Izvestiya: Mathematics **88**:3 590–600 *Izvestiya RAN: Ser. Mat.* **88**:3 192–202

DOI: https://doi.org/10.4213/im9485e

The length of the cut locus on convex surfaces

Liping Yuan and T. Zamfirescu

Abstract. In this paper, we prove the conjecture stating that, on any closed convex surface, the cut locus of a finite set M with more than two points has length at least half the diameter of the surface.

Keywords: closed convex surface, cut locus, finite set, diameter.

§ 1. Introduction

It is notorious that the cut locus is kind of an "enfant terrible" in differential geometry. The reason is clear. While the concept is quite natural, the classical methods of differential geometry are not very helpful. This paper is about its length, in the most basic case, that of a compact convex surface in 3-space.

All *surfaces* appearing in this paper are compact 2-dimensional Alexandrov spaces with curvature bounded below and without boundary, as defined in [1]. Thus, they are equipped with an intrinsic metric and are topological 2-manifolds [1].

For any surface S and closed subset $M \subset S$, ρ denotes the intrinsic metric of S and $d(M)$ the intrinsic diameter of M. A *segment* ab is a shortest path from a to b (of length $\rho(a,b)$).

A point $x \in S$ such that some shortest path xy from x to M (called a *segment from x to M*) cannot be extended as a shortest path to M beyond x is called a *cut point* with respect to M in the direction of yx. The set $C(M)$ of all cut points with respect to M is called the *cut locus* of M. If M contains a single point x, we write $C(x)$ for $C(M)$. Let λ denote the *length*, that is, the 1-dimensional Hausdorff measure.

Consider now surfaces S of the same diameter.

It is easily seen that $C(M)$ is connected if M is finite. It was shown in [2] and [3] (and already followed from [4]) that the length $\lambda C(x)$ of $C(x)$ may be infinite. $C(x)$ may even fail to have locally finite length: there are convex surfaces S on which, for any point x, every open set in S contains a compact subset of $C(x)$ with infinite length [5]. Although if in the Riemannian case this cannot happen (see [6], [2]), $\lambda C(M)$ still has no upper bound depending only on card M.

The authors gratefully acknowledge financial support by NSF of China (12271139, 11871192), NSF of Hebei Province (A2023205045), the High-end Foreign Experts Recruitment Program of People's Republic of China (G2023003003L), the Program for Foreign Experts of Hebei Province (2024), the Special Project on Science and Technology Research and Development Platforms of Hebei Province (22567610H), and the Program for 100 Foreign Experts Plan of Hebei Province.

AMS 2020 Mathematics Subject Classification. 53C45, 53C22.

Thus, we are interested in establishing a lower bound for the length of the cut locus. The case when card $M = 1$ and the surface is the sphere $\mathbf{S}^2$ shows that the bound vanishes. For surfaces not homeomorphic to $\mathbf{S}^2$, but where still card $M = 1$, several results have been obtained in [7].

For infinite M, the bound vanishes again: take M to be a great circle on $\mathbf{S}^2$. If card $M = 2$, the bound is also zero: take S to be an ellipsoid of revolution with one axis much longer than the others, and take M to consist of the two endpoints of the long axis.

Even if M is a 3-point set, $C(M)$ can have length as small as wished, see Figure 1. But the surface in Figure 1 is not convex. This motivated the following conjecture, which was formulated in 2005.

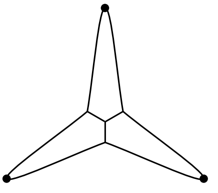

Figure 1

Conjecture (Itoh–Zamfirescu). For any smooth convex surface S, if $M \subset S$ is finite and contains more than two points, then $\lambda C(M) \geqslant d(S)/2$ (see [8]).

We prove here this conjecture for an arbitrary convex surface.

§ 2. Prerequisites

We shall use several lemmas, in which S is always a surface in $\mathbb{R}^3$ and $M \subset S$ is finite.

Lemma 1. *If* card $M \geqslant 3$ *and* $d(M) \leqslant d(S)/2$, *then* $\lambda C(M) \geqslant d(S)/2$.

A set in S is called *concyclic* if it is included in some intrinsic circle (the set of all points at a fixed positive distance from a fixed point) of S. The following lemma refers to a set in S which is not concyclic.

Lemma 2. *Let S be homeomorphic to $\mathbf{S}^2$, $M = \{p, p_a, p_b\} \subset S$, $a, b \in S$, and $\rho(a, b) = d(S)$. Assume that p_a, p_b are the points of M closest from a, respectively, b ($p_a \neq p_b$), and any point of S equidistant from p_a and p_b has smaller distance from p. If $\rho(p, p_a) + \rho(p, p_b) \leqslant d(S)$, then $\lambda C(M) \geqslant d(S)/2$.*

Lemma 3. *If S is convex, M is concyclic, and* card $M = 3$, *then* $\lambda C(M) \geqslant d(S)/2$.

Lemma 4. *If S is convex, $u, v, w \in S$, and $u^*, v^*, w^* \in \mathbb{R}^2$ are such that $\rho(u, v) = \|u^* - v^*\|$, $\rho(v, w) = \|v^* - w^*\|$, $\rho(w, u) = \|w^* - u^*\|$, then, for any points $u' \in uw$, $v' \in vw$, and points $u'^* \in u^*w^*$, $v'^* \in v^*w^*$ corresponding via the isometries between sides, we have $\rho(u', v') \geqslant \|u'^* - v'^*\|$.*

Lemmas 1–3 are from [8], while Lemma 4 is Alexandrov's convexity condition (see [9], pp. 39, 40, or [10], p. 47).

Lemma 5. *Let u, v be points on a convex surface S, and let Γ', Γ'' be two segments from u to v. Then, by gluing Γ' to Γ'', the closure of each of the two components of $S \setminus (\Gamma' \cup \Gamma'')$ becomes a convex surface.*

This is a consequence of the well-known Gluing Theorem of Alexandrov (see [10], p. 362, or [11], p. 154) and his Realization Theorem.

Lemma 6. *Let S be convex, $M \subset S$, $a, b \in S$, and $\rho(a,b) = d(S)$. Assume that p_a, p_b are closest in M from a, respectively, b. If $\operatorname{card} M \geqslant 2$ and $p_a = p_b$, then $\lambda C(M) \geqslant d(S)/2$.*

Proof. Let us scale the metric such that $d(S) = 2$.

Let p be a point in $M \setminus \{p_a\}$ closest to p_a. Let c_a and c_b be the cut points with respect to M in the directions of the segments $p_a a$ and $p_a b$, respectively. Then $\rho(c_a, p_a) \leqslant \rho(c_a, p)$ and $\rho(c_b, p_a) \leqslant \rho(c_b, p)$. Let p' be the midpoint of some segment $p_a p$. See Figure 2.

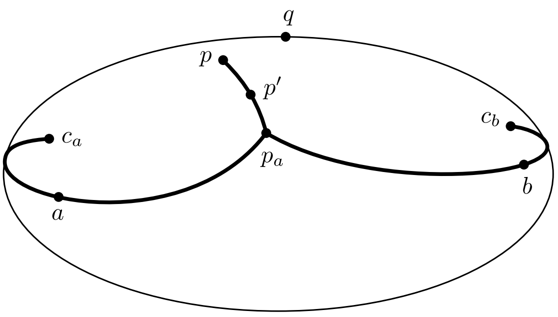

Figure 2

If $\rho(c_a, c_b) \geqslant 1$, we are done. If not, we have $\rho(a, c_a) + \rho(b, c_b) \geqslant 1$; otherwise

$$\rho(a,b) \leqslant \rho(a, c_a) + \rho(c_a, c_b) + \rho(c_b, b) < 2,$$

which is false.

Since $\rho(p_a, a) + \rho(p_a, b) \geqslant 2$, we have $\rho(p_a, c_a) + \rho(p_a, c_b) \geqslant 3$. Hence $\rho(p_a, c_a)$ and $\rho(p_a, c_b)$ cannot be both less than $3/2$. Assume without loss of generality that $\rho(p_a, c_a) \geqslant 3/2$.

We shall compare the triangle $c_a p_a p$ with the Euclidean triangle $c_a^* p_a^* p^*$ having the same side lengths, shown in Figure 3, (a). In $c_a^* p_a^* p^*$ we have $\|p^* - p_a^*\| \leqslant 2$, $\|p^* - c_a^*\| \geqslant \|p_a^* - c_a^*\| \geqslant 3/2$.

Consider the isosceles triangle $\Delta \subset \mathbb{R}^2$ with vertices $(-1,0)$, $(1,0)$, $(0, \sqrt{5}/2)$, see Figure 3, (b). It has one side of length 2, and two sides of length $3/2$ each. Also, consider the isosceles triangle $p^* p_a^* s$ with $\|p^* - s\| = \|p_a^* - s\| = \|p_a^* - c_a^*\|$. Let p'^* be the midpoint of $p_a^* p^*$. We obviously have $\|s - p'^*\| \leqslant \|c_a^* - p'^*\|$.

Comparing in $\mathbb{R}^2$ the triangles $p^* p_a^* s$ and Δ, we see that $\|s - p'^*\| \geqslant \sqrt{5}/2$. Hence $\|c_a^* - p'^*\| \geqslant \sqrt{5}/2$.

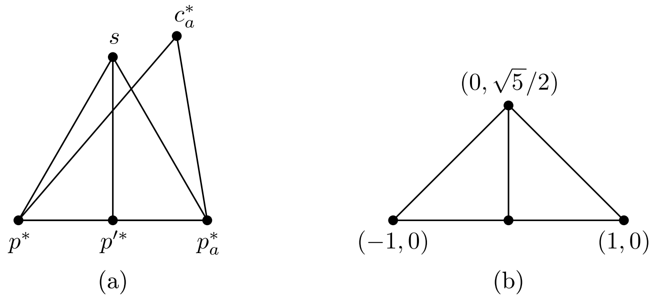

Figure 3

Now Lemma 4 tells us that

$$\rho(c_a, p') \geqslant \|c_a^* - p'^*\| \geqslant \frac{\sqrt{5}}{2}.$$

Thus, any of the arcs of $C(M)$ from c_a to p' has length at least $\sqrt{5}/2$.

Lemma is completely proved.

The *ambiguous locus* $A(M)$ of M is the set of all points $x \in S$ admitting at least two segments xa, xb from x to M with distinct $a, b \in M$ (see [12]).

We shall also use a straightforward generalization of the cyclic part $C^{\mathrm{cp}}(z)$ of the cut locus of a point $z \in S$ (see [7]). It is defined as $C^{\mathrm{cp}}(M) = S \setminus S'$, where S' is the set of all points $x \in S$ such that, for any segments σ_1, σ_2 from x to M, either $\sigma_1 = \sigma_2$ or $\sigma_1 \cup \sigma_2$ is a null-homotopic closed curve in $(S \setminus M) \cup (\sigma_1 \cap \sigma_2)$.

We have $A(M) \subset C^{\mathrm{cp}}(M) \subset C(M)$.

§ 3. The case $n = 3$

In this section, we consider the case of a 3-point set M.

Proposition. *Let $S \subset \mathbb{R}^3$ be a convex surface and $M \subset S$. If card $M = 3$, then $\lambda C(M) \geqslant d(S)/2$.*

Proof. We start by scaling the metric so that $d(S) = 2$.

If M is concyclic, then, by Lemma 3, $\lambda C(M) \geqslant 1$. So, in the rest of the proof, we assume that M is not concyclic.

Let $a, b \in S$ satisfy $\rho(a, b) = 2$, and let $p_a, p_b \in M$ be the points closest to, respectively, a and b.

For $p_a = p_b$, the proposition follows from Lemma 6. Suppose hence that $p_a \neq p_b$.

Let $M = \{p, p_a, p_b\}$. Since M is not concyclic, the set Γ_a of all points equidistant from p and p_a does not meet the set Γ_b of all points equidistant from p and p_b, but the cut locus $C(M)$ also includes an arc γ from some point $a' \in \Gamma_a$ to some point $b' \in \Gamma_b$. See Figure 4.

If $\rho(p, p_a) + \rho(p, p_b) \leqslant 2$, the conclusion follows from Lemma 2. So, suppose that $\rho(p, p_a) + \rho(p, p_b) > 2$.

Let q_a, q_b, q' be the midpoints of the three segments pp_a, pp_b, $p_a p_b$. We put $\{p'_a\} = p_a p_b \cap \Gamma_a$ and $\{p'_b\} = p_a p_b \cap \Gamma_b$.

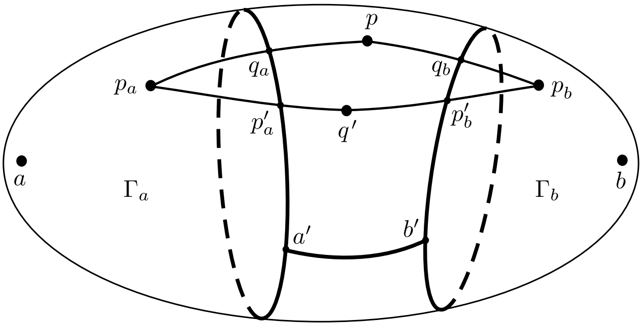

Figure 4

By Lemma 4, $\rho(q', q_a) \geqslant \rho(p, p_b)/2$ and $\rho(q', q_b) \geqslant \rho(p, p_a)/2$. It follows that $\rho(q', q_a) + \rho(q', q_b) > 1$.

Denote by α the arc of Γ_a from q_a to a' which contains p'_a, and by α', its subarc from p'_a to a'. Let β and β' be defined analogously. We have

$$\begin{aligned}\lambda C(M) &\geqslant \lambda\alpha + \lambda\beta + \lambda\gamma \geqslant \rho(q_a, p'_a) + \lambda\alpha' + \rho(q_b, p'_b) + \lambda\beta' + \lambda\gamma \\ &\geqslant \rho(q_a, p'_a) + \rho(p'_a, p'_b) + \rho(p'_b, q_b) \geqslant \rho(q_a, q') + \rho(q', q_b) > 1.\end{aligned}$$

This proves the proposition.

§ 4. More preparation

Let $S \subset \mathbb{R}^3$ be a surface and $M \subset S$ finite. For every point $x \in M$,

$$V(x) = \{z \in S \colon \forall\, y \in M \setminus \{x\} \quad \rho(x, z) \leqslant \rho(y, z)\}$$

is the (Voronoi) *cell of* x. Two such cells, $V(x)$ and $V(y)$, are *neighbours* if their boundaries meet.

We shall need the following strengthening of Lemma 2.

Lemma 7. *Let S be homeomorphic to $\mathbf{S}^2$, $M \subset S$, $p, p_a, p_b \in M$, and $\rho(a, b) = d(S)$. Assume that p_a, p_b are closest in M from a, respectively, b ($p_a \neq p_b$), and $V(p)$ or $V(p_b)$ is the only neighbour of $V(p_a)$. If $\rho(p, p_a) + \rho(p, p_b) \leqslant d(S)$, then $\lambda C(M) \geqslant d(S)/2$.*

Proof. Let q_a be the intersection point of a segment $p_a p$ with $\operatorname{bd} V(p_a)$. Further, if $p_a \neq a$, let c_a be the cut point of M in the direction of a segment $p_a a$; if $p_a = a$, set $c_a = q_a$. The point c_b is defined analogously, while $\{q_b\} = pp_b \cap \operatorname{bd} V(p_b)$.

We may suppose that $\rho(a, c_a) + \rho(b, c_b) > d(S)/2$, otherwise $\rho(c_a, c_b) \geqslant d(S)/2$, and we are done.

From the inequality in the statement, we get

$$\rho(p, p_a) + \rho(p, p_b) - \rho(p_a, p_b) \leqslant \rho(a, b) - \rho(p_a, p_b) \leqslant \rho(a, p_a) + \rho(b, p_b).$$

This implies

$$\begin{aligned}\rho(p, q_a) + \rho(p, q_b) - \rho(q_a, q_b) &= \rho(p, p_a) + \rho(p, p_b) - (\rho(p_a, q_a) + \rho(q_a, q_b) + \rho(q_b, p_b)) \\ &\leqslant \rho(p, p_a) + \rho(p, p_b) - \rho(p_a, p_b) \leqslant \rho(a, p_a) + \rho(b, p_b).\end{aligned}$$

Since $\rho(q_a, c_a) \geqslant \rho(p_a, c_a) - \rho(p_a, q_a)$ and $\rho(q_b, c_b) \geqslant \rho(p_b, c_b) - \rho(p_b, q_b)$, we have

$$\rho(q_a, c_a) + \rho(q_b, c_b) \geqslant \rho(p_a, a) + \rho(a, c_a) + \rho(p_b, b) + \rho(b, c_b) - \rho(p_a, q_a) - \rho(p_b, q_b).$$

Let $c_a c_a'$ be the shortest arc in $C(M)$ with $c_a' \in \operatorname{bd} V(p_a)$. Let c_b' be defined analogously. Since $V(p)$ or $V(p_b)$ is the only neighbour of $V(p_a)$, there is a unique shortest arc $a'b' \subset C(M)$ with $a' \in \operatorname{bd} V(p_a)$ and $b' \in \left(\bigcup_{v \in M} \operatorname{bd} V(v)\right) \setminus \operatorname{bd} V(p_a)$.

Let $A \subset C(M)$ be an arc joining q_a to q_b and missing c_a' if $c_a' \neq a'$ and c_b' if $c_b' \neq b'$.

We also find arcs A_a and A_b joining c_a to q_a and c_b to q_b, and meeting A only in q_a, q_b, and perhaps a', b'. Finally,

$$\begin{aligned}
\lambda C(M) &\geqslant \lambda A + \lambda A_a + \lambda A_b \geqslant \rho(q_a, q_b) + \rho(q_a, c_a) + \rho(q_b, c_b) \\
&\geqslant \rho(q_a, q_b) + \rho(p_a, a) + \rho(a, c_a) + \rho(p_b, b) + \rho(b, c_b) - \rho(p_a, q_a) - \rho(p_b, q_b) \\
&\geqslant \rho(p, q_a) + \rho(p, q_b) + \rho(a, c_a) + \rho(b, c_b) - \rho(p_a, q_a) - \rho(p_b, q_b) \\
&\geqslant \rho(a, c_a) + \rho(b, c_b) > \frac{d(S)}{2}.
\end{aligned}$$

Lemma 7 is proved.

In the case of convex S, we obtain the conclusion of Lemma 7 without using the inequality from its hypotheses.

Lemma 8. *Let S be convex, $M \subset S$, $p, p_a, p_b \in M$, $a, b \in S$ and $\rho(a, b) = d(S)$. Assume that p_a, p_b are closest in M from, respectively, a, and b ($p_a \neq p_b$), and $V(p)$ or $V(p_b)$ is the only neighbour of $V(p_a)$. Then $\lambda C(M) \geqslant d(S)/2$.*

Proof. If $\rho(p, p_a) + \rho(p, p_b) \leqslant d(S)$, then the conclusion follows, by Lemma 7. So, suppose that $\rho(p, p_a) + \rho(p, p_b) > d(S)$.

Case I. $V(p)$ is the only neighbour of $V(p_a)$.

Let q_a, q, q_b be the midpoints of $p_a p$, $p_a p_b$, $p p_b$, respectively. Considering the triple $\{p^*, p_a^*, p_b^*\} \subset \mathbb{R}^2$ isometric to $\{p, p_a, p_b\} \subset S$, we have in the triangle $p^* p_a^* p_b^*$

$$\|q^* - q_a^*\| + \|q^* - q_b^*\| = \frac{\|p^* - p_b^*\|}{2} + \frac{\|p^* - p_a^*\|}{2} > \frac{d(S)}{2},$$

where q_a^*, q^*, q_b^* correspond to q_a, q, q_b, respectively.

Let xy be the minimal subsegment of $p_a p_b$ such that $q \in xy$, $x \in \operatorname{bd} V(p_a)$, and $y \in C(M)$. Also, let vw be the minimal subsegment of $p p_b$ such that $q_b \in vw$, and $v, w \in C(M)$. Using again the above isometry, and denoting by x^*, y^*, v^*, w^* the corresponding points in $\mathbb{R}^2$, we note that either $\|y^* - v^*\| \geqslant \|y^* - q_b^*\|$, or $\|y^* - w^*\| \geqslant \|y^* - q_b^*\|$. Suppose the latter is true (the proof in the other case is similar). Define a', b' as in the previous proof. Let $A \subset C(M)$ be an arc joining q_a to x and missing a' if $x \neq a'$. Further, let $D \subset C(M)$ be an arc joining x to y, and $B \subset C(M)$ an arc joining y to w disjoint from D.

Hence

$$\begin{aligned}
\lambda C(M) &\geqslant \lambda A + \lambda B + \lambda D \geqslant \lambda A + \rho(x, y) + \rho(y, w) \\
&\geqslant \rho(q_a, x) + \rho(x, q) + \rho(q, y) + \|y^* - w^*\| \\
&\geqslant \|q_a^* - x^*\| + \|x^* - q^*\| + \|q^* - y^*\| + \|y^* - q_b^*\| \\
&\geqslant \|q_a^* - q^*\| + \|q^* - q_b^*\| > \frac{d(S)}{2},
\end{aligned}$$

and the proof of Case I is finished.

Case II. $V(p_b)$ is the only neighbour of $V(p_a)$.

We may suppose that p is a point of M different from p_a and p_b, closest to p_b. If $M = \{p, p_a, p_b\}$, the conclusion follows from the proposition. So, let $s \in M \setminus \{p, p_a, p_b\}$ be closest to p.

We reconsider the proof of Case I, reversing the roles of p and p_b. Using the same notation, we obtain

$$\lambda C(M) \geqslant \frac{\|p^* - p_b^*\|}{2} + \frac{\|p_a^* - p_b^*\|}{2}.$$

Let q' be the midpoint of a segment ps. In the Euclidean realization $s^*p^*p_b^*$ of spp_b, we have $\|q'^* - q_b^*\| = \|s^* - p_b^*\|/2 \geqslant \|p^* - p_b^*\|/2$.

Since q' and q_b can be joined by an arc of $C(M)$ disjoint from the arcs used above, we have

$$\begin{aligned} C(M) &\geqslant \|p^* - p_b^*\| + \frac{\|p_a^* - p_b^*\|}{2} \geqslant \|p^* - p_b^*\| + \frac{\|p^* - p_a^*\| - \|p^* - p_b^*\|}{2} \\ &= \frac{\|p^* - p_a^*\| + \|p^* - p_b^*\|}{2} = \frac{\rho(p, p_a) + \rho(p, p_b)}{2} > \frac{d(S)}{2}. \end{aligned}$$

Lemma 8 is proved.

§ 5. The general result

The main result of this paper is as follows.

Theorem. *For any convex surface S, if $M \subset S$ is finite and contains more than two points, then $\lambda C(M) \geqslant d(S)/2$.*

Proof. If $d(M) \leqslant d(S)/2$, the result follows from Lemma 1. So, from now on, we assume that $d(M) > d(S)/2$.

We distinguish between two cases.

Case I. $C^{\mathrm{cp}}(M)$ is 2-connected.

Let $u, v \in M$ realize the diameter of M. Let the point $w \in M \setminus \{u\}$ be closest from u and $w' \in M \setminus \{v\}$ be closest from v. These points may coincide. Assume without loss of generality that $\rho(u, w) \leqslant \rho(v, w')$.

Let u', v' be midpoints of uw, respectively, vw. The point u' belongs to $A(M)$. If v' is not in $A(M)$, then it does not belong to the cell $V(v)$ of v; let $v'' \in v'v \cap \operatorname{bd} V(v) \subset A(M)$. If v' belongs to $A(M)$, we put $v'' = v'$ (see Figure 5).

In the Euclidean plane, consider the triangle $u^*v^*w^*$ with

$$\rho(u, v) = \|u^* - v^*\|, \quad \rho(v, w) = \|v^* - w^*\|, \quad \rho(w, u) = \|w^* - u^*\|.$$

We also consider the points $u'^* \in u^*w^*$ and $v'^*, v''^* \in v^*w^*$, corresponding to u', v', v'' according to the isometries.

Since $\rho(u, w) \leqslant \rho(v, w') \leqslant \rho(v, w)$, we have $\|u'^* - w^*\| \leqslant \|v'^* - w^*\|$ and, therefore, $\angle u'^*v'^*w^* \leqslant \angle v'^*u'^*w^*$. This yields $\angle u'^*v'^*w^* < \pi/2$, whence

$$\|u'^* - v''^*\| \geqslant \|u'^* - v'^*\| = \frac{\|u^* - v^*\|}{2} = \frac{d(M)}{2}.$$

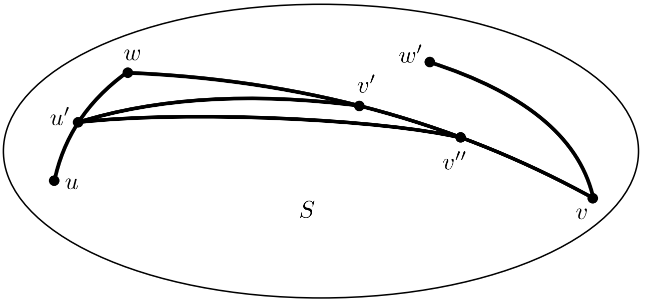

Figure 5

By Lemma 4,

$$\rho(u', v'') \geqslant \|u'^{*} - v''^{*}\| \geqslant \frac{d(M)}{2} > \frac{d(S)}{4}.$$

Since $C^{\mathrm{cp}}(M)$ is 2-connected, there exist two paths from u' to v'' in $C^{\mathrm{cp}}(M)$ having only their endpoints u', v'' in common. Thus, each of them having length at least $\rho(u', v'')$, together have length exceeding $d(S)/2$. Consequently, $\lambda C(M) > d(S)/2$.

Case II. $C^{\mathrm{cp}}(M)$ is not 2-connected.

We use induction on $n = \operatorname{card} M$. For $n = 3$, the statement of the theorem is true by the proposition. We assume now that $n > 3$ and that the statement of the theorem is true if $2 < \operatorname{card} M < n$.

Since $C^{\mathrm{cp}}(M)$ is not 2-connected, there exists a point $x \in M$ whose cell $V(x)$ disconnects S, that is, $S \setminus V(x)$ is not connected (see Figure 6).

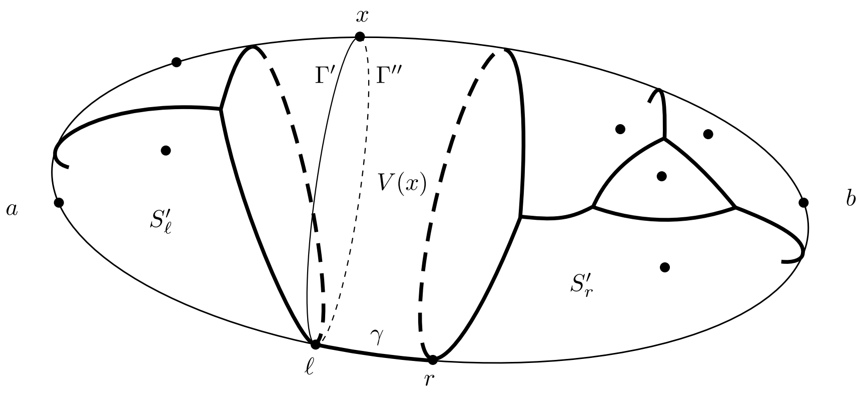

Figure 6

Let $\gamma \subset C(M)$ be the arc (possibly reduced to one point) of those points in S joined with x by two segments with non-null homotopic union in $(S \setminus M) \cup \{x\}$. Thus, γ traverses $V(x)$ if $\operatorname{card} \gamma \neq 1$, and $C(M) \setminus \gamma$ is disconnected. Let ℓ and r be the “left” and the “right” endpoint of γ. Let Γ', Γ'' be the two segments from ℓ

to x. The closed Jordan curve $\Gamma = \Gamma' \cup \Gamma''$ cuts S into two pieces, the left-hand one S'_ℓ and the right-hand one S'_r.

Now we glue Γ' to Γ''. By Lemma 5, we obtain from S'_ℓ a convex surface S_ℓ containing all points of $M_\ell = M \cap S'_\ell$. In S_ℓ, the cut locus $C(M_\ell)$ of M_ℓ is equal to the left component of $C(M) \setminus \gamma$ plus $\{\ell\}$. Indeed, the gluing process does not change the Voronoi cell $V(x)$. To prove this, suppose that, for some points $s \in M_\ell \setminus \{x\}$, $t \in S'_\ell \cap V(x)$ and segment st, after gluing, st crosses $\Gamma' = \Gamma''$ at a point q obtained by identifying $q' \in \Gamma'$ with $q'' \in \Gamma''$. We necessarily have $sq \cap \operatorname{bd} V(x) \neq \varnothing$, say $sq'' \cap \operatorname{bd} V(x) = \{e\}$. Hence

$$\begin{aligned}\lambda(st) &= \lambda(sq) + \lambda(qt) = \lambda(se) + \lambda(eq'') + \lambda(q't)\\ &\geqslant \lambda(xe) + \lambda(eq'') + \lambda(q't) > \lambda(xq'') + \lambda(q't)\\ &= \lambda(xq') + \lambda(q't) \geqslant \lambda(xt).\end{aligned}$$

Of course, the Voronoi cells of the other points of M_ℓ also remain unchanged.

Assume that $\operatorname{card} M_\ell \geqslant 3$ and $\operatorname{card} M_r \geqslant 3$.

As $\operatorname{card} M_\ell < \operatorname{card} M$, the induction hypothesis says that $\lambda C(M_\ell) \geqslant d(S_\ell)/2$. Proceeding similarly on the right-hand side, we obtain $\lambda C(M_r) \geqslant d(S_r)/2$. Notice that $\gamma \subset C(M_r)$.

Now let $a, b \in S$ realize the diameter of S. Suppose first that $a \in S_\ell$, $b \in S_r$. Then

$$\begin{aligned}\lambda C(M) = \lambda C(M_\ell) + \lambda C(M_r) &\geqslant \frac{d(S_\ell) + d(S_r)}{2}\\ &\geqslant \frac{\rho(a,x) + \rho(b,x)}{2} \geqslant \frac{\rho(a,b)}{2} = \frac{d(S)}{2}.\end{aligned}$$

Suppose now that $a, b \in S_r$. Then

$$\lambda C(M) \geqslant \lambda C(M_r) \geqslant \frac{d(S_r)}{2} = \frac{d(S)}{2}.$$

The case $a, b \in S_\ell$ is in fact included in the preceding case; we can see this by choosing Γ', Γ'' to join r to x, and reasoning symmetrically.

Assume now that $\operatorname{card} M_\ell = 2$. If $a, b \in S_r$, we proceed as above. If not, suppose $a \notin S_r$. Let $M_\ell = \{x, y\}$, where y is the second point of M in S_ℓ. Clearly, by construction, the only neighbour cell to $V(y)$ is $V(x)$. If $a \in V(y)$, then the hypotheses of Lemma 8 are satisfied, because $y = p_a$ and the only neighbour cell of $V(p_a)$ is $V(x)$, where x is either p_b or some third point of M. If $a \notin V(y)$, then $x = p_a$, and as long as $b \in V(y)$, we have $y = p_b$, and one can apply Lemma 8 with swapped a and b. It remains to elucidate the case where neither a nor b is in $V(y)$.

Obviously, $a \in V(x)$, whence $x = p_a$. Consider a segment by and put $\{f\} = by \cap \operatorname{bd} V(y)$. We now revive c_a, the cut point of M in the direction of a segment $p_a a$, beyond a. Let $A \subset C(M_\ell)$ be an arc joining c_a with f, avoiding the left-hand endpoint ℓ of γ. We intend to show that $\lambda A \geqslant \rho(f, a)/2$.

Consider the triangle $f p_a c_a$, with the Euclidean realization $f^* p_a^* c_a^*$. If $\rho(f, c_a) \geqslant \rho(f, a)/2$, we are done. If not, then $\rho(a, c_a) \geqslant \rho(f, c_a)$, but we shall see that this is in fact impossible. In $f^* p_a^* c_a^*$, $\angle f a c_a < \pi/2$. Hence $\rho(f, p_a) > \rho(f, a)$. Since $\rho(a, b) \geqslant \rho(y, b)$, we have $\rho(f, a) \geqslant \rho(f, y)$, and therefore, $\rho(f, p_a) > \rho(f, y)$.

Consider now the triangle byp_a. We have

$$\rho(b,y) = \rho(b,f) + \rho(f,p_a) > \rho(b,f) + \rho(f,a) \geqslant \rho(b,a),$$

a contradiction.

So, we proved that $\lambda A \geqslant \rho(f,a)/2$. Further, we have

$$\lambda C(M_\ell) \geqslant \lambda A + \rho(f,\ell) \geqslant \frac{\rho(f,a)+\rho(f,\ell)}{2} \geqslant \frac{\rho(\ell,a)}{2}.$$

Finally,

$$\lambda C(M) = \lambda C(M_\ell) + \lambda C(M_r) \geqslant \frac{\rho(\ell,a)}{2} + \frac{\rho(\ell,b)}{2} \geqslant \frac{\rho(a,b)}{2},$$

proving the theorem.

As already observed in [8], p. 103, the inequality of the Theorem is best possible.

§ 6. Outlook

Let $S \subset \mathbb{R}^3$ be an Alexandrov surface, and denote now by $C_S(M)$ the cut locus of a compact set $M \subset S$. Since, for convex S, the lower bound for $\lambda C_S(M)$ jumps from 0 to $d(S)/2$ as card M changes from 2 to 3, and then stays there for all finite M, but returns to 0 as M becomes infinite, it is interesting to determine the more exact jumping moment in the latter case. So we have the following problem.

Problem 1. Determine the lower bound of $\lambda C_S(M)$ for countable M.

Is it ∞? Is it still $d(S)/2$? Does smoothness of S make any difference? (See Theorem 8 in [8].)

While uncountable M brings the lower bound back to 0, it seems that the number of components of M makes again a difference. So we have the following problem.

Problem 2. Does $\lambda C_S(M) \geqslant d(S)/2$ hold again for convex S if M has more than two, but finitely many, simply connected components whose diameter is bounded in some way?

Note that in Problem 2 some boundedness condition on the components of M is necessary, otherwise the inequality is false: if $x_1 \in \mathbf{S}^2$, $\varepsilon > 0$, $M'_\varepsilon = \{y \in \mathbf{S}^2 \colon \|x_1 - y\| \geqslant \varepsilon\}$ and $x_2, \dots, x_n \in \mathbf{S}^2 \setminus (M'_\varepsilon \cup \{x_1\})$, then, for $M_\varepsilon = M'_\varepsilon \cup \{x_1, \dots, x_n\}$, it is clear that $\lambda C_{\mathbf{S}^2}(M_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Acknowledgements. We want to thank the referee of this paper for his/her careful work.

Bibliography

[1] Y. Burago, M. Gromov, and G. Perelman, “A. D. Alexandrov spaces with curvature bounded below”, *Russian Math. Surveys* **47**:2 (1992), 1–58.

[2] J. Itoh, “The length of a cut locus on a surface and Ambrose’s problem”, *J. Differential Geom.* **43**:3 (1996), 642–651.

[3] K. Shiohama and M. Tanaka, "Cut loci and distance spheres on Alexandrov surfaces", *Actes de la table ronde de géométrie différentielle* (Luminy 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris 1996, pp. 531–559.

[4] T. Zamfirescu, "Many endpoints and few interior points of geodesics", *Invent. Math.* **69**:2 (1982), 253–257.

[5] T. Zamfirescu, "Extreme points of the distance function on convex surfaces", *Trans. Amer. Math. Soc.* **350**:4 (1998), 1395–1406.

[6] J. J. Hebda, "Metric structure of cut loci in surfaces and Ambrose's problem", *J. Differential Geom.* **40**:3 (1994), 621–642.

[7] J. Itoh and T. Zamfirescu, "On the length of the cut locus on surfaces", *Stochastic geometry, convex bodies, empirical measures and applications to engineering science*, vol. II (Tropea 2001), Rend. Circ. Mat. Palermo (2) Suppl., vol. 70, Circ. Mat. Palermo, Palermo 2002, pp. 53–58.

[8] J. Itoh and T. Zamfirescu, "On the length of the cut locus for finitely many points", *Adv. Geom.* **5**:1 (2005), 97–106.

[9] A. D. Alexandrov, *Selected works*, vol. 2: *Intrinsic geometry of convex surfaces*, Chapman & Hall/CRC, Boca Raton, FL 2006.

[10] A. D. Alexandrow, *Die innere Geometrie der konvexen Flächen*, Akademie-Verlag, Berlin 1955.

[11] H. Busemann, *Convex surfaces*, Interscience Tracts in Pure and Applied Mathematics, vol. 6, Intersci. Publ., New York–London 1958.

[12] T. Zamfirescu, "On the cut locus in Alexandrov spaces and applications to convex surfaces", *Pacific J. Math.* **217**:2 (2004), 375–386.

Received 10/APR/23

Liping Yuan
School of Mathematical Sciences,
Hebei Normal University, P. R. China;
Hebei International Joint Research Center
for Mathematics and Interdisciplinary Science,
P. R. China;
Hebei Key Laboratory of Computational
Mathematics and Applications, P. R. China
E-mail: lpyuan@hebtu.edu.cn

Tudor Zamfirescu
School of Mathematical Sciences,
Hebei Normal University, P. R. China;
Hebei International Joint Research Center
for Mathematics and Interdisciplinary Science,
P. R. China;
Fachbereich Mathematik,
Technischen Universität Dortmund,
Dortmund, Germany;
Romanian Academy, Bucharest, Romania
E-mail:
tudor.zamfirescu@mathematik.tu-dortmund.de