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Generous Sets

Augustin Fruchard
Université de Haute Alsace, IRIMAS, Mulhouse, France
augustin.fruchard@uha.fr

Liping Yuan
School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, P.R.China
and: Hebei Int. Research Center, Mathematics and Interdisciplinary Science, Shijiazhuang, P.R.China
lpyuan@hebtu.edu.cn

Tudor Zamfirescu
Fachbereich Mathematik, Universität Dortmund, Germany, Germany
and: Romanian Academy, Bucharest, Romania,
and: School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, P.R.China
tuzamfirescu@gmail.com

[Abstract-pdf]

\def\Rk{{\mathbb{R}^k}} We investigate the notion of generosity, a particular case of non-selfishness. Let $\cal F$ be a family of sets in $\Rk$. A set $M \subset \Rk$ is called $\cal F$-{\it convex} if for any points $x,y\in M$ there is a set $F\in \cal F$ such that $x,y\in F$ and $F\subset M$. We call a family $\cal F$ of compact sets {\it complete} if $\cal F$ contains all compact $\cal F$-convex sets. A single convex body K will be called {\it generous}, if the family of all convex bodies isometric to K is not complete. We investigate here the generosity of convex bodies.

Keywords: F-convex, complete, generous, grateful.

MSC: 52A10, 52A20.

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Generous Sets

Augustin Fruchard
Université de Haute Alsace, IRIMAS, Mulhouse, France
augustin.fruchard@uha.fr

Liping Yuan*
School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, P. R. China;
and: Hebei Key Lab. of Comp. Mathematics and Applications, Shijiazhuang, P. R. China
lpyuan@hebtu.edu.cn

Tudor Zamfirescu
Fachbereich Mathematik, Universität Dortmund, Germany;
and: Roumanian Academy, Bucharest, Roumania; and: School of
Mathematical Sciences, Hebei Normal University, Shijiazhuang, P. R. China
tuzamfirescu@gmail.com

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We investigate the notion of generosity, a particular case of non-selfishness. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}$. A set $M \subset \mathbb{R}$ is called $\mathcal{F}$*-convex* if for any points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. We call a family $\mathcal{F}$ of compact sets *complete* if $\mathcal{F}$ contains all compact $\mathcal{F}$-convex sets. A single convex body K will be called *generous*, if the family of all convex bodies isometric to K is not complete. We investigate here the generosity of convex bodies.

Keywords: $\mathcal{F}$-convex, complete, generous, grateful.

2010 Mathematics Subject Classification: 52A10, 52A20.

1. Introduction

Let $\mathcal{F}$ be a family of sets in $\mathbb{R}$ $(k \geq 2)$. A set $M \subset \mathbb{R}$ with card$M \geq 2$, is called $\mathcal{F}$*-convex* if for any pair of points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

The third author proposed at the 1974 meeting on Convexity in Oberwolfach the investigation of $\mathcal{F}$-convexity, for various families $\mathcal{F}$. Obviously, usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are all examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the third author [1], and also Böröczky Jr [2], investigated the rectangular convexity, the case when $\mathcal{F}$ contains all non-degenerate rectangles.

Magazanik and Perles dealt with staircase connectedness, a special kind of polygonal connectedness [5].

In [10] the third author studied the case when $\mathcal{F}$ is the family of all right triangles in $\mathbb{R}$.

* Corresponding author.

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In the last two authors' paper [7], the type of convexity studied in [10] was generalized and the right triple convexity was introduced, where $\mathcal{F}$ is the family of all triples $\{x, y, z\}$ such that $\angle xyz = \pi/2$. See also [6].

We call a family $\mathcal{F}$ of compact sets *complete*, if $\mathcal{F}$ contains all compact $\mathcal{F}$-convex sets. A single compact set L is called *selfish*, if the family $\mathcal{F}_L$ of all sets similar to L (resulting from an isometry followed by a homothety, i.e. dilation/contraction) is complete [8]. Further, a compact set L will be called *generous*, if the family $\mathcal{G}_L$ of all sets congruent, i.e. isometric, to L is not complete. If K is a compact $\mathcal{G}_L$-convex set not belonging to $\mathcal{G}_L$, we say that L *is generous towards* K.

The set of all selfish sets is disjoint from the set of all generous sets, of course, but there are sets which are neither selfish, nor generous.

We investigate here the generosity of compact convex sets.

For distinct $x, y \in \mathbb{R}$, let $\overline{xy}$ be the line through x, y and xy the line-segment from x to y.

A k-dimensional compact convex set in $\mathbb{R}$ is called a *convex body*.

For $M \subset \mathbb{R}$ with $k \geq 2$, $\mathrm{cl} M$ denotes its topological closure and $\mathrm{bd}\, M$ its boundary, while the convex hull and the affine hull of M, denoted by $\mathrm{conv}\, M$ and $\overline{M}$ respectively, are the intersection of all convex sets, respectively affine subspaces, including M.

We also denote by $x_1 x_2 \dots x_n$ the convex hull of the finite set $\{x_1, x_2, \dots, x_n\}$. Such a set is called a *polytope*. An extreme point of the polytope P, i.e. a point not belonging to the relative interior of any line-segment included in P, is called a *vertex* of P. A polytope the vertices of which are among the vertices of another polytope P, is called a *subpolytope* of P.

The Euclidean distance between two points $a, b \in \mathbb{R}$ will be denoted by $|ab|$. So, $|ab| = \|a - b\|$. Also, $m(ab) = (a + b)/2$ is the midpoint of ab. If ab and cd are line-segments, we write $ab \| cd$, if $\overline{ab}$ and $\overline{cd}$ are parallel.

For any compact set M, put $\mathrm{diam}\, M = \max_{x,y \in M} |xy|$. If K is a convex body, and $x, y \in \mathrm{bd}\, K$, then xy is a *chord* of K. A chord $xy \subset K$ is a *diameter* of K, if $|xy| = \mathrm{diam}\, K$.

Angles, denoted by $\widehat{xyz}$, are always unoriented. Sometimes the term "angle" will refer to the measure $\angle xyz$ of the angle $\widehat{xyz}$, a number between 0 and π.

The closed unit ball in $\mathbb{R}$ is denoted by $\mathbb{B}_k$, and $\mathrm{bd}\, \mathbb{B}_k = \mathbb{S}_{k-1}$.

If the sets $L, L' \subset \mathbb{R}$ are congruent, we write $L \sim L'$ and say that L' is *a copy of* L.

A planar convex body is called *n-symmetric*, if it is invariant under some rotation of angle $\frac{2\pi}{n}$.

2. Examples of generous convex bodies

We start with a simple and general argument which will be used several times in our work.

Proposition 2.1. *Let $K, L \subset \mathbb{R}$ be non-congruent convex bodies. If, for any $x, y \in \operatorname{bd} K$, there exists $L' \sim L$ with $x, y \in L'$ and $L' \subset K$, then L is generous towards K.*

Proof. Let $x', y' \in K$. Put $xy = \overline{x'y'} \cap K$. Since $x, y \in \operatorname{bd} K$, there exists $L' \sim L$ with $x, y \in L'$ and $L' \subset K$. Because L' is convex, $xy \subset L'$, whence $x', y' \in L'$. □

Now we present several examples of generous convex bodies.

It is immediately seen that the (circular) disc $\mathbb{B}_2$ is selfish in $\mathbb{R}^2$, but is it so in $\mathbb{R}^3$, too? No, it is even generous, because $\mathbb{B}_3$ is $\mathcal{G}_{\mathbb{B}_2}$-convex! By Theorem 1.1 of [8], the square is selfish, too, in $\mathbb{R}^2$. More generally, every rectangle is selfish in $\mathbb{R}^2$ [9]. But are they so in $\mathbb{R}^3$? The answer is again no. For any rectangle R, $\mathbb{B}_3$ is $\mathcal{F}_R$-convex. But, unlike the discs, rectangles are also not generous, by Theorem 4.4 of the present paper.

Theorem 2.2. *If A is an arc of endpoints a, b (with $a, b \in A$) and of length $\pi/2$ in $\mathbb{S}_1$, then the ball $\mathbb{B}_k$ is $\mathcal{G}_M$-convex, where $M = \operatorname{conv}(A \cup \{-a\})$.*

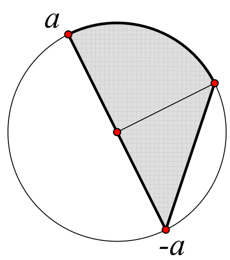
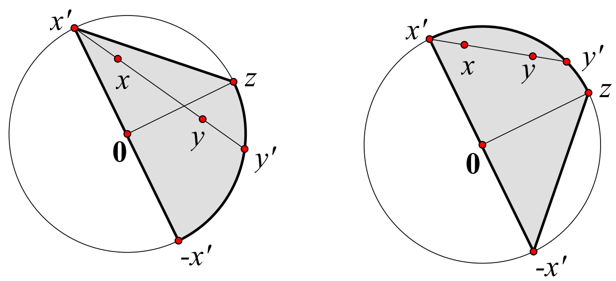


Figure 2.1: The set M.

Proof. Let $x, y \in \mathbb{B}_k$ be distinct from $\mathbf{0}$. Consider the intersection points x', y' of $\overline{xy}$ with $\mathbb{S}_{k-1}$.

We find a point $z \in \mathbb{S}_{k-1} \cap \overline{\mathbf{0}xy}$ at distance $\sqrt{2}$ from x', such that $\overline{\mathbf{0}x'}$ does not separate y' from z in $\overline{\mathbf{0}xy}$, as shown in Figure 2.2.



(*a*) M' (*b*) M''

Figure 2.2: M' and M''.

If $|x'y'| \geq \sqrt{2}$, then we take the subset M' of $\mathbb{B}_k$ congruent to M such that $-x'$ corresponds to a, z corresponds to b, and x' to $-a$; we have

$$\{x, y\} \subset x'y' \subset M' \subset \mathbb{B}_k.$$

If $|x'y'| < \sqrt{2}$, then we consider the subset M'' of $\mathbb{B}_k$ congruent to M such that x' corresponds to a, z corresponds to b, and $-x'$ to $-a$; we again have

$$\{x, y\} \subset x'y' \subset M'' \subset \mathbb{B}_k.$$

For the points $x, \mathbf{0} \in \mathbb{B}_k$, we may choose arbitrarily $y \in \mathbb{B}_k \setminus \{x, \mathbf{0}\}$, and find as before a subset of $\mathbb{B}_k$ congruent to M and containing x, y. It automatically contains $\mathbf{0}$, too. □

Theorem 2.3. *Assume that* $v_1v_2...v_{16} \subset \mathbb{R}^2$ *is a regular* 16*-gon. Then the octagon* $v_1v_2v_3v_4v_5v_8v_9v_{10}$ *is generous.*

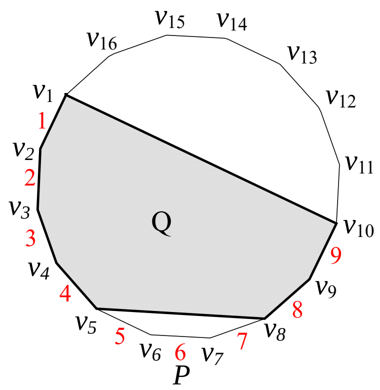


Figure 2.3: P and Q.

Proof. Let $P = v_1v_2...v_{16}$ and $Q = v_1v_2v_3v_4v_5v_8v_9v_{10}$. We show that P is $\mathcal{G}_Q$-convex.

We consider the family $\mathcal{T}$ of all isosceles triangles and trapezoids with vertices among the vertices of P, each of which having at least two sides of length $|v_1v_2|$. They belong to eight equivalence classes, regarding congruence. For example, $v_1v_2v_4v_5 \sim v_3v_4v_6v_7$. If the nine consecutive edges v_iv_{i+1} $(i = 1, ..., 9)$ get numbers 1, 2,..., 9, then each element of $\mathcal{T}$ can be identified with a pair of numbers, for example $v_1v_2v_4v_5$ with (1, 4). Clearly, $v_iv_{i+1}v_jv_{j+1} \sim v_{i'}v_{i'+1}v_{j'}v_{j'+1}$, i.e., $(i, j) \sim (i', j')$, if and only if $j - i = j' - i'$.

We want to verify that Q displays polygons from $\mathcal{T}$ of all types, i.e., from all equivalence classes. Indeed, Q has 1, 2, 3, 4, 8, 9 among its edges. Since $1 = 2 - 1$, $2 = 3 - 1$, $3 = 4 - 1$, $4 = 8 - 4$, $5 = 8 - 3$, $6 = 8 - 2$, $7 = 8 - 1$, $8 = 9 - 1$, all classes are represented in Q.

Now, take $x, y \in P$. They lie in a triangle $v_iv_{i+1}v_{i+2}$ or trapezoid $v_iv_{i+1}v_jv_{j+1}$; call it T. Since a polygon congruent with T can be found in Q, according to the discussion above, this means that Q can be rotated to contain x, y. □

In the way described by Theorem 2.3 one can find (infinitely) many examples of generous polygons. As they are anyway just examples, we preferred to give here just a specific one, and resisted the temptation of trying to be exhaustive.

Theorem 2.4. *If* $D = v_1 \dots v_{20}$ *is a regular dodecahedron in* $\mathbb{R}^3$ *and* $P = v_1 \dots v_{19}$*, then* P *is generous towards* D*.*

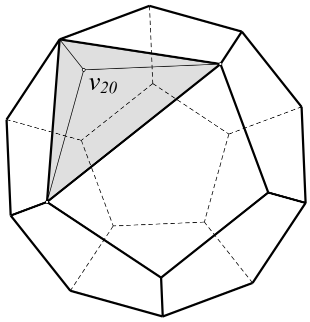


Figure 2.4: D and P.

Proof. The faces of P are nine regular pentagons, three trapezoids and one equilateral triangle. Among the nine pentagonal faces, one can find a pair of neighbouring ones, a pair of opposite ones, and a pair that are neither neighbouring nor opposite.

Suppose D is centred at $\mathbf{0}$. Let $x, y \in D$. For $\mathbf{0} \notin \{x, y\}$, denote by x', y' the points of $\operatorname{bd} D$ such that $x \in \mathbf{0}x'$ and $y \in \mathbf{0}y'$. (The case $\mathbf{0} \in \{x, y\}$ is obvious.) Let F be the face of D containing x' (any of them if x' is a vertex or on an edge of D) and F' the face containing y'.

Three cases can occur: Either F and F' are neighbouring, or they are opposite, or neither neighbouring nor opposite. In the three cases, one can find a subpolytope of D congruent to P and admitting F and F' as faces. Since $x, y \in \mathbf{0}x'y'$, the proof is finished. □

Theorem 2.5. *Suppose $abcda'b'c'd' \subset \mathbb{R}^3$ is a cube, with an upper face $abcd$, a lower one $a'b'c'd'$, and aa', bb', cc', dd' as edges. Then the polytope $aa'bb'ec'fd'$ is generous, where e, f are the midpoints of cc', dd', respectively.*

Proof. Set $C = abcda'b'c'd'$ and $A = aa'bb'ec'fd'$. Assume C is centred at $\mathbf{0}$. We prove that C is $\mathcal{G}_A$-convex.

Let $x, y \in C$. Put $\{x', y'\} = \overline{xy} \cap \operatorname{bd} C$.

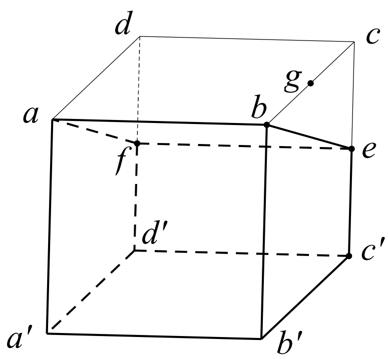


Figure 2.5: C and A.

Case 1. x', y' belong to opposite faces of C.

Suppose without loss of generality that $x' \in aa'd'$. If $y' \in bb'c'$, then $\{x', y'\} \subset A$. Suppose now $y' \in bcc'$.

If $y' \in bec'$, then again $\{x', y'\} \subset A$. If $y' \in bc'g$, where $g = m(bc)$, the symmetry with respect to $\overline{cda'b'}$ leaves $aa'd'$ invariant and brings $bc'g$ into $c'be$, a situation which was just settled. If $y' \in cc'g$, another automorphism of C sends $cc'g$ into $c'd'e$ and $aa'd'$ into baa', which are both in A. (The mentioned automorphism is a rotation about the z-axis bringing a to b, followed by a symmetry with respect to $\overline{a'bcd'}$.)

Case 2. x', y' belong to neighbouring faces of C.

This case is solved by the remark that $abb'a' \cup a'b'c'd' \subset A$.

Case 3. x', y' belong to the same face of C.

This case is trivial. □

We now present families of convex bodies partly displaying increased symmetry.

Theorem 2.6. *Suppose $k \geq 3$, $E \subset \mathbb{R}$ is a convex body invariant under any rotation about the k-th axis $\overline{x_k}$, H is a hyperplane including $\overline{x_k}$, H^+ is a closed half-space bounded by H, and K is a convex body different from E. If $E \cap H^+ \subset K \subset E$, then K is generous towards E.*

Proof. Let $x, y \in E$. Choose a hyperplane $J \supset \overline{x_k}$, which does not separate x from y. Let J^+ be the closed half-space determined by J which contains both x, y. The rotation mapping H^+ into J^+ will transform K into a congruent copy containing x, y. □

Theorem 2.7. *Let $K \subset \mathbb{R}^3$ be a convex body symmetric with respect to both axes $\overline{x_1}$ and $\overline{x_2}$. Consider the set $L = \{y = (y_1, y_2, y_3) \in K : y_2 \geq 0 \quad \vee \quad y_3 \geq 0\}$. Any convex body C distinct from K and satisfying $L \subset C \subset K$ is generous.*

Proof. We prove that K is $\mathcal{G}_C$-convex.

Let $x, y \in K$. If $x, y \in L$, we are done. If $x, y \notin L$, the set L' symmetric to L with respect to $\overline{x_1}$ contains x, y. Suppose now that $x \in L$, $y \notin L$. We may assume without loss of generality that $x_3 \geq 0$. Then the set L'' symmetric to L with respect to $\overline{x_3}$ contains x, y.

The symmetries carrying L to L' or L'' will transform C into a convex body containing x, y and included in K. □

Theorems 2.6 and 2.7 allow us to uncover the large degree to which a generous convex body can be prescribed.

3. General results

Let $\mathbf{G}$ be the set of all generous convex compact sets in $\mathbb{R}$.

Proposition 3.1. *Let L be a convex body and $K \in \mathbf{G}$. If K is $\mathcal{G}_L$-convex, then $L \in \mathbf{G}$, too.*

Proof. Since $K \in \mathbf{G}$, some C non-congruent with K is $\mathcal{G}_K$-convex. Then, C is $\mathcal{G}_L$-convex, too, because, for $x, y \in C$, there exists $K' \subset C$, congruent with K with $x, y \in K'$, and since K' is $\mathcal{G}_L$-convex, there exists $L' \subset K'$ congruent with L, such that $x, y \in L'$. Since $L' \subset K' \subset C$, and $K' \neq C$, L' and C are non-congruent. Hence, $L' \in \mathbf{G}$. □

Theorem 3.2. *Let K, L be two non-congruent convex bodies.*

(a) *If L is generous towards K, then the number of different copies of L included in K is at least three.*

(b) *If L is generous towards K and has exactly three copies L_1, L_2, L_3 in K, then necessarily $L_1 \cup L_2 = L_2 \cup L_3 = L_3 \cup L_1 = K$.*

(c) *Conversely, if $L_1, L_2, L_3 \subset K$ are three copies of L such that $L_1 \cup L_2 = L_2 \cup L_3 = L_3 \cup L_1 = K$, then L is generous towards K.*

Proof. (a) Assume only two copies L_1, L_2 of L exist in K. Then, for $x \in K \setminus L_1$ and $y \in K \setminus L_2$, no copy of L can at the same time contain $\{x, y\}$ and be included in K.

(b) Indeed, if $L_1 \cup L_2 \neq K$, choose $x \in K \setminus (L_1 \cup L_2)$ and $y \in K \setminus L_3$; then no copy of L can at the same time contain $\{x, y\}$ and be included in K.

(c) Let $x, y \in K$. If $\{x, y\} \not\subset L_1$, say $x \notin L_1$, then $x \in L_2$ and $x \in L_3$. If, moreover, $\{x, y\} \not\subset L_2$, then $y \notin L_2$; then $y \in L_3$, and we have $\{x, y\} \subset L_3$. □

Proposition 3.3. *If L is generous towards K and $L_1, L_2 \subset K$ are two copies of L, then* $\operatorname{diam} L = \operatorname{diam}(L_1 \cup L_2) = \operatorname{diam} K$. *Consequently, L_2 cannot be a non-trivial translate of L_1.*

Proof. Let Δ be a diameter of K; since L is generous towards K, there is a copy L' of L containing Δ and included in K, yielding

$$\operatorname{diam} K = \operatorname{diam} \Delta \leq \operatorname{diam} L' \leq \operatorname{diam}(L_1 \cup L_2) \leq \operatorname{diam} K.$$

So, we have equalities above.

Suppose L_2 is a non-trivial translate of L_1. Let $a_1 b_1$ be a diameter of L_1 and $a_2 b_2$ the diameter of L_2 obtained by translation. At least one of the diagonals of the parallelogram $a_1 b_1 b_2 a_2$ is longer than its sides, hence $\operatorname{diam}(L_1 \cup L_2) > \operatorname{diam} L_1$. This contradicts our previous findings. □

Let $P \subset \mathbb{R}^2$ be a polygon. A broken line $\langle abcd \rangle$ (possibly $a = d$) is called a *zyggy*, if ab and cd are edges of P. If bc is an edge, too, then $\langle abcd \rangle$ is called a *boundary zyggy of P*. If bc is a diameter of P, then $\langle abcd \rangle$ is called a *diametral zyggy of P*.

Theorem 3.4. *If L is a convex polygon generous towards a convex body $K \subset \mathbb{R}^2$, then K is a polygon, and every zyggy of K includes a set congruent to a zyggy of L. More precisely, for any zyggy $\langle abcd \rangle$ of K there exist $a_1 \in ab$ and $d_1 \in cd$ such that $\langle a_1 b c d_1 \rangle$ is a zyggy of a copy of L.*

In particular, this holds for boundary and diametral zyggies, too.

Proof. We denote by $\Delta(L)$ the set of all distances between any two different vertices of L.

If x and y are two different extremal points of K, and L' congruent to L is such that $\{x, y\} \subset L' \subset K$, then x, y are vertices of L', hence the distance between two extremal points of K is bounded below by $\min \Delta(L)$, whence K is a polygon.

Let $\langle abcd \rangle$ be a zyggy of L. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ be two sequences of points $a_n \in ab$, $d_n \in cd$, such that:

(i) $a_n \to b$ and $d_n \to c$,

(ii) the sequences $\{|a_n b|\}_{n\in\mathbb{N}}$ and $\{|d_n c|\}_{n\in\mathbb{N}}$ are decreasing, and

(iii) the sequence $\{|a_n d_n|\}_{n\in\mathbb{N}}$ is monotone.

Therefore, for n large enough, $|a_n d_n|$ does not belong to $\Delta(L)$.

Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of bodies congruent to L such that $\{a_n, d_n\} \subset L_n \subset K$. Extracting a subsequence if necessary, we assume that all L_n have the same orientation in the plane, i.e. the isometries bringing L to L_n are either all orientation preserving or all orientation reversing.

Our aim is to prove that, extracting a subsequence if necessary, $\{L_n\}_{n\in\mathbb{N}}$ is a constant sequence. We partition $\mathbb{N}$ in four subsets depending upon whether a_n or d_n is a vertex of L_n: Let

$$N_1 = \{n \in \mathbb{N}\,;\, a_n \in V(L_n) \text{ and } d_n \in V(L_n)\},$$

$$N_2 = \{n \in \mathbb{N}\,;\, a_n \in V(L_n) \text{ and } d_n \notin V(L_n)\},$$

$$N_3 = \{n \in \mathbb{N}\,;\, a_n \notin V(L_n) \text{ and } d_n \in V(L_n)\},$$

$$N_4 = \{n \in \mathbb{N}\,;\, a_n \notin V(L_n) \text{ and } d_n \notin V(L_n)\}.$$

Since a_n and d_n belong to $\operatorname{bd} K$, they belong to $\operatorname{bd} L_n$; therefore a_n belongs either to $V(L_n)$ or to an edge of L_n, and the same holds for d_n.

N_1 is finite since $|a_n d_n| \notin \Delta(L)$ for n large enough.

We now prove that N_2 and N_3 are also finite. Let m and n be two different integers of N_2, i.e. $a_m \in V(L_m)$, $a_n \in V(L_n)$, and d_m and d_n are on edges of L_m and L_n respectively. If d_m and d_n belong to copies of the same edge of L, then a_m and a_n would correspond to the same vertex of L, hence L_m would be a non-trivial translate of L_n, which is excluded by Proposition 3.3. To sum up, if m and n are two different integers of N_2, then d_m and d_n belong to copies of two different edges of L, hence N_2 is finite. For the same reason N_3 is also finite.

Since $N_1 \cup N_2 \cup N_3 \cup N_4 = \mathbb{N}$, we conclude that N_4 is infinite. Extracting a subsequence of $\{(a_n, d_n)\}_{n\in\mathbb{N}}$ if necessary, we may assume that $N_4 = \mathbb{N}$. In this manner, for all $n \in \mathbb{N}$, a_n belongs to an edge of L_n, which corresponds to some edge of L. Since the number of edges of L is finite, extracting a subsequence if necessary, we may assume that this edge of L is the same for the whole sequence $\{a_n\}_{n\in\mathbb{N}}$. Since two distinct copies of L within K cannot be translates of each other, this means that $\{L_n\}_{n\in\mathbb{N}}$ is a constant sequence.

Now, b and c are vertices of L_1, and $\langle a_1 bcd_1 \rangle$ is a zyggy of L_1. If $\langle abcd \rangle$ is a boundary or diametral zyggy of K, then $\langle a_1 bcd_1 \rangle$ is a boundary, respectively diametral, zyggy of L_1. □

4. Beginning of classification

In this section we work in $\mathbb{R}^2$.

Theorem 4.1. *There is no generous triangle.*

Proof. Indeed, if L is a triangle and K is $\mathbb{G}_L$-convex, then every boundary zyggy of K must have angles summing up to π (since this is the case for boundary zyggies of a triangle). But this can happen only if K is itself a triangle, one congruent to L. □

Let us call an *n-vase* a broken line $\langle a_0 \dots a_n\rangle$ such that:

- no three among the points a_i are collinear,
- $a_0 \neq a_n$,
- the polygon with consecutive vertices $a_0, \dots, a_n$ is convex,
- the ray starting from a_1 and containing a_0 does not cross the ray starting from a_{n-1} and containing a_n.

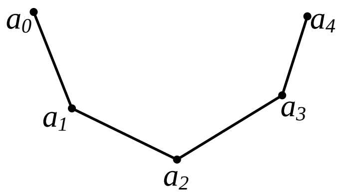


Figure 4.1: A 4-vase $\langle a_0a_1a_2a_3a_4\rangle$.

For example, a 4-vase is shown in Figure 6.

Observe that the last condition is equivalent to asking that the angles α_i at a_i satisfy $\sum_{i=1}^{n-1} \alpha_i \geq (n-2)\pi$.

Lemma 4.2. *If a convex polygon K contains a 4-vase in its boundary, then no convex quadrilateral is generous towards K.*

Proof. Assume that $\operatorname{bd} K$ contains a 4-vase $\langle a_0a_1a_2a_3a_4\rangle$ and $L = abcd$ is generous towards K. Then K contains at least three zyggies which are 3-vases: $\langle a_0a_1a_2a_3\rangle$ and $\langle a_1a_2a_3a_4\rangle$, which are boundary zyggies, and $\langle a_0a_1a_3a_4\rangle$, which is not a boundary zyggy (such a zyggy will be called a *non-boundary zyggy*).

We first assume that L has no two parallel sides; then among the four boundary zyggies of L, two are 3-vases and consecutive, say $\langle dabc\rangle$ and $\langle abcd\rangle$, and two are not: $\langle bcda\rangle$ and $\langle cdab\rangle$, see Figure 7.

By Theorem 3.4, each of the three aforementioned 3-vases of K must share the same angles with one of the 3-vases of L.

However the non-boundary zyggy of K cannot share the same two angles with one of the boundary zyggies of K. For example, if

$$\{\angle a_0a_1a_2, \angle a_1a_2a_3\} = \{\angle a_0a_1a_3, \angle a_1a_3a_4\},$$

then a_2a_3 and a_3a_4 would be parallel, a contradiction. It follows that the two boundary zyggies of K share the angles with one zyggy of L, say $\langle dabc\rangle$, and the non-boundary zyggy of K shares the angles with $\langle abcd\rangle$.

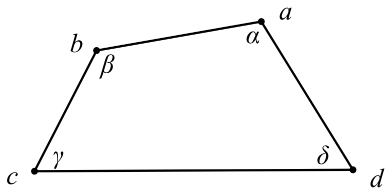


Figure 4.2: The quadrilateral L.

Let α, β, γ and δ be the angles at a, b, c and d respectively on $\operatorname{bd} L$. Let $\alpha_1' = \angle(a_0a_1a_3)$ and $\alpha_3' = \angle(a_1a_3a_4)$, the angles of the non-boundary zyggy of K. The above discussion sums up as:

$$\{\alpha_1, \alpha_2\} = \{\alpha_2, \alpha_3\} = \{\alpha, \beta\} \text{ and } \{\alpha_1', \alpha_3'\} = \{\beta, \gamma\}.$$

Since $\alpha_1' < \alpha_1$ and $\alpha_3' < \alpha_3$ and one of the angles α_1', α_3' has to be β, one of the angles α_1, α_3 is not β, hence we must have $\alpha_2 = \beta$, whence $\alpha_1 = \alpha_3 = \alpha$ (and $\alpha \neq \beta$).

By Theorem 3.4, we also have $|ab| = |a_1a_2| = |a_2a_3|$, hence the triangle $a_1a_2a_3$ is isosceles at a_2. Thus, the angles $\angle a_3a_1a_2 = \alpha_1 - \alpha_1'$ and $\angle a_1a_3a_2 = \alpha_3 - \alpha_3'$ are equal, yielding $\alpha_1' = \alpha_3'$. Then, the equality $\{\alpha_1', \alpha_3'\} = \{\beta, \gamma\}$ implies $\alpha_1' = \alpha_3' = \beta = \gamma$. Since $\langle a_0a_1a_3a_4\rangle$ is a vase, we also have $\alpha_1' + \alpha_3' \geq \pi$. To sum up, we have

$$\alpha_1 = \alpha_3 = \alpha > \alpha_2 = \alpha_1' = \alpha_3' = \beta = \gamma \geq \frac{\pi}{2} > \delta.$$

Now, by Theorem 3.4, we have $|bc| = |a_1a_3|$ and $|bc| \leq |a_2a_3|$, whence $\alpha_2 \leq \frac{\pi}{3}$, and a contradiction is obtained.

If L has two parallel sides then, among its four boundary zyggies, more than two are 3-vases but only one of them is related with the two boundary zyggies $\langle a_0a_1a_2a_3\rangle$ and $\langle a_1a_2a_3a_4\rangle$ of K, and the rest of the proof is the same. □

Theorem 4.3. *If K and L are convex bodies such that L is a convex quadrilateral generous towards K, then we are up to isometries in one of the following three situations.*

(a) *K is an equilateral triangle $v_1v_2v_3$ and L is of the form v_1v_2xy with $x \in v_2v_3$, $y \in v_3v_1$, with $\angle xoy \leq \frac{2\pi}{3}$, where o is the centre of K. The smallest quadrilaterals L in the sense of inclusion are those for which $\angle xoy = \frac{2\pi}{3}$; the L with least area is obtained for $x = m(v_2v_3)$ and $y = m(v_3v_1)$.*

(b) *K is a rectangle $v_1v_2v_3v_4$ and L is of the form $v_1v_2v_3x$ with $x = (1-t)v_4 + tv_1$, $0 < t \leq \frac{1}{2}$. The smallest L (in the sense of both inclusion and area) has $t = \frac{1}{2}$.*

(c) *K is a regular pentagon $v_1v_2v_3v_4v_5$ and $L \sim v_1v_2v_3v_4$.*

Proof. We already know from Theorem 3.4 that K is a convex polygon.

It is easy to see that an n-gon with $n \geq 6$ always contains a 4-vase, hence by Lemma 4.2 K is either a triangle, or a quadrilateral, or a pentagon.

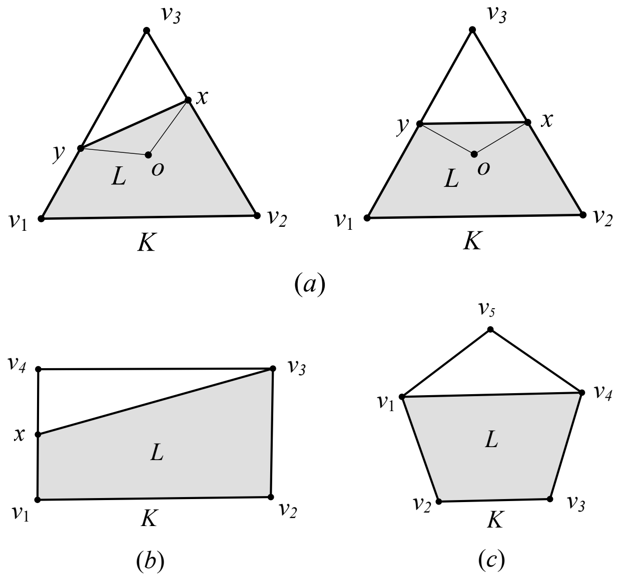


Figure 4.3: K and L in Theorem 4.3.

The case where K is a triangle will be treated in the next section in a more general setting, see Theorem 5.6: Not only no quadrilateral can be generous towards a non-equilateral triangle, but no convex body at all.

The case where K is a quadrilateral will be treated by Theorem 5.7.

We now assume that K is a pentagon. By Lemma 4.2, we can also assume that $\mathrm{bd}\, K$ has no 4-vase. This implies that all five boundary zyggies of K are 3-vases and that K has no pair of parallel sides. Each boundary zyggy of K must contain a congruent copy of a boundary zyggy of L which therefore has to be a 3-vase with non-parallel sides.

We now prove that only one boundary zyggy of L serves for all five boundary zyggies of K. Already, only two boundary zyggies of L can play this role and are moreover consecutive, say $\langle dabc \rangle$ and $\langle abcd \rangle$. If both zyggies serve, then there are two consecutive zyggies of K using both zyggies of L (because K has an odd number of vertices). Then one easily sees that K has to be included into L, a contradiction.

A a consequence, one boundary zyggy of L, say $\langle dabc \rangle$, has a congruent copy included into each boundary zyggy of K. Since the number of vertices of K is odd, the angles $\alpha = \angle dab$ and $\beta = \angle abc$ must be equal, and the sides of K are all of equal length $|ab|$, hence K is regular.

Now let $K = abcde$ be a regular pentagon and let L be a convex quadrilateral generous towards K. By Theorem 3.4 applied to the zyggy $\langle abcd \rangle$, L is congruent to $a'bcd'$ with some $a' \in ab$ and $d' \in cd$. Applying Theorem 3.4 to the zyggy $\langle badc \rangle$, we then obtain that L has to be congruent to $ab'c'd$ with some $b' \in ab$ and $c' \in cd$. Only one quadrilateral (up to isometries) can satisfy both constraints: $abcd$.

Conversely the quadrilateral $abcd$ is generous towards K. □

It is easy to verify that, for each $n = 3, \ldots, 7$, there is a pentagon generous to $\mathbb{P}_n$. It can also be seen that no other integer n satisfies this property. Are there pentagons generous towards other polygons than $\mathbb{P}_n$, where $3 \leq n \leq 7$? Yes, towards any non-regular 3-symmetric hexagon!

Theorem 4.4. *A planar polygon cannot be generous to a convex body of dimension* 3.

Proof. Suppose the planar polygon L is generous to a three-dimensional convex body K. Then any two different extreme points of K have to be at distance at least $\delta(L)$ from each other, where $\delta(L)$ is the smallest edge-length of L; hence, K has finitely many extreme points, and is therefore a polytope.

Let ab be a diameter of K. Let ξ be the largest length different from $|ab|$ of a chord between vertices of L (if it exists). Consider now two points $x, y \in \mathrm{bd}\, K$, close to a, b, respectively, such that $\xi < |xy| < |ab|$. They belong to a zyggy of a copy L' of L included in K. That zyggy is included in a diametral zyggy of L'. There are just four such zyggies (for each diameter), with at most four distinct angles. But the angle $\widehat{xab}$ varies, when x varies on $\mathrm{bd}\, K$, taking values in a whole interval (notice that all faces at a make an acute angle with the diameter ab). Hence, there are many values of $\angle xab$, which are not among those of diametral zyggies of L. It follows that those points x do not belong to any diametral zyggy of L'. This contradicts the generosity of L to K. □

Question 4.5. Is there any generous tetrahedron?

Question 4.6. Which convex pentahedra are generous?

5. Gratefulness

A convex body K in $\mathbb{R}$ is said to be *grateful* if there exists a convex body L which is generous towards K, i.e. such that K is $\mathbb{G}_L$-convex, but not congruent to L. Every $\mathbb{P}_n$ is grateful, for all $n \geq 3$. Are there other grateful polygons in $\mathbb{R}^2$? Yes: any convex non-regular 3-symmetric hexagon. More generally, the following holds.

Theorem 5.1. *For all $n \geq 3$, every planar n-symmetric convex body is grateful.*

Proof. Let K be an n-symmetric convex body for some $n \geq 3$ and denote by r the rotation of angle $\frac{2\pi}{n}$ leaving K unchanged. Choose an extremal point x of K. For any $y \in \mathbf{0}x$ distinct from and x, let D be the line orthogonal to $\mathbf{0}x$ and containing y. Then D cuts K into two pieces L and L'. (We take both L and L' compact and such that $L \cup L' = K$ and $L \cap L' \subset D$.) Let L be the piece not containing x. If y is close enough to x then L' and $r(L')$ are disjoint. It follows that the copies $L_1 = L$, $L_2 = r(L)$ and $L_3 = r^2(L)$ satisfy the assumptions of Theorem 3.2(c), yielding the generosity of L towards K. □

Theorem 5.2. *If $K \subset \mathbb{R}^2$ is a grateful convex body having a unique diameter, then K is symmetrical with respect to this diameter and also with respect to the mediator of this diameter.*

Proof. Assume $(-x)x$ is the only diameter of K (with midpoint at the origin $\mathbf{0}$) and $L \subset K$ is generous towards K. Since K and L have the same diameter, L can be included in K in at most four ways: as L, $-L$, L' or $-L'$, where the prime denotes the symmetry with respect to $\overline{(-x)x}$.

Suppose $K \neq -K$ and consider some $u \in K \setminus (-K)$. To fix ideas assume that $u \in L$; then $L \not\subset -K$, hence only three copies of L fit in $-K$: $-L$, L' and $-L'$. By

Theorem 3.2 (b), we then have $-K = L' \cup (-L') = -(L' \cup (-L')) = K$, absurd. We obtain in the same manner $K = K'$. □

We denote by $\mathcal{G}(K)$ the family of all compact convex subsets of K which are generous towards K.

Lemma 5.3. *If K is a convex body in the plane, then the set $\mathcal{G}(K) \cup \{K\}$, endowed with the Hausdorff-Pompeiu metric, is compact.*

Proof. By the Blaschke selection theorem, the set $\mathcal{C}(K)$ of all compact subsets of K is compact, hence it suffices to prove that $\mathcal{G}(K) \cup \{K\}$ is closed in $\mathcal{C}(K)$. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}(K) \cup \{K\}$ that converges to some $L \in \mathcal{C}(K)$. Clearly, L is convex. Let us show that L is generous towards K or equals K. Given $x, y \in K$, for any $n \in \mathbb{N}$ there is an isometry $\varphi_n : K \to \mathbb{R}^2$ such that $x, y \in \varphi_n(L_n)$ and $L_n \subset K$. All these isometries φ_n are in the compact subset of all isometries $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(K) \cap K \neq \emptyset$, hence there is a subsequence $\{\varphi_{n_k}\}_{k\in\mathbb{N}}$ which converges to some isometry φ. Now, the sequence $\{\varphi_{n_k}(L_{n_k})\}_{k\in\mathbb{N}}$ converges to $\varphi(L)$, which contains x, y and is included in K. □

Theorem 5.4. *If $K \subset \mathbb{R}^2$ is a grateful polygon, then there exists a polygon generous towards K.*

Proof. The area function $\mathcal{A} : \mathcal{G}(K) \to \mathbb{R}$ is continuous. By Lemma 5.3, it attains its minimum on $\mathcal{G}(K) \cup \{K\}$.

Let $L \subset K$ realize this minimum area $\mathcal{A}(L)$. Clearly, $L \neq K$, because any compact convex subsets of K generous towards K has area smaller than $\mathcal{A}(K)$. We prove that L is a polygon.

Since K is an n-gon, it has a finite number d of diameters (actually $d \leq n$, see [3]). Let us fix a diameter ab of L. Then, for any isometry φ such that $\varphi(L) \subset K$, $\varphi(ab)$ is a diameter of K, and for each diameter xy there are at most four isometries which send ab to xy. In other words, the number m of isometries φ such that $\varphi(L) \subset K$ is finite and at most $4d$. Let us denote these isometries by $\{\varphi_1, \ldots, \varphi_m\}$. Since each side of K contains at most two extremal points of each $\varphi_i(L)$, at most $2mn$ extremal points a of L are such that $\varphi_i(a) \in \operatorname{bd} K$ for some $i \in \{1, \ldots, m\}$.

If L is not a polygon, then it has infinitely many extremal points, hence there is some extremal point a of L such that $\varphi_i(a) \notin \operatorname{bd} K$ for all $i \in \{1, \ldots, m\}$.

Let ε be the minimum of all distances $|f_i(a)z|$, where $1 \leq i \leq m$ and $z \in \operatorname{bd} K$. Take $a', a'' \in \operatorname{bd} L$, such that $|aa'| < \varepsilon$ and $|aa''| < \varepsilon$, on each side of a on $\operatorname{bd} L$. Let P be the closed half-plane not containing a and bounded by $\overline{a'a''}$. Then the compact convex subset $L \cap P$ of K is generous towards K and has an area less than $\mathcal{A}(L)$, a contradiction. □

Theorem 5.5. *If K is a planar convex body different from a rhombus, symmetric with respect to $\overline{x_1}$ and $\overline{x_2}$, then K is grateful.*

Proof. Put $Q^+ = \{(x_1, x_2) : x_1 \geq 0 \quad \wedge \quad x_2 \geq 0\}$. Let $a_1(-a_1) = K \cap \overline{x_1}$ and $a_2(-a_2) = K \cap \overline{x_2}$. Obviously, K is $\mathbb{G}_L$-convex, where $L = \operatorname{conv}(K \setminus Q^+)$. Since $K \neq L$, K is grateful. □

Remark that a convex body as described in Theorem 5.5 may have any number of diameters, even (countably or uncountably) infinitely many.

If the number of diameters is odd, then either $a_1(-a_1)$ or $a_2(-a_2)$ is a diameter, but not both.

Theorem 5.6. *The only grateful triangle is the equilateral one.*

Proof. If a convex body L is generous to a triangle K, then L has to fit at least three times in K. Since K is a triangle, its diameters are necessarily edges, but for every edge of K only two distinct copies of L can have it as a diameter. So, in $K = abc$, we must have, say, $|ab| = |ac| \geq |bc|$.

Suppose $|ab| > |bc|$. Any copy L' of L, included in K and containing b, c, must also contain either ab or ac, in order to have $\operatorname{diam} L' = \operatorname{diam} K$. But then, $L' = K$, a contradiction. □

Theorem 5.7. *The only grateful quadrilaterals in* $\mathbb{R}^2$ *are the rectangles.*

Proof. Let K be a grateful quadrilateral, and L a convex body generous to K. By Theorem 5.4, we may suppose L to be a polygon.

Suppose first that $K = abcd$ has only one diameter. Then L, too, has a single diameter, of same length. If the diameter of K is an edge of K, then there exist no three distinct copies of L inside of K, which contradicts Theorem 3.2. So, assume the diameter is the diagonal ac. Then some copy of L must contain both b, d, and have diameter $\operatorname{diam} K$. This implies $L = K$, a contradiction.

Hence, K has (at least) two diameters.

Assume first they have a common endpoint, say a. Then both diameters are edges (and they are ab, ad in $K = abcd$), or one of them is a diagonal (and they are ac, ad in K).

If the diameters are ab, ad, then the copy of L containing b, c must also contain ab, therefore the whole triangle abc; analogously, another copy of L contains acd. But there is no third copy of L inside K having ab or ad as a diameter. (Some vertex of such a copy would lie outside of K.)

If the diameters are ac, ad, then the copy of L containing b, d must also include ad, but not $\{c\}$ (otherwise it equals K). So, L has just one diameter. Let L' be the copy of L in K containing c, d. Then $L' \supset acd$, and it has more than one diameter, a contradiction.

Hence, K has two diameters, which cross each other, ac, bd.

Take $x \in bc$, $y \in ad$. Some copy of L must include the zyggy $\langle xcay \rangle$ or $\langle xbdy \rangle$, because $\operatorname{diam} L = \operatorname{diam} K$. Let $\lambda = |xc|/|bc|$ and $\mu = |ya|/|da|$. Call $\min\{\lambda, \mu\}$ and $\max\{\lambda, \mu\}$ the *small* and *big ratio* of the zyggy $\langle xcay \rangle$. We want to determine the small and the big ratio of a zyggy which covers any pair of points x, y with $x \in bc$ and $y \in ad$. By taking x, y close to b, a, we see that the big ratio must be 1. By taking x close to b, and $y \in am(ad)$ close to the midpoint $m(ad)$ of ad, we see that the small ratio is $1/2$.

These zyggies must be included in L. If the two angles of a zyggy are not equal, then the previous argument gives 1 as small ratio, which means L is congruent to K. So, the two angles are equal, and $bc \| ad$. Analogously, $ab \| cd$. Hence, K is a parallelogram. Having equally long diagonals, it is a rectangle.

Indeed, any convex body L between $m(ad)abc$ and K (i.e. such that $m(ad)abc \subset L \subset K$), distinct from K, is generous towards the rectangle K. □

Question 5.8. Is $\mathbb{P}_5$ the only grateful pentagon in $\mathbb{R}^2$?

For every non-prime number $n \geq 6$, there are non-regular grateful n-gons: choose a divisor $d \geq 3$ of n, and take any non-regular d-symmetric n-gon.

Let n be a prime number. Is $\mathbb{P}_n$ the only grateful n-gon? The answer is positive for $n = 3$ (Theorem 5.6), unknown for $n = 5$ (Question 5.8), and negative for $n \geq 7$. It suffices to consider $\mathbb{P}_n = v_1 ... v_n$, and modify it by taking a point $v_2' \in v_2 v_3$ and the polygon $v_1 v_2' v_3 ... v_n$ instead of $\mathbb{P}_n$.

Proposition 5.9. *There exist grateful convex bodies without any symmetry.*

Proof. Cut a small piece of $\mathbb{P}_7$ containing v_1 and not symmetric with respect to $\overline{v_1}$; this is K. In K, reproduce the same cut at v_2 and v_4 to obtain L. Three copies of L fit into K: $L_1 = L$ itself, $L_2 = r(L)$, where r is the rotation about of angle $\frac{2\pi}{7}$ sending v_2 to v_1, and $L_3 = r^3(L)$. The small cuts are at the vertices v_1, v_2, v_4 for L_1, v_1, v_3, v_7 for L_2, and v_1, v_5, v_6 for L_3, hence we have $L_1 \cup L_2 = L_2 \cup L_3 = L_3 \cup L_1 = K$. Then, L is generous towards K, due to Theorem 3.2 (c). □

Theorem 5.10. *If a grateful polygon in $\mathbb{R}^2$ has an angle less than $\pi/3$, then it has a second angle of equal size, the two angles being at the endpoints of a single diameter, which is a bisector of both.*

Proof. Suppose $K \subset \mathbb{R}^2$ is a grateful polygon with an angle $\alpha = \angle abc < \pi/3$. Let L be generous to K.

Let ef be a diameter of K. If $b \notin \{e, f\}$, then be or bf is longer than ef, because $\angle ebf < \pi/3$. This being impossible, it follows that $b \in \{e, f\}$, say $b = f$. We claim that be is the unique diameter of K.

Indeed, assume that K has a second diameter be'. We may assume that all diameters of K lie in $\widehat{ebe'}$. We have $\angle bee' > \pi/3$, because $\angle ebe' < \pi/3$. There exists $L^* \sim L$ with $e, e' \in L^*$, whence $ebe' \subset L^*$. Now, only L^* and the set symmetrical to it with respect to the bisector of $\widehat{ebe'}$ are copies of L included in K, and a third copy is missing. This is, by Theorem 3.2 (a), impossible, and the claim is proven.

Now, the conclusion follows from Theorem 5.2. □

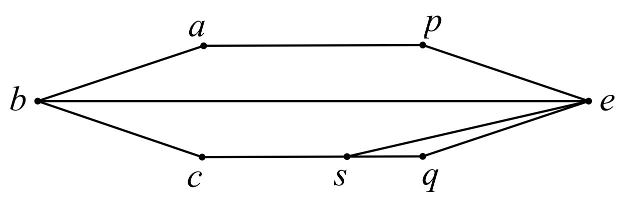


Figure 5.1: A grateful hexagon *abcqep*.

For example, the hexagon *abcqep* of Figure 5.1 is grateful, as the hexagon *abcsep* is generous to it.

Question 5.11. Is Theorem 5.10 more generally true for any convex polygon with some acute angle?

Acknowledgements

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