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EXTENDING AND PARALLELING STECHKIN'S CATEGORY THEOREM

Tudor Zamfirescu

Abstract. We strengthen one of Stechkin's theorems. We also obtain results in the same spirit regarding the farthest point mapping. We work in length spaces, sometimes without bifurcating geodesics, sometimes with geodesic extendability.

Let $K \subset \mathbb{R}^d$ be a closed set and p_K the *nearest point mapping*, which associates to every point $x \in \mathbb{R}^d$ the set of all points in K closest to x. Asplund and Stechkin have been the pioneers and founders of the smallness theory for the set of points with unique nearest points from a given compact set. It was already well-known that p_K is single-valued almost everywhere, when Stechkin [14] proved in 1963 that, from the point of view of Baire categories, too, p_K is single-valued at most points of $\mathbb{R}^d$. See also Cobzaş [6] and the surveys of Konyagin [13] and Vlasov [15].

We always say that *most* elements of a Baire space have property $\mathscr{P}$, if those not enjoying $\mathscr{P}$ form a first category set, i.e. a countable union of nowhere dense sets.

We showed in [18] that, in any Alexandrov space with curvature bounded below, p_K is properly multivalued on a σ-porous set (which is in general "smaller" than a set of first Baire category). We also extended Stechkin's result to more general metric spaces in [19]. Theorem 3 from [18] is a generalization of an

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old result of the author [16], which had been repeatedly strengthened in various ways, for example in [7]–[11], [20].

Here we introduce the notion of a *last non-ambiguous point* with respect to (a given) closed set K, and prove that, for most K, most points are such points. Moreover, we extend several results to the farthest point mapping. Analogues of Stechkin's result were proved for the farthest points (in some classes of Banach spaces) by Edelstein [12] and Asplund [1]. Related to our topic is the well-known problem of convexity of the Chebyshev sets and its dual problem.

1. Definitions and notation

Let (X, ρ) be a metric space. For $a, b \in X$, the point $m \in X$ is called *midpoint of* $\{a, b\}$ if $\rho(a, m) = \rho(m, b) = \rho(a, b)/2$.

We say that X has *property* Z if the following holds: Any point m is simultaneously midpoint of both $\{a, b\}$ and $\{a, c\}$ only if $b = c$. In length spaces it coincides with the property of not having bifurcating geodesics, as defined in [19]. A *length space* is a metric space (X, ρ) in which any pair of points a, b are joined by at least one path ab of length $\rho(a, b)$, called *segment*. The set of all such paths is denotet by $\Sigma(a, b)$. A length space with property Z will be called a Z*-space*.

Let $\mathcal{K}$ be the space of nonempty compact subsets of X and h the Pompeiu–Hausdorff distance in $\mathcal{K}$, i.e. for $K, K' \in \mathcal{K}$,

$$h(K, K') = \max\Big\{\max_{a\in K} \rho(a, K'), \max_{b\in K'} \rho(b, K)\Big\},$$

where $\rho(x, K) = \min_{z\in K} \rho(x, z)$. The space $(\mathcal{K}, h)$ is complete as soon as (X, ρ) is itself complete. Let $K \in \mathcal{K}$. For $x \in X$, the *nearest point mapping* is defined by

$$p_K(x) = \{y \in K : \rho(x, y) = \rho(x, K)\},$$

and the *farthest point mapping* by

$$F_K(x) = \{y \in K : \rho(x, y) = h(\{x\}, K)\}.$$

Any segment between x and some point of $p_K(x)$ is caled a *segment from* x *to* K.

The open ball of centre x and radius $\varepsilon > 0$ will be denoted by $B(x, \varepsilon)$.

We proved in [19] the following.

THEOREM 1.1. *For any compact set* K *in a complete* Z*-space* (X, ρ)*, the mapping* p_K *is single-valued at most points of* X*.*

With respect to the compact set K in the metric space (X, ρ), the set $M(K)$ of all points $x \in X$ admitting more than one segment from x to K is called the *multijoined locus of* K.

2. Last non-ambiguous points

A point $x \in X$ for which $p_K(x)$ is a single point will be called a *last non-ambiguous point* if x is not interior to any segment from a point of X to K.

At a first glance, the sheer existence of such points seems questionable. It appears that the condition on y yields p_K to be properly multi-valued at x, or yields at least the existence of two segments from x to K. This is, however, wrong, and Theorem 1 will show that we can have, we even usually have, many last non-ambiguous points.

The following lemma is Theorem 2 in [17].

Lemma 2.1. *If K is closed and $M(K)$ dense in a complete locally compact length space X, then most points of X are last non-ambiguous points of K.*

A metric space will be called *locally nonlinear* if each open set has Hausdorff dimension larger than 1.

The next lemma is Theorem 3 in [19].

Lemma 2.2. *In a complete separable locally nonlinear Z-space X, for most $K \in \mathcal{K}$, p_K is properly multi-valued at a dense set of points.*

Theorem 2.3. *For most compact sets in the complete separable locally compact locally nonlinear Z-space X, most points of X are last non-ambiguous points.*

Proof. By Lemma 2.2, for most $K \in \mathcal{K}$, the set $M(K)$ is dense in X. By Lemma 2.1, most points of X are last non-ambiguous points. $\square$

3. On the farthest point mapping in Z-spaces

There exist many papers about farthest points in various Banach spaces, for example [3], [2], [4].

We say that a length space has *extendable segments* if, for any segment xy, there exists a point $y' \neq y$, and a segment xy', such that $xy' \supset xy$.

We now prove an analogue to Theorem 1.1 for the farthest point mapping.

Theorem 3.1. *For any compact set K in a complete Z-space (X, ρ) with extendable segments, the mapping F_K is single-valued at most points of X.*

Proof. Suppose the conclusion of the theorem is false. Then F_K is not single-valued on a set of second category, which means that $\bigcup_{n=1}^{\infty} A_n$ is of second category, where

$$A_n = \{x \in X : \operatorname{diam} F_K(x) \geq 1/n\}.$$

Thus, for some index n, A_n is dense in a ball $B(a, r)$.

Let $y \in F_K(a)$. Consider a segment ay and some point $x \in B(a,r)$, such that $ay \subset xy$. Let $y' \in F_K(x)$. Since $\rho(a,y') \le \rho(a,y)$, $\rho(x,y') \le \rho(x,a) + \rho(a,y')$, and

$$\rho(a,y') \ge \rho(x,y') - \rho(x,a) \ge \rho(x,y) - \rho(x,a) = \rho(a,y),$$

we must have everywhere the equality sign, and the segments $xa \cup ay$ and $xa \cup ay'$ would bifurcate, if $y \ne y'$. It follows that $F_K(x) = \{y\}$.

Since F_K is upper semi-continuous (see Blatter [5]), there is some neighbourhood N of x such that $F_K(u) \subset B(y, 1/(3n))$ for all $u \in N$. Then $\operatorname{diam} F_K(u) < 1/n$ for all these u, whence A_n is not dense in $B(a,r)$, and a contradiction is obtained. □

Theorem 3.2. *For most compact sets in a separable complete locally nonlinear Z-space, the farthest point mapping is properly multi-valued at a dense set of points.*

Proof. Let $B(x_0, \varepsilon)$ be an open ball in the given metric space X. We prove that the compact sets $K \subset X$, for which F_K is single-valued on $B(x_0,\varepsilon)$, form a nowhere dense set. Indeed, in any open set $\mathcal{O} \subset K$, there exists a compact set K not containing x_0.

Let $y_0 \in F_K(x_0)$. For $\eta > 0$, such that $\eta < \rho(x_0, K)/2$ and $\eta < \varepsilon$, let

$$K_\eta = \{z \in \sigma : \sigma \in \Sigma(x_0, y),\ y \in K,\ \rho(y,z) = \eta\}.$$

We have $h(K, K_\eta) < \eta$.

Take y_1 on a segment σ_0 from x_0 to y_0, so that $\rho(y_0, y_1) = \eta/2$. The whole ball $B(y_1, \eta/4)$ is disjoint from K_η, and for any finite set $F \subset B(y_1, \eta/4)$, still $h(K, K_\eta \cup F) < \eta$, and so, for η small enough, $K_\eta \cup F$ lies in $\mathcal{O}$.

Since $\dim B(y_1, \eta/4) > 1$, we can choose $y_2 \in B(y_1, \eta/4) \setminus \sigma_0$. Consider the point $y_3 \in \sigma_0$ with $\rho(x_0, y_2) = \rho(x_0, y_3)$. Then $y_3 \in B(y_1, \eta/4)$ too. Let σ_2, σ_3 be segments from x_0 to y_2, y_3, respectively. Since X has property Z, $\sigma_3 \subset \sigma_0$.

We now choose the points $x_2 \in \sigma_2$, $x_3 \in \sigma_3$ such that $\rho(x_0, x_2) = \rho(x_0, x_3) < \eta$. Property Z yields $\rho(x_2, y_3) > \rho(x_2, y_2)$ and $\rho(x_3, y_2) > \rho(x_3, y_3)$. Let

$$\nu < \min\{\rho(x_2, y_3) - \rho(x_2, y_2),\ \rho(x_3, y_2) - \rho(x_3, y_3)\}.$$

If $h(K', K_\eta \cup \{y_2, y_3\}) < \nu/2$ in $\mathcal{K}$, then K' meets both $B(y_2, \nu/2)$ and $B(y_3, \nu/2)$. Therefore the point of K' farthest from x_2 lies in $B(y_3, \nu/2)$ and the point of K' farthest from x_3 lies in $B(y_2, \nu/2)$.

The function

$$f(x) = h(\{x\}, K' \cap B(y_2, \nu/2)) - h(\{x\}, K' \cap B(y_3, \nu/2))$$

is continuous; moreover, $f(x_2) < 0$ and $f(x_3) > 0$. Therefore, there exists a point $x \in x_2x_0 \cup x_0x_3$ with $f(x) = 0$, which yields $\operatorname{card} F_{K'}(x) > 1$. Hence, the set $\mathcal{K}_{m,n} \subset \mathcal{K}$ of all compact sets K, for which F_K is single-valued on $B(x_m, 1/n)$, is nowhere dense. Since X is separable, $\{x_m\}_{m=1}^{\infty}$ can be chosen to be dense in X, and then the set $\bigcup_{m,n=1}^{\infty} \mathcal{K}_{m,n}$ of all compact sets K with F_K single-valued on any non-degenerate ball is of first Baire category. □

4. On the farthest point mapping in absence of property Z

Without assuming that X has property Z, Theorem 3.1 is not valid. Take, for example, X to be the union of the two coordinate axes in $\mathbb{R}^2$, and take K to be the set $\{(1,0),(0,1)\}$. Then F_K is not single-valued at all points of X with non-positive coordinates.

As in the case of the nearest point mapping, we can only prove the following theorem.

THEOREM 4.1. *For most compact sets K in a complete separable length space (X,ρ), $F_K(x)$ is single-valued at most points $x \in X$.*

PROOF. Fix an open ball $B(x_0,\varepsilon) \subset X$. We claim that the set $\mathcal{K}_n$ of all compact sets $K \subset X$ for which $\{x : \operatorname{diam} p_K(x) \geq 1/n\}$ is dense in $B(x_0,\varepsilon)$ is nowhere dense in $\mathcal{K}$.

We use the beginning of the preceding proof, up to (and including) the inequality $h(K, K_\eta \cup F) < \eta$, and the remark that $K_\eta \cup F$ lies in $\mathcal{O}$, for η small enough. We now choose $F = \{y_1\}$. Clearly, $F_{K_\eta \cup \{y_1\}}(x_0) = \{y_1\}$. If $K' \in \mathcal{K}$ is close enough to $K_\eta \cup \{y_1\}$ and x close enough to x_0, then

$$\operatorname{diam} F_{K'}(x) < \frac{1}{n}.$$

So, our claim is proved.

Let K be such that F_K is not single-valued on a set of second category. Then

$$\{x : \operatorname{diam} p_K(x) > 0\} = \bigcup_{n=1}^{\infty} A_n$$

is of second category. (We recall that $A_n = \{x \in X : \operatorname{diam} F_K(x) \geq 1/n\}$.) This implies that, for some n, the set A_n is dense in some ball.

Since X is separable, there is a countable set $\{x_i\}_{i=1}^{\infty}$ dense in X. Hence A_n is dense in some ball $B(x_i, r_j)$ with rational radius r_j. Hence all compact sets K for which p_K is not single-valued on a set of second category belong to the union $\bigcup_{n,i,j=1}^{\infty} \mathcal{K}_{n,i,j}$, where $\mathcal{K}_{n,i,j}$ is the set of those $K \in \mathcal{K}$ for which A_n is dense in $B(x_i, r_j)$.

Since we showed that each $\mathcal{K}_{n,i,j}$ is nowhere dense, the union above is of first category and the theorem is proved. □

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