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Generalized Rectangular Convexity

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Abstract In this paper we investigate the tr-convexity and the rectangular biconvexity in Euclidean spaces.

Keywords tr-convex, thin rectangular convexity, rectangular biconvexity

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1 Introduction

A *plane rectangle* is the Cartesian product $I_1 \times I_2 \subset \mathbb{R}^2$ of two line-segments $I_1, I_2 \subset \mathbb{R}$ of positive length. A *plane thin rectangle* is the boundary of a plane rectangle. In $\mathbb{R}^d$, a *rectangle* (a *thin rectangle*) is a set congruent to a plane rectangle (respectively a plane thin rectangle).

Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$*-convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

The second author proposed at the 1974 meeting on Convexity in Oberwolfach the investigation of this very general kind of convexity [1]. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

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Blind, Valette and the second author [1], and also Böröczky Jr. [2], investigated the rectangular convexity, Magazanik and Perles dealt with staircase connectedness [7], the second author studied the right convexity [11], then we generalized the latter type of convexity and investigated the right triple convexity (see [8, 9]). All these concepts (and many more) are particular cases of $\mathcal{F}$-convexity. The rectangular convexity is obtained if $\mathcal{F}$ is the family of all (non-degenerate) rectangles in $\mathbb{R}^d$.

In [6] we presented a discretization of rectangular convexity, the right quadruple convexity (rq-convexity, for short), which constitutes a generalization of rectangular convexity.

Here we investigate another generalization of rectangular convexity, by taking the family of all thin rectangles as $\mathcal{F}$, thus obtaining the *thin rectangular convexity.* In this case, $\mathcal{F}$-convex sets are called *tr-convex.*

Both rq-convexity and tr-convexity are generalizations of rectangular convexity. One could think that rq-convexity is a generalization of the tr-convexity. This is, however, false; the reason resides in the non-rq-convexity of a thin rectangle!

Now the old general concept of $\mathcal{F}$-convexity can also be further generalized. After considering the family $\mathcal{F}$, call a set $M \subset \mathbb{R}^d$ $\mathcal{F}$-*biconvex* if, for any pair of distinct points $x, y \in M$, there exist sets $F_x, F_y \in \mathcal{F}$, such that $x \in F_x$, $y \in F_y$, $F_x \cap F_y \neq \emptyset$ and $F_x \cup F_y \subset M$. The union $F_x \cup F_y$ is called a *biset.*

For the family $\mathcal{F}$ corresponding to the usual convexity, the $\mathcal{F}$-biconvexity has already been investigated (see [3]). If $\mathcal{F}$ is the family of all non-degenerate rectangles in $\mathbb{R}^d$, and $\mathcal{F}$-convexity is the rectangular convexity, $\mathcal{F}$-biconvexity will be naturally called *rectangular biconvexity.*

A second intention of this paper is to start the investigation of this other generalization of rectangular convexity.

For $M \subset \mathbb{R}^d$, we denote by $\overline{M}$ its affine hull, by $\operatorname{diam} M = \sup_{x,y \in M} \|x - y\|$ its diameter, and by $\operatorname{conv} M$ its convex hull; further, $\operatorname{int} M$, $\operatorname{cl} M$, $\operatorname{bd} M$ denote its relative interior, closure, and boundary, respectively, which means in the topology of $\overline{M}$. $\complement M$ denotes the complement of M in $\mathbb{R}^d$.

A set of four points $w, x, y, z \in \mathbb{R}^d$ (always $d \geq 2$) forms a *rectangular quadruple* if $\operatorname{conv}\{w, x, y, z\}$ is a non-degenerate rectangle. If $\mathcal{F}$ is the family of all rectangular quadruples, we obtain the *rq-convexity.*

If $\operatorname{conv}\{a_1, a_2, \ldots, a_n\} \in \mathbb{R}^2$ is a polygonal convex set with consecutive vertices $a_1, a_2, \ldots, a_n$, then $a_1 a_2 \cdots a_n$ denotes the set and $[a_1 a_2 \cdots a_n]$ its boundary polygon.

For distinct $x, y \in \mathbb{R}^d$, let $\overline{xy}$ be the line through x, y, xy the line-segment from x to y, and H_{xy} the hyperplane through x orthogonal to $\overline{xy}$. If L_1, L_2 are affine subspaces of $\mathbb{R}^d$, $L_1 \| L_2$ means that they are parallel.

For $M_1, M_2 \subset \mathbb{R}^d$, let $d(M_1, M_2) = \inf\{d(x, y) \mid x \in M_1, y \in M_2\}$ denote the *distance between* M_1 *and* M_2. The *distance from a point* x *to a set* M is defined as $d(\{x\}, M)$.

The d-dimensional unit ball (centred at $\mathbf{0}$) is denoted by B_d ($d \geq 2$).

For $x \in \mathbb{R}^d$ and $r > 0$, $B_r(x)$ denotes the compact ball of centre x and radius r.

2 Not Simply Connected tr-convex Sets

In $\mathbb{R}^2$, all compact rectangularly convex sets are conjectured to be extremely circular and symmetric. A planar convex set is *extremely circular* if its set of extreme points lies on a circle. Analogously, it is reasonable to conjecture that all compact rq-convex or tr-convex sets have an extremely circular and symmetric convex hull. Consequently, when investigating compact connected rq-convex or tr-convex sets M, we may begin by assuming that $\operatorname{conv} M$ is extremely circular and symmetric. In [6] we focused on rq-convexity and took $\operatorname{bd}(\operatorname{conv} M)$ to be a circle. Here we treat tr-convexity, and shall start in the same way. But the conclusion will be quite different.

Theorem 2.1 *If M is compact and* tr*-convex, and* $\operatorname{conv} M$ *is a disc, then* $M = \operatorname{conv} M$.

Proof Suppose, on the contrary, that $(\operatorname{conv} M) \setminus M \neq \emptyset$. Then there exists a disc $B_\varepsilon(x) \subset \operatorname{conv} M$ disjoint from M. Let ab be a diameter of $\operatorname{conv} M$ passing through x. Let $b' \in \operatorname{bd}(\operatorname{conv} M)$ be close to b, so that $ab' \cap B_\varepsilon(x) \neq \emptyset$. For the points a, b', there exists a thin rectangle T containing them and included in M. Obviously, ab' cannot be the diagonal of T, because then a vertex would be outside $\operatorname{conv} M$. It cannot be a side of T, as it is not included in M. No other possibility remains. □

It seems that the same conclusion holds for any extremely circular symmetric $\operatorname{conv} M$, except for a single case: that of $\operatorname{conv} M$ being a rectangle.

Thus, we have now two conjectures, one about rq-convex sets (see also [1, 6]), and another one about tr-convex sets.

Conjecture 2.2 *Each compact simply connected rq-convex set in* $\mathbb{R}^2$ *is an extremely circular symmetric convex set.*

Conjecture 2.3 *Each compact* tr*-convex set M in* $\mathbb{R}^2$ *is an extremely circular symmetric convex set, or else* $\operatorname{conv} M$ *is a rectangle and* $\operatorname{cl}((\operatorname{conv} M) \setminus M)$ *is a union of rectangles with pairwise parallel sides* (*see Figure* 1).

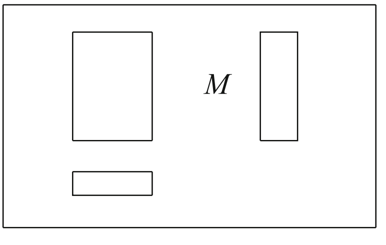

Figure 1 A tr-convex set with holes

3 Unbounded tr-convex Sets

Looking for an analog of Theorem 3.2 in [6], we obtain a characterization of the rectangles.

Theorem 3.1 *The complement of a connected open bounded set M in* $\mathbb{R}^2$ *is* tr*-convex, if and only if* $\operatorname{cl} M$ *is a rectangle.*

Proof The "if" part is obvious.

Suppose now $\complement M$ is tr-convex. Let $Q = \operatorname{conv}(\operatorname{cl} M)$.

We first show that $\operatorname{bd} Q \subset \operatorname{bd} M$. Suppose this is not true. Choose $x \in \operatorname{bd} Q \setminus \operatorname{bd} M$.The point x lies in a line-segment yz with $y, z \in \operatorname{bd} M$, and $\overline{yz}$ is a supporting line of Q. The line H_{xy} through x perpendicular to $\overline{yz}$ cuts $\operatorname{bd} M$ in at least two points. Let uv be the longest line-segment with $u, v \in H_{xy} \cap \operatorname{bd} M$. Then any thin rectangle containing u, v separates y from z in $M \cup \{y, z\}$, and therefore must meet M, generating a contradiction. Hence, $\operatorname{bd} Q \subset \operatorname{bd} M$.

We now prove that M is convex. Assume it is not. Then $\complement M$ has a component inside Q, whence $\complement M$ is not connected. However, this contradicts its tr-convexity.

Let L, L' be two parallel supporting lines of Q, such that the distance between them be minimal (and equal to the width of Q). It is well-known that there exist points $x \in Q \cap L$, $x' \in Q \cap L'$ such that L and $\overline{xx'}$ are orthogonal.

At least one of the two arcs on $\operatorname{bd} Q$ between x and x' is different from xx'. Call it A. Let $y' \in A$ be, among the points farthest from $\overline{xx'}$, the closest to L'. We claim that $y' \in L'$.

Assume that $y' \notin L'$.

Take $w \in A \setminus L'$ between y and x'. We make the choice such that $\operatorname{bd} Q$ is differentiable at w.

Then, the tangent L_w at w to $\operatorname{bd} Q$ is neither parallel, nor orthogonal to L. The existence in $\complement M$ of a thin rectangle containing x', w implies that the line L'' through x' orthogonal onto L_w supports Q. Now, consider a point $w' \in \operatorname{bd} Q \setminus (A \cup L'')$ close to L''. Clearly, no supporting line of Q at w' is orthogonal onto L_w. Moreover, no supporting line of Q at w' is parallel to L_w, if $(\operatorname{bd} Q) \cap L''$ is too short, in particular if it equals $\{x'\}$. Hence, $\complement M$ includes no thin rectangle containing w and w', unless $(\operatorname{bd} Q) \cap L''$ has positive length.

Since the smooth point w has been arbitrarily chosen in the subarc of A from y' to x', the condition about the positive length of $(\operatorname{bd} Q) \cap L''$ shows that L_w remains the same for all w, i.e., A includes a line-segment from y' to some point x'' of L'. This point x'' is different from x', otherwise L, L' do not realize the width of Q.

Now, no thin rectangle in $\complement M$ contains points in the relative interiors of $y'x''$ and $x'x''$.

Thus, $y' \in L'$ and $y'x' \subset \operatorname{bd} M$.

Reasoning symmetrically, there must exist a point $y \in A \cap L$ such that $yy' \| xx'$ and $y'y \cup yx \subset \operatorname{bd} M$.

If $xx' \subset \operatorname{bd} Q$, then $Q = xx'y'y$. If not, another symmetric reasoning leads to the existence of points $z \in L \setminus A$ and $z' \in L' \setminus A$, such that $zz' \| xx'$ and $[zz'y'y] \subset \operatorname{bd} Q$.

In conclusion, $\operatorname{bd} M$ is a thin rectangle. Being convex, $\operatorname{cl} M$ must be a rectangle. □

Theorem 3.2 *The complement of a connected open bounded set M in $\mathbb{R}^d$, where $d \geq 3$, is* tr-*convex, if M is either convex or a cylinder $K \times I$, where $K \subset \mathbb{R}^{d-1}$ and $I \subset \mathbb{R}$ are open.*

Proof Let $x, y \notin M$. Assume M to be open, bounded and convex. Let H_x, H_y be two hyperplanes disjoint from M, with $x \in H_x$, $y \in H_y$. They are supporting $\operatorname{cl} M$ in case x, y are boundary points. Take two parallel lines $L_x \subset H_x$, $L_y \subset H_y$ through x, y. They can be obtained by cutting H_x and H_y with a 2-dimensional plane, if $H_x \| H_y$, or by choosing them parallel to any line in $H_x \cap H_y$.

If M is a cylinder, we proceed like in the proof of Theorem 4.1 in the next section. □

Theorem 3.3 *The complement of a convex open set M in $\mathbb{R}^d$, where $d \geq 3$, is* tr-*convex, if* $\operatorname{cl} M$ *is strictly convex.*

Proof If M is bounded, the conclusion follows from Theorem 3.2.

Suppose M is unbounded and take $x, y \notin M$. We assume $x, y \in \operatorname{bd} M$, the general case being reducible to this one by taking the metric projections onto $\operatorname{cl} M$.

Consider the supporting planes $H_x \ni x$ and $H_y \ni y$. As $\operatorname{bd} M$ contains no half-lines, H_x and H_y are not parallel. Let L be a line in $H_x \cap H_y$. Take the lines $L_x \ni x$, $L_y \ni y$ parallel to L. Clearly, the recession cone K of $\operatorname{cl} M$ does not contain any half-line included in $L_x - x$.

Let P be a 2-dimensional plane through x, y, orthogonal to $H_x \cap H_y$. The convex curve $P \cap \operatorname{bd} M$ contains a Jordan arc A from x to y. Take arbitrarily $u \in \operatorname{int} A$.

Let $x_n \in L_x$, $y_n \in L_y$, with $x_n y_n \| xy$ $(n \in \mathbb{N})$, such that $x_n \in x x_{n+1}$ for every n and $\|x - x_n\| \to \infty$, if $n \to \infty$. We claim that, for n large enough, $x_n y_n \cap M = \emptyset$.

Indeed, suppose $z_n \in x_n y_n \cap M$, for all n. This implies that the half-line L_u starting at u and parallel to L lies in $\operatorname{cl} M$, which, in turn, implies that the recession cone of K does contain $L_u - u \subset L_x - x$, and a contradiction is obtained.

Thus, $x_n y_n \cap M = \emptyset$, for some n. As this can be done in both directions on L_x and L_y, we obtain a thin rectangle in $\complement M$ containing x, y. □

4 tr-convexity of Cylinders

As already remarked in [1], for $d \geq 3$, there is not even any conjectured characterization of rectangularly convex sets in $\mathbb{R}^d$. Among the sets mentioned in [1] as rectangularly convex we find the cylinder $K \times [0,1]$ with a $(d-1)$-dimensional compact convex set K as basis. In particular, any *right parallelotope*, i.e., the cartesian product of d pairwise orthogonal line-segments, is rectangularly convex and, a fortiori, rq-convex.

In [6] it is established that not all convex cylinders have rq-convex boundaries. Concerning the tr-convexity, we shall see that, on the contrary, all convex cylinders have a tr-convex boundary. We prove the following stronger result.

Theorem 4.1 *If L is a $(d-1)$-dimensional convex body, J is a compact interval, and $K \subset L$, $I \subset J$ are open or empty, then the compact set $(L \times J) \setminus (K \times I) \subset \mathbb{R}^d$ is* tr*-convex.*

Proof Let $\{x, y\} \subset M = (L \times J) \setminus (K \times I)$.

Putting $J = [0,1]$, let x_i, y_i be the orthogonal projections of x and y (respectively) on $\mathbb{R}^{d-1} \times \{i\}$ $(i = 0, 1)$.

Case 1 $x, y \in (L \setminus K) \times J$.

Clearly, in this case, $\{x, y\} \subset [x_0 x_1 y_1 y_0] \subset M$.

Case 2 $x, y \in K \times J$.

Consider the points x_0, y_0. Take the line Q to be $\overline{x_0 y_0}$ if $x_0 \neq y_0$, or any line through x_0 if $x_0 = y_0$. The line through x parallel to Q meets $\operatorname{bd}(L \times J)$ at u_x, v_x, say. Similarly, the line through y parallel to Q meets $\operatorname{bd}(L \times J)$ at u_y, v_y, chosen such that $\overline{u_x u_y}$ and $\overline{v_x v_y}$ are parallel. Then $\{x, y\} \subset [u_x v_x v_y u_y] \subset M$.

Case 3 $x \in (L \setminus K) \times J$ and $y \in K \times J$.

Assume without loss of generality that y is closer to $L \times \{0\}$ than to $L \times \{1\}$. The line through y parallel to $\overline{x_0 y_0}$ meets $\operatorname{bd}(L \times J)$ at u_y, v_y. Put $\{z\} = x_0 x_1 \cap u_y v_y$.

Now, $y \in zu_y$ or $y \in zv_y$; assume without loss of generality the latter. Let w be the projection of v_y on $L \times \{1\}$. Then $\{x, y\} \subset [zx_1wv_y] \subset M$. □

Corollary 4.2 *Every convex cylinder has a* tr*-convex boundary.*

5 Rectangular Biconvexity in the Plane

The very special class of sets to which all rectangularly convex bodies in the plane seemingly must belong suggests the question whether taking rectangular biconvexity instead would significantly enlarge the class. Indeed, this is so.

A convex body K has at every boundary point x a tangent cone T_x. In the planar case, the cone is bounded by two half-lines, which make an angle $\alpha(x)$. If $\alpha(x) = \pi$ for some $x \in \mathrm{bd}K$, then K is said to be *smooth at* x. It is called *smooth* if it is smooth at every boundary point. We shall say that K is *obtuse*, if $\alpha(x) > \pi/2$ at all $x \in \mathrm{bd}\, K$.

A *chord* of K is a line-segment xy joining two points $x, y \in \mathrm{bd}\, K$. A chord xy is called a *normal of* $K \subset \mathbb{R}^2$ *at* x if it is orthogonal to a supporting line of K at x. So, a chord xy is a normal of K, if it is a normal at (at least) one of its endpoints.

For a convex body to be rectangularly convex or even biconvex, it is necessary that $\alpha(x) \geq \pi/2$ at all boundary points x.

Theorem 5.1 *If the convex body* $K \subset \mathbb{R}^2$ *is obtuse and, for each normal* xy *of* K*,* K *is smooth at* x *or* y*, then* K *is rectangularly biconvex.*

Proof It suffices to prove the rectangular biconvexity property for the endpoints of chords. So, let xy be a chord of K. Let $[xx')$ and $[xx'')$ denote the two half-lines bounding T_x. Analogously, consider $[yy')$ and $[yy'')$. Assume without loss of generality that x' and y' are not separated by $\overline{xy}$.

Case 1 xy is not normal.

Assume without loss of generality $\angle x'xy > \pi/2$. Now, we look at y. If $\angle xyy' > \pi/2$, then a rectangle of small width with xy as a side exists in K. Now, suppose $\angle xyy' \leq \pi/2$. Then, since xy is not a normal, $\angle xyy'' > \pi/2$. In this case, a rectangle of small width having xy as diagonal lies in K.

Case 2 xy is a normal of K, say, at x.

By hypothesis, K is smooth at x or y. Assume first it is smooth at y.

The condition $\alpha(x) > \pi/2$ guarantees the existence of a rectangle $R_x \subset K$ of, perhaps, a very small diameter, with $x \in R_x$.

Now, if $\angle xyy' > \pi/2$, then there exists a rectangle $R_y \subset K$ of small width, with y as a vertex, with one side strictly included in xy, and meeting R_x. Thus, $x, y \in R_x \cup R_y$ and $R_x \cup R_y \subset K$.

We proceed analogously if $\angle xyy'' > \pi/2$.

Now, consider the case $\angle xyy' = \pi/2$, meaning that xy is a double normal. For a sequence $\{y_n\}_{n=1}^{\infty}$, with $y_n \in \mathrm{bd}\, K$, convergent to y, take chords y_nx_n normal at y_n. Then $x_n \to x$. Consider the chord $y_nx'_n$ verifying $\angle yy_nx'_n = \pi/2$. As $x'_n \to x$, too, for n large enough, $y_nx'_n$ meets R_x.

Take the point $s_n \in y_n x_n' \cap R_x$ closest to y_n, and consider the rectangle $R(n) = yy_n s_n t_n$. For n large enough, $R(n) \subset K$, $R_x \cap R(n) \neq \emptyset$ and $x, y \in R_x \cup R(n)$.

Finally, assume K is smooth at x. If xy is normal at y, too, then we are in the case just discussed. If xy is not normal at y, suppose without loss of generality $\angle xyy' > \pi/2$. Then there exists a rectangle of small width in K having xy as a diagonal. □

Corollary 5.2 *Every smooth convex body in* $\mathbb{R}^d$ *is rectangularly biconvex.*

Let K, L be convex bodies in $\mathbb{R}^2$ satisfying $K \subset \operatorname{int} L$, and consider the compact set $M = L \setminus \operatorname{int} K$. Is M rectangularly biconvex?

Even in the case of L alone ($K = \emptyset$), Theorem 5.1 established its rectangular biconvexity only under some additional conditions. So, too much we cannot expect regarding the rectangular biconvexity of M.

If K is smooth, M is certainly not rectangularly biconvex. Indeed, take $x, y \in \operatorname{bd} K$, with parallel supporting lines (for K) at these points. Then the strip bounded by these lines separates any rectangle containing x from any rectangle containing y. Thus, an obvious condition on K is that, for no pair of boundary points, their Gauss images are single, opposite vectors. But even asking for this condition is far from guaranteeing the rectangular biconvexity. Lengthy sets of conditions would have to be added. We present, however, a case which avoids this.

Theorem 5.3 *Suppose* K *is a triangle and* $L \subset \mathbb{R}^2$ *a smooth convex body such that* $K \subset \operatorname{int} L$. *Then* $M = L \setminus \operatorname{int} K$ *is rectangularly biconvex.*

Proof Put $K = abc$ and $\{c_a, a_c\} = \overline{ac} \cap \operatorname{bd} L$, with c_a closer to c and a_c closer to a. Analogously, consider $\{a_b, b_a\}$ and $\{b_c, c_b\}$. See Figure 2.

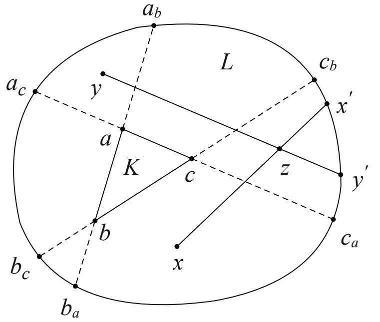

Figure 2 Rectangular biconvexity of a not simply connected set

Let $x, y \in M$. The lines $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ divide M in 6 regions, R_{ab}, R_{bc}, R_{ca}, adjacent to ab, bc, ca respectively, a region R_a between R_{ab} and R_{ca}, and other two analogous regions, R_b and R_c.

Suppose $x \in R_{bc}$ and $y \in R_a$. After having treated this case, all other cases become analogous or easier. Take a point $z \in \operatorname{int} R_c$.

Let $x' \in \overline{zx} \cap \operatorname{bd} L$, such that $z \in xx'$, and $y' \in \overline{zy} \cap \operatorname{bd} L$, such that $z \in yy'$.

Claim There exists a non-degenerate rectangle $R_x \subset M$ containing x, with small width and small distance from x'.

This is obvious if $x \notin \operatorname{bd} M$. Now, suppose $x \in \operatorname{bd} M$. If zx is not normal at x for $\operatorname{bd} L$, one of the two angles between zx and the tangent line at x is larger than $\pi/2$. Then, a rectangle of small width with a vertex at x can be found in L, having a side xx'' on xx', with x'' close or equal to x'. If zx is normal at x for $\mathrm{bd}\, L$, we take a point $z' \notin xx'$ close to z, remark that $z'x$ is not normal at x, and proceed as before. The Claim is proven.

Analogously, there exists a non-degenerate rectangle $R_y \subset M$ containing y, with very small distance from y'. As x, y', x', y lie in this order on $\mathrm{bd}\, L$, the rectangles R_x, R_y cross each other. So, we obtain $\{x, y\} \subset R_x \cup R_y \subset M$, with $R_x \cap R_y \neq \emptyset$. □

6 Rectangular Biconvexity in *d*-space

By Corollary 5.2, smooth convex bodies in $\mathbb{R}^d$ are rectangularly biconvex. This is strengthened by the next result.

Theorem 6.1 *If $K, L \subset \mathbb{R}^d$ are smooth convex bodies with* $\mathrm{card}(K \cap L) > 1$*, then $K \cup L$ is rectangularly biconvex.*

Proof Let $x, y \in K \cup L$. The case that $x, y \in K$ or $x, y \in L$ is solved by Theorem 5.1. The only interesting case is $x \in \mathrm{bd}\, K$, $y \in \mathrm{bd}\, L$.

Choose two points $u, v \in K \cap L$. Let $z \in \mathrm{int}\, uv$. Close to z we find a point $z' \in \mathrm{int}\, uv$, such that neither is xz' a normal of K at x, nor is yz' a normal of K at y.

Since $z' \in \mathrm{int}(K \cup L)$, for some $\varepsilon > 0$, $B_\varepsilon(z') \subset K \cup L$. Now, we can exhibit a thin rectangle $R_x \subset K \cup B_\varepsilon(z')$ having xz' as a side, and another one $R_y \subset L \cup B_\varepsilon(z')$ having yz' as a side. Thus, $x, y \in R_x \cup R_y$, while $R_x \cup R_y \subset K \cup L$ and $R_x \cap R_y \ni z'$. □

A *cone* in $\mathbb{R}^d$ is the convex hull $\mathrm{conv}(\{v\} \cup K)$, where K is a smooth $(d-1)$-dimensional compact convex set, and $v \in \mathbb{R}^d \setminus \overline{K}$. It is called *right*, if $\angle avb \geq \pi/2$ for some pair of points $a, b \in \mathrm{bd}\, K$.

Theorem 6.2 *Every right cone is rectangularly biconvex.*

Proof Let $L = \mathrm{conv}(\{v\} \cup K)$ be a right cone. Consider the points $x, y \in L$.

If $x, y \in K$, then the Corollary 5.2 settles the case.

Suppose now $x \in K$, $y = v$. Let $a, b \in \mathrm{bd}\, K$ satisfy $\angle avb \geq \pi/2$. Notice that $ab \subset \mathrm{bd}\, K$ is not excluded. Clearly, for every point $z \in \mathrm{int}(ab)$, there exists a rectangle $R_z \subset avb$ containing both v and z. If $x \in \mathrm{int}\, K$, then there exists a rectangle in K containing both x and z, which together with R_z provides the biset. Assume $x \in \mathrm{bd}\, K \setminus \{a, b\}$. Even if $\overline{zx}$ happens to be normal to $\mathrm{bd}\, K$ at x, for any point $z' \in \mathrm{int}(ab)$ close enough to z, $\overline{z'x}$ will not be normal to $\mathrm{bd}\, K$. Then there exists a rectangle R' containing x and z', either using xz' as a side or as a diagonal; $R' \cup R_{z'}$ provides the required biset. See Figure 3.

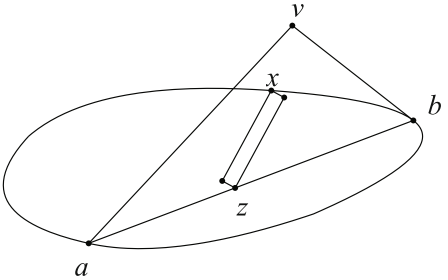

Figure 3 Rectangular biconvexity of a right cone

If $x \in \{a, b\}$, say $x = a$, then take a 2-dimesional plane $\Pi \supset ab$ and $a_1, a_2 \in (\Pi \cap \operatorname{bd} K)$ on both sides of $\Pi \cap \operatorname{bd} K$, close to a. If they are close enough, $\angle a_1 a a_2 > \pi/2$. Take a small square Q inside $\Pi \cap \operatorname{bd} K$, containing a. Choose $z \in \operatorname{int} Q \cap ab$. Now, the biset $Q \cup R_z$ is suitable.

Suppose now that x, y are in other positions. If x is not closer from $\overline{K}$ than y, take the hyperplane Ξ through y parallel to $\overline{K}$. The proof above now works in $\operatorname{conv}(\{x\} \cup (L \cap \Xi))$. □

Let $S \subset \mathbb{R}^d$ be starshaped, and $K \subset \mathbb{R}^{d'}$ be a convex body. Here $d, d' \geq 1$. The Cartesian product $S \times K \subset \mathbb{R}^{d+d'}$ is called an *sc-cylinder.*

Theorem 6.3 *Every sc-cylinder is rectangularly biconvex.*

Proof Let $L = S \times K$ be an sc-cylinder, with S starshaped in the subspace $x_{d+1} = \cdots = x_{d+d'} = 0$ of $\mathbb{R}^{d+d'}$, and K a convex body in $x_1 = \cdots = x_d = 0$.

Let $x, y \in L$.

Case 1 $x, y \in S \times \{u\}$, for some $u \in K$.

Thus, $x = x_u \times u$, $y = y_u \times u$, with $x_u, y_u \in S$. Take $k \in \ker S$ and $w \in K \setminus \{u\}$. We have the convenient biset

$$x(x_u \times w)(k \times w)(k \times u) \cup y(y_u \times w)(k \times w)(k \times u).$$

Case 2 $x \in S \times \{u\}$, $y \in S \times \{v\}$, with distinct $u, v \in K$.

Thus, $x = x_u \times u$, $y = y_v \times v$, with $x_u, y_v \in S$. Take again $k \in \ker S$. The biset

$$x(x_u \times v)(k \times v)(k \times u) \cup y(y_u \times v)(k \times v)(k \times u)$$

is suitable again.

Case 2 also includes the case $x, y \in \{s\} \times K$, where $s \in S$. □

A set is *smooth* if its boundary is a differentiable surface. It is smooth at some boundary point, if the boundary is a surface, which is differentiable at that point. Further, we say that a set $M \subset \mathbb{R}^d$ is *locally concave* at $x \in \operatorname{bd} M$, if there exists a neighbourhood V of x and a hyperplane H through x, such that $H^+ \cap V \subset M$, where H^+ is one of the half-spaces bounded by H.

Theorem 6.4 *Let $S \subset \mathbb{R}^2$ be starshaped. If S is at every boundary point either smooth or locally concave, and* $\dim \ker S \geq 2$, *then S is rectangularly biconvex.*

Proof Assume without loss of generality that $\mathbf{0} \in \ker S$.

Let $x, y \in S$. Interesting is only the case $x, y \in \operatorname{bd} S$. If S is locally concave at x, or $\operatorname{bd} S$ is smooth and $\overline{\mathbf{0}x}$ is not a normal of $\operatorname{bd} S$ at x, then we find a rectangle $R_x \subset S$ containing x and $\mathbf{0}$. The same is true about y.

Now, suppose $\operatorname{bd} S$ is smooth at x and $\overline{\mathbf{0}x}$ is a normal of $\operatorname{bd} S$ at x. Since $\dim \ker S > 1$, we find a point $z \in \ker S$ close to $\mathbf{0}$ and not collinear with $\mathbf{0}$ and x. Clearly, $\overline{zx}$ is not normal to $\operatorname{bd} S$ at x, and we find a rectangle $R_x \subset S$ containing both x and z.

Now, if $\overline{zy}$ is not normal to $\operatorname{bd} S$ at y, we find a rectangle $R_y \subset S$ containing y and z, and we are done. If $\overline{zy}$ is normal to $\operatorname{bd} S$ at y, then we find a point $z' \in \ker S \setminus \mathbf{0}x$ close to z and not collinear with z and y. Again, $\overline{z'y}$ is not normal to $\operatorname{bd} S$ at y, and we find a rectangle $R_y \subset S$ containing both y and z'. As $\overline{z'x}$ is also not normal to $\operatorname{bd} S$ at x, there exists a rectangle $R'_x \subset S$ containing both x and z'. The rectangles R'_x and R_y will do the job. □

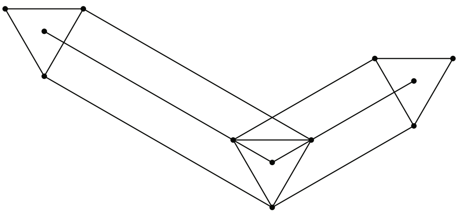

Figure 4 $S + K$ is not rectangularly biconvex

Most starshaped sets are nowhere dense, by the Corollary to Theorem 1 in [10]. So we cannot expect them to be rectangularly biconvex. By taking the Minkowski sum of a starshaped set S and a convex body K, there is place for hope. However, it is seen that, without additional requirements, rectangular biconvexity is impossible, see Figure 4.

Perhaps unexpectedly, taking a convex body from most of them will suffice!

Corollary 6.5 *Let $S \subset \mathbb{R}^2$ be starshaped. For most convex bodies $K \subset \mathbb{R}^2$, the set $S + K$ is rectangularly biconvex.*

Proof Most convex bodies are smooth, as Klee [5] has shown (see also [4]). It is seen that all hypotheses of Theorem 6.4 are verified. □

Conflict of Interest

The authors declare no conflict of interest.

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