Liping Yuan*, Tudor Zamfirescu and Yanxue Zhang

Tetrahedral cages for unit discs

https://doi.org/10.1515/advgeom-2021-0016. Received 4 May, 2019; revised 7 December, 2019

Abstract: A cage is the 1-skeleton of a convex polytope in $\mathbb{R}^3$. A cage is said to hold a set if the set cannot be continuously moved to a distant location, remaining congruent to itself and disjoint from the cage. In how many positions can (compact 2-dimensional) unit discs be held by a tetrahedral cage? We completely answer this question for all tetrahedra.

Keywords: Tetrahedra, tetrahedral cage, unit disc.

2010 Mathematics Subject Classification: 52B10

Communicated by: E. Bannai

1 Introduction

A *cage* is the 1-skeleton of a (convex) polytope in $\mathbb{R}^3$. If P is the polytope, the cage is denoted by cage(P). A cage G is said to *hold a compact set K* disjoint from G, if no rigid continuous motion can bring K in a position far away without meeting G on its way. A compact 2-dimensional ball in $\mathbb{R}^3$ will be called a *disc*. The subject of holding (3-dimensional) balls in cages has been treated by Coxeter [3], Besicovitch [2], Aberth [1] and Valette [4]. The first two authors proved that there are tetrahedral cages holding n discs, for every $n \le 16$ except for $n \in \{7, 9, 11, 13, 14, 15\}$, and there is no such cage for any other n. In this paper the discs to be held are all of the same size. The question we answer, asked in [5], is the same, but in the new context. It is about the number of positions in which the unit disc can be held by tetrahedral cages. A priori we expect to have more exceptions. We shall see that the number of exceptions is a little larger, indeed!

For distinct $x, y \in \mathbb{R}^3$, let $\overline{xy}$ be the line through x, y and xy the line-segment from x to y. We denote by Π_{xy} the plane through x orthogonal to $\overline{xy}$, and by Π_{xy}^+ the open half-space containing y determined by Π_{xy}. For non-collinear $x, y, z \in \mathbb{R}^3$, let $C(xyz)$ be the circumscribed circle of the triangle xyz in its plane $\overline{xyz}$, and o_{xyz} its centre.

Following [5], for any cage G, let $\mathcal{D}(G)$ be the space of all discs held by G, endowed with the Pompeiu–Hausdorff metric. Let $\mathcal{D}_r(G)$ be the set of all discs in $\mathcal{D}(G)$ of radius at least r. Assume that, for some component $\mathcal{E}$ of $\mathcal{D}_r(G)$ and any number $s > r$, $\mathcal{D}_s(G) \cap \mathcal{E}$ is connected or empty. We call such a component $\mathcal{E}$ an *end-component of* $\mathcal{D}(G)$. If n is the maximal number of pairwise disjoint end-components of $\mathcal{D}(G)$, we say that G *holds n discs*. In fact, intuitively, G does not hold n pairwise disjoint discs simultaneously; merely there are n positions (ways) in which a disc can be held. Let the component $\mathcal{E}$ of $\mathcal{D}_r(G)$ be an end-component of $\mathcal{D}(G)$. Put $\sigma(\mathcal{E}) = \sup\{s : \mathcal{D}_s(G) \cap \mathcal{E} \neq \emptyset\}$. Choose an increasing sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers satisfying $s_n > r$ and $\lim_{n\to\infty} s_n = \sigma(\mathcal{E})$. Consider a disc $D_n \in \mathcal{D}_{s_n}(G)$ for each n. If $\{D_n\}_{n=1}^{\infty}$ converges to some disc $D(\mathcal{E})$ independent of the choice of the numbers s_n and the discs D_n, we call $D(\mathcal{E})$ the *limit disc* of $\mathcal{E}$. Several end-components may have the same limit disc. If the limit disc of an end-component $\mathcal{E}$ lies in the plane of a face F of conv G, we say that G *holds a disc at the face F*. For each end-component, we have a disc held, even if the limit discs

***Corresponding author: Liping Yuan,** School of Mathematical Sciences, Hebei Normal University, 050024 Shijiazhuang, P.R. China, Hebei Key Laboratory of Computational Mathematics and Applications, 050024 Shijiazhuang, P.R. China, email: lpyuan@hebtu.edu.cn
Tudor Zamfirescu, Fachbereich Mathematik, Technische Universität Dortmund, 44221 Dortmund, Germany, Roumanian Academy, Bucharest, Roumania, College of Mathematics and Information Science, Hebei Normal University, 050024 Shijiazhuang, P.R. China, email: tuzamfirescu@gmail.com
Yanxue Zhang, School of Mathematical Sciences, Hebei Normal University, 050024 Shijiazhuang, P.R. China, Hebei Key Laboratory of Computational Mathematics and Applications, 050024 Shijiazhuang, P.R. China, email: ??

coincide. So, a cage may hold several discs at the same face. Also if a face F is not triangular, several distinct limit discs can be coplanar with F.

If we briefly say that the cage G *holds* n *unit discs*, this means that G holds n discs, i.e. the maximal number of pairwise disjoint end-components is n, and $\sigma(\mathcal{E})$ does not depend on the chosen end-component $\mathcal{E}$.

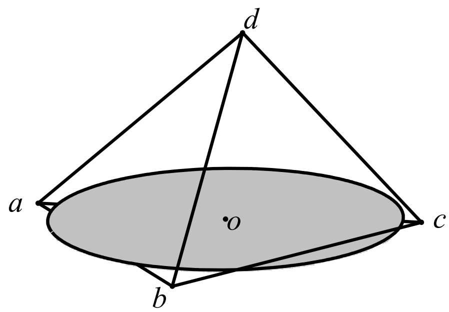

Figure 1: A unit disc held by a regular tetrahedral cage.

Figure 1 illustrates one of the 16 positions in which a disc can be held by a regular tetrahedral cage. For other tetrahedra the number of positions can be much smaller. It is relevant whether the disc partly lies below some edge, as in Figure 1; for such a position it is further needed that $\angle dao < \pi/2$ where $o = o_{abc}$, in order for the disc to be held, as one easily verifies. Such arguments will be used in the following sections.

2 Auxiliary material

We present here several results preparing our main result in the next section.

Lemma 1 ([5]). *If for* $a, b, c, x, o \in \mathbb{R}^3$, $\angle axb \leq \pi/2$, $\angle cxa < \pi/2$ *and* o *lies in the relative interior of* bxc, *then* $\angle axo < \pi/2$.

Lemma 2 ([5]). *If a polytopal cage holds at least one disc at some triangular face, then that triangle is acute.*

Lemma 3 ([5]). *If a tetrahedral cage has an acute face, then it has one, two, or four discs held at that face. More precisely, suppose that* $T = abcd$ *is a tetrahedron with the acute face* abc.

i) *If* $\angle dao_{abc} < \pi/2$, $\angle dbo_{abc} \geq \pi/2$ *and* $\angle dco_{abc} \geq \pi/2$, *then* cage($T$) *holds one disc at the face* abc.
ii) *If* $\angle dao_{abc} < \pi/2$, $\angle dbo_{abc} < \pi/2$ *and* $\angle dco_{abc} \geq \pi/2$, *then* cage($T$) *holds two discs at* abc.
iii) *If* $\angle dao_{abc} < \pi/2$, $\angle dbo_{abc} < \pi/2$ *and* $\angle dco_{abc} < \pi/2$, *then* cage(T) *holds four discs at* abc.

For a proof of Lemma 3, see [5], proof of Lemma 2.3.

Lemma 4 ([5]). *It is not possible that at some face precisely one disc is held, and at at most one face no disc is held.*

Proof. Suppose there exists at most one face at which no disc is held. Then at most one of the 12 angles (of the 4 triangles), say acd, is non-acute. It follows that the triangles abc, bcd and abd are acute, and all angles at a, b, d are acute, too.

By Lemma 1, $\angle o_{abc}ad < \pi/2$ and $\angle o_{abc}bd < \pi/2$; thus, at least two discs are held at abc. Similarly, at least two discs are held at bcd. At abd exactly four discs are held, as all cage angles at a, b, d are acute.

At acd, either 0 or 4 discs are held. Hence, at no face exactly one disc is held. □

Lemma 4 has already been used inside the proof of Theorem 2.7 in [5].

Theorem 1 ([5]). *There are tetrahedral cages holding n discs for every* $n \in \{1, 2, 3, 4, 5, 6, 8, 10, 12, 16\}$, *and there is no such cage for any other n.*

Lemma 5. *If all faces of a tetrahedral cage are acute triangles, then the cage holds* 16 *discs. This means that, if the number of discs held at each face is positive, then that number is* 4 *for every face.*

Proof. Assume that all faces of the tetrahedral cage cage($abcd$) are acute triangles. We have $\angle abd < \pi/2$, $\angle abc < \pi/2$ and $\angle cbd < \pi/2$. Using Lemma 1 we get $\angle abo_{bcd} < \pi/2$. Analogously, $\angle aco_{bcd} < \pi/2$ and $\angle ado_{bcd} < \pi/2$. Thus, the face bcd holds 4 discs.

Analogously, the faces abc, acd and abd hold 4 discs each. □

Lemma 6. *Let* $C \subset \mathbb{R}^3$ *be a circle and let* $H^+ \subset \mathbb{R}^3$ *be an open half-space. If* $a, b \in C \cap H^+$, *then at least one of the two arcs determined by* a, b *on* C *lies in* H^+.

The proof is straightforward.

3 Tetrahedral cages for unit discs

Our main result is the following.

Theorem 2. *There are tetrahedral cages holding n unit discs for every* $n \in \{1, 2, 3, 4, 6, 8, 12, 16\}$, *and there is no such cage for any other n.*

Proof. The cases $n = 0, 1, 2, 3, 4$ are already mentioned and settled in [5]. For example, for $n = 0$ just take a tetrahedral cage, all faces of which are obtuse triangles. By Theorem 1, no $n \in \{7, 9, 11, 13, 14, 15\}$ can be realized. So, it remains to consider $n \in \{5, 6, 8, 10, 12, 16\}$.

Case $n = 5$. If a tetrahedral cage T has an acute face, then it has one, two, or four discs held at that face, by Lemma 3. In order to obtain exactly 5 unit discs held by T, there are 3 possibilities for the number of discs held at the four faces, namely, $5 = 0 + 1 + 2 + 2$, $5 = 1 + 1 + 1 + 2$, $5 = 0 + 0 + 1 + 4$. By Lemma 4, the first two possibilities cannot be realized.

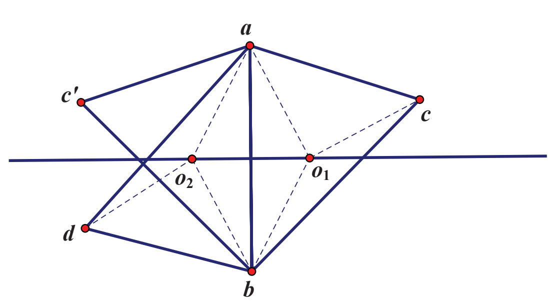

Figure 2: $n = 5$.

Now we discuss the Case $5 = 0 + 0 + 1 + 4$. Assume that a tetrahedral cage T holds 5 unit discs in this way. We assume without loss of generality that the face abc holds one unit disc, and the face abd four unit discs. Let o_1 be the centre of $C(abc)$, and o_2 the centre of $C(abd)$. Regarding abc, we have $\angle dao_1 \geq \pi/2$, $\angle dbo_1 \geq \pi/2$ and $\angle dco_1 < \frac{\pi}{2}$ (note that the latter inequality is imposed by the fact that d belongs to the torus Θ obtained by rotating $C(abc)$ about $\overline{ab}$, and Θ is tangent to Π_{co_1}, while the other inequalities are required by Lemma 4). Regarding abd, we have $\angle cao_2 < \pi/2$, $\angle cbo_2 < \pi/2$ and $\angle cdo_2 < \pi/2$, by Lemma 4. See Figure 2.

The two faces have the common edge ab. Rotate abc about $\overline{ab}$ decreasing the dihedral angle between abc and abd, until it reaches the plane $\overline{abd}$. Let abc' be its new position. By symmetry, $\angle cao_2 = \angle c'ao_1 < \pi/2$ and $\angle cbo_2 = \angle c'bo_1 < \pi/2$. The points a, b, c', d are concyclic. If A is the arc of $C(abd)$ from a to b containing

d, then $c' \in A$, since both triangles abc and abd are acute. Thus, d lies in one of the two subarcs $\widehat{ac'}$, $\widehat{bc'}$ of A, say in the second. The inequalities just obtained show that $c' \in H^+_{ao_1}$. Since $\angle bao_1 < \pi/2$, too, both b and c' lie in $H^+_{ao_1}$. Lemma 6 together with $a \in H_{ao_1}$ imply $\widehat{bc'} \subset H^+_{ao_1}$. Hence $d \in H^+_{ao_1}$, which contradicts $\angle dao_1 \geq \pi/2$.

Case $n = 6$. Let o_1 be the centre of $C(abc)$, o_2 the centre of $C(abd)$, $\angle bao_1 = \angle bao_2 = \angle abo_1 = \angle abo_2 = 5\pi/36$, $\angle o_2ad = \angle o_2da = \pi/9$, $\angle o_2bd = \angle o_2db = \pi/4$, and $\angle o_1ac = \angle o_1ca = \angle o_1bc = \angle o_1cb = 13\pi/72$. Rotate slightly abd about $\overline{ab}$ up to a new position abd'; now, let o'_2 be the centre of $C(abd')$. See Figure 3.

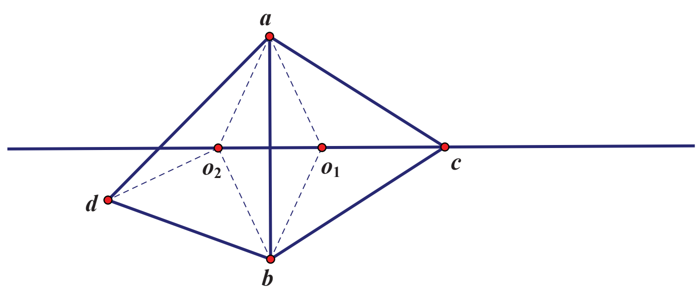

Figure 3: $n = 6$.

Because $\angle dao_1 = 7\pi/18 < \pi/2$, $\angle dbo_1 = 19\pi/36 > \pi/2$, $\angle dco_1 < \pi/2$; $\angle cao_2 = \angle cbo_2 = 11\pi/24 < \pi/2$, $\angle cdo_2 < \pi/2$, we have $\angle d'ao_1 < \pi/2$, $\angle d'bo_1 > \pi/2$, $\angle d'co_1 < \pi/2$; also, $\angle cao'_2 = \angle cbo'_2 < \pi/2$ and $\angle cd'o'_2 < \pi/2$. There are precisely two unit discs held by cage($abcd'$) at abc, and four unit discs held at abd'. Since acd' and bcd' are obtuse triangles, no discs are held there. Thus, this tetrahedral cage holds 6 unit discs.

Case $n = 8$. The tetrahedral cage constructed in [5] already holds 8 unit discs.

Case $n = 10$. In order to obtain exactly 10 unit discs held by T, there are 3 possibilities for the number of discs held at the four faces, namely, $10 = 1 + 1 + 4 + 4$, $10 = 2 + 2 + 2 + 4$, $10 = 0 + 2 + 4 + 4$. By Lemma 5, the first two cases are in fact impossible.

Assume that a tetrahedral cage T holds 10 unit discs such that three faces of T do hold discs, and the radii of their circumscribed circles are equal. The fourth face does not hold any disc. Without loss of generality, assume that the triangle bcd is not acute, say $\angle cbd \geq \pi/2$.

Let o_1 be the centre of $C(abc)$, o_2 the centre of $C(abd)$ and o_3 the centre of $C(acd)$. First assume that $\angle cbd = \pi/2$. We have $\angle bca < \pi/2$, $\angle bcd < \pi/2$, $\angle acd < \pi/2$; by Lemma 1, $\angle bco_3 < \pi/2$. Analogously, $\angle bao_3 < \pi/2$, $\angle bdo_3 < \pi/2$. The face acd holds 4 unit discs. Analogously, the faces abc and abd hold 4 unit discs each. This is too much! Hence $\angle cbd > \pi/2$.

Because all angles at a, c, d are acute, four unit discs are held at acd. Regarding abc, we have $\angle dao_1 < \pi/2$ and $\angle dco_1 < \pi/2$, whence at least two unit discs are held at abc. The same is true regarding abd. In order to obtain exactly 10 unit discs held by T, there are 2 possibilities: abc holds two unit discs and abd four, or vice-versa.

Without loss of generality, assume that the first case is true. So, T holds two discs at abc. Since all angles at a and c are acute, $\angle dao_1 < \pi/2$ and $dco_1 < \pi/2$; hence, $\angle dbo_1 \geq \pi/2$. Rotate abd about $\overline{ab}$ up to abd_1 such that d_1 is on $\overline{abc}$, separated from c by $\overline{ab}$. See Figure 4. Rotate acd about $\overline{ac}$ up to abd_2 such that d_2 is on $\overline{abc}$, separated from b by $\overline{ac}$. Let o'_1 be the centre of $C(abd_1)$, o'_2 the centre of $C(acd_2)$, $\angle bao'_1 = \angle bao_1 = \angle abo'_1 = \angle abo_1 = \alpha$, $\angle o'_1ad_1 = \angle o'_1d_1a = \angle o'_2ad_2 = \angle o'_2d_2a = \beta$. Then $\angle o'_1d_1b = \angle o'_1bd_1 = \frac{\pi}{2} - \alpha - \beta$, $\angle o_1ac = \angle o_1ca = \angle o'_2ac = \angle o'_2ca = \gamma$, $\angle o_1bc = \angle o_1cb = \frac{\pi}{2} - \alpha - \gamma$, and $\angle o'_2cd_2 = \angle o'_2d_2c = \frac{\pi}{2} - \beta - \gamma$.

We have $\|b - d_1\| = 2\cos(\frac{\pi}{2} - \alpha - \beta) = 2\sin(\alpha + \beta)$, $\|b - c\| = 2\cos(\frac{\pi}{2} - \alpha - \gamma) = 2\sin(\alpha + \gamma)$, $\|c - d_2\| = 2\cos(\frac{\pi}{2} - \beta - \gamma) = 2\sin(\beta + \gamma)$ and eventually

$$\cos\angle cbd = \frac{\|b - d_1\|^2 + \|b - c\|^2 - \|c - d_2\|^2}{2\|b - d_1\|\|b - c\|}.$$

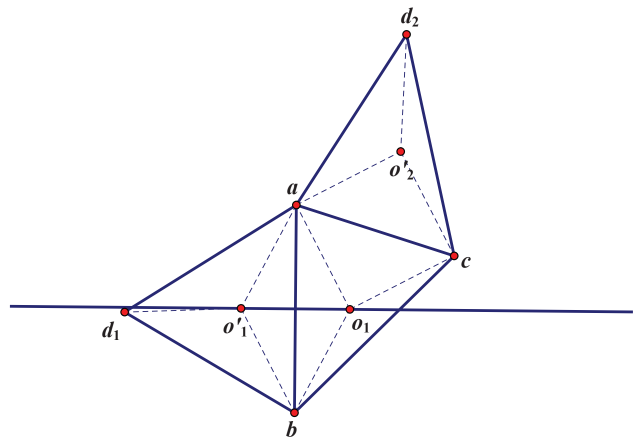

Figure 4: $n = 10$.

Because $\cos \angle cbd < 0$, we must have

$$\sin^2(\alpha + \beta) + \sin^2(\alpha + \gamma) < \sin^2(\beta + \gamma). \tag{1}$$

But $\angle dbo_1 \geq \pi/2$, whence $\angle d_1bo_1 > \pi/2$. We have $\angle d_1bo_1 = \angle d_1bo_1' + \angle o_1'bo_1 = \frac{\pi}{2} - \alpha - \beta + 2\alpha > \frac{\pi}{2}$. Hence $\alpha > \beta$, which yields $\sin^2(\alpha + \gamma) > \sin^2(\beta + \gamma)$, contradicting equation (1).

In conclusion, it is impossible for the tetrahedral cage T to hold 10 unit discs.

Case $n = 12$. We construct a suitable cage cage($abcd$). Consider the triangle abd_1 obtained from abd exactly like in the case $n = 10$. Analogously, consider bcd_2.

Let o be the centre of $C(abc)$, o_1 the centre of $C(abd_1)$, o_2 the centre of $C(bcd_2)$. See Figure 5. Then o is symmetric with o_1 about $\overline{ab}$, o is symmetric with o_2 about $\overline{bc}$; take $\angle bao = \angle abo = \angle bao_1 = \angle abo_1 = \pi/60$, $\angle o_1ad_1 = \angle o_1d_1a = 13\pi/36$, whence $\angle o_1bd_1 = \angle o_1d_1b = \angle o_2bd_2 = \angle o_2d_2b = 11\pi/90$. Also, take $\angle oac = \angle oca = \pi/6$. It follows that $\angle obc = \angle ocb = \angle o_2bc = \angle o_2cb = 19\pi/60$ and $\angle o_2cd_2 = \angle o_2d_2c = 11\pi/180$.

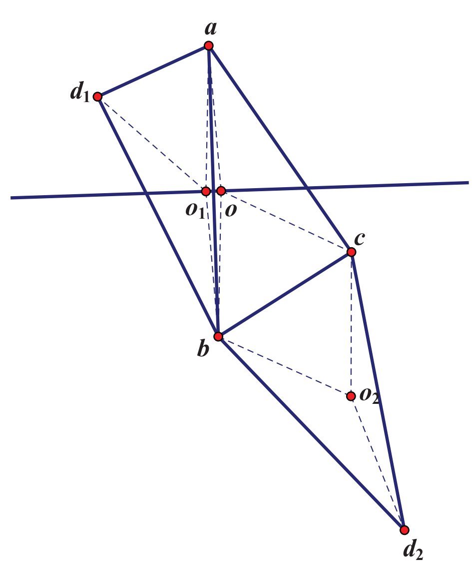

Figure 5: $n = 12$.

Then $\|a-d\| = \|a-d_1\| = 2\cos\frac{13\pi}{36}$, $\|c-d\| = \|c-d_2\| = 2\cos\frac{11\pi}{180}$ and $\|a-c\| = 2\cos\frac{\pi}{6}$. We obtain

$$\cos\angle dac = \frac{\| a-d \|^2 + \| a-c \|^2 - \| c-d \|^2}{2 \| a-d \| \| a-c \|} < 0,$$

which implies that dac is an obtuse triangle. This cage holds 12 unit discs.

Case $n = 16$. This is clear. □

One can wish to have a characterization of all tetrahedral cages holding n unit discs, and this for every n for which they exist. While this is easy to accomplish for very small n or $n = 16$, in other cases it seems more complicated. We choose to leave this to the enthusiastic reader.

4 A pentahedral cage for unit discs

The smallest n for which there are no tetrahedral cages holding n discs is 7, by Theorem 2.7 in [5]. This prompted the authors of [5] to look for a pentahedral cage holding 7 discs. Theorem 3.5 in [5] presents such a cage. But, based on that example, we cannot find a cage holding 7 unit discs. So, the natural question arises whether a cage holding 7 unit discs does or does not exist.

Theorem 3. *There exists a quadrilateral pyramid such that the associated cage holds 7 unit discs.*

Proof. We consider the unit circle $\mathbb{S}_1 \subset \mathbb{R}^2 = P$ and take on it the points a, b, c, d such that $\lambda(\widehat{ab}) = (\pi/2)+3\varepsilon$, $\lambda(\widehat{bc}) = (\pi/2)-\varepsilon$, $\lambda(\widehat{cd}) = (\pi/2)-\varepsilon$, $\lambda(\widehat{da}) = (\pi/2)-\varepsilon$, where λ denotes length and ε is small.

Take $e' \in ac$, at distance ε from c. The triangles $e'ab$, $e'bc$, $e'cd$, $e'da$ are obtuse. By choosing $e \in \mathbb{R}^3 \setminus P$ above P and close enough to e', the triangles eab, ebc, ecd, eda will be obtuse, too. Hence, cage($abcde$) holds no disc at any triangular face. How many discs are held at $abcd$?

First note that $\angle ea\mathbf{0} < \pi/2$, $\angle eb\mathbf{0} < \pi/2$, $\angle ec\mathbf{0} < \pi/2$, $\angle ed\mathbf{0} < \pi/2$, where $\mathbf{0}$ is the origin of $\mathbb{R}^3$. Consequently, the discs held are: one above the whole $abcd$, one below ab and above bc, cd, and da, one below bc and above the other three, one below cd and above the other three, one below da and above the other three, one below bc and cd and above the other two, and one below cd and da and above the other two. These are the 7 unit discs held by cage($abcde$). □

Funding: The authors gratefully acknowledge financial support by NSF of China (11871192, 11471095) and the Program for Foreign experts of Hebei Province (No. 2019YX002A). The research of the second author was also partly supported by the GDRI ECO-Math.

References

[1] O. Aberth, An isoperimetric inequality for polyhedra and its application to an extremal problem. *Proc. London Math. Soc.* (3) **13** (1963), 322–336. MR149371 Zbl 0113.16204

[2] A. S. Besicovitch, A cage to hold a unit-sphere. In: *Proc. Sympos. Pure Math., Vol. VII*, 19–20, Amer. Math. Soc. 1963. MR0155236 Zbl 0141.19903

[3] H. Coxeter, Review 1950. *Math. Reviews* **20** (1959), 322. MR0095448

[4] G. Valette, À propos des cages circonscrites à une sphère. *Bull. Soc. Math. Belg.* **21** (1969), 124–125. MR262927 Zbl 0189.52803

[5] L. Yuan, T. Zamfirescu, Tetrahedral and pentahedral cages for discs. *Ars Math. Contemp.* **17** (2019), 255–270. MR4031523 Zbl 07172042