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Thin Right Triangle Convexity

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Abstract: Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ (always $d \geq 2$). A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-*convex*, if for any pair of distinct points $x, y \in M$, there is a set $F \in \mathcal{F}$, such that $x, y \in F$ and $F \subset M$. A *thin right triangle* is the boundary of a non-degenerate right triangle in $\mathbb{R}^2$. The aim of this paper is to introduce and begin investigating the thin right triangle convexity for short *trt*-convexity, which is obtained when $\mathcal{F}$ is the family of all thin right triangles. We investigate the *trt*-convexity of unbounded sets, convex surfaces and planar geometric graphs.

Keywords: thin right triangles; *trt*-convexity; complements

MSC: 52A01; 52A37

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1. Introduction

Let $\mathcal{F}$ be a family of sets in a space $\mathbb{R}^d$ $(d \geq 2)$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x, y \in M$, there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

The investigation of this very general kind of convexity was proposed in 1974 at a meeting on convexity in Oberwolfach by the third author.

Usual convexity, affine linearity, arc-wise connectedness and polygonal connectedness are examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the third author [1], and Böröczky Jr. [2] investigated the rectangular convexity, the case when $\mathcal{F}$ contains all non-degenerate rectangles. The last two authors [5] presented a discretization of rectangular convexity, the right quadruple convexity (for short, *rq*-convexity), where $\mathcal{F}$ is the family of all rectangular quadruples, which are vertex sets of rectangles. They are also currently studying another generalization, the thin rectangular convexity, by taking all thin rectangles (boundaries of rectangles) as $\mathcal{F}$.

It became clear that these generalizations revealed many interesting families of sets, leading far beyond the horizon of convexity.

In [6], the third author studied right convexity, the case when $\mathcal{F}$ consists of all right triangles, and the last two authors [4] investigated the right triple convexity for short *rt*-convexity, where $\mathcal{F}$ contains all triples $\{x, y, z\} \subset \mathbb{R}^d$ with $\angle xyz = \pi/2$.

Wang and the last two authors [3], generalizing in another way, investigated the poidge-convexity (the case with $\mathcal{F}$ consisting of all unions $\{x\} \cup \sigma$, called poidges, where x is a point, σ a line segment, and $\mathrm{conv}(\{x\} \cup \sigma)$ a right triangle).

A *thin right triangle* is the boundary of a non-degenerate right triangle in $\mathbb{R}^2$. The aim of this paper is to introduce and begin investigating the thin right triangle convexity for short *trt*-convexity, which is obtained when $\mathcal{F}$ is the family of all thin right triangles.

Two points in M are said to enjoy the *trt-property in M*, if they belong to some thin right triangle included in M. Thus, M is *trt-convex* if any pair of its points enjoy the *trt*-property. Obviously, *trt*-convexity implies *rt*-convexity.

For convex sets, right convexity, *rt*-convexity, poidge-convexity and *trt*-convexity are equivalent. Not every poidge-convex set is *trt*-convex, not even *rt*-convex, but every rightly convex set is *rt*-convexity, poidge-convex and *trt*-convexity.

In order to obtain results describing *trt*-convex sets different from those provided by the initial study of right convexity, we must focus on sets which are essentially non-convex.

2. Definitions and Notation

For $M \subset \mathbb{R}^d$, we denote by $\complement M$ its complement, by $\overline{M} = \text{aff}M$ its affine hull and by conv$M$ its convex hull; further, int$M$, cl$M$ and bd$M$ denote its topological interior, closure and boundary, respectively, considered in aff$M$.

For distinct $x, y \in \mathbb{R}^d$, let $\overline{xy}$ be the line through x, y, xy the line segment from x to y, and H_{xy} the hyperplane through x orthogonal to $\overline{xy}$. If L, $L^{'}$ are affine subspaces of $\mathbb{R}^d$, $L \parallel L^{'}$ means that they are parallel and $L \perp L^{'}$ that they are orthogonal. In addition, $\pi_L(x)$ denotes the orthogonal projection of x onto L.

For any compact set $C \subset \mathbb{R}^d$, let S_C be the smallest hypersphere containing C in its convex hull.

Let M be a closed convex set in $\mathbb{R}^d$ ($d \geq 2$). A point x in M is called an *extreme point* of M if there exists no non-degenerate line segment in M that contains x in its relative interior. The set of extreme points of M is denoted by ext$M$.

A *convex body* is a compact convex set in $\mathbb{R}^d$ with a non-empty interior. The intersection of a convex body with a supporting hyperplane is called a *face*. A *facet* is a face of dimension $d - 1$.

Let h denote the Pompeiu–Hausdorff distance (also called Hausdorff distance) between compact sets.

For $a_1, \cdots, a_n \in \mathbb{R}^d$, put $a_1 \cdots a_n = \text{conv}\{a_1, \cdots, a_n\}$.

3. Unbounded *trt*-Convex Sets

We now investigate the *trt*-convexity of unbounded sets in $\mathbb{R}^d$.

Theorem 1. *The complement of a connected bounded set $M \subset \mathbb{R}^2$ which equals* intcl$M$ *is trt-convex if and only if* cl$M$ *is a right or an acute triangle.*

Proof. The "if" part is obvious.

Suppose $\complement M$ is *trt*-convex. Let $Q = \text{conv cl}M$.

First, we show that bd$Q \subset$ bd$M$. If the inclusion is not true, we choose $x \in (\text{bd}Q) \setminus \text{bd}M$. There are $y, z \in \text{bd}M$ such that x lies in the line segment yz, and $\overline{yz}$ is a supporting line of Q. The line H_{xy} is orthogonal to yz and cuts bd$M$ in at least two points. Let uv be the longest line segment with $u, v \in H_{xy} \cap \text{bd}M$. Then any thin right triangle containing u, v meets M, contradiction.

Now we prove that M is convex. Indeed, otherwise we choose $x \in (\text{int}Q) \setminus M$, $y \in M$. The line $\overline{xy}$ cuts bd$Q$ in two points, u, v. Assume that the order on $\overline{xy}$ is u, x, y, v. Then, x, v do not enjoy the *trt*-property in $\complement M$.

We claim that bd$M$ does not contain parallel non-degenerate line segments.

Indeed, if bd$M$ contains such line segments, then there is no thin right triangle in $\complement M$ containing their midpoints.

Let L_1, L_2, L_3, L_4, L_5 be supporting lines of cl$M$, parallel to the sides of a regular pentagon. Choose $a_i \in L_i \cap \text{cl}M$ ($i = 1, \cdots, 5$). Let A_1 be the arc in bd$M$ from a_1 to a_2 not containing a_3. If $A_1 \neq a_1a_2$, for any $b \in A_1$ distinct from a_1, a_2, $\angle a_1ba_2 \geq 3\pi/5$. Let $c_i \in A_1$ lie between a_i and b ($i = 1, 2$), such that a supporting line at c_i be parallel to $\overline{a_ib}$. Obviously, c_1, c_2 do not enjoy the *trt*-property in $\complement M$. Hence, cl$M = a_1a_2a_3a_4a_5$, where some of the a_i's may not be distinct.

Assume that an angle of the polygon cl$M$ is obtuse. Then, the midpoints of its sides do not enjoy the *trt*-property in $\complement M$. Hence, cl$M$ has no obtuse angle and must consequently be a rectangle or a non-obtuse triangle. As bd$M$ contains no parallel line segments, the proof is finished. □

Theorem 2. *The complement of a connected bounded open set M in $\mathbb{R}^d$, where $d \geq 3$, is trt-convex, if for every $x, y \in \mathrm{bd}M$, there exist two non-parallel hyperplanes H_x, H_y disjoint from M, with $x \in H_x$, $y \in H_y$.*

Proof. We claim that M is convex.

Indeed, otherwise there exist two points $u, v \in M$ such that $uv \cap \mathrm{bd}M \neq \emptyset$. Choose $w \in uv \cap \mathrm{bd}M$. Since $u \notin H_w$, H_w separates u from v. So $H_w \cap M \neq \emptyset$, a contradiction.

Now, we choose $x, y \in \complement M$.

If $x, y \in \mathrm{bd}M$, consider the two non-parallel hyperplanes H_x, H_y at x and y, respectively. Take a line $L \subset H_x \cap H_y$. Choose $z \in L$, such that $\angle xzy < \pi/2$. There exist $x' \in \overline{zx}$, $y' \in \overline{zy}$ far away, yielding a right triangle $zx'y'$ with $x \in zx'$, $y \in zy'$ and $x'y' \cap M = \emptyset$. Then, $x'y'$, $x'z$, $y'z$ form a thin right triangle containing x, y.

If $x \in \mathrm{bd}M$, $y \notin \mathrm{bd}M$, consider the point $z \in \mathrm{cl}M$ closest from y. Let $H_x \ni x$ and $H_z \ni z$ be the hyperplanes given by the statement. Let $H \ni y$ be parallel to H_z. It follows that H_x and H are not parallel. The proof continues as before. We proceed similarly if $x, y \notin \mathrm{bd}M$. □

The connectedness condition in Theorem 2 is in fact not necessary. We prove the following strengthening.

Theorem 3. *The complement of a bounded open set M in $\mathbb{R}^d$, with at most two components and $d \geq 3$, is trt-convex, if for every $x, y \in \mathrm{bd}M$, there exist two non-parallel hyperplanes H_x, H_y disjoint from M, with $x \in H_x$, $y \in H_y$.*

Proof. For M connected, we apply Theorem 2.

Suppose now that M has the components M_1, M_2. Like in the proof of Theorem 2, it can be shown that both M_1, M_2 are convex.

Consider $x, y \in \complement M$. If $x, y \in \mathrm{bd}M$, we proceed like in the preceding proof.

Assume now $x \notin \mathrm{bd}M$.

Case 1. $xy \cap M = \emptyset$.

Then, close to x a point z can be found such that $\angle xzy = \pi/2$ and $xyz \cap M = \emptyset$.

Case 2. $xy \cap M_1 \neq \emptyset$.

Let $\{u, v\} = xy \cap \mathrm{bd}M_1$. (It is easily seen that this intersection has two points.) By hypothesis, there exist non-parallel hyperplanes $H_u \ni u$, $H_v \ni v$ disjoint from M. Put $H = H_u \cap H_v$.

M_2 can lie in only one of the four half spaces determined by $H_u \cup H_v$ in $\mathbb{R}^d$. This implies that at most one of the sets $\mathrm{conv}(\{x\} \cup H)$, $\mathrm{conv}(\{y\} \cup H)$ meets M_2, say the latter.

Consider an arbitrary line $L \subset H$ and the plane $P = \mathrm{aff}(\{y\} \cup L)$. Clearly, $y \notin Y$, where

$$Y = M_2 \cap \mathrm{conv}(\{y\} \cup H).$$

There exists a half-line $L' \subset L$ such that, for each $z \in L'$, $\overline{yz} \cap M_2 = \emptyset$. Now, choose $z' \in L'$ far away, so that $\angle xz'y < \pi/2$. Another pair of points x', y' can be chosen on $\overline{xz'}$, $\overline{yz'}$, respectively, such that $x \in x'z'$, $y \in y'z'$, $\angle x'y'z' = \pi/2$, and $\mathrm{bd}(x'y'z') \cap M = \emptyset$. □

Remark 1. *The $d \geq 3$ in Theorem 3 is the best possible. When $d = 2$, we may consider $M = M_1 \cup M_2$, where both $\mathrm{cl}M_1$ and $\mathrm{cl}M_2$ are right triangles. However, $x \in \mathrm{bd}M_1$ and $y \in \mathrm{bd}M_2$ do not enjoy the trt-property in $\complement M$. See Figure 1.*

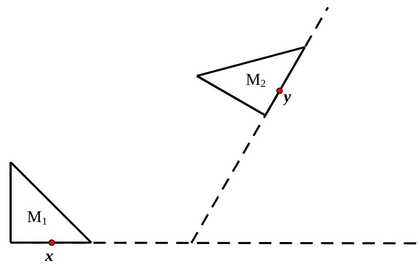

Figure 1. Illustration of Remark 1.

A planar convex set having the union of two half-lines with a common end point as boundary is called a *digon*.

Theorem 4. *The complement of a bounded open set M in* $\mathbb{R}^d$*, with at most three components and* $d \geq 4$*, is trt-convex, if for every* $x, y \in \mathrm{bd}M$*, there exist two non-parallel hyperplanes* H_x, H_y *disjoint from M, with* $x \in H_x,\ y \in H_y$.

Proof. We proceed like in the preceding proof until Case 1, including it.

Case 2. We again take u, v, H_u and H_v as before. The affine subspace $H = H_u \cap H_v$ has dimension $d - 2 \geq 2$. Consider an arbitrary plane $P \subset H$ and the 3-dimensional affine subspaces $E_x = \mathrm{aff}(\{x\} \cup P)$, $E_y = \mathrm{aff}(\{y\} \cup P)$.

It is again true that at most one of $\mathrm{conv}(\{x\} \cup H)$, $\mathrm{conv}(\{y\} \cup H)$ meets M_2, say the first.

If $E_x \cap M_2 \neq \emptyset$ (notice that this may not be the case), consider $K_x = \mathrm{cl}(E_x \cap M_2)$. Let Π be the plane through x parallel to P. If $\Pi \cap K_x \neq \emptyset$ (again, this may not happen), take the two supporting lines $L_1,\ L_2$ of $\Pi \cap K_x$ passing through x. Take $x_1 \in L_1 \cap K_x,\ x_2 \in L_2 \cap K_x$. Let P_{x_i} be the supporting plane of K_x at x_i $(i = 1,\ 2)$.

The set $(P_{x_1} \cap P) \cup (P_{x_2} \cap P)$ includes two half-lines determining a digon Δ_x with the following property. Every half-line starting at x and containing a point of K_x either misses P or meets P inside Δ_x.

We proceed in the same way with M_3. It meets $\mathrm{conv}(\{x\} \cup H)$ or $\mathrm{conv}(\{y\} \cup H)$ or none of them. In the first case, we consider E_x, in the second E_y. Going analogously ahead, we find a digon $\Delta_y \subset P$, such that every half-line from y through a point of K_y either misses P or meets P inside Δ_y.

Now, choose a half-line $L' \subset P \setminus (\Delta_x \cup \Delta_y)$, and continue as in the preceding proof. □

Question 1. *Is Theorem 2 valid without any condition regarding connectedness?*

4. *trt*-Convexity of Convex Surfaces

Can convex surfaces be *trt*-convex?

A tetrahedron in $\mathbb{R}^3$ having a vertex, at which the angles of all three facets are right, will be called *right*.

Theorem 5. *For a convex body* $P \subset \mathbb{R}^3$, $\mathrm{bd}P$ *is trt-convex if and only if P is a right tetrahedron.*

Proof. For the "if" implication, let $\mathbf{0}abc$ be a tetrahedron with all three angles at $\mathbf{0}$ right. We show that $\mathrm{bd}(\mathbf{0}abc)$ is *trt*-convex.

Let $x, y \in \mathrm{bd}(\mathbf{0}abc)$. We have four essentially different situations.

Case 1. $x, y \in \mathbf{0}ab$.

This case follows from the right convexity of any right triangle.

Case 2. $x, y \in abc$.

Similarly, abc being an acute triangle, it must be rightly convex.

Case 3. $x \in \mathbf{0}ab$, $y \in \mathbf{0}bc$.

Assume without loss of generality that y is not closer than x from $\overline{\mathbf{0}ac}$. Then,

$$\overline{\pi_{\overline{\mathbf{0}b}}(x)\, y} \cap \mathrm{bd}(\mathbf{0}bc) \subset bc.$$

Denote by z this intersection. Put $\{u\} = \overline{x\pi_{\overline{\mathbf{0}b}}(x)} \cap ab$. The points u, $\pi_{\overline{\mathbf{0}b}}(x)$ and z are the vertices of a thin right triangle included in $\mathrm{bd}(\mathbf{0}abc)$.

Case 4. $x \in \mathbf{0}ab$, $y \in abc$.

Consider the point u from Case 3 and $\{v\} = \overline{x\pi_{\overline{\mathbf{0}a}}(x)} \cap ab$. Clearly,

$$abc = acv \cup bcu.$$

If $y \in acv$, then $\overline{vy} \cap ac \neq \emptyset$. If $y \in bcu$, then $\overline{uy} \cap bc \neq \emptyset$.

So, if $y \in acv$, then x, y lie in a thin right triangle with vertices v, $\pi_{\overline{\mathbf{0}a}}(x)$, $\overline{vy} \cap ac$, included in $\mathrm{bd}(\mathbf{0}abc)$. Analogously, for $y \in bcu$.

Now, let us prove the "only if" implication.

Let $Q = \mathrm{ext}P$.

Claim 1. If two crossing line segments belong to $\mathrm{bd}P$, then they lie in a facet of P.

Indeed, let uv, xy be the two line segments, and $\{z\} = uv \cap xy$. Being a convex body, P is not included in the plane $\overline{xuz}$, supposed horizontal. Let $s \in P \setminus \overline{xuz}$, below $\overline{xuz}$, say. Take $t \in \mathrm{int}(xuz)$ and assume that $t \in \mathrm{int}(ss')$ for some $s' \in P$. Then, $z \in \mathrm{int}(ss'vy)$, which contradicts $z \in \mathrm{bd}P$. Hence, there exist no such points s', and consequently $xuz \subset \mathrm{bd}P$.

Analogously, $uzy \subset \mathrm{bd}P$, $yzv \subset \mathrm{bd}P$, $vzx \subset \mathrm{bd}P$ and $uxvy$ is a facet of P. Claim 1 is proven.

For any $u, v \in Q$, we have $uv \subset \mathrm{bd}P$. Indeed, otherwise there is no thin right triangle in $\mathrm{bd}P$ containing u, v.

We now prove that P has a facet. Assume that P has no facet.

Consider three extreme points $u_1, u_2, u_3 \in Q$. We have $u_1u_2 \cup u_2u_3 \cup u_3u_1 \subset \mathrm{bd}P$. Put $\overline{u_1u_2u_3} = \Pi$. Then, $u_1u_2u_3 \subset \Pi \cap P$.

Since P contains no facet, Π is not a supporting plane of P. Hence, Π divides P into two parts P_1, P_2.

Choose $v_1 \in Q \cap P_1 \setminus \Pi$, $v_2 \in Q \cap P_2 \setminus \Pi$. We have $v_1v_2 \subset \mathrm{bd}P$.

Consider the non-degenerate polytope $u_1u_2u_3v_1v_2$. By Radon's theorem, either one vertex is in the tetrahedron determined by the other four, or one line segment Σ joining two vertices meets the triangle Δ formed by the other three. The first possibility is excluded, the vertices being in Q. Since $\Sigma \subset \mathrm{bd}P$, $\Sigma \cap \Delta$ avoids $\mathrm{int}P$, whence Σ meets $\mathrm{bd}\Delta$, and Claim 1 provides a facet of P, which contradicts our assumption.

Let E be a facet of P.

Claim 2. $Q \cap \mathrm{bd}E$ is nowhere dense in $\mathrm{bd}E$.

Suppose, in contrast, there exists a non-degenerate arc $A \subset Q \cap \mathrm{bd}E$. Choose $w \in Q \setminus E$ and put $\{x\} = \pi_{\overline{E}}(w)$. For $y, z \in A$ close to each other, but different from x, $\angle ywz$ is small. If $\angle wyz = \pi/2$, then $\angle xyz = \pi/2$ and $\angle wzy < \pi/2$. Then, choose $z' \in A$ close to z, such that still $\angle wz'y < \pi/2$. Since y, z, z' are not collinear, being in Q, $\angle xyz' \neq \pi/2$, whence $\angle wyz' \neq \pi/2$.

Hence, either the triangle wyz is not right, or we find the triangle wyz' which is not right. Obviously, the midpoints of the two long sides do not have the *trt*-property.

Claim 3. There are no disjoint line segments in $\mathrm{bd}E$.

Suppose, on the contrary, S, T are such line segments, assumed maximal (with respect to inclusion). Let S^* be the component of $(\mathrm{bd}P) \setminus \mathrm{bd\,conv}(\{w\} \cup S)$ not containing T, and T^* the component of $(\mathrm{bd}P) \setminus \mathrm{bd\,conv}(\{w\} \cup T)$ not containing S. Since points in S^* and T^* have the *trt*-property, we must have $\mathrm{cl}S^* = \mathrm{conv}(\{w\} \cup S)$ and $\mathrm{cl}T^* = \mathrm{conv}(\{w\} \cup T)$.

Thus, the triangles $wab = \mathrm{cl}S^*$ and $wcd = \mathrm{cl}T^*$ are facets of P.

Now, we easily find $p \in \mathrm{int}(wab)$, $q \in \mathrm{int}(wcd)$, such that $\angle pwq \neq \pi/2$. Put $\{p'\} = \overline{wp} \cap ab$ and $\{q'\} = \overline{wq} \cap cd$. We can arrange the triangle $wp'q'$ not to be right. Indeed, if, for example $\angle wp'q' = \pi/2$, $\overline{wp'} \perp \overline{E}$ or not. In the first case, for any choice of a point $p'' \neq \pi_{\overline{ab}}(q')$ very close to p' on ab, the triangle $wp''q'$ is not right. In the second case, for any choice of a point q'' very close to q on cd, the triangle $wp'q''$ is not right.

Hence, we may suppose the triangle $wp'q'$ not to be right. However, then p and q do not enjoy the *trt*-property. Claim 3 is verified.

From Claims 2 and 3, it follows that E is a triangle ijk. In fact, every facet of P is a triangle. Moreover, wi, wj, wk are edges of P. Suppose wij is not a facet of P. Then, some point $m \in Q$ is separated from k by $\overline{wij}$. However, then mk meets wij, which means that $mk \cap \mathrm{bd}(wij) \neq \emptyset$, as all four points belong to Q and $mk \subset \mathrm{bd}P$. By Claim 1, P has a quadrilateral facet, and a contradiction is found. Therefore, $P = wijk$.

It remains to prove that the tetrahedron P is right.

We call a tetrahedron *abcd quasiright*, if for any pair of opposite edges, such as ab, cd, we have $\overline{ab} \perp \overline{acd}$ or $\overline{ab} \perp \overline{bcd}$ or $\overline{cd} \perp \overline{abc}$ or $\overline{cd} \perp \overline{abd}$. Notice that a right tetrahedron is quasiright. We first show that P is quasiright.

Choose arbitrarily the pair of opposite edges ij and kw of P. Choose $x \in \mathrm{int}(ij)$ and $y \in \mathrm{int}(wk)$. Then, the only triangle boundaries in $\mathrm{bd}P$ containing x, y are $\mathrm{bd}(wxk)$ and $\mathrm{bd}(ijy)$.

If $\overline{ij} \perp \overline{iwk}$ or $\overline{ij} \perp \overline{jwk}$, then the condition for P to be quasiright is fulfilled at ij, kw. Otherwise, $\mathrm{card}(H_{ij} \cap wk) \leq 1$, $\mathrm{card}(H_{ji} \cap wk) \leq 1$ and $\mathrm{card}(S_{ij} \cap wk) \leq 2$, so ijy is not right for any $y \in \mathrm{int}(wk) \setminus (H_{ij} \cup H_{ji} \cup S_{ij})$. Fix such a point y. Since x, y have the *trt*-property in $\mathrm{bd}P$, wxk is right for any $x \in \mathrm{int}(ij)$. So $\angle kwx = \pi/2$ or $\angle wkx = \pi/2$ for all $x \in \mathrm{int}(ij) \setminus S_{wk}$. Suppose without loss of generality $\angle kwx = \pi/2$ and $\angle wkx < \pi/2$. Choose $x' \in \mathrm{int}(ij)$ close to x such that $\angle wkx' < \pi/2$. We also have $\angle kwx' = \pi/2$. So $\overline{wk} \perp \overline{ijw}$. Again, the condition for P to be quasiright is fulfilled at ij, kw.

Hence, P is quasiright. If it is not right, it must look like in Figure 2. However, then, the midpoint of ij and a point close to w on wk do not enjoy the *trt*-property. □

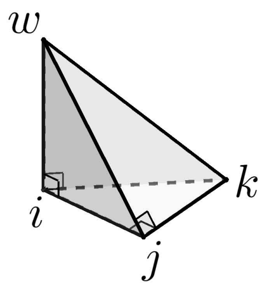

Figure 2. A tetrahedron *wijk*

5. *trt*-Convexity of Planar Geometric Graphs

A *Jordan curve* in $\mathbb{R}^d$ is the image of an injective continuous map of a circle into $\mathbb{R}^d$.

A *Jordan arc* in $\mathbb{R}^d$ is the image of an injective continuous map of an interval $[a,\ b]$ into $\mathbb{R}^d$.

A *geometric graph D* is a set containing finitely many points in $\mathbb{R}^d$ called *vertices*, which form a set $V(D)$, and including the union of finitely many Jordan arcs, each joining two vertices, called *edges*.

A *planar geometric graph* is a geometric graph in $\mathbb{R}^2$, the edges of which have pairwise no points in common, other than vertices of both.

Theorem 6. *A planar geometric graph is trt-convex if and only if it is a thin right triangle or such a triangle plus its height corresponding to the hypotenuse.*

Proof. Let $D \subset \mathbb{R}^2$ be a planar geometric graph. The bounded components of $\mathbb{R}^2 \setminus D$ will be called *regions*.

The "if" part is obvious. Let us prove the other implication.

Clearly, D is connected. Each region F of D has a closed Jordan curve as boundary, consisting of two or more edges.

A broken line with at most three line segments and no obtuse angle will be called a 3-*path*.

Claim 1. Each edge is a 3-path .

Indeed, let E be an edge between the vertices v, w. Take $v' \in E$ close to v and $w' \in E$ close to w. The *trt*-property of v', w' yields the existence of a thin right triangle $T \subset D$, to which v', w' belong. Hence, the subarc $E' \subset E$ from v', w' is included in T. Therefore it is a 3-path . By letting v', w' converge to v, w, respectively, we obtain the existence of a 3-path from v to w in E, whence E is a 3-path .

Let F be a region of D and $E \subset \mathrm{bd}F$ an edge.

Claim 2. If the 3-path E has an angle (measured toward F) $\alpha \neq \pi$ at some point, then $\alpha < \pi$.

Indeed, suppose at some point $y \in E$, the angle is $\alpha > \pi$. Take $x \in (\mathrm{bd}F) \setminus \{y\}$ such that yx bisects that angle. The *trt*-property at x, y is clearly violated.

An important consequence of Claim 2 is that each edge lying between two regions is a line segment.

Let a, b determine the diameter of D. The *trt*-property of a, b yields the existence of a thin right triangle $T \subset D$ with vertices a, b, c and the right angle at c.

For $D = T$, we have already the first case of the statement.

Another important consequence of Claim 2 is that each region is convex. Let Y be the polygonal boundary of a region F or of the unbounded component D' of $\complement D$. Let v be a vertex of Y, not necessarily in $V(D)$, and β the angle at v toward F, respectively $\complement D'$.

Claim 3. $\beta \leq \pi/2$.

Indeed, assume $\beta > \pi/2$. Take u, u' on the two sides of Y which meet at v, close to v. Clearly, the *trt*-property is violated at u, u'.

From Claim 3 it follows that Y has at most four sides. Moreover, if Y has four sides, then $\mathrm{conv}Y$ must be a rectangle. However, then, the *trt*-property is violated at midpoints of opposite sides. Hence, Y is a triangle. So $\complement D' = abc$, and D is a triangulation.

Claim 4. If the edges J', J'' of D have a common vertex v, then the angle at v equals π or is not obtuse.

Indeed, if that angle is obtuse but not π, the *trt*-property at points on J', J'', close to v is not enjoyed.

Remark 2. If D has a vertex $v \in \mathrm{bd}(abc)$ distinct from a, b, c, then, by Claim 4, v has degree 3 and some edge vw of D is orthogonal to $\mathrm{bd}(abc)$ at v.

Remark 3. If D has a vertex $v \in \mathrm{int}(abc)$, then v has degree four and the four edges at v are pairwise collinear or orthogonal. This also follows from Claim 4.

It is clear that a belongs to a triangle apq of the triangulation, with $p \in ab$.

By Remark 2, apq has its right angle at p, so either $q \in \mathrm{int}(abc)$ and Remark 3 is contradicted, or $q \in ac$ and Remark 2 is contradicted, except for the case when $q = c$, which corresponds to the second case of the statement.

The proof is complete. □

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