# Poidge-convexity

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**Abstract**

We investigate in this paper the poidge-convexity, which is a generalization of right convexity, introduced by one of the present authors in 2014. Although not every convex body is poidge-convex, there are many families of compact sets, some of them very different from convex bodies, which are poidge-convex. We present here several such families.

**Keywords:** poidge-convexity; starshaped sets; topological discs; polyhedral surfaces.

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## 1 Introduction

At the 1974 meeting about convexity in Oberwolfach, the third author proposed the investigation of the following very general kind of convexity. Let $\mathcal{F}$ be a family of sets in a space $\mathscr{X}$. A set $M \subset \mathscr{X}$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x,\ y \in M$ there is a set $F \in \mathcal{F}$ such that $x,\ y \in F$ and $F \subset M$. In this paper, $\mathscr{X}$ will be $\mathbb{R}^n$; we always assume $n \geq 2$. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the third author [1], and also Böröczky Jr. [2], investigated the rectangular convexity, the case when $\mathcal{F}$ is the family of all non-degenerate rectangles.

Bruckner and Bruckner [7], and also Magazanik and Perles [11] investigated $L_n$ sets, which are $\mathcal{F}$-convex sets, $\mathcal{F}$ consisting of all polygonal paths with at most $n$ edges in the plane. Magazanik and Perles [10] and Breen [3, 4, 5, 6] dealt with staircase connectedness, which is also a kind of $\mathcal{F}$-convexity, $\mathcal{F}$ being the family of all staircases. The third author studied the right convexity [18] (the case with $\mathcal{F}$ consisting of all right triangles), and the last two authors, generalizing it, investigated the *rt*-convexity, i.e. the right triple convexity [12, 13].

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A set $M \subset \mathbb{R}^n$ is called *rt-convex* if it is $\mathcal{F}$-convex, $\mathcal{F}$ being the family of all triples $\{x, y, z\} \subset \mathbb{R}^n$ with $\angle xyz = \pi/2$. If, for $x, y \in M$, there exists $z \in M$ such that the triangle $xyz$ is right, we say that *the pair $x, y$ has the rt-property.*

A set $M$ will be called *poidge-convex* if it is $\mathcal{F}$-convex, $\mathcal{F}$ being the family of all unions $\{x\} \cup \sigma$, called *poidges*, where $x$ is a point, $\sigma$ a line-segment, and $\text{conv}(\{x\} \cup \sigma)$ a right triangle (see Figure 1). If, for points $u, v \in M$, there is a poidge containing them included in $M$, we say that *the pair $u, v$ has the poidge-property.*


Figure 1: Poidges

For convex sets, right convexity, *rt*-convexity and poidge-convexity are equivalent. Obviously, in general, not every *rt*-convex set is poidge-convex, but every rightly convex set is both *rt*-convex and poidge-convex. Less obvious, but true, is that not every poidge-convex set is *rt*-convex either.

The poidge-convexity also generalizes the thin rectangular convexity (for short *tr-convexity*, where $\mathcal{F}$ is the family of all boundaries of non-degenerated rectangles), and a fortiori the rectangular convexity itself.

In this paper, we start the investigation of poidge-convexity. Our main goal will be to identify large classes of poidge-convex sets. On the one hand, not all convex bodies are poidge-convex: take, for example, the convex hull of an ellipse different from a circle. On the other hand, sets looking quite different from being convex, like meandering topological discs or single-point-kernel starshaped sets, can be found among the poidge-convex sets.

## 2 Notation

For distinct $x,\ y \in \mathbb{R}^n$, let $xy$ denote the line-segment from $x$ to $y$, and $\overline{xy}$ the line through $x, y$.

For any compact set $M \subset \mathbb{R}^n$, let $S_M$ be the smallest hypersphere containing $M$ in its convex hull; also, $\overline{M}$ means the affine hull of $M$, $\text{int}\, M$ the relative interior of $M$ (i.e., in the topology of $\overline{M}$) and bd $M$ the relative boundary of $M$.

By $\dim M$ we denote the Hausdorff dimension of $M$. Also, denote by $\mathrm{p}_A(x)$ the orthogonal projection of $x$ onto the affine subspace $A$. The distance from a point $x$ to a compact set $M$ will be denoted by $\rho(x, M) = \min\{\|x - y\| : y \in M\}$.

Furthermore, for distinct $x,\ y \in \mathbb{R}^n$, we denote by $H_{xy}$ the hyperplane through $x$ orthogonal to $\overline{xy}$, by $H_{xy}^+$ the closed half-space containing $y$ determined by $H_{xy}$, and by $H_{xy}^-$ the other closed half-space determined by $H_{xy}$.

We denote by $[xy)$ the half-line starting at $x$ and passing through $y$. Also, we set $\widehat{xky} = \operatorname{conv}([kx) \cup [ky))$, if $k \notin xy$.

The compact ball of centre $\omega$ and radius $r$ is denoted by $B_r(\omega)$. $\mathbf{0}$ is the origin of $\mathbb{R}^n$.

In a Baire space, we say that *most* of its elements have property **P** if those not enjoying **P** form a set of first Baire category.

# 3 Starshaped sets

## 3.1 One-sided starshaped sets

Let $\mathcal{S}$ be the Baire space of all starshaped sets in $\mathbb{R}^n$, always considered compact here. For $M \in \mathcal{S}$, a line-segment $kx \subset M$ is called a *ray* if $k \in \ker M$ and $kx$ cannot be extended beyond $x$ in $M$.

It is known (see the corollary to Theorem 1 in [17]) that most starshaped sets have a single-point kernel. Moreover, in $\mathcal{S}$, the set of those starshaped sets $M$ with $\operatorname{card}(M \cap S_M) = 2$ is nowhere dense. For a proof, the reader can adapt the first part of the proof of Theorem 1 in [16], which deals with convex curves instead of starshaped sets.

More generally, we call any compact set $M$ *ordinary* if $\operatorname{card}(M \cap S_M) \geq 3$.

Clearly, being ordinary is a necessary condition for any compact set to be poidge-convex.

Let $M \subset \mathbb{R}^2$ be a starshaped set with more than one point, and let $k \in \ker M$. The set $M$ will be called *one-sided*, if there exist half-lines $[kx)$ and $[ky)$ such that

(i) $M \subset \widehat{xky}$, or

(ii) $[kx)$ and $[ky)$ are opposite, i.e. $[kx) \cup [ky)$ is a line, and $M$ is included in one of the two half-planes with boundary $[kx) \cup [ky)$.

Now, taking $\angle xky$ to be minimal over all $x, y \in \mathbb{R}^2 \setminus \{k\}$ and $k \in \ker M$ in case (i), we call $\omega(M) = \angle xky$ the *opening* of $M$. In case (ii), put $\omega(M) = \pi$. Thus, $0 \leq \omega \leq \pi$. (Note that $M \neq \{k\}$ because $\operatorname{card} M > 1$.) See Figure 2.

We call $M$ *acute*, if there exists no non-degenerate triangle $kxy \subset M$, where $kx$ is a ray of $M$.

**Theorem 3.1.** *Let $M \subset \mathbb{R}^2$ be an ordinary one-sided starshaped set. If $0 < \omega(M) \leqslant \frac{\pi}{2}$, then $M$ is poidge-convex. If $M$ is acute and $\frac{\pi}{2} < \omega(M) < \pi$, then $M$ is not poidge-convex.*

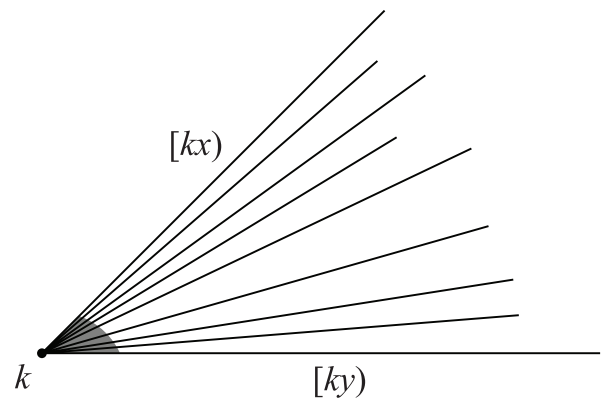


Figure 2: The opening of an one-sided starshaped set

*Proof.* Take the point $k$ from the definition of the opening to be $\{\mathbf{0}\}$. Assume $0 < \omega(M) \leqslant \frac{\pi}{2}$. Let $x, y \in M \setminus \{\mathbf{0}\}$, with $\|x\| \leqslant \|y\|$. If $x, y$ are linearly independent, consider $z = \mathrm{p}_{\overline{\mathbf{0}y}}(x)$. Then $\{x\} \cup zy \subset M$. If $x, y$ are linearly dependent, there exists $u \in M \setminus \overline{\mathbf{0}y}$, since $\omega(M) > 0$. Consider a point $v \in \mathbf{0}u$ close to $\mathbf{0}$ and $w = \mathrm{p}_{\overline{\mathbf{0}y}}(v)$. Then $x, y$ belong to $\{v\} \cup wy \subset M$. It remains to consider the pairs $(\mathbf{0}, x) \in M$. Let $\mathbf{0}y$ be the ray containing $x$. From $\mathrm{card}(M \cap S_M) \geqslant 3$ it follows that $M$ has a point $z' \notin \mathrm{int\,conv}\, S_{\mathbf{0}y}$ different from $\mathbf{0}$ and $y$. One possibility is that $\angle z'\mathbf{0}y < \frac{\pi}{2}$. Then $\mathbf{0}z' \cap S_{\mathbf{0}y} \neq \{\mathbf{0}, y\}$. Take $z \in \mathbf{0}z' \cap S_{\mathbf{0}y} \setminus \{\mathbf{0}, y\}$; possibly $z = z'$. Obviously, the points $\mathbf{0}, x$ belong to the poidge $\{z\} \cup \mathbf{0}y \subset M$. The other possibility is that $\angle z'\mathbf{0}y = \frac{\pi}{2}$. In this case, $\mathbf{0}, x$ belong to the poidge $\{z'\} \cup \mathbf{0}y \subset M$.

Now assume $\frac{\pi}{2} < \omega(M) < \pi$. There exist two rays $\mathbf{0}x, \mathbf{0}y$ of $M$ such that $\frac{\pi}{2} < \angle x\mathbf{0}y < \pi$. We claim that $x, y$ do not enjoy the poidge-property. Since every line-segment in $M$ starting at $x$ or $y$ must be collinear with $\mathbf{0}$, otherwise $M$ would not be acute, the only way for $x, y$ to have the poidge-property would be to belong to a poidge with its line-segment $\sigma$ along one of the rays $\mathbf{0}x, \mathbf{0}y$. But since $\angle x\mathbf{0}y > \frac{\pi}{2}$, $\sigma$ would contain $\mathbf{0}$ in its relative interior. This contradicts the inequality $\omega(M) < \pi$. □

It is easily seen that, for $\omega(M) = 0$, $M$ is not poidge-convex and, for $\omega(M) = \pi$, both poidge-convexity and non-poidge-convexity are possible.

We consider the family $\mathcal{S}^\kappa$ of all one-sided $M \in \mathcal{S}$ with bounded opening $\omega(M) \leq \kappa$. This family is closed in $\mathcal{S}$, and is therefore itself complete and consequently a Baire space.

**Theorem 3.2.** *For $0 < \kappa \leq \pi/2$, most sets in $\mathcal{S}^\kappa$ are poidge-convex.*
*For $\pi/2 < \kappa < \pi$, most sets in $\mathcal{S}^\kappa$ are not poidge-convex.*

*Proof.* For any $\kappa \in ]0, \pi[$, most sets in $\mathcal{S}^\kappa$ are ordinary. The proof parallels the one for $\mathcal{S}$ or for the space of all convex curves instead of $\mathcal{S}^\kappa$. Moreover, the sets $M \in \mathcal{S}^\kappa$ with $\omega(M) = 0$

form a nowhere dense family. Hence, by the first part of Theorem 3.1, most sets in $\mathcal{S}^\kappa$ are poidge-convex.

For any $\kappa < \pi$, most sets in $\mathcal{S}^\kappa$ are nowhere dense (the proof works like in the case of $\mathcal{S}$ instead of $\mathcal{S}^\kappa$, see the corollary to Theorem 1 in [17]), hence acute. By the second part of Theorem 3.1, most sets in $\mathcal{S}^\kappa$ are not poidge-convex. □

### 3.2 Symmetric starshaped sets

Clearly, the space $\mathcal{S}^*$ of all (point) symmetric starshaped sets in $\mathbb{R}^n$ is closed in $\mathcal{S}$, so it is itself a Baire space.

Like most starshaped sets, most symmetric starshaped sets have a single-point kernel.

If a starshaped set is symmetric, then its kernel is also symmetric, with the same centre of symmetry, say $\mathbf{0}$. Being convex, the kernel contains $\mathbf{0}$.

**Theorem 3.3.** *A set $M \in \mathcal{S}^*$ is poidge-convex if and only if it is ordinary.*

*Proof.* Suppose $M \in \mathcal{S}^*$ is ordinary. Take $x, y \in M \setminus \{\mathbf{0}\}$, assuming without loss of generality that $\|x\| \leqslant \|y\|$. Suppose first that $x, y$ are linearly independent. Taking $z = \mathrm{p}_{\overline{\mathbf{0}y}}(x)$, we have $\{x, y\} \subset \{x\} \cup zy \subset M$. If $x, y$ are linearly dependent, then $x \in y(-y)$. If $S_{y(-y)} \cap M = \{y, -y\}$, then $M$ is not ordinary. Hence, $S_{y(-y)} \cap M$ contains some point $z \notin \{y, -y\}$, and $x, y$ belong to the poidge $\{z\} \cup y(-y)$.

Suppose now that $M$ is not ordinary, whence it has a single diameter $x(-x)$. There is no point $z \in S_{x(-x)} \cap M \setminus \{x, -x\}$. Moreover, the hyperplanes $H_{x(-x)}$ and $H_{(-x)x}$ meet $M$ in $x, -x$ only. So, $x, -x$ don't belong to any poidge in $M$. □

The next result has a corresponding one for convex bodies (with a similar proof), and uncovers a major discrepancy in comparison with the non-symmetric case.

**Theorem 3.4.** *Most sets belonging to $\mathcal{S}^*$ are not ordinary.*

*Proof.* Let $\mathcal{S}^*_m$ be the set of all $M \in \mathcal{S}^*$ having a pair of diameters at Pompeiu-Hausdorff distance at leat $1/m$ from each other. We prove that $\mathcal{S}^*_m$ is nowhere dense in $\mathcal{S}^*$.

Let $\mathcal{O} \subset \mathcal{S}^*$ be open and $M \in \mathcal{O}$. Choose a diameter of $M$. This is also a diameter of $S_M$. Extend it in both directions by $\varepsilon > 0$ to a line-segment $\Delta_\varepsilon$. Then $M_\varepsilon = M \cup \Delta_\varepsilon$ belongs to $\mathcal{S}^*$ and has the unique diameter $\Delta_\varepsilon$. If $\varepsilon$ is small enough, $M_\varepsilon \in \mathcal{O}$, too. There exists an open neighbourhood $\mathcal{N}$ of $M_\varepsilon$ in $\mathcal{O}$ such that every set belonging to $\mathcal{N}$ has all its diameters at distance less than $1/2m$ from $\Delta_\varepsilon$. This shows that they are not in $\mathcal{S}^*_m$. Thus, $\mathcal{S}^*_m$ is nowhere dense, and the set $\cup_{m=1}^\infty \mathcal{S}^*_m$ of all $M \in \mathcal{S}^*$ having more than one diameter is of first Baire category. □

**Corollary 3.5.** *Most sets belonging to $\mathcal{S}^*$ are not poidge-convex.*

### 3.3 Starshaped sets close to convex

We introduce a measure of non-convexity for non-convex starshaped sets, which might be of independent interest. Line-segments $xy$ joining $x, y \in \mathrm{bd}\, M$ are called *chords* of the set $M \subset \mathbb{R}^n$. Let $k$ belong to the kernel $\ker M$ of $M \in \mathcal{S}$. Let $\Gamma(M)$ be the set of chords $xy$ of $M$ not included in $M$. Let

$$\xi(M) = \sup_{xy \in \Gamma(M)} \inf_{k \in \ker M} \angle xky.$$

See Figure 3. If the sequence $\{M_m\}_{m=1}^{\infty}$ of starshaped sets converges, and $\xi(M_m) \to 0$, then the limit set is convex. Thus, $\xi$ indicates how close to being convex a starshaped set is. We shall see that starshaped sets "closer" to convex ones, i.e. with smaller value of $\xi$, are more likely to be poidge-convex.

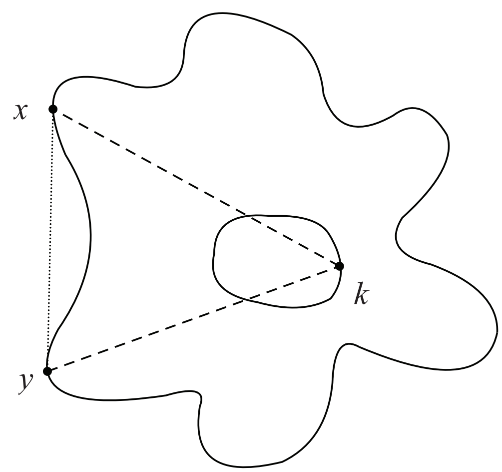


Figure 3: The function $\xi$.

**Theorem 3.6.** *If* $M \subset \mathbb{R}^2$, $M \in \mathcal{S}$, $M$ *is ordinary,* $\dim \ker M > 1$, *and* $\xi(M) < \frac{\pi}{2}$, *then* $M$ *is poidge-convex.*

*Proof.* Let $x, y \in M$.

Case 1. $xy \subset M$.

Being ordinary, $M$ must have a point $z \notin (\mathrm{int\,conv}\, S_{xy}) \cup \{x, y\}$.

If $z \in S_{xy}$, then $\{z\} \cup xy$ is a suitable poidge. If $z \in H_{xy} \cup H_{xy}$, then again $\{z\} \cup xy$ is a suitable poidge. So, let $z$ lie in one of the four components of $\mathbb{R}^2 \setminus (H_{xy} \cup H_{yx} \cup \mathrm{conv}\, S_{xy})$, namely $H_{xy}^-$, $H_{yx}^-$, $E_1$, $E_2$.

Take $k \in (\ker M) \setminus (\overline{xy} \cup \overline{xz} \cup \overline{yz})$. This is possible, because $\dim \ker M > 1$.

If $k \in (H_{xy} \cup H_{yx} \cup S_{xy})$, we are done.

If $k \in E_1 \cup E_2$, then $kx \cap S_{xy} \neq \emptyset$, and we are done again.

If $k \in H^-_{xy}$, then $ky$ meets $H_{xy}$ and the intersection point is not $x$. For $k \in H^-_{yx}$, we have an analogous situation. So, assume $k$ lies in $\operatorname{int}\operatorname{conv} S_{xy}$.

Now, we consider again the positions that $z$ may take.

If $z \in H^-_{xy}$, then $kz \cap H_{xy} \neq \emptyset$ and the intersection point is not $x$.

If $z \in H^-_{yx}$, then $kz \cap H_{yx} \neq \emptyset$ and the intersection point is not $y$.

Finally, if $z \in E_1 \cup E_2$,, then $kz \cap S_{xy} \neq \emptyset$.

Thus, in all cases a poidge in $M$ including $xy$ can be found.

<u>Case 2.</u> $xy \not\subset M$.

Extend $xy$ until we obtain a chord $x'y' \in \Gamma(M)$. Since $\inf_{k \in \ker M} \angle x'ky' \leq \xi(M) < \pi/2$, we find a point $k^* \in \ker M$ satisfying $\angle x'k^*y' < \pi/2$, which implies $\angle xk^*y < \pi/2$.

Assume without loss of generality that $\|k^* - x\| \leq \|k^* - y\|$. Now, if $z = \mathrm{p}_{\overline{k^*y}}(x)$, then $\{x\} \cup zy$ is a suitable poidge. □

# 4 Topological discs

## 4.1 Smooth topological discs

For a set $M \subset \mathbb{R}^2$ with differentiable boundary, a chord $xy$ is called a *double normal of* $M$ if it is orthogonal to both tangent lines of $\operatorname{bd} M$ at $x$ and $y$ (see Figure 4).

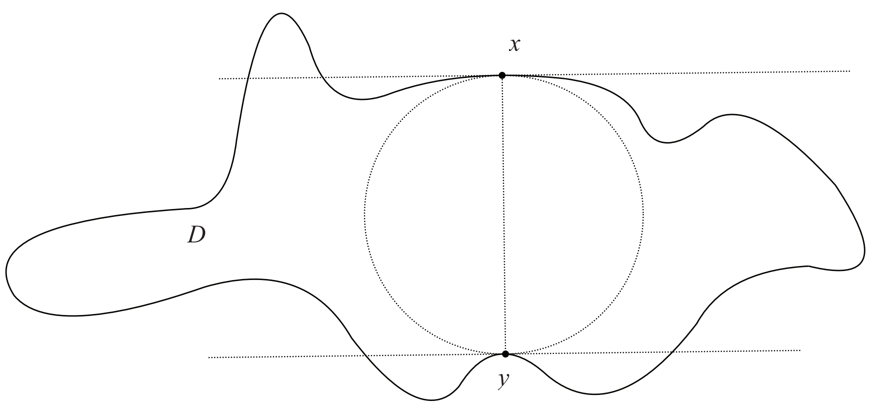


Figure 4: A double normal $xy$.

Let $\mathfrak{D}_m$ be the space of all planar topological discs with a $C^m$-boundary.

**Theorem 4.1.** *A topological disc $D \in \mathfrak{D}_2$ is poidge-convex if for any double normal $xy$ of* $\operatorname{bd} D$*, the curvature of* $\operatorname{bd} D$ *at $x$ or $y$ is less than* $\frac{2}{\|x-y\|}$.

*Proof.* Clearly, $x, y$ have the poidge-convex property if one of them is interior to $D$. So, let $x, y \in \operatorname{bd} D$. If $xy$ is not a double normal, then the tangent line at $x$ or $y$ is not orthogonal to $\overline{xy}$. Suppose the tangent line $T$ at $x$ is not orthogonal to $\overline{xy}$. Then there exists a small line-segment $xz \subset D$ orthogonal to $\overline{xy}$. Thus, $\{y\} \cup xz$ is a suitable poidge.

If $xy$ is a double normal, by hypothesis, at $x$ or $y$, say at $x$, the curvature radius satisfies $\rho(x) > \frac{\|x-y\|}{2}$. This implies that $S_{xy}$ has a small arc starting at $x$, inside of $D$. If that arc is $\widehat{xz}$, then the poidge $\{y\} \cup xz$ is as required. (Notice that the case that $D$ is not locally convex at $x$ is not excluded. In that case, the curvature condition is even superfluous.) $\square$

**Remarks.** A topological disc must have at least one double normal: take its diameter!

The condition, which works for *rt*-convexity, that $D$ has at least two diameters, or that $D$ is ordinary (i.e. card $(D \cap S_D) \geqslant 3$), is not sufficient for the poidge-convexity.

We offer now an extension of Theorem 4.1 to topological discs with boundaries of class $C^1$.

Suppose $D \subset \mathbb{R}^2$ is a topological disc, locally convex at some boundary point $x$. Then, if bd $D$ is differentiable at $x$, a lower and an upper curvature, $\gamma_i(x)$ and $\gamma_s(x)$, at $x$ can be defined, as in [8] (page 14). The preceding theorem can be (in the same way) proven in the following more general form.

**Theorem 4.2.** *A topological disc $D \in \mathfrak{D}_1$ is poidge-convex if, for any double normal $xy$ of* bd $D$*, at one of its endpoints, say at $x$, either $D$ is locally convex and $\gamma_s(x) < \frac{2}{\|x-y\|}$, or $\mathbb{R}^2 \setminus \operatorname{int} D$ is locally convex.*

## 4.2 Unions of convex sets

Here, we consider unions of compact convex sets in $\mathbb{R}^n$, possibly of distinct dimensions.

**Theorem 4.3.** *Every ordinary connected union of two ordinary compact convex sets is poidge-convex.*

*Proof.* Let $M = A \cup B$, with $A, B$ compact convex sets, satisfying the hypothesis. Let $x, y \in M$. If both $x, y$ lie in $A$, or both in $B$, then we have a right triangle $xyz$ in $A$ or in $B$ because they are rightly convex, by Theorem 3 in [18]. So, assume without loss of generality that $x \in A \setminus B$ and $y \in B \setminus A$. If inside $S_{xy}$ there is no point of $M$, then $A \cap B = \emptyset$ and $M$ is not connected. If $M \subset \operatorname{conv} S_{xy}$, then, $M$ being ordinary, there exists a point $z \in S_{xy} \setminus \{x, y\}$. If $M \not\subset \operatorname{conv} S_{xy}$, but $S_{xy} \cap M = \{x, y\}$, then either $A \subset \operatorname{conv} S_{xy}$ and $B \cap \operatorname{conv} S_{xy} = \{y\}$, or vice-versa. In both cases $M$ becomes disconnected, or one of the sets $A, B$ is not ordinary, which is false.

Hence, in any (possible) case, there exists $z \in S_{xy} \cap M \setminus \{x, y\}$. Now, if $z \in A$ then $\{y\} \cup xz$ is a suitable poidge, and if $z \in B$, a suitable poidge is $\{x\} \cup yz$. $\square$

For $m \geq 3$, an ordinary connected union of $m$ ordinary compact convex sets may not be poidge-convex, see Figure 5.

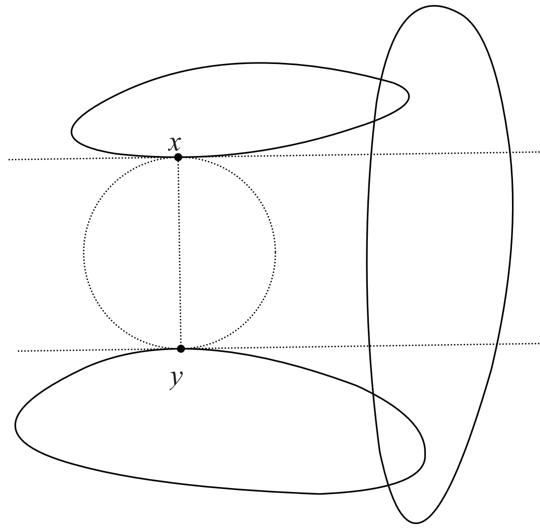


Figure 5: A connected union of 3 convex bodies.

# 5 Not simply connected sets

## 5.1 Convex bodies with spherical holes

It is easily seen that there are poidge-convex sets which are not simply connected. So, for example, the set $B_1(\mathbf{0}) \setminus \operatorname{int} B_\varepsilon(\mathbf{0})$ in $\mathbb{R}^2$, for any $\varepsilon \in ]0,1[$.

**Theorem 5.1.** *Let $K \subset \mathbb{R}^2$ be an ordinary convex body, and assume that $\mathbf{0}$ is the centre of $S_K$. The set $K \setminus \operatorname{int} B_\alpha(\mathbf{0})$ is poidge-convex, if $\alpha$ satisfies the following conditions. For any double normal $ab$ of* bd $K$*,*

*(i) if $ab$ is not a chord of $S_K$, then $\alpha \le \rho(\mathbf{0}, ab)$, and*

*(ii) if $ab$ is a chord of $S_K$ and $c \in S_K \cap K \setminus \{a, b\}$, then $\alpha \le \max\{\rho(\mathbf{0}, bc), \rho(\mathbf{0}, ca)\}$.*

*Proof.* Let $L = K \setminus \operatorname{int} B_\alpha(\mathbf{0})$, and $x, y \in L$. We need to investigate only the case when $xy$ is a double normal of bd $K$, which we now assume.

Case 1. $xy$ is not a chord of $S_K$.

In this case, $xy \subset L$, because $xy \cap \operatorname{int} B_\alpha(\mathbf{0}) = \emptyset$, by condition (i). Since $K$ is ordinary, $S_{xy} \cap K \ne \{x, y\}$. Choose $w \in S_{xy} \cap K \setminus \{x, y\}$. So, $\{w\} \cup xy$ is a poidge in $L$.

Case 2. $xy$ is a chord of $S_K$.

Since $K$ is ordinary, $S_K \cap K \setminus \{x, y\}$ contains a point $z$. Now, we have $yz \cap \operatorname{int} B_\alpha(\mathbf{0}) = \emptyset$ or $zx \cap \operatorname{int} B_\alpha(\mathbf{0}) = \emptyset$, by condition (ii) of the hypothesis. Assume without loss of generality that $yz \cap \operatorname{int} B_\alpha(\mathbf{0}) = \emptyset$.

Since $\angle yzx = \pi/2$, we have the poidge $\{x\} \cup yz \subset L$. □

Is always the existence of such a number $\alpha$ guaranteed?

**Theorem 5.2.** *If $K \subset \mathbb{R}^2$ is an ordinary convex polygon, in which every double normal passing through the centre* $\mathbf{0}$ *of $S_K$ is a chord of $S_K$, then a number $\alpha$ satisfying the conditions in Theorem 5.1 does exist.*

*Proof.* Let $\mathcal{D}_1$ be the family of those double normals of $K$ which do not contain $\mathbf{0}$, and $\mathcal{D}_2$ the complementary family of double normals, which must be chords of $S_K$.

Since no double normal in $\mathcal{D}_1$ contains $\mathbf{0}$, $\beta = \min\{\rho(\mathbf{0}, N) : N \in \mathcal{D}_1\} > 0$.

The set of double normals passing through $\mathbf{0}$ is finite. For each double normal $N = ab \in \mathcal{D}_2$, consider

$$\gamma_N = \max\{\max\{\rho(\mathbf{0}, ax), \rho(\mathbf{0}, bx)\} : x \in S_K \cap \text{bd } K \setminus \{a, b\}\}$$

and $\gamma = \min\{\gamma_N : N \in \mathcal{D}_2\} > 0$.

By choosing $\alpha = \min\{\beta, \gamma\}$, both conditions of Theorem 5.1 are satisfied, condition (i) for the double normals of $\mathcal{D}_1$, and condition (ii) for the double normals of $\mathcal{D}_2$. □

**Corollary 5.3.** *If $K \subset \mathbb{R}^2$ is an ordinary convex polygon, in which no double normal passes through the centre* $\mathbf{0}$ *of $S_K$, then a number $\alpha$ satisfying the conditions in Theorem 5.1 does exist.*

Theorem 5.2 presents only sufficient conditions for a convex polygon to acquire an appropriate number $\alpha$. All regular polygons are poidge-convex, although they do not satisfy the conditions of Theorem 5.2.

### 5.2 Tetrahedral surfaces

Starting with this section, we investigate the poidge-convexity of convex surfaces in $\mathbb{R}^n$, for $n = 3$ or for larger $n$.

The following lemma is straightforward.

**Lemma 5.4.** *Every non-obtuse triangle is poidge-convex.*

**Lemma 5.5.** *If $P$ is a $(n-1)$-dimensional polytope in $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $y \in P \setminus (V(P) \cup \{x\})$, then there exists a poidge in $P \cup \{x\}$ containing both $x$ and $y$.*

*Proof.* Indeed, $H_{yx} \cap P$ includes a non-degenerate line-segment $\sigma$ with an endpoint in $y$. Thus, $\{x\} \cup \sigma$ is a suitable poidge. □

**Theorem 5.6.** *Suppose abcd is a tetrahedron in $\mathbb{R}^3$ with a non-obtuse facet (i.e. 2-dimensional face) abc. Then* bd $(abcd)$ *is poidge-convex if and only if*

$$d \in \left(H^+_{ba} \cup H^+_{ca}\right) \bigcap \left(H^+_{ab} \cup H^+_{cb}\right) \bigcap \left(H^+_{ac} \cup H^+_{bc}\right).$$

*The boundary of a tetrahedron with all facets obtuse is not poidge-convex.*

*Proof.* Suppose $abc$ and $d$ are as stipulated in the first part of the theorem. We prove that bd $(abcd)$ is poidge-convex.

Let $x, y \in$ bd $(abcd)$ be any two distinct points.

Case 1. There is a face $F$ of $abcd$ such that $x, y \in F$.

If $F$ is a non-obtuse face, then there exists a poidge containing $x, y$ and contained in $F$, by Lemma 5.4. If $F$ is an obtuse face, then we obviously have to consider only the case that $x, y$ are vertices of the longest edge of bd $F$. If that edge is included in bd $(abc)$, then we find a poidge containing $x$ and $y$ in the face $abc$, by Lemma 5.4. So, suppose the obtuse angle of $F$ is not at $d$. Let $d'$ be the orthogonal projection of $d$ on the plane $\overline{abc}$.

Subcase 1.1. $d' \in H^+_{ba} \cap H^+_{ca} \cap H^+_{ab} \cap H^+_{cb} \cap H^+_{ac} \cap H^+_{bc}$. (See Figure 6.)

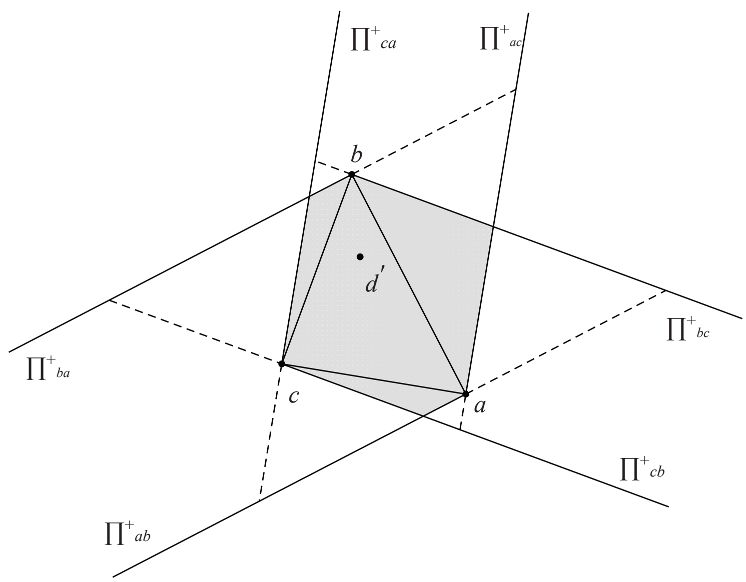


Figure 6: $d' \in H^+_{ba} \cap H^+_{ca} \cap H^+_{ab} \cap H^+_{cb} \cap H^+_{ac} \cap H^+_{bc}$.

All the angles $\widehat{dac}$, $\widehat{dca}$, $\widehat{dcb}$, $\widehat{dbc}$, $\widehat{dba}$, $\widehat{dab}$ are non-obtuse. Then the obtuse face $F$ must have an obtuse angle at $d$, and we obtain a contradiction.

Subcase 1.2. $d' \in (H^+_{ca} \cap H^+_{ac}) \setminus (H^+_{ba} \cap H^+_{bc})$. (See Figure 7.)

All angles $\widehat{dac}$, $\widehat{dca}$, $\widehat{dcb}$, $\widehat{dab}$ are non-obtuse. Clearly, $F \neq cad$. If $F = bcd$, it must have its obtuse angle at $b$. If $d''$ is the orthogonal projection of $d$ onto $\overline{ac}$, then $d'' \in ac$ and $x, y$ belong to the poidge $cd'' \cup \{d\}$. The case $F = abd$ is analogous.

The cases $d' \in (H^+_{bc} \cap H^+_{cb}) \setminus (H^+_{ac} \cap H^+_{ab})$ and $d' \in (H^+_{ba} \cap H^+_{ab}) \setminus (H^+_{ca} \cap H^+_{cb})$ are analogous to subcase 1.2.

Case 2. There are two distinct faces $F_1$ and $F_2$ such that $x \in F_1$ and $y \in F_2$.

If $x \in \operatorname{int} F_1$ or $y \in \operatorname{int} F_2$, then they have the poidge-property, by Lemma 5.5. Otherwise, $x \in$ bd $F_1$ and $y \in$ bd $F_2$, which means that we are in Case 1, unless $x$ and $y$ are in opposite edges. Without loss of generality, we can assume that $x \in ad$ and $y \in bc$. Let $w$ be the

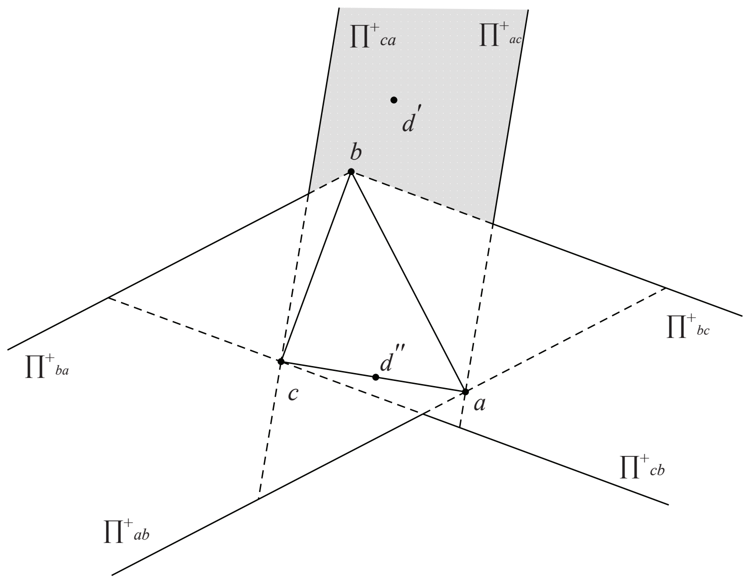


Figure 7: $d' \in (H^+_{ca} \cap H^+_{ac}) \setminus (H^+_{ba} \cap H^+_{bc})$.

orthogonal projection of $x$ onto $\overline{bc}$. Then $w \in bc$, and $x, y$ will be contained in the poidge $\{x\} \cup wb$ or $\{x\} \cup wc$.

Now, let us show that, if $abc$ is as required, but $d$ not (see Figure 8), then bd $(abcd)$ is not poidge-convex. This will be proved by showing that $d$ and one of the other three vertices do not enjoy the poidge-property.

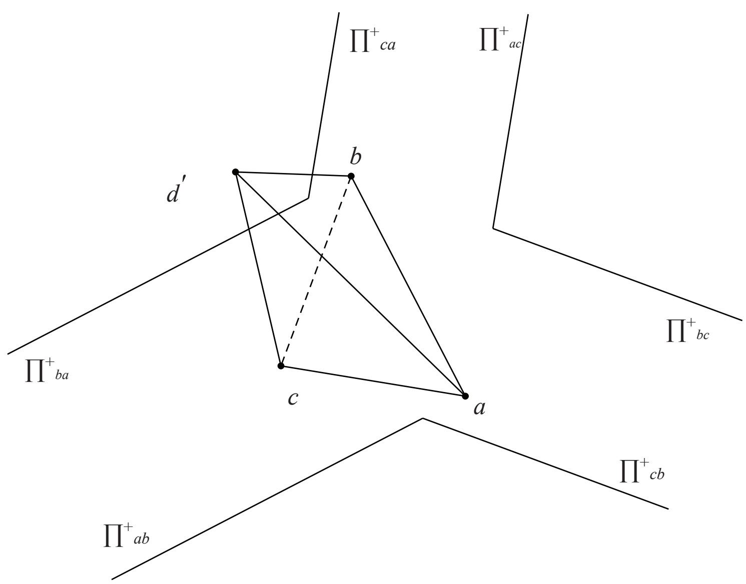


Figure 8: bd $(abcd)$ is not poidge-convex.

If $d \notin H^+_{ba} \cup H^+_{ca}$, then the angles $\widehat{dca}$ and $\widehat{dba}$ are obtuse. Then, obviously, $a$ and $d$ do not enjoy the poidge-property. The cases $d \notin H^+_{ba} \cup H^+_{ca}$ and $d \notin H^+_{ab} \cup H^+_{cb}$ are analogous.

The proof of the first part of the theorem is now complete.

For the second part, assume all facets of the tetrahedron $abcd$ are obtuse. Assume without loss of generality that $ab$ is a diameter of $abcd$. The pair of points $a, b$ has not the poidge-property. Indeed, $S \cap abcd = \{a, b\}$, $H_{ab} \cap abcd = \{a\}$ and $H_{ba} \cap abcd = \{b\}$. $\square$

### 5.3 Boundaries of Cartesian products and cones

As we could see in the preceding section, there exist poidge-convex convex surfaces. However, many convex surfaces are not poidge-convex: for example, no strictly convex body has a poidge-convex boundary. Therefore, most convex surfaces are not poidge-convex (see [9]). We present now some classes of convex surfaces which are poidge-convex.

We consider the Cartesian product in $\mathbb{R}^n$ of a $k$-dimensional compact convex set $K$ and an $(n-k)$-dimensional compact convex set $L$, hence with $\overline{K}$ orthogonal to $\overline{L}$.

**Theorem 5.7.** *Every Cartesian product of compact convex sets of positive dimensions has a poidge-convex boundary.*

*Proof.* We consider $\overline{K}$ to be spanned by the first $k$ axes, and $\overline{L}$ by the last $n-k$ axes. Let $S = \text{bd}\,(K \times L)$.

Let $x, y \in S$. We have $x = u \times t_u$ and $y = v \times t_v$, where $u, v \in K$ and $t_u, t_v \in L$.

Consider the chord $t'_u t'_v = \overline{t_u t_v} \cap L$ of $L$.

<u>Case 1.</u> $x, y \in \{u\} \times L$, where $u \in \text{bd}\, K$.

Take $u' \in K \setminus \{u\}$ arbitrarily. Then the poidge $\{u' \times t'_u\} \cup (\{u\} \times t'_u t'_v)$ contains $x, y$ and lies in $S$.

<u>Case 2.</u> $x \in \{u\} \times L$, $y \in \{v\} \times L$,where $u, v \in \text{bd}\, K$ are distinct.

If $t'_u, t_u, t_v, t'_v$ lie in this order on $\overline{t_u t_v}$, we have $t_u t_v \subset t_u t'_v$. Then $\{x\} \cup (\{v\} \times t_u t'_v)$ is a suitable poidge in $S$.

<u>Case 3.</u> $x, y \in K \times \{t\}$, where $t \in \text{bd}\, L$.

For this case we have the poidge $(u \times t') \cup xy \subset S$, where $t' \in (\text{bd}\, L) \setminus \{t\}$.

<u>Case 4.</u> $x \in K \times \{t_u\}$, $y \in K \times \{t_v\}$, where $t_u, t_v \in \text{bd}\, L$ are distinct.

If $u \neq v$, a good poidge is $\{y\} \cup x(v \times t_u)$. Otherwise, $\{y\} \cup xz$ is a good one, $z$ being an arbitrary point of $K \times \{t_u\}$ different from $x$.

<u>Case 5.</u> $x \in K \times \{t_u\}$, $y \in (\text{bd}\, K) \times L$, where $t_u \in \text{bd}\, L$.

Now, if we are not in Case 1 or 4, a suitable poidge is $\{x\} \cup (\{v\} \times I)$, where $I$ is an arbitrary line-segment in $L$ starting at $t_u$. $\square$

Let $K$ be an $(n-1)$-dimensional convex body in $\mathbb{R}^n$, and $v \in \mathbb{R}^n \setminus K$. Put $v' = \mathrm{p}_{\overline{K}}(v)$. We call the cone $C = \text{conv}\,(\{v\} \cup K)$ *right*, if $v' \in K$ and, for any pair of points $a, b \in \text{bd}\, K$, $\angle avb \leq \pi/2$.

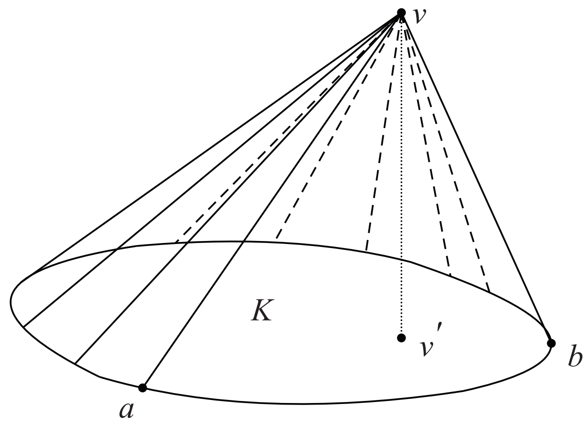


Figure 9: A conical surface.

The boundary of a right cone will be called here a *conical surface*. See Figure 9.

**Theorem 5.8.** *Every conical surface is poidge-convex.*

*Proof.* Let $S = \text{bd}\, C$, where $C = \text{conv}\,(\{v\} \cup K)$ is a right cone. We observe the preceding notation.

Let $x, y \in S$. We assume without loss of generality that $\|v - x\| \le \|v - y\|$.

<u>Case 1.</u> $x, y \in K$.

If $x, y$ are not both boundary points of $K$, then they clearly have the poidge-property (in $K$). Suppose now $x, y \in \text{bd}\, K$. Since $\angle xvy \le \pi/2$ and $\|v - x\| \le \|v - y\|$, $z = \mathbb{p}_{\overline{vy}}(x)$ belongs to $vy$. Thus, $\{x\} \cup yz$ is a suitable poidge.

<u>Case 2.</u> The points $v, x, y$ are collinear.

The point $x$ is closer from $v$ than $y$, possibly $x = v$.

Let $\{y'\} = \overline{xy} \cap K$.

If $y' = y$, then we consider the poidge $\{x\} \cup yz$, where $z = \mathbb{p}_{\overline{K}}(x)$.

If $y' \ne y$, then take $u \in (\text{bd}\, K) \setminus \{y'\}$ so close to $y'$ that $\|v - u\| > \|v - y\|$. Then $\angle vuy < \pi/2$, and the projection $z = \mathbb{p}_{\overline{vu}}(x)$ belongs to $vu$. So, we obtain the poidge $\{z\} \cup xy \subset S$.

<u>Case 3.</u> $x \in vx'$, $y \in vy'$, with $x', y' \in \text{bd}\, K$.

We may suppose $x \ne v$, $y \ne v$, and $x' \ne y'$, otherwise we are in Case 2. Since $\angle xvy \le \pi/2$, we have $z \in vy \setminus \{y\}$, where $z = \mathbb{p}_{\overline{vy}}(x)$. Thus, $\{x\} \cup yz$ is a suitable poidge.

<u>Case 4.</u> $x \in K$, $y \notin K$.

Let $z = \mathbb{p}_{\overline{K}}(y)$. If $z \ne x$, then $\{y\} \cup xz$ is a good poidge. If $z = x$, take any point $z' \in K$ different from $z$. The poidge $\{y\} \cup zz'$ will do it. □

For results on right convexity in cylinders and cones, see [18].

### 5.4 Polyhedral surfaces

We have already seen that some tetrahedral surfaces are poidge-convex, some are not. The same is true in the larger frame of all polyhedral surfaces.

The *star* $S_x$ at a vertex $x$ of a polytope in $\mathbb{R}^n$ is the union of all facets (i.e. $(n-1)$-dimensional faces) having $x$ as a vertex. Two vertices of a polytope in $\mathbb{R}^n$ will be called *opposite* if $H_{xy}$ and $H_{yx}$ are supporting hyperplanes of the polytope.

Recall that a compact set $M \subset \mathbb{R}^n$ is called ordinary, if $\mathrm{card}(S_M \cap M) \geq 3$. This is equivalent to the property that, for any pair of points $x, y \in M$, not all points of $M$ different from $x, y$ lie inside $S_{xy}$, i.e. $M \setminus \mathrm{int\,conv}\, S_{xy} \neq \{x, y\}$.

Now, we say that a polytope $P \in \mathbb{R}^n$ is *extraordinary*, if, for any pair of opposite vertices $x, y \in P$, not all points of $S_x \cup S_y$ different from $x, y$ lie inside $S_{xy}$, i.e. $(S_x \cup S_y) \setminus \mathrm{int\,conv}\, S_{xy} \neq \{x, y\}$.

Of course, every extraordinary polytope is ordinary, but not vice-versa.

Among the extraordinary polytopes we find all those admitting a circumscribed sphere, i.e. a sphere containing all vertices. The Platonic and the Archimedean Solids are well-known examples.

**Theorem 5.9.** *Every extraordinary polytope in $\mathbb{R}^n$ has a poidge-convex boundary.*

*Proof.* By Lemma 5.5, we only need to verify the poidge-property for pairs of vertices $x, y$ of the given extraordinary polytope $P$.

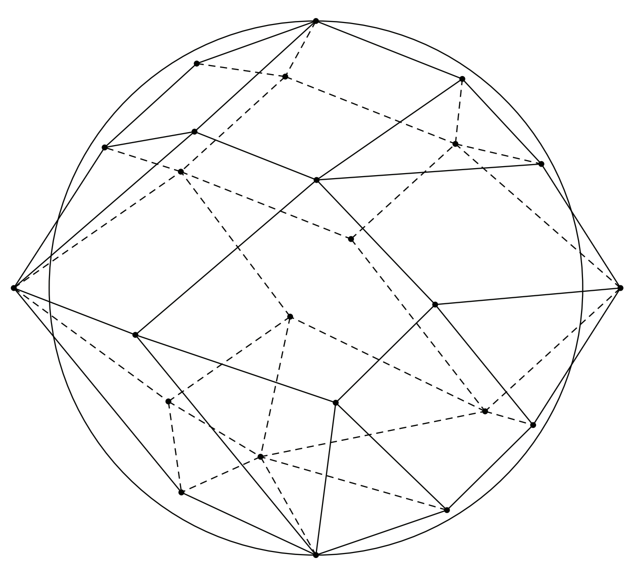

Figure 10: An ordinary, not extraordinary polytope.

Take $x, y \in V(P)$. Let $H_x$ be a supporting hyperplane of $P$ at $x$. Consider the hyperplane $H_y \ni y$ parallel to $H_x$.

If $H_y$ is supporting $P$, then our hypothesis implies the existence of some point $z \in (S_x \cup S_y) \setminus \operatorname{int}\operatorname{conv} S_{xy}$ different from $x$ and $y$. Assume without loss of generality that $z \in S_x$. Then $zx \cap S_{xy} \neq \{x\}$. If $z' \in zx \cap S_{xy} \setminus \{x\}$, then $xz' \subset S_x$, as $\{x\}$ is the kernel of the starshaped set $S_x$. Moreover, $\angle xz'y = \pi/2$. Thus, $\{y\} \cup xz'$ is a suitable poidge.

If $H_y$ is not supporting $P$, then both $H_y$ and $S_{xy}$ are locally cutting bd $P$ at $y$. Locally, the intersection $S_{xy} \cap S_y$ is a union of pieces of spheres of dimension $n-2$. Take a point $z$ in $S_{xy} \cap S_y$ different from $y$. Then $\{x\} \cup yz$ is a suitable poidge. □

Figure 10 shows an ordinary polytope which is not extraordinary.

Sometimes even the 1-skeleta of polytopes are poidge-convex.

In $\mathbb{R}^3$, let $\mathbf{T}_1$, $\mathbf{C}_1$, $\mathbf{O}_1$, $\mathbf{D}_1$, $\mathbf{I}_1$ be the boundary 1-complexes of the regular tetrahedron, cube, regular octahedron, regular dodecahedron, and regular icosahedron, respectively.

**Theorem 5.10.** $\mathbf{T}_1$, $\mathbf{C}_1$, $\mathbf{O}_1$ *are poidge-convex, while* $\mathbf{D}_1$, $\mathbf{I}_1$ *are not.*

*Proof.* We leave to the reader the first part of the statement. Consider $\mathbf{D}_1$. Let $abcde$ be a face of $\mathbf{D}_1$, and $aa'$ an edge different from $ab$ and $ea$. Consider the points $x \in aa'$ and $y \in bc$. Obviously, $H_{xy} \cap aa' = \{x\}$, and $H_{yx} \cap bc = \{y\}$. Moreover, $xy \not\subset \mathbf{D}_1$. Hence, $x, y$ have not the poidge-property.

For $\mathbf{I}_1$, the proof is similar. □

## 6 Problems

We end the paper with two problems about the relationship between various $\mathcal{F}$-convexities.

**Problem 6.1.** *Which poidge-convex sets are not rt-convex?*

We mentioned (and it is easily verified) that *tr*-convexity implies poidge-convexity.

**Problem 6.2.** *Which poidge-convex sets are not tr-convex?*

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## References


[1] R. Blind, G. Valette, T. Zamfirescu, Rectangular convexity, *Geom. Dedicata*, **9** (1980) 317-327.

[2] K. Böröczky, Jr., Rectangular convexity of convex domains of constant width, *Geom. Dedicata*, **34** (1990) 13-18.

[3] M. Breen, Characterizing certain staircase convex sets in $\mathbb{R}^n$, *Beitr. Algebra Geom.*, **51** (2010) 251-261.

[4] M. Breen, Generating a staircase starshaped set and its kernel in $\mathbb{R}^3$ from certain staircase convex subsets, *Beitr. Algebra Geom.*, **53** (2012) 515-528.

[5] M. Breen, Suitable families of boxes and kernels of staircase starshaped sets in $\mathbb{R}^n$, *Aequat. Math.*, **87** (2014) 43-52.

[6] M. Breen, Staircase convex results for a certain class of orthogonal polytopes, *Aequat. Math.*, **89** (2015) 1213-1221.

[7] A. M. Bruckner, J. B. Bruckner, Generalized convex kernels, *Israel J. Math.*, **2** (1964) 27-32.

[8] H. Buseman, *Convex Surfaces*, New York: Interscience 1958.

[9] V. Klee, Some new results on smoothness and rotundity in normed linear spaces, *Math. Ann.* **139** (1959) 51-63.

[10] E. Magazanik, M. A. Perles, Staircase connected sets, *Discrete Comp. Geom.*, **37** (2007) 587-599.

[11] E. Magazanik, M. A. Perles, Generalized convex kernels of simply connected $L_n$ sets in the plane, *Israel J. Math.*, **160** (2007) 157-171.

[12] L. Yuan, T. Zamfirescu, Right triple convex completion, *J. Convex Anal.*, **22** (2015) 291-301.

[13] L. Yuan, T. Zamfirescu, Right triple convexity, *J. Convex Anal.*, **23** (2016) 1219-1246.

[14] L. Yuan, T. Zamfirescu, Selfishness of convex bodies, *J. Nonlinear Convex Anal.*, **18** (2017) 525-534.

[15] L. Yuan, T. Zamfirescu, Y. Zhang, Isosceles triple convexity, *Carpathian J. Math.*, **33** (2017) 127-139.

[16] T. Zamfirescu, Inscribed and circumscribed circles to convex curves, *Proc. Amer. Math. Soc.* **80** (1980) 455-457.

[17] T. Zamfirescu, Typical starshaped sets, *Aequationes Math.* **36** (1988) 188-200.

[18] T. Zamfirescu, Right convexity, *J. Convex Anal.*, **21** (2014) 253-260.

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