x in S the set Q_x. Similarly, M_x is the set of all relative maxima of ρ_x, F_x the set of all farthest points from x (i.e., absolute maxima of ρ_x) and M, respectively F, are the corresponding set-valued mappings.

Properties of the mappings Q, M and F on Alexandrov spaces have previously been obtained in [9] and [19]. See the survey [17] for various results concerning the mapping F on convex surfaces.

We proved in [2], together with Imre Bárány, that the set Q_y^{-1} of all points with respect to which y is critical is never empty. It is also shown in [2] that Q_y^{-1} is single-valued for all $y \in S$ if and only if the genus of S is 0. We continue this study in the following.

Let $\mathcal{G}$ denote the space of all Riemannian metrics on the surface S; it is viewed as the space of sections of the bundle of positive definite symmetric matrices over S, endowed with the $\mathcal{C}^\infty$ Whitney topology [4].

In a topological space $\mathcal{T}$, a property P is called *generic* if the set of all elements in $\mathcal{T}$ without property P is of first Baire category. We obtain an even stronger sense of genericity if "nowhere density" replaces "first Baire category", and this is the meaning we use in this paper. Several results and open questions about generic Riemannian metrics are presented in [3], see also the references therein. We mention next only one.

M.A. Buchner [4] showed that, on a surface, the set of metrics which are cut locus stable is open and dense in $\mathcal{G}$; moreover, for any such metric, every ramification point of the cut locus has degree three. We get, and later use, a slightly improved result, see §2 for the definitions and Theorem 4.1 for the precise statement.

Our Theorem 4.2 contributes to this topic, too. It states that any point y in any orientable surface S is critical with respect to an odd number of points in S, for a generic metric on S. This result is sharp, as Theorem 5.2 shows. Theorem 4.2 is also useful for the proof of our Theorem 6.1.

Theorem 6.1 provides, for orientable Riemannian surfaces, an upper bound for $|Q_y^{-1}|$. It is based on its counter-part for Alexandrov surfaces, Theorem 3.2, which strengthens Theorem 2 in [24]. We apply Theorem 6.1 to estimate the cardinality of diametrally

opposite sets on S (Corollary 7.2). Thus, our results also contribute to a description of farthest points H. Steinhaus had asked for (see §A35 in [6]).

The case of points y in orientable Alexandrov surfaces, which are common maxima of several distance functions, is treated in [18]; for an introduction to Alexandrov spaces with curvature bounded below, see [5]. See also [13], [14], for results in a direction somewhat similar to ours.

2. Preliminaries

The length (1-dimensional Hausdorff measure) of the set A is denoted by $\lambda(A)$.

Let S be a surface. By T_x we denote the circle of all tangent directions at $x \in S$; we have $\lambda(T_x) = 2\pi$.

Let $x \in S$. For every $\tau \in T_x$, a point $c(\tau)$ called *cut point* is associated, defined by the requirement that the arc $xc(\tau) \subset \Gamma$ is a segment which cannot be extended further (as a segment) beyond $c(\tau)$; here, Γ is the geodesic through x of tangent direction τ at x. The set of all these cut points is the *cut locus $C(x)$* of the point x. The cut locus was introduced by H. Poincaré in 1905 [12] and became, since then, an important tool in Global Riemannian Geometry, see for example [11], [15], or [16].

It is known that $C(x)$, if it is not a single point, is a *local tree* (i.e., each of its points z has a neighbourhood V in S such that the component $K_z(V)$ of z in $C(x) \cap V$ is a tree), even a tree if S is homeomorphic to the sphere. If S is not a topological sphere, the *cyclic part of $C(x)$* is the minimal (with respect to inclusion) subset $C^{cp}(x)$ of $C(x)$, whose removal from S produces a topological (open) disk. It is easily seen that $C^{cp}(x)$ is a local tree with finitely many ramification points and no extremities (see [10]).

Recall that a *tree* is a set T any two points of which can be joined by a unique Jordan arc included in T. The *degree* of a point y of a local tree is the number of components of $K_y(V) \setminus \{y\}$ if the neighbourhood V of y is chosen such that $K_y(V)$ is a tree. A point y of the local tree T is called an *extremity* of T if it has degree 1, and a *ramification point* of T if it has degree at least 3. A local tree is *finite* if it has finitely many points of degree different from 2. An *internal edge* of the finite tree T is a Jordan arc in T in which the endpoints and no other points are ramification points of T.

All these notions admit obvious extensions to Alexandrov surfaces. The main result in [1], Theorem 4 in [23] and Theorem 1 in [21] yield the existence of Alexandrov surfaces S on which the set of all extremities of any cut locus is residual in S.

It is, however, known that $C(x)$ has an at most countable set $C_3(x)$ of ramification points [16]. Let $C_3^{cp}(x)$ be the set of points of degree at least 3 in the finite local tree $C^{cp}(x)$. We stress that the degree is not taken in $C(x)$, but in $C^{cp}(x)$. It is known that $C_3^{cp}(x)$ is a finite set [10].

Let S be a surface and $x \in Q_y^{-1}$; put $i(x) = 2$ if there are precisely 2 segments from y to x, and $i(x) = 3$ if there are at least 3 segments from y to x. For $j = 2, 3$, we say that the *point x is of type j* if $i(x) = j$. Put $\sharp_y^j = |\{x \in Q_y^{-1} : i(x) = j\}|$; clearly, $|Q_y^{-1}| = \sharp_y^2 + \sharp_y^3$.

In [2] the authors proved together with Imre Bárány, in the framework of Alexandrov surfaces, the following three results. (See [7] for a variational proof of the first one, valid for finite dimensional Riemannian manifolds.)

Lemma 2.1. *Every point on every surface is critical with respect to some point of the surface.*

Lemma 2.2. *Assume S is a Riemannian surface, y a point in S, and $x \in Q_y^{-1}$ is such that the union U of two segments from x to y disconnects S. If a component S' of $S \setminus U$ meets no segment from x to y then $Q_y^{-1} \cap S' = \emptyset$. In particular, if the union of any two segments from x to y disconnects S then $Q_y^{-1} = \{x\}$.*

Lemma 2.2 shows, in particular, that on many surfaces there are points which are critical with respect to precisely one other point.

Lemma 2.3. *An orientable surface S is homeomorphic to the sphere S^2 if and only if each point in S is critical with respect to precisely one other point of S.*

3. A general result

We prove in this section a result for arbitrary Alexandrov surfaces, which in particular holds for (Riemannian) surfaces. Before giving it, we recall a result in graph theory. All graphs we consider in the following are finite, connected, and may have loops and multiple edges.

Lemma 3.1. *Let G be a connected graph with m edges, n vertices and q generating cycles. Then*
i) $m - n + 1 = q$;
ii) $m \leq 3(q - 1)$ and $n \leq 2(q - 1)$, with equality if and only if G is cubic.

Proof. The equality (i) is well known. For the inequalities (ii), fix q. It follows from the first part that m and n are maximal if and only if G is cubic. In this case we have $3n = 2m$, and we obtain $n = 2(q - 1)$, $m = 3(q - 1)$. $\square$

Recall that a point y in an Alexandrov surface is called *smooth* if $\lambda(T_y) = 2\pi$, where T_y is the space of tangent directions at y (as defined, for example, in [5]).

For the simplicity of our exposition, we see every graph G as a 1-dimensional simplicial complex.

Theorem 3.2. *Let y be a smooth point on an orientable Alexandrov surface S of genus g.*
If $g = 0$, then $|Q_y^{-1}| = 1$.
If $g \geq 1$, then $\sharp_y^2 \leq 6g - 3$ and $\sharp_y^3 \leq 4g - 2$; this yields $|Q_y^{-1}| \leq 10g - 5$.

For any point y on the standard projective plane, $Q_y^{-1} = Q_y$ is a circle, so one cannot drop the orientability condition in Theorem 3.2.

The restriction to smooth points in Theorem 3.2 is essential, too. Indeed, for any surface S with a conical point y, if $\lambda(T_y) \leq \pi$ then $Q_y^{-1} = S \setminus \{y\}$. See [18] for properties of the sets M_y^{-1} and Q_y^{-1} in case $\pi \leq \lambda(T_y) < 2\pi$.

Proof. The case $g = 0$ is covered by Lemma 2.3, so we may assume $g \geq 1$. And in virtue of Lemma 2.2, we may consider only points $y \in S$ with $Q_y^{-1} \subset C^{cp}(y)$.

Assume for simplicity of the exposition that $C(y) = C^{cp}(y)$.

Consider $C^{cp}(y) = (V, E)$ as a graph, with $V = C_3^{cp}(y)$ and E the set of components of $C^{cp}(y) \setminus V$. Call the elements of V *vertices*, and the elements of E *edges*.

We claim that the interior of each edge I of $C^{cp}(y)$ contains at most one point $x \in Q_y^{-1}$. To see this, assume there exists some point $x \in Q_y^{-1}$ interior to I. Then there are two segments from x to y, making at y the angle π. Since y is smooth, $\lambda(T_y) = 2\pi$ and therefore the two images x', x'' of x on T_y are diametrally opposite. Let $x_* \neq x$ be another point in the interior of I, with images x'_*, x''_* on T_y. Since S is orientable, the order on T_y is either x', x'_*, x''_*, x'' or x', x''_*, x'_*, x''. In both cases x'_*, x''_* cannot be diametrally opposite, hence $x_* \notin Q_y^{-1}$.

Then, since $C^{cp}(y)$ has $2g$ generating cycles, Lemma 3.1 gives $\sharp_y^2 \leq 6g - 3$ and $\sharp_y^3 \leq 4g - 2$, which together imply $|Q_y^{-1}| = \sharp_y^2 + \sharp_y^3 \leq 10g - 5$. □

Notice that this upper bound on $|Q_y^{-1}|$ is only imposed by the topology of S. We shall refine it in Section 6 by local geometrical considerations.

4. Two generic results

For the proof of Theorem 4.2, we shall make use of the main result in [4], that we complete in the following with a new statement of independent interest. Notice that this result doesn't require orientability of the surface. See [4] for the definition of cut locus stable metrics.

Theorem 4.1. *Let S be a surface and y a point in S.*

The set $\mathcal{C}^y$ of cut locus (with respect with y) stable metrics on S is open and dense in $\mathcal{G}$. For any $\mathfrak{g}$ in $\mathcal{C}^y$, every ramification point of the cut locus $C(y)$ with respect to $\mathfrak{g}$ is joined to y by precisely three segments.

There exists a set $\tilde{\mathcal{C}}^y$ of cut locus (with respect with y) stable metrics on S, open and dense in $\mathcal{G}$, such that for any $\tilde{\mathfrak{g}}$ in $\tilde{\mathcal{C}}^y$, every ramification point x of the cyclic part of the cut locus $C(y)$ with respect to $\mathfrak{g}$ is joined to y by precisely three segments, no two of them of opposite tangent directions at x or at y.

Proof. The first part of the theorem is proved by Buchner in [4].

Consider now a metric $\mathfrak{g}$ in $\mathcal{C}^y$ and a point $x \in C_3^{cp}(y)$, hence it is joined to y by precisely three segments, say $\gamma^1, \gamma^2, \gamma^3$. Assume that the tangent directions of γ^1 and

γ^2 at x are opposite, so they form a *geodesic loop*. Since the limit of geodesic loops is a geodesic loop, the set of all such metrics is closed in $\mathcal{G}$, and its complement $\tilde{\mathcal{C}}_x^y$ is open.

We prove now the density of $\tilde{\mathcal{C}}_x^y$ in $\mathcal{G}$. In order to do it, we approximate $\mathfrak{g}$ in two steps.

First we "put a bump" to slightly cover γ^1, assymetrically with respect to the left and right parts of γ^1. Consequently, in a neighbourhood of the image set of γ^1 (on S), there is a unique shortest path $\tilde{\gamma}^1$ from x to y with respect to the new metric $\mathfrak{g}'$; $\tilde{\gamma}^1$ is a little longer than γ^1 and, more importantly, it makes no angle of π with γ^2 or γ^3.

Second, we put bumps on γ^2 and γ^3, such that the obtained metric $\mathfrak{g}''$ has the following properties. In respective neighbourhoods of the image sets of γ^2 and γ^3 (on S), there are unique shortest paths $\tilde{\gamma}^2$, $\tilde{\gamma}^3$ from x to y, with respect to $\mathfrak{g}''$. They have the same respective tangent directions at x as γ^2 and γ^3 and, moreover, they have the same length as $\tilde{\gamma}^1$.

One can proceed similarly for the tangent directions at y.

Since $C_3^{cp}(x)$ is a finite set, after finitely many such procedures we get a metric $\tilde{\mathfrak{g}} \in \tilde{\mathcal{C}}^y$ approximating $\mathfrak{g}$, with the desired properties. □

Theorem 4.2. *If S is an orientable surface and y a point in S then, for a generic Riemannian metric on S, y is critical with respect to an odd number of points in S.*

During our proof we shall refer to the proof of Theorem 1 in [2].

Proof. Consider a metric on S as in Theorem 4.1. We will identify here T_y with a Euclidean circle of centre $\mathbf{0}$ and length $\lambda(T_y) = 2\pi$.

If S is homeomorphic to the sphere then the statement follows from Lemma 2.3. Assume this is not the case.

A finite number of cycles were defined in [2] to prove Lemma 2.1, by joining points in T_y corresponding to the vertices in $C_3^{cp}(y)$ by line segments or arcs in T_y. Next we indicate a geometrical interpretation (i.e., an equivalent definition) for some of those cycles, useful for our purpose.

The injectivity radius $\text{inj}(S)$ is positive. Therefore, for any $\varepsilon > 0$ sufficiently small, there is a natural identification Φ of T_y to the boundary $\text{bd}N_\varepsilon$ of the ε-neighbourhood N_ε of $C(y)$ in S. Choose a point $x \in C_3^{cp}(y)$. For each segment γ_x from x to y, take the (first) point z_{γ_x} in $\gamma_x \cap \text{bd}N_\varepsilon$. The set of all these points z_{γ_x} has $\deg x = |c^{-1}(x)|$ components, each of which is a point or an arc. (Recall that c is the restriction of the exponential map to T_y.) Join with segments the extremities of consecutive – with respect to some circular order – components. The simple closed curve C_x thus constructed corresponds, by the use of Φ^{-1}, to the cycle C_i determined by $c^{-1}(x)$, and is called a *vertex-cycle*. Moreover, the boundary of every component of $N_\varepsilon \setminus \bigcup_{x \in C_3^{cp}(x)} \text{int}C_x$ yields, again by the use of Φ^{-1}, a cycle C_i determined by consecutive points α, β in $c^{-1}(C_3^{cp}(y))$, and is called an *edge-cycle*.

Let $C_1, ..., C_n$ be all these cycles.

If $\mathbf{0} \in \cup_{j=1}^n C_j$ then, for some $x \in C_3^{cp}(y)$, there are two segments of diametrally opposite tangent directions in T_y, see [2].

If $\mathbf{0} \notin \cup_{j=1}^n C_j$ consider, as in [2], the winding number $w(C_j) = w(\mathbf{0}, C_j)$ of every cycle C_j with respect to $\mathbf{0}$. We have

$$\sum_{i=1}^n w(C_i) = w\left(\sum_{i=1}^n C_i\right) = w(T_y) = 1 \pmod 2,$$

because each edge not in T_y is used exactly twice. This shows that $w(C_i) \neq 0$ for some cycle C_i.

If this cycle C_i is an edge-cycle then, because S is orientable, a semi-continuity argument shows that its corresponding edge in $C_3^{cp}(y)$ contains at least one point in Q_y^{-1}, see [2] for details.

If C_i is a vertex-cycle, $w(C_i) \neq 0$ means that $\mathbf{0}$ is surrounded by C_i, which is impossible if $\mathbf{0} \notin \operatorname{conv} C_i$. By construction, $\operatorname{conv} C_i = \operatorname{conv} c^{-1}(x)$ for some $x \in C_3^{cp}(y)$.

By Theorem 4.1, all the vertices of $C(y)$ have degree three. This and the orientability of S show now that all cycles C_i considered above are simple closed curves, hence $w(C_i) \in \{0, \pm 1\}$. Therefore, because $\sum_{i=1}^n w(C_i) = 1 \pmod 2$, the number of cycles C_i with $w(C_i) \neq 0$ is odd. Each such C_i intersects Q_y^{-1}.

We claim that, if non-zero, $|C_i \cap Q_y^{-1}| = 1$. This is clear for the cycles determined by vertices of $C_3^{cp}(y)$, because these cycles have precisely three sides. Consider now a cycle C_i determined by an edge e of $C_3^{cp}(y)$. Then, because S is orientable, C_i has the form $\alpha_+ \beta_- \beta^- \alpha^+$, with $\alpha_+ \beta_-$ and $\alpha^+ \beta^-$ of contrary orientations on T_y. Take $x \in C_i \cap Q_y^{-1} \neq \emptyset$ and define $l(x) = c^{-1}(x) \cap \alpha_+ \beta_-$ and $r(x) = c^{-1}(x) \cap \alpha^+ \beta^-$. Of course, $l(x)$ and $r(x)$ contain each a single tangent direction for $x \in C(y) \setminus C_3(y)$, and at least one of them has at least two tangent directions for $x \in C_3(y)$. In any case, let $l_\alpha(x)$ be the tangent direction in $l(x)$ closest to α_+ along the arc $\alpha_+ \beta_-$, and let $l_\beta(x)$ be the tangent direction in $l(x)$ closest to β_+ along the same arc $\alpha_+ \beta_-$; possibly $l_\alpha(x) = l_\beta(x)$. Similarly, let $r_\alpha(x)$, $r_\beta(x)$ be the tangent directions in $r(x)$ closest to α^+, respectively β^-, along the arc $\alpha^+ \beta^-$. By definition, the angle between $l_\alpha(x)$ and $r_\alpha(x)$ towards α_+ is at most π, as is the angle between $l_\beta(x)$ and $r_\beta(x)$ towards β_+. Because S is orientable, for $z \in e \setminus \{x\}$ both $l(z)$ and $r(z)$ are inside precisely one of the above two angles, hence $z \notin Q_y^{-1}$ and the claim follows.

The metric we considered is, by Theorem 4.1, such that for any $x \in C_3^{cp}(y)$ and any two segments γ, γ' joining y to x, the angle of γ, γ' at y satisfies $\angle \gamma \gamma' \neq \pi$. Therefore, for any two cycles C_i with $w(C_i) \neq 0$, the points $Q_y^{-1} \cap C_i$ are different and thus $|Q_y^{-1}|$ is odd. $\quad \square$

5. Torus case

In this section we show that the statement of Theorem 4.2 is sharp, in the sense pointed out by Theorem 5.2.
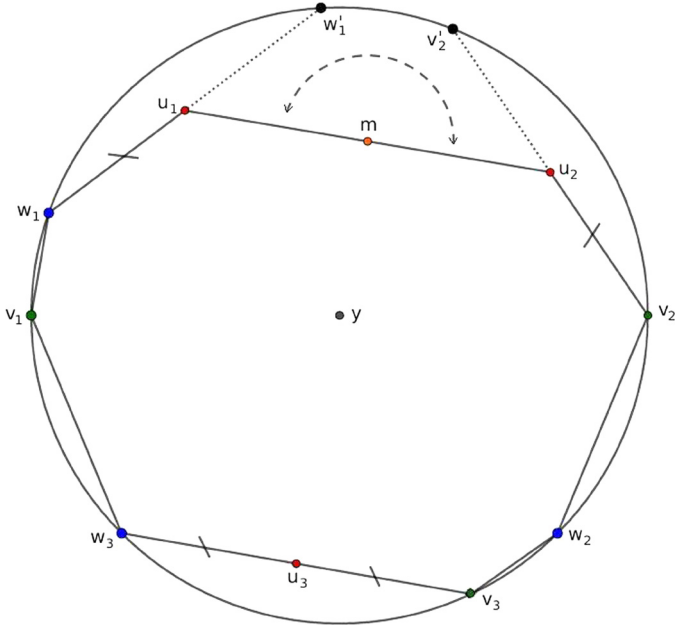
Fig. 1. Construction for a point y on a torus, with $|Q_y^{-1}| = 2$.

We will use the following result of A. D. Weinstein (Proposition C in [20]).

Lemma 5.1. *Let M be a d-dimensional Riemannian manifold and D a d-disc embedded in M. There exists a new metric on M agreeing with the original metric on a neighbourhood of M minus the interior of D, such that, for some point p in D, the exponential mapping at p is a diffeomorphism of the unit disc about the origin in the tangent space at p to M, onto D.*

Theorem 5.2. *For any point y on the torus T, there exist sets of metrics $\mathcal{E}_i^y$ on T ($i \in \{1, 2, 3, 4, 5\}$), such that $|Q_y^{-1}| = i$ with respect to any metric $\mathfrak{g} \in \mathcal{E}_i^y$ and, moreover, $\mathrm{int}\mathcal{E}_j^y \neq \emptyset$ for $j \in \{1, 3, 5\}$, while $\mathcal{E}_k^y$ contains continuous families of metrics for $k \in \{2, 4\}$.*

Proof. We indicate next a construction to get $|Q_y^{-1}| = 2$; it can be easily adapted to obtain the conclusion. (The stability of the respective constructions under small perturbations, for $j \in \{1, 3, 5\}$, provides the non-empty interior.)

Consider, in the hyperbolic plane $\mathbb{H}$ of constant curvature -1, a circle C of centre y and radius r, with $r > 0$ a parameter to be chosen later. Fig. 1 illustrates in the plane our construction.

Consider points v_1 and v_2 diametrically opposite on C. On one of the half-circles bounded by v_1 and v_2 consider points w_3, u_3, w_2 such that v_2, w_2, v_3, w_3, v_1 are in circular order and $\lambda(v_1 w_3) = \lambda(v_2 w_2) > \lambda(v_3 w_3)$. On the other half-circle consider points

w_1, w_1', v_2' such that $\lambda(v_1 w_1) = \lambda(v_3 w_2)$ and $\lambda(w_1 w_1') = \lambda(v_3 w_3) = \lambda(v_2 v_2')$. Of course, we may choose w_2 such that $v_1, w_1, w_1', v_2', v_2$ are in circular order. Let u_1 be the mid-point of $w_1 w_1'$, u_2 the mid-point of $v_2 v_2'$, m the mid-point of $u_1 u_2$, and u_3 the mid-point of $v_3 w_3$.

We may choose r such that the total angle θ_v at v verifies $\theta_v := \angle w_1 v_1 w_3 + \angle u_2 v_2 w_2 + \angle w_2 v_3 w_3 = 2\pi$.

Cut the polygon $v_1 w_1 u_1 m u_2 v_2 w_3 v_3 u_3 w_3 v_1$ out from $\mathbb{H}$ and naturally identify (glue along) the edges mu_1 and mu_2. Further naturally identify the edges in the following pairs: $v_1 w_1$ and $v_2 w_2$, $v_2 w_2$ and $v_1 w_3$, $w_1 u_1$ and $w_3 u_3$, and $v_2 u_2$ and $v_3 u_3$.

Denote by v the common image of v_1, v_2, v_3, by w the common image of w_1, w_2, w_3, by u the common image of u_1, u_2, u_3, by y the image of y, and by m the image of m, via the above glueing procedure.

The resulting closed surface is a torus T'' with conical singularities at the points m (where $\theta_m = \pi$), u (where $\theta_u = \angle w_1 u_1 m + \angle m u_2 v_2 + \pi > 2\pi$), and w_1 (where $\theta_{w_1} = \angle v_1 w_1 u_1 + \angle v_2 w_2 v_3 + \angle v_3 w_3 v_1 > 2\pi$).

Smoothen first T'' locally around m and w to obtain a surface T' with unique singularity at u. Of course, small changes around those points do not affect the segments from y to v or u. Moreover, because the directions of the segments from w to y were all included in an open half-circle of T_y, this property will remain true for all points in $T' \setminus T''$.

Next we show how to smoothen T' around u. Consider a metrical ε-neighbourhood U_ε of u on T', of boundary length $l = l(\varepsilon, \theta_u) = \lambda(\partial U_\varepsilon)$. Consider some $\alpha < -1$ such that, on the hyperbolic plane of constant curvature α, the geodesic ball D of radius ε has boundary length precisely l. Cut U_ε off T' and replace it by D. Also denote by u the center of D after the replacement. By Lemma 5.1, there exists a torus T whose metric outside a neighbourhood of D coincides with the metric on T', and such that the directions of the segments from u to $p = y$ remain the same as those on T'. Therefore, on the obtained Riemannian surface T we have $Q_y^{-1} = \{v, u\}$.

Of course, continuous changes of the positions of w_1, w_2, w_3 yield continuous families of metrics with the desired property. $\quad\square$

6. An upper bound for $|Q_y^{-1}|$

Theorem 6.1. *Let S be an orientable (Riemannian) surface of genus g and y a point in S.*

If $g = 0$, then $|Q_y^{-1}| = 1$, and if $g = 1$, then $|Q_y^{-1}| \leq 5$.
If $g \geq 2$, then $|Q_y^{-1}| \leq 8g - 5$.

Proof. The proof consists of two steps. First we directly prove that $|Q_y^{-1}| \leq 8g - 4$, and afterward we invoke Theorem 4.2 to decrease that upper bound by 1.

Step 1. In virtue of Lemma 2.2, we may consider only points $y \in S$ with $Q_y^{-1} \subset C^{cp}(y)$.

As in the proof of Theorem 3.2, we consider $C^{cp}(y) = (V, E)$ as a graph, with vertex set $V = C_3^{cp}(y)$ and edge set E the set of components of $C^{cp}(y) \backslash V$. By Theorem 3.2, each edge of $C^{cp}(y)$ contains at most one interior point $x \in Q_y^{-1}$, so $\sharp_y^2 \le 6g - 3$, $\sharp_y^3 \le 4g - 2$, and $\sharp_y^2 + \sharp_y^3 \le 10g - 5$.

Notice that this upper bound on $|Q_y^{-1}|$ is only imposed by the topology of S. We refine it next by local geometrical considerations.

For the graph $C^{cp}(y) = (V, E)$, call an edge *white* if it intersects Q_y^{-1}, and *black* if it doesn't. A vertex is *white* if it belongs to Q_y^{-1}, and *black* otherwise. A Y is a subgraph of $C^{cp}(y)$ formed by a vertex x of degree three and three edges issuing at x.

Assume first that $C^{cp}(y)$ is a cubic graph.

We claim that, if there exists a white Y in $C^{cp}(y)$, then no other edge is white. To see this, assume the edges e_{kl}, e_{km} and e_{kn} are white and share a common extremity, say v_k. Then the images on T_y of the vertices incident to these edges respect the circular order $v_l, v_k, v_m,\ v_n, v_k, v_l,\ v_m, v_k, v_n$. Since the images of each edge in the white Y contain opposite points with respect to the centre of T_y, there is no place for other white edges.

Thus, if $|E| = 3$, then S has genus 1 (because it is orientable) and we get the upper bound $|Q_y^{-1}| \le |V| + |E| = 5$. This is sharp, as one can easily see for a flat torus whose fundamental domain is a parallelogram.

If $g > 1$ then, by our claim, at least one third of the edges are black. Assuming all vertices are white, we obtain $|Q_y^{-1}| \le 8g - 4$.

So we have obtained an upper bound $|Q_y^{-1}| \le B_3(g) = 8g - 4$, if $C^{cp}(y)$ is a cubic graph. We treat now the general case, in order to obtain an upper bound $|Q_y^{-1}| \le B(g)$ with no restriction on the degree of vertices in V.

Slightly modify the metric $\mathfrak{g}$ of S around the vertices of $C^{cp}(y)$ of degree larger than three to obtain a new metric $\mathfrak{g}'$ on S close to $\mathfrak{g}$, with the following properties: every vertex in $C^{cp}(y)(\mathfrak{g}')$ has degree three, and every white edge of $C^{cp}(y)(\mathfrak{g})$ is still white in $C^{cp}(y)(\mathfrak{g}')$. This is possible by small perturbations of $\mathfrak{g}$ around (some of) the vertices $x \in C^{cp}(y)(\mathfrak{g})$ with $\deg x > 3$ (see Theorem 4.1). Notice that, for $\mathfrak{g}'$ close enough to $\mathfrak{g}$, there cannot be more white edges in $C^{cp}(y)(\mathfrak{g})$ than in $C^{cp}(y)(\mathfrak{g}')$. As for the vertices, two or more black neighbours in $C^{cp}(y)(\mathfrak{g}')$ may correspond to a white vertex of degree larger than 3 in $C^{cp}(y)(\mathfrak{g})$, which reduces to repaint in white at most half of the non isolated black vertices of $C^{cp}(y)(\mathfrak{g})$. Thus, we get

$$B(g) \le B_3(g) + \frac{1}{2} \left| V \backslash \left(Q_y^{-1} \cup \{v \in V : v \text{ is black and isolated}\} \right) \right|. \tag{1}$$

Since our upper bound $B_3(g)$ assumes all vertices to be white, the inequality (1) gives

$$B(g) \le B_3(g) = 8g - 4$$

and the proof of Step 1 is complete.

Step 2. Consider now metrics on S as in Theorem 4.2, hence the upper bound in this case is odd, namely $B_3(g)^{\mathrm{odd}} = 8g - 5$.

The proof of Theorem 4.2 also shows that, if its cardinality is not odd, Q_y^{-1} contains "double" points; i.e., points corresponding to several cycles. This, of course, implies that, in case $|Q_y^{-1}|$ is even, Q_y^{-1} doesn't have maximum number of elements. Therefore, $B(g) \leq B_3(g) \leq B_3(g)^{\mathrm{odd}} = 8g - 5$ and the proof is complete. $\quad\square$

7. Applications

With the special case of mutually critical points deals [24]. A yet more particular case is that of pairs of points at distance equal to the largest distance on S,

$$d(S) = \max_{x,y \in S} \rho(x, y).$$

For any $x \in S$, we call $\rho_x^{-1}(d(S))$ the *diametrally opposite set* of x, if it is not void. In this case, the point x itself is called *diametral*; of course, not every point is necessarily diametral.

Notice that any diametrally opposite set verifies $\rho_x^{-1}(d(S)) \subset Q_x \cap Q_x^{-1}$ for any $x \in S$.

If S is homeomorphic to S^2, every diametrally opposite set contains a single point, by Theorem 1 in [2] (see also [19]). In the standard projective plane, every point is diametral and any diametrally opposite set is a circle. If S is orientable, every diametrally opposite set is finite, by Theorem 3.2 or Theorem 6.1.

In analogy with the characterization provided by Theorem 2 in [2] (given here as Lemma 2.3), we may think of a similar one imposing cardinality 1 for all diametrally opposite sets. But this condition is weaker. Although surfaces homeomorphic to S^2 satisfy, by Theorem 1 in [2], the imposed condition, there are further examples of surfaces verifying it: any flat torus with a rectangular fundamental domain has only single-point diametrally opposite sets.

In any flat torus without a rectangular fundamental domain, the diametrally opposite set of every point x has exactly 2 points.

A direct consequence of Theorem 6.1 is the following.

Corollary 7.1. *For any point y on an orientable surface of genus $g \geq 2$, the set F_y^{-1} has at most $8g - 5$ points. Hence any diametrally opposite set has at most $8g - 5$ points.*

Concerning the tightness of Theorem 6.1 (and Corollary 7.1) we obtain the following.

Theorem 7.2. *There exist orientable surfaces $\tilde{T}_g$ of genus g with diametrally opposite sets consisting of $4g + 1$ points, where $2g + 1$ points are of type 2 and $2g$ points are of type 3.*

Proof. For the case of surfaces homeomorphic to S^2, see Theorem 1 in [2].

Take now a flat torus with a parallelogram, union of two equilateral triangles with a common edge, as fundamental domain.

In such a torus, for any point y, $C(y)$ is a Θ-shaped graph. Cut along $C(y)$ and unfold to obtain a regular hexagon $v_1 v_2 v_1 v_2 v_1 v_2$. (If y is taken to be the identified vertices of the
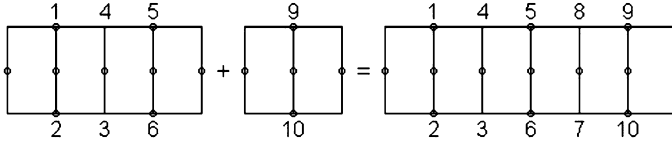
Fig. 2. Inductive construction for tower graphs. Glueing a surface T_2 of genus 2 to a torus T_1, to obtain the surface T_3 of genus 3: the right-most edge of $C(y)$ on T_2 is identified to the left-most edge of $C(y)$ on T_1, to get $C(y)$ on T_3. The points in Q_y^{-1} are marked by small circles.

parallelogram, then v_1 and v_2 are the centres of the two triangles.) Replace small discs of radius ε about the midpoints m_1, m_2, m_3 of the three distinct edges of the hexagon and about the centre (also denoted by) y of the hexagon, by congruent bumps, all bounded by circles of length $2\pi\varepsilon$. The bumps have centres $\tilde{m}_i$ and $\tilde{y}$ at distance $\frac{1}{\sqrt{3}} - \frac{1}{2} + \varepsilon$ from the respective boundaries. In this way we obtain a torus $\tilde{T}_1$, on which

$$\rho(\tilde{y}, v_1) = \rho(\tilde{y}, v_2) = \rho(\tilde{y}, \tilde{m}_1) = \rho(\tilde{y}, \tilde{m}_2) = \rho(\tilde{y}, \tilde{m}_3) = \frac{1}{\sqrt{3}} = d(\tilde{T}_1).$$

Thus, $\{v_1, v_2, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3\}$ is a diametrally opposite set of $\tilde{y}$.

Next we define inductively surfaces T_g for all $g \geq 2$, with the following properties.

The domain $D_g = T_g \setminus C(y)$ is a regular $6(2g-1)$-gon of centre y in the hyperbolic plane of constant curvature -1, with the property that all its angles are $2\pi/3$.

The cut locus of y in T_g is a –horizontally sitting– tower-shape graph with $2g+1$ levels. Each level-edge provides a point in Q_y^{-1} of type 2, where from $\sharp_y^2 = 2g+1$, and each vertex of even level is of type 3, so $\sharp_y^3 = 2g$. Fig. 2 shows the case $g = 2$, as well as where to attach a handle to T_2 in order to obtain the order of vertices on D_3.

To see that we can realize the tower-shape graphs as cut loci, it needs to specify how to identify (i.e., the order of) vertices and edges on D_g.

The domain $D_2 = T_2 \setminus C(y)$ is a regular 18-gon whose vertices, given in circular order, are 1, 2, 3, 4, 5, 6, 5, 4, 1, 2, 1, 4, 3, 6, 5, 6, 3, 2. The edges, following the above order of vertices, are $a, b, c, d, e, f, d, g, h, a, g, c, i, e, f, i, b, h$, see Fig. 3. Clearly, only the points 1, 2, 5, 6 are of type 3, and only the edges a, c, e, f, h of $C(y)$ contain each a point of type 2, hence $\sharp_y^2 = 5$, $\sharp_y^3 = 4$, and $|Q_y^{-1}| = 9$.

Assume we have T_g and $y \in T_g$ as above. Choose the right-most edge of $C(y)$, say e, and attach along it a handle. This reduces to locate the two images of e on $\mathrm{bd}D_g$ and to insert between their extremities (labeled $4g-2$ and $4g-3$) the points $4g-1, 4g+2, 4g+1, 4g+2, 4g-1, 4g$, and respectively $4g-1, 4g, 4g+1, 4g+2, 4g+1, 4g$, see again Fig. 3. Label the vertices of D_{g+1} with the new obtained order. Identify the edges in the obvious way to obtain T_{g+1}, and notice that $\sharp_y^2 = 2g+1$, $\sharp_y^3 = 2g$.

Finally, replace (as in the case $g = 1$) small disks about the midpoints of the distinct edges of $\mathrm{bd}D_{g+1}$ and about the centre y of D_{g+1}, by congruent bumps of centres $\tilde{x}_i, \tilde{y}$ in order to obtain $\rho(\tilde{y}, \tilde{x}_i) = d(\tilde{T}_{g+1})$, where $\tilde{T}_{g+1}$ is the constructed surface. □
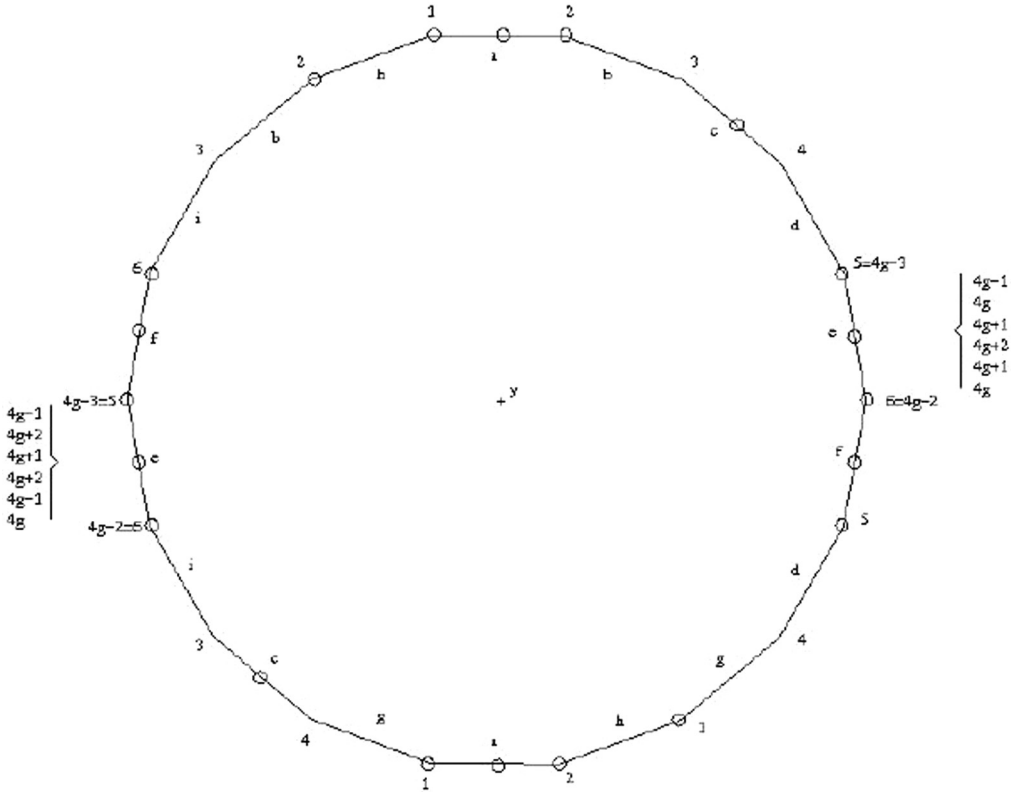
Fig. 3. Domain D_2. The points in Q_y^{-1} are marked by small circles.

8. Open questions

Our approach leaves open several problems, among which we state in the following only three that we find particularly interesting.

1. The number of points with respect to which a point y on a flat torus is critical, does not depend on y. This and Theorem 1 in [2] lead us to the following question. Find all surfaces S with the property that $|Q_y^{-1}|$ does not depend on $y \in S$.
2. For the first step in the proof of Theorem 6.1, we considered points $x \in Q_y^{-1}$ which are vertices of $C^{cp}(y)$, and white subgraphs Y of $C^{cp}(y)$ centered at x. In other words, if we endow the graph $C^{cp}(y)$ with the discrete natural metric, we considered 1-neighbourhoods of the points in $Q_y^{-1} \cap C_3^{cp}(y)$. Would the use of k-neighbourhoods, with $k \geq 2$, improve the upper bound?
3. Every orientable surface of genus $g > 0$ possesses points x, y such that $y \in Q_x$ and there are two segments from y to x with opposite tangent directions at y (see the proof of Theorem 2 in [2]).

Is the same true for all surfaces homeomorphic to the sphere? Or, at least, is it true for densely many surfaces homeomorphic to the sphere?

For a similar – still open – problem concerning convex surfaces, see [22].

Acknowledgments

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