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Zero forcing density of Archimedean tiling graphs

by
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Warmly dedicated to the old friend Ionică at his reaching of eight decades of very successful, happy life

Abstract

This paper mainly discusses the zero forcing density of infinite graphs. When G is an infinite graph, arrange all distinct finite subgraphs of G in a sequence $\{G_n\}$. The *zero forcing density* of G is defined by $\rho_G = \liminf_{n\to\infty} \frac{Z(G_n)}{|V_{G_n}|}$, where $Z(G_n)$ is the zero forcing number of G_n. When $\rho_G = \liminf_{n\to\infty} \frac{Z(G_n)}{|V_{G_n}|} = 0$, then we define the *second density* as $\rho'_G = \liminf_{n\to\infty} \frac{Z(G_n)}{\sqrt{|V_{G_n}|}}$. Considering the eleven Archimedean tiling graphs, we get upper bounds of zero forcing density of the tilings $(3^4, 6)$, $(3^2, 4, 3, 4)$, $(4, 8^2)$, $(3, 6, 3, 6)$, $(3, 12^2)$. The zero forcing density of the other six graphs is 0. Then we obtain upper bounds of the second density of these six Archimedean tiling graphs.

Key Words: Zero forcing density, zero forcing set, Archimedean tiling.
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1 Introduction

The concepts of zero forcing set and number were first introduced in [1]. One of the primary aims of these concepts is to study the minimum rank of a graph.

Let G be a finite graph, the vertices of which are coloured black or white.

Definition 1. *Colour-change rule: If x is a black vertex and exactly one neighbour y of x is white, then change the colour of y into black.*

Definition 2. *Given an initial colouring of G, if after repeatedly applying the colour-change rule all vertices become black, we say that the set of initial black vertices is a zero forcing set of G.*

Definition 3. *If U is a zero forcing set of G with the minimum number of elements, then $|U|$ is the zero forcing number $Z(G)$ of G.*

Consider, for example, a path P of length 4, see Figure 1. Initially, let a be a black vertex and the other vertices white. After repeatedly applying the colour-change rule, all vertices become black. So $\{a\}$ is a zero forcing set of the path P, as shown in Figure 1 (a). If, initially, b and c are black vertices, and the other vertices are white, after the repeated application of the colour-change rule, again all vertices become black. So, the set of vertices

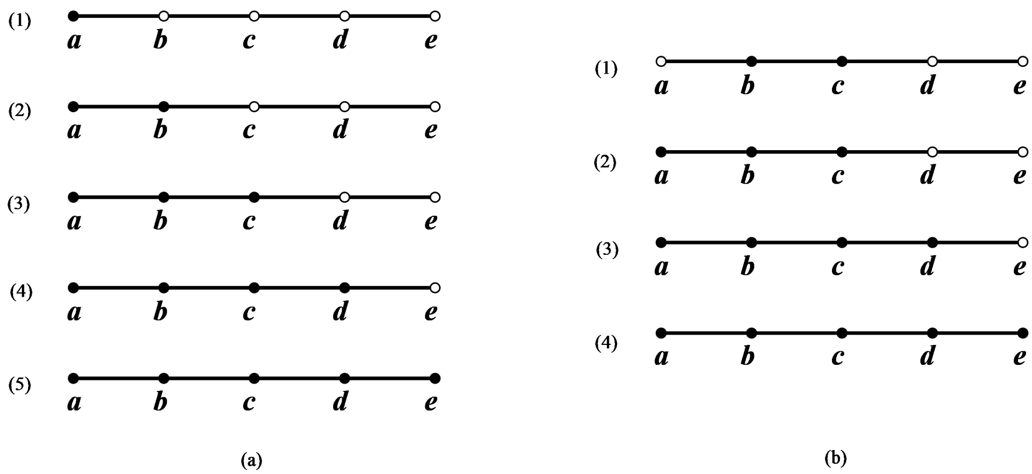

Figure 1: The colour-change of a path.

$\{b,\ c\}$ is also a zero forcing set of P, see Figure 1 (b). Zero forcing sets of the same graph are not unique, and the zero forcing number of P is 1.

Let B be a symmetric matrix in $\mathbb{R}^{n\times n}$. The graph of B, denoted by $\mathcal{G}(B)$, is the graph with vertex set $\{1, 2..., n\}$ and edge set $\{ij : b_{ij} \neq 0 \text{ for } 1 \leq i < j \leq n\}$. Let

$$\mathcal{S}(G) = \left\{B \in \mathbb{R}^{n\times n} : \mathcal{G}(B) = G\right\}.$$

The *minimum rank* and the *maximum nullity* of a graph G with order n are defined, respectively, as:

$$mr(G) = \min\{rank(B) : B \in \mathcal{S}(G)\} \text{ and } M(G) = \max\{nul(B) : B \in \mathcal{S}(G)\},$$

where $nul(B)$ is the dimension of the null space of the matrix B. Clearly, $mr(G)+M(G) = n$. For a graph G, it is shown in [1] that $M(G) \leq Z(G)$.

For any graph G, V_G denotes its set of vertices and E_G its set of edges. $\delta := \min\{d(v) : v \in V_G\}$ is the *minimum degree* of G, and $\Delta := \max\{d(v) : v \in V_G\}$ is its *maximum degree.* The *girth* g of a graph G is the minimum of the length of the cycles in G. A vertex of degree 1 is a *pendant vertex*, whereas a vertex of degree at least 3 is a *major vertex.* Let $p(G)$ denote the number of pendant vertices in G. Given a graph G, a pendant vertex u is said to be a *terminal vertex* of a major vertex x of G if $d(u, x) < d(u, y)$ for all other major vertices y of G. The *terminal degree ter(x)* of a major vertex x is the number of terminal vertices of x.

Davila and Kenter [4] conjectured that $Z(G) \geq (g-3)(\delta-2)+\delta$ for a graph with girth $g \geq 3$ and minimum degree $\delta \geq 2$. Gentner et al. [7] showed that $Z(G) \geq 2\delta - 2$ for every triangle-free graph with minimum degree $\delta \geq 2$. Later, Davila and Kenter's conjecture was proved to be true for all graphs with girth $g \in \{5,\ 6,\ 7,\ 8,\ 9,\ 10\}$ and minimum degree $\delta \geq 2$ (see Davila and Henning [5], Gentner and Rautenbach [8]). Using techniques in [5], in 2018 Davila et al. [6] provided a proof for all graphs with $g \geq 5,\ \delta \geq 2$.

Amos et al. [2] showed that

$$Z(G) \leq \frac{\Delta n}{\Delta + 1}$$

if G is a graph with $\delta \geq 1$; if G is a connected graph with more than two vertices, then

$$Z(G) \leq \frac{(\Delta - 2)n}{\Delta - 1}.$$

The case of equality was characterized, independently, by Gentner et al. [7] and Lu et al. [11]. Caro and Pepper [3] obtained

$$Z(G) \leq \frac{(\Delta - 2)n - (\Delta - \delta) + 2}{\Delta - 1},$$

where G is a connected non-regular graph with $\Delta \geq 2$. This gives a better bound for graphs with $\Delta - \delta \geq 3$.

A major vertex x is an *exterior major vertex* if $ter(x) > 0$. The number of exterior major vertices of G is denoted by $ex(G)$. For a graph G, let $\Phi(G) = |E_G| - |V_G| + c(G)$ be the *cyclomatic number* of G, where $c(G)$ is the number of connected components of G.

Wang et al. [12] showed $Z(G) \in [\, p(G) - ex(G) - 1,\ p(G) + 2\Phi(G)]$, and characterized graphs satisfying $Z(G) = p(G) - ex(G)$ and $Z(G) = p(G) + 2\Phi(G) - 1$ respectively. Later, Li et al. [10] characterized graphs satisfying $Z(G) = p(G) - ex(G) + 1$ and $Z(G) = p(G) + 2\Phi(G) - 2$ respectively.

Definition 4. *If G is a finite graph, then the zero forcing density of G is defined as*

$$\rho_G = \frac{Z(G)}{|V_G|}.$$

If G is an infinite graph, arrange all distinct finite subgraphs of G in a sequence $\{G_n\}$. Then, the zero forcing density of G is defined as

$$\rho_G = \liminf_{n\to\infty} \frac{Z(G_n)}{|V_{G_n}|}.$$

When $\rho_G = \liminf_{n\to\infty} \frac{Z(G_n)}{|V_{G_n}|} = 0$, then the second density is defined as

$$\rho'_G = \liminf_{n\to\infty} \frac{Z(G_n)}{\sqrt{|V_{G_n}|}}.$$

To explore the zero forcing density of infinite graphs, we start with Archimedean tiling graphs. They are divided into monohedral tilings, dihedral tilings and trihedral tilings according to the number of non-congruent tiles [9]. Considering finite subgraphs of a tiling graph, we find for them zero forcing sets, and then we get the limit of the ratio between the number of vertices in zero forcing sets and the total number of vertices. So we obtain an upper bound of the zero forcing density of the tiling graph.

2 Zero forcing density of monohedral tiling graphs

Theorem 1. *The zero forcing density of the* (4^4) *tiling is* 0. *Its second density is bounded above by* $2\sqrt{2}$.

Proof. The degree of each vertex in the (4^4) tiling is 4. First, pick a vertex u and make it black. Then consider the four adjacent vertices of u, and make three of them black and the remaining one white, as shown in Figure 2 (1). At this time, the set of black vertices is a zero forcing set of the graph in Figure 2 (1). Consider the adjacent vertices of those vertices on the black dotted line in Figure 2 (1), and the vertices on the black dotted line in Figure 2 (2) are obtained. The set of black and gray vertices forms a zero forcing set in Figure 2 (2), where the set of black vertices is the zero forcing set in Figure 2 (1), and the set of gray vertices is added to it. By analogy, the vertices in Figure 2 (n) are obtained by adding the adjacent vertices of the black dotted line in Figure 2 $(n-1)$. The additional vertices of the zero forcing set of Figure 2 (n) are these gray vertices in Figure 2 (n).

Then we get a family of finite subgraphs. For any positive integer n, compare the Figure 2 (n) with the Figure 2 $(n-1)$. The additional vertices of the Figure 2 (n) are the vertices on the black dotted line, hence, the total number of vertices increases by $4n$. The number of vertices of the zero forcing set, for any positive integer n, increases by 4. Therefore, we get that the total number of vertices of Figure 2 (n) is $2n^2+2n+1$. The number of elements in the zero forcing set in Figure 2 (n) is $4n$. Thus, the zero forcing density of the (4^4) tiling is 0. But we get

$$\lim_{n\to\infty}\frac{4n}{\sqrt{2n^2+2n+1}}=2\sqrt{2}.$$

Therefore, its second density is bounded above by $2\sqrt{2}$. ∎

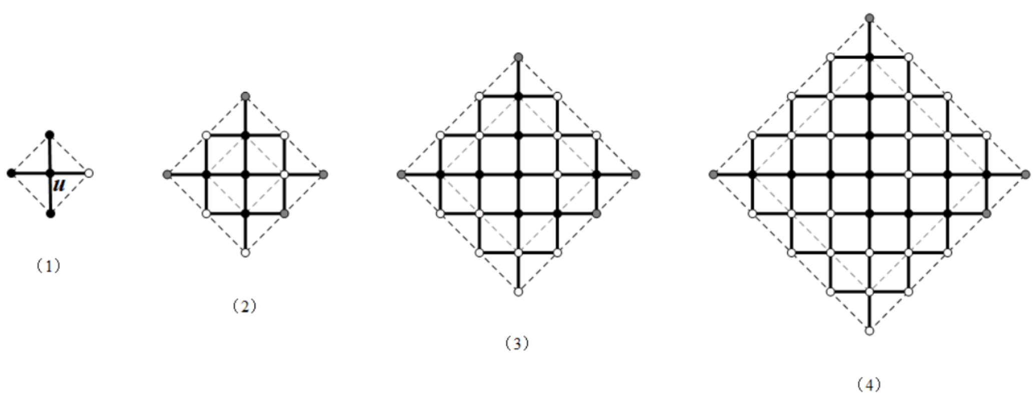

Figure 2: A family of finite subgraphs of the tiling (4^4).

Theorem 2. *The zero forcing density of the* (3^6) *tiling is* 0. *Its second density is at most* $2\sqrt{3}$.

Proof. Considering the tiling of (3^6), we also get a family of finite subgraphs, see Figure 3. The gray shadow part of each graph is the previous graph; the set of black and gray vertices is a zero forcing set, where the set of black vertices is the previous zero forcing set, and the set of gray vertices is added to it. For any positive integer n, compare the Figure 3 (n) with the Figure 3 ($n-1$). The additional vertices are the vertices on the maximal regular hexagon in Figure 3 (n). So, the total number of vertices increases by $6n$; the number of vertices of a zero forcing set increases by 6 in every Figure 3 (n). So the total number of vertices of Figure 3 (n) is $3n^2+3n+1$. The number of vertices in the zero forcing set of Figure 3 (n) is $6n$. Thus, $\rho_{(3^6)}=0$. But

$$\rho'_{(3^6)} \leq \lim_{n\to\infty} \frac{6n}{\sqrt{3n^2+3n+1}} = 2\sqrt{3}.$$

□

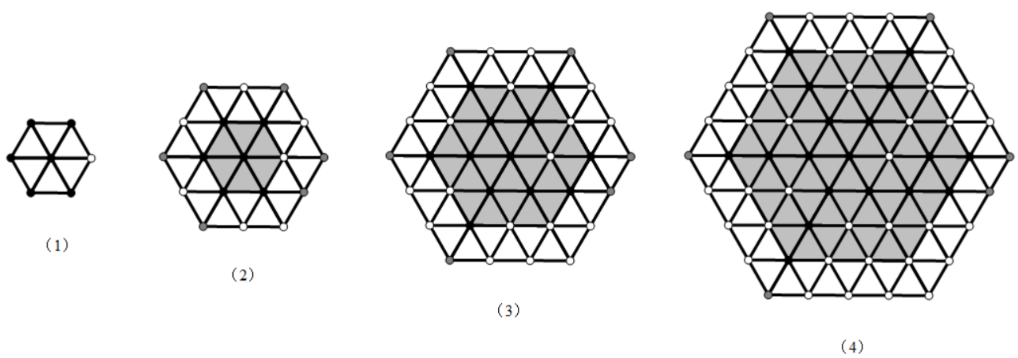

Figure 3: A family of finite subgraphs of the tiling (3^6).

The proofs of the next four theorems are similar. Observe the attached figures.

Theorem 3. *For the* (6^3) *tiling,* $\rho_{(6^3)}=0$ *and* $\rho'_{(6^3)} \leq \sqrt{6}$.

Proof.

$$\rho'_{(6^3)} \leq \lim_{n\to\infty} \frac{6n-9}{\sqrt{6n^2-12n+4}} = \sqrt{6},$$

where $n \geq 3$, see Figure 4. □

3 Zero forcing density of dihedral tiling graphs

Theorem 4. *For the* ($3^4, 6$) *tiling,* $\rho_{(3^4,6)} \leq \frac{1}{6}$.

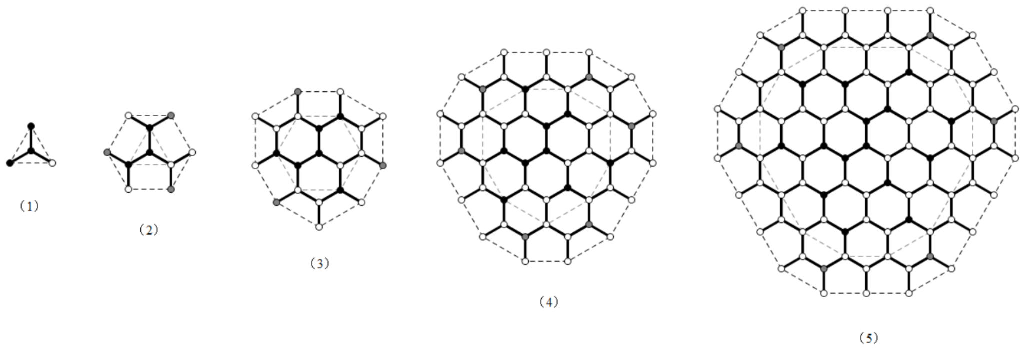

Figure 4: A family of finite subgraphs of the tiling (6^3).

Proof.

$$\rho_{(3^4,6)} \leq \lim_{n\to\infty} \frac{3n^2-7n+10}{18n^2-48n+28} = \frac{1}{6},$$

where $n \geq 3$, see Figure 5. □

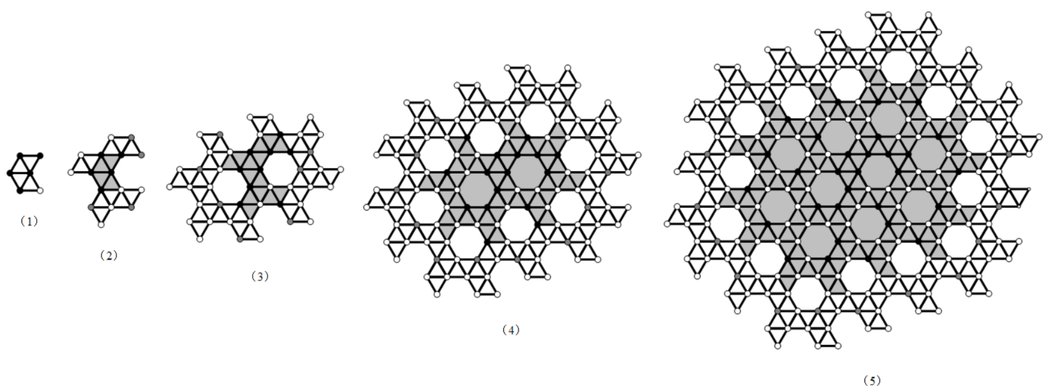

Figure 5: A family of finite subgraphs of the tiling $(3^4, 6)$.

Theorem 5. *An upper bound of the zero forcing density of the* $(3^2, 4, 3, 4)$ *tiling is* $\frac{1}{4}$.

Proof. Consider Figure 6 (n) for $n \geq 3$. We have

$$\rho_{(3^2,4,3,4)} \leq \lim_{n\to\infty} \frac{2n^2-n+3}{8n^2-8n+1} = \frac{1}{4}.$$

□

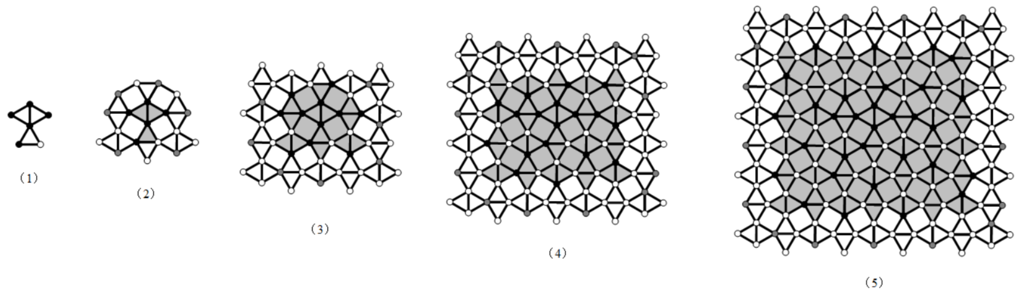

Figure 6: A family of finite subgraphs of the tiling $(3^2, 4, 3, 4)$.

Theorem 6. *The zero forcing density of the* $(4, 8^2)$ *tiling is at most* $\frac{1}{4}$.

Proof. Consider the subgraphs of the tiling $(4, 8^2)$ in Figure 7 (n) for $n \geq 5$. We have

$$\rho_{(4,8^2)} \leq \lim_{n\to\infty} \frac{4n^2 - 24n + 43}{16n^2 - 72n + 80} = \frac{1}{4}.$$

□

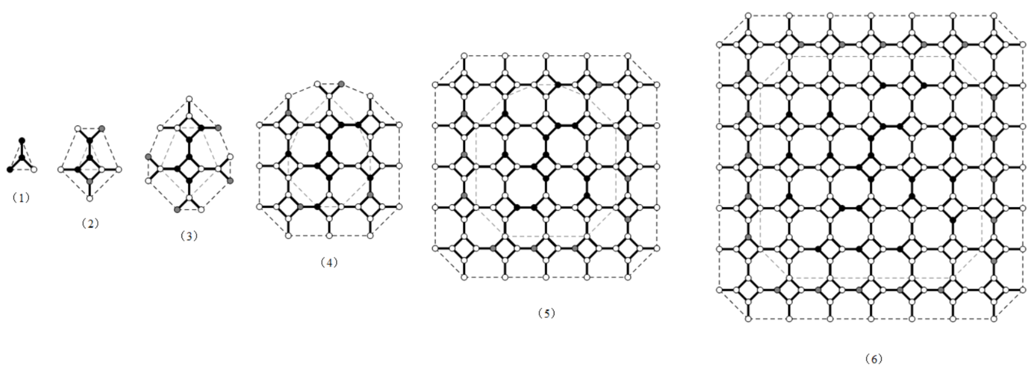

Figure 7: A family of finite subgraphs of the tiling $(4, 8^2)$.

Theorem 7. *The zero forcing density of the* $(3, 6, 3, 6)$ *tiling is at most* $\frac{1}{3}$.

Proof. Considering the $(3, 6, 3, 6)$ tiling, we obtain a family of finite subgraphs, as shown in Figure 8. In each graph, the set of black and gray vertices is a zero forcing set. Now we compute the number of vertices in Figure 8 (n). When $n = 3m$, the number of vertices of the zero forcing set is $9m^2 + 5m$, and the total number of vertices is $27m^2 + 3m - 1$; when $n = 3m + 1$, the number of vertices of the zero forcing set is $9m^2 + 11m + 4$, whereas the

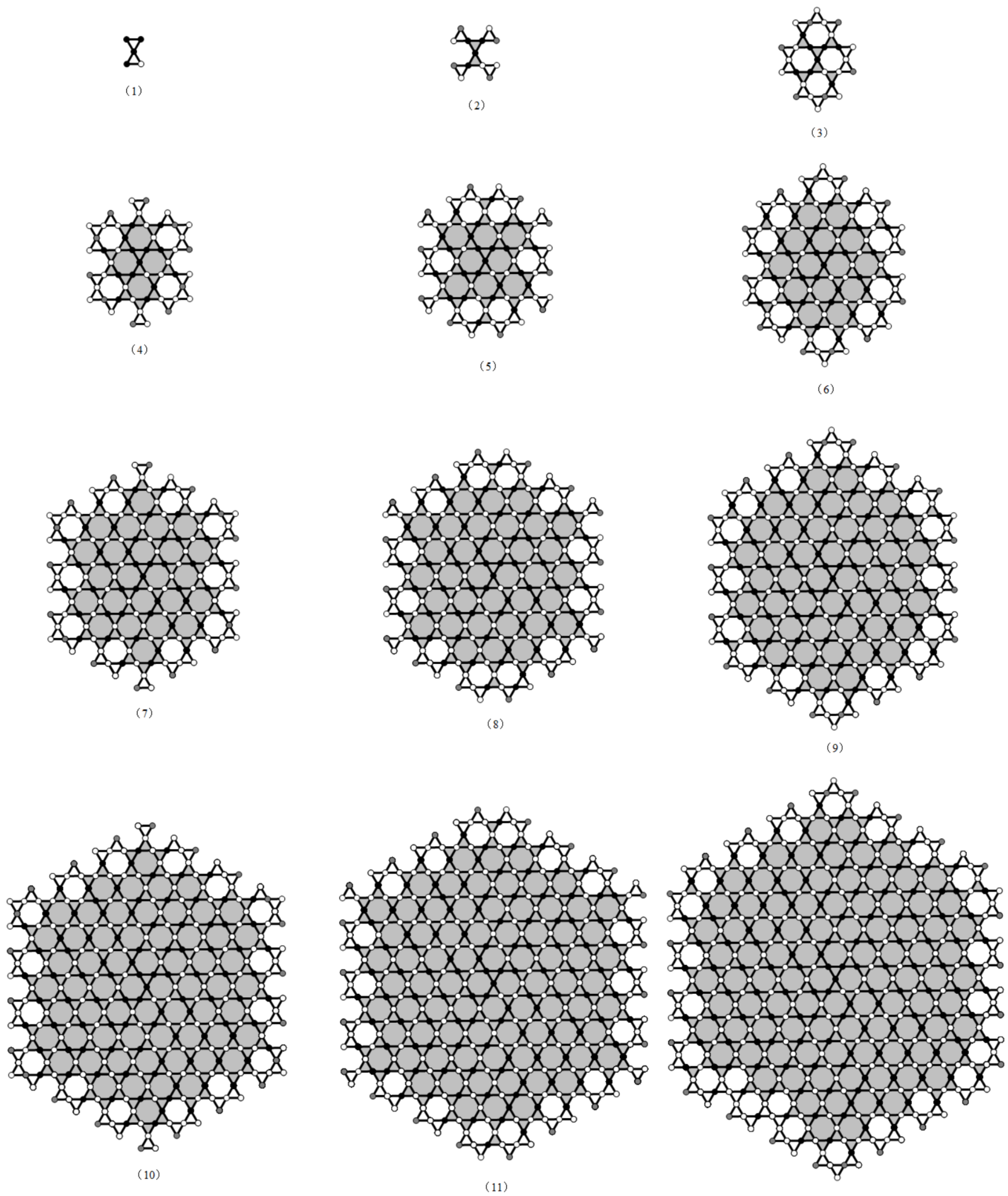

Figure 8: A family of finite subgraphs of the tiling $(3,6,3,6)$.

total number of vertices is $27m^2 + 21m + 5$; when $n = 3m + 2$, the number of vertices of the zero forcing set is $9m^2 + 17m + 8$, and the total number of vertices $27m^2 + 39m + 13$. We have

$$\rho_{(3,6,3,6)} \leq \begin{cases} \lim\limits_{m\to\infty} \dfrac{9m^2+5m}{27m^2+3m-1} = \dfrac{1}{3}; \\ \lim\limits_{m\to\infty} \dfrac{9m^2+11m+4}{27m^2+21m+5} = \dfrac{1}{3}; \\ \lim\limits_{m\to\infty} \dfrac{9m^2+17m+8}{27m^2+39m+13} = \dfrac{1}{3}. \end{cases}$$

□

The next two proofs are similar to that of Theorem 7. We observe the attached figures.

Theorem 8. *The zero forcing density of the* $(3, 12^2)$ *tiling is bounded above by* $\frac{1}{6}$.

Proof. See Figure 9. In Figure 9 (n), we have

$$\rho_{(3,12^2)} \leq \begin{cases} \lim\limits_{m\to\infty} \dfrac{9m^2+3m}{54m^2-6m} = \dfrac{1}{6}, \text{ for } n = 3m; \\ \lim\limits_{m\to\infty} \dfrac{9m^2+9m+3}{54m^2+30m+6} = \dfrac{1}{6}, \text{ for } n = 3m+1; \\ \lim\limits_{m\to\infty} \dfrac{9m^2+15m+6}{54m^2+66m+18} = \dfrac{1}{6}, \text{ for } n = 3m+2. \end{cases}$$

□

Theorem 9. *The zero forcing density of the* $(3^3, 4^2)$ *tiling is* 0. *An upper bound of its second density is* $\frac{4\sqrt{6}}{3}$.

Proof. Considering Figure 10 (n), we have

$$\rho'_{(3^3,4^2)} \leq \begin{cases} \lim\limits_{m\to\infty} \dfrac{16m-1}{\sqrt{24m^2+8m}} = \dfrac{4\sqrt{6}}{3}, \text{ if } n = 3m; \\ \lim\limits_{m\to\infty} \dfrac{16m+5}{\sqrt{24m^2+24m+6}} = \dfrac{4\sqrt{6}}{3}, \text{ if } n = 3m+1; \\ \lim\limits_{m\to\infty} \dfrac{16m+10}{\sqrt{24m^2+40m+16}} = \dfrac{4\sqrt{6}}{3}, \text{ if } n = 3m+2. \end{cases}$$

□

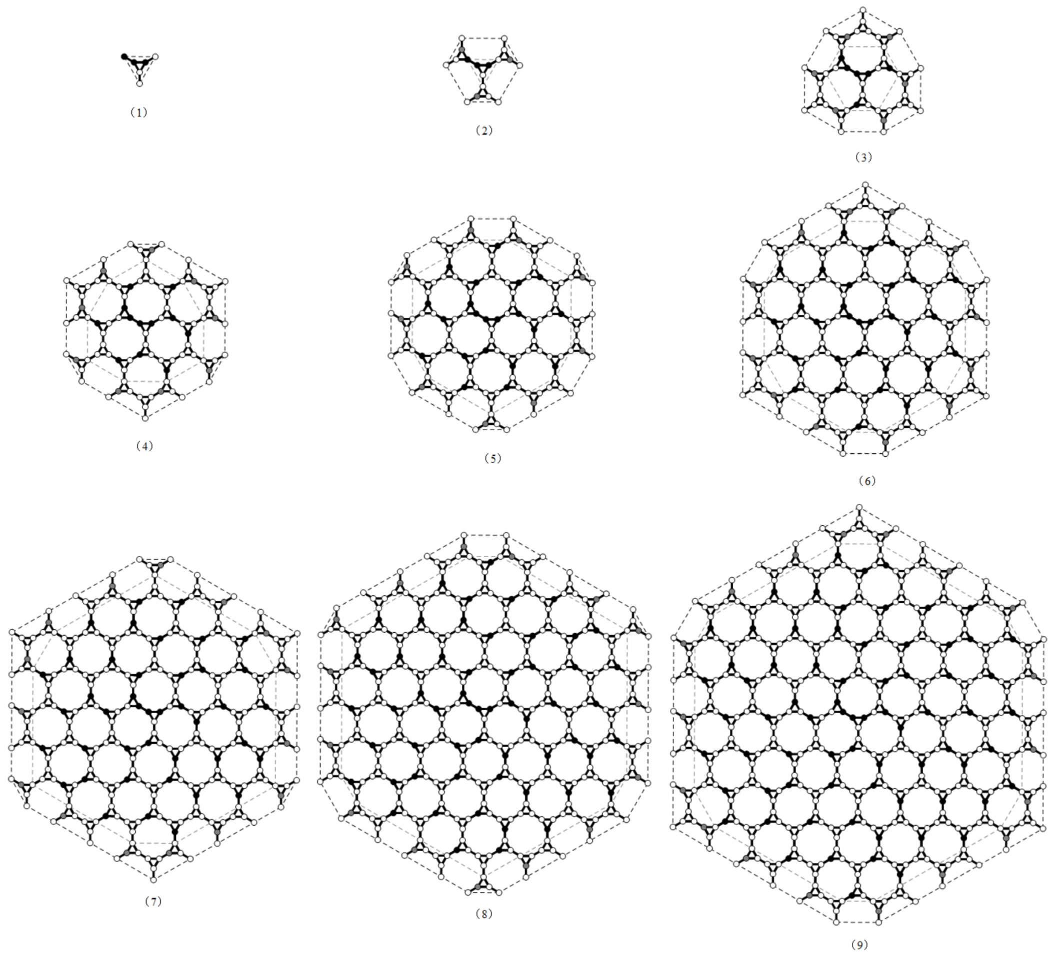

Figure 9: A family of finite subgraphs of the tiling $(3, 12^2)$.

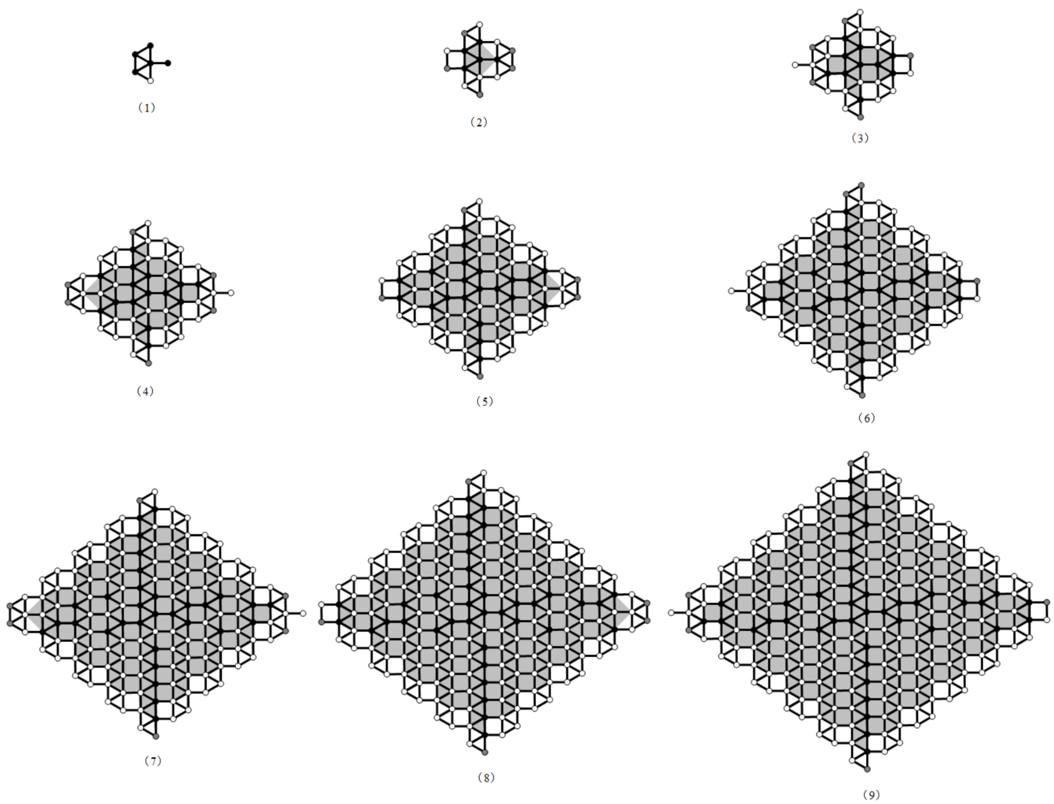

Figure 10: A family of finite subgraphs of the tiling $(3^3, 4^2)$.

4 Zero forcing density of trihedral tiling graphs

Theorem 10. *The zero forcing density of the* $(3, 4, 6, 4)$ *tiling is* 0. *Its second density is bounded above by* $2\sqrt{2}$.

Proof. Observe the family of finite subgraphs of the $(3, 4, 6, 4)$ tiling, as shown in Figure 11. Clearly, the number of vertices of the zero forcing set of Figure 11 (2) is 8. For any positive integer $n \geq 3$, the number of vertices of the zero forcing set of Figure 11 (n) is $6n-4$. When n is odd, the total number of vertices of the Figure 11 (n) is $\frac{9}{2}n^2 - 2n - \frac{3}{2}$; when n is even, that number is $\frac{9}{2}n^2 - n - 1$. Then, the zero forcing density of the $(3, 4, 6, 4)$ tiling is 0. But we have

$$\rho'_{(3,4,6,4)} \leq \lim_{n\to\infty} \frac{6n-4}{\sqrt{\frac{9}{2}n^2 - 2n - \frac{3}{2}}} = \lim_{n\to\infty} \frac{6n-4}{\sqrt{\frac{9}{2}n^2 - n - 1}} = 2\sqrt{2}.$$

□

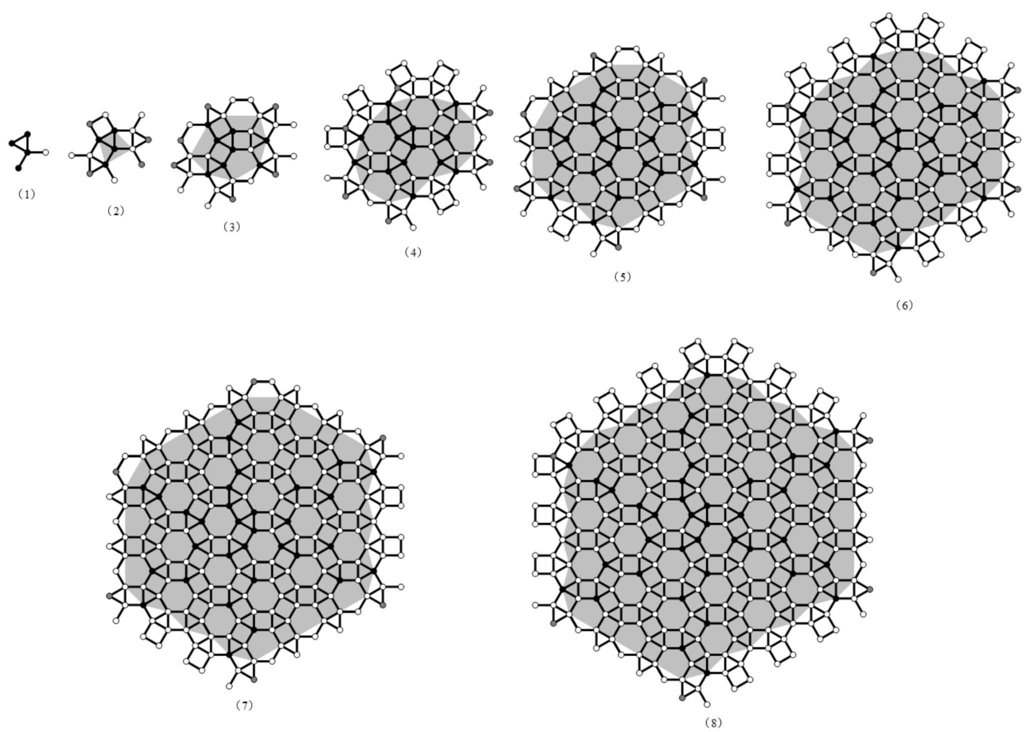

Figure 11: A family of finite subgraphs of the tiling $(3, 4, 6, 4)$.

Theorem 11. *The zero forcing density of the* $(4, 6, 12)$ *tiling is* 0, *whereas its second density is at most* $\frac{2\sqrt{30}}{5}$.

Proof. By a method similar to that used in the proof of Theorem 1, we have

$$\rho'_{(4,6,12)} \leq \lim_{n\to\infty} \frac{12n-32}{\sqrt{30n^2-118n+84}} = \frac{2\sqrt{30}}{5},$$

where $n \geq 4$, see Figure 12. ∎

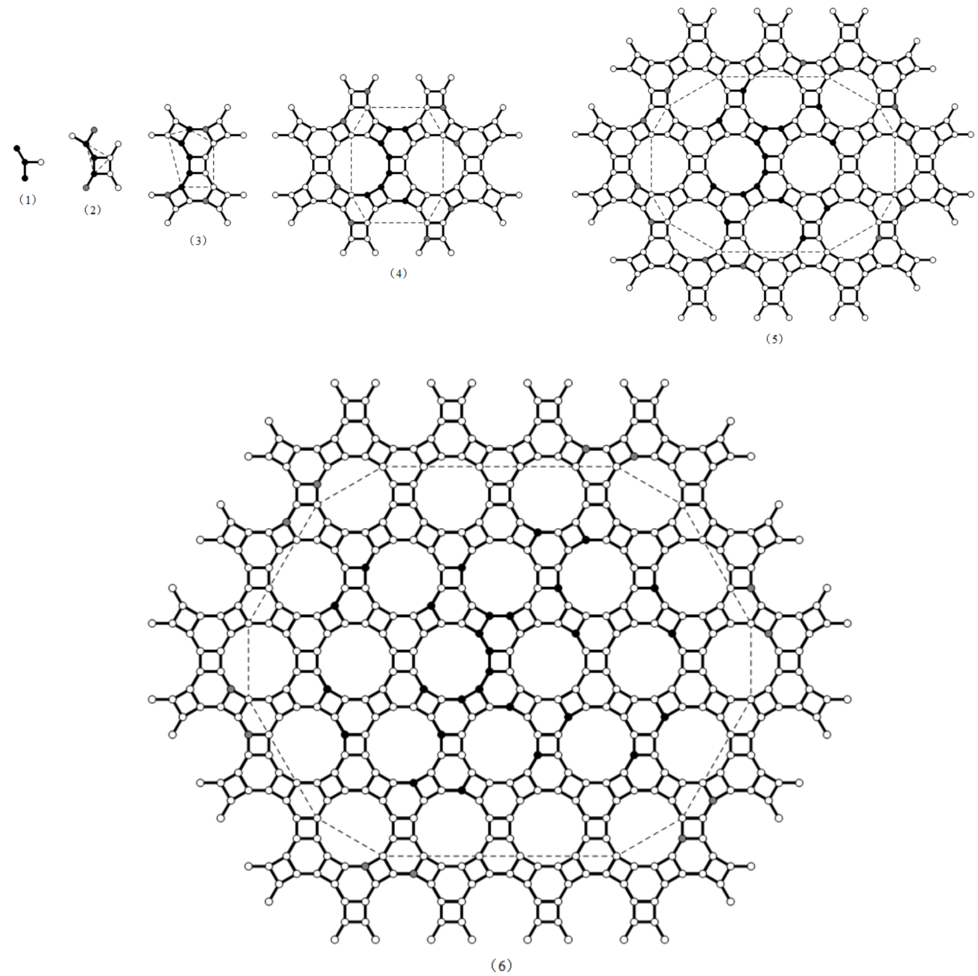

Figure 12: A family of finite subgraphs of the tiling $(4, 6, 12)$.

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