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Poidge-convexity in triangular lattices

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ABSTRACT

We adapt the notion of poidge-convexity to subgraphs of the triangular lattice, and investigate the poidge-convexity of graphs belonging to two families of such subgraphs: paths and grid graphs.

1. Introduction

At the 1974 meeting on convexity in Oberwolfach, the third author proposed the investigation of the following very general kind of convexity, also see [1]. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-convex if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

For $M \subset \mathbb{R}^2$, $\operatorname{conv} M$ and $\operatorname{bd} M$ denote the convex hull and the boundary of M, respectively.

In 1980, Blind, Valette and the third author [1] first investigated rectangular convexity, the case when $\mathcal{F}$ is the family of all (2-dimensional) rectangles, which was also studied by Böröczky, Jr. [2]. In 2014, the third author [10] studied the right convexity, the case with $\mathcal{F}$ consisting of all (2-dimensional) right triangles. Later, the last two authors [8,9] investigated the *rt*-convexity, which is a discrete generalization of the right convexity. Recently, the authors [7] studied poidge-convexity, another generalization of the right convexity.

But $\mathcal{F}$ can be taken to be any family of sets, not just subsets of $\mathbb{R}^d$. For a subset of the vertex set of a graph, the g-convexity investigated by Farber and Jamison [6], the T-convexity studied by Changat and Mathew [4], Duchet's M-convexity [5], and the P_3-convexity considered by Centeno et al. [3] can also be regarded as examples of $\mathcal{F}$-convexity for suitable families $\mathcal{F}$.

Planar lattices, square, triangular or hexagonal, play a crucial role in many real life problems. There is no need of enumerating them here. These problems are converted into mathematical ones. In this mathematical paper, we propose a mathematical problem, and solve it, hoping that it will be useful one day.

A *lattice graph* is a finite subgraph of the infinite planar triangular lattice $\mathcal{T}$ in $\mathbb{R}^2$. The triangles in $\mathcal{T}$ are equilateral, of side 1.

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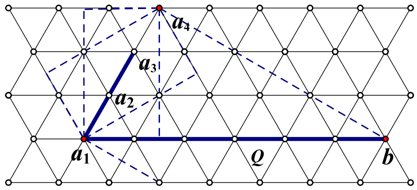

Fig. 1. $a_1 \in V(P)$ and $a_3a_4 \notin E(P)$.

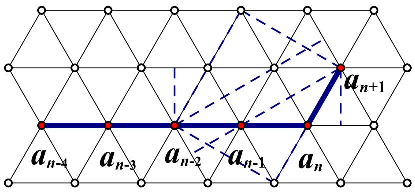

Fig. 2. $\angle a_{n-1}a_na_{n+1} = \frac{2\pi}{3}$.

A path in a lattice graph is called *linear* if all its vertices are on the same lattice line.

Now, we adapt to lattice graphs the notion of poidge-convexity, introduced for planar sets in [7].

A graph in $\mathcal{T}$ consisting of an isolated vertex v of $\mathcal{T}$ plus a linear path P in $\mathcal{T}$, such that v and the endvertices of P are the vertices of a non-degenerate right triangle, is called a *poidge*. A lattice graph G is *poidge-convex* if, for any pair $x, y \in V(G)$, there is a poidge with x, y as vertices, subgraph of G.

For any vertices a, b, c of $\mathcal{T}$, we denote by $\widehat{abc}$ the angle of abc at b, and by $\angle abc$ its measure. Also, we denote by $\delta(a, b)$ the length of the shortest path in $\mathcal{T}$ between a and b.

For any two vertices a, b, we denote by $\overline{ab}$ the lattice line through a, b, and by ab the line-segment from a to b. If a, b are neighbours in $\mathcal{T}$, then $ab \in E(\mathcal{T})$. Put $a_1a_2 \dots a_n = a_1a_2 \cup a_2a_3 \cup \dots a_{n-1}a_n$ and $[a_1a_2 \dots a_n] = a_1a_2 \dots a_n \cup a_na_1$.

For any two parallel lattice lines L_1 and L_2, we define the intrinsic distance $\rho(L_1, L_2)$ to be the Euclidean distance between them multiplied by $\frac{2}{\sqrt{3}}$. Let $\pi(a, L)$ be the orthogonal projection of a onto the line L.

Let C be a (finite) cycle in $\mathcal{T}$. We denote by $\mathrm{set}(C)$ the set of all points lying on the edges and at the vertices of C. All vertices and edges lying on C or in the bounded component of the complement of $\mathrm{set}(C)$ in $\mathbb{R}^2$ form a graph called *grid graph* (of *boundary* C).

In this paper, our goal is to investigate the poidge-convexity of paths and grid graphs in $\mathcal{T}$. We achieve full characterizations.

2. Poidge-convex paths

First, we discover that linear paths of a poidge-convex path cannot be too long.

Lemma 1. *Suppose that a_1, a_2, a_3 and a_4 are four consecutive vertices on a lattice line, and P is a poidge-convex path, such that the length of any linear subpath of P is at most* 3. *If $a_1 \in V(P)$ and at least one of the three edges a_1a_2, a_2a_3 and a_3a_4 is not in $E(P)$, then $a_4 \notin V(P)$* (see Fig. 1).

Proof. By the poidge-convexity of P, if $a_4 \in V(P)$, then there exists a poidge in P containing a_1 and a_4. Then there exists a linear subpath Q of P, such that Q only contains one of the two vertices a_1 and a_4, say a_1, and $Q \cup \{a_4\}$ is a poidge. However, Q is a subpath of length 6, which is not possible, see Fig. 1. □

Lemma 2. *If P is a poidge-convex path, then the length of any linear subpath of P is at most two.*

Proof. Let P be a poidge-convex path. Suppose without loss of generality that $a_0a_1 \cdots a_{n-1}a_n$ ($n \geq 3$) is a maximal linear subpath of P.

If $\angle a_{n-1}a_na_{n+1} = \frac{2\pi}{3}$, as shown in Fig. 2, then there is no poidge in P containing a_{n-2} and a_{n+1}. Hence, $\angle a_{n-1}a_na_{n+1} = \frac{\pi}{3}$.

Case 1. $n = 3$, as shown in Fig. 3. By the poidge-convexity of P, there exists a poidge in P containing a_0 and a_4. This implies that $a_0b_1b_2 \subset P$, see Fig. 3. By the poidge-convexity of P, there exists a poidge in P containing a_3 and b_1. Therefore, we have $a_4b_3 \in E(P)$. Then $b_2b_3 \notin E(P)$. The poidge in P containing b_2 and b_3 must be $a_4b_3b_4 \cup \{b_2\}$ or $b_1b_2b_4 \cup \{b_3\}$. So, $b_4 \in V(P)$, but only one of the two edges b_2b_4 and b_3b_4 belongs to P. This contradicts Lemma 1.

Case 2. $n > 3$, as shown in Fig. 4. Then no poidge in P contains a_{n-3} and a_{n+1}.

Hence, the length of any linear subpath of P is at most two. □

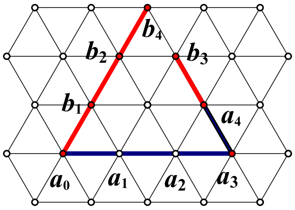

Fig. 3. $n = 3$.

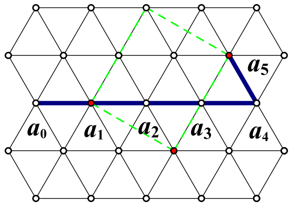

Fig. 4. $n > 3$.

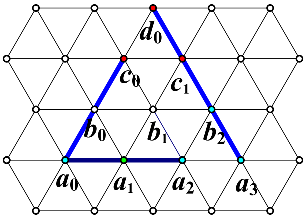

Fig. 5. $b_2 \in V(P)$.

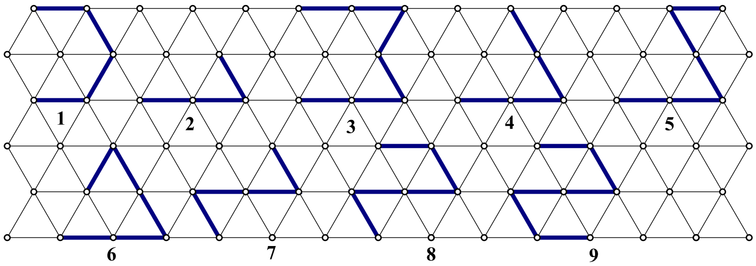

Fig. 6. The 9 kinds of poidge-convex paths in $\mathcal{T}$.

Lemma 3. *Let $P \subset \mathcal{T}$ be a poidge-convex path such that the length of a maximal linear subpath of P is 2. Suppose that $a_0a_1a_2$ is a linear subpath of P. If $u \in V(P)$ is a neighbour of a_2 in $\mathcal{T}$ distinct from a_1, then $\angle ua_2a_1 = \frac{\pi}{3}$.*

Proof. Suppose, on the contrary, that $u = b_2 \in V(P)$, see Fig. 5. Due to the poidge-convexity of P, there exists a poidge in P containing a_0 and b_2, which means that $a_0b_0c_0 \subset P$. Since there exists a poidge in P containing a_1 and b_2, and according to Lemma 1, we have $b_2c_1 \in E(P)$. Then $a_2b_2 \notin E(P)$ or $c_0c_1 \notin E(P)$. Assume without loss of generality that $c_0c_1 \notin E(P)$. Since there exists a poidge in P containing c_0 and c_1, $c_1d_0 \in E(P)$. This contradicts Lemma 1; Lemma 3 is proven. □

Now, we present all poidge-convex paths of $\mathcal{T}$.

Theorem 1. *There are precisely 9 kinds of poidge-convex paths in $\mathcal{T}$* (see Fig. 6).

Proof. Let P be a poidge-convex path and label some vertices of $\mathcal{T}$ as shown in Fig. 7. Due to Lemma 2, the length of a maximal linear subpath of P is at most 2.

First, suppose the length of a maximal linear subpath of P is 1.

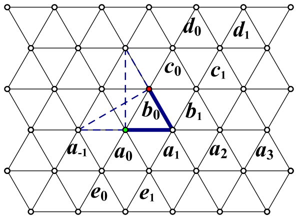

Fig. 7. $a_1b_0 \in E(P)$.

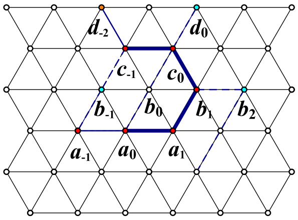

Fig. 8. $a_1b_1 \in E(P)$.

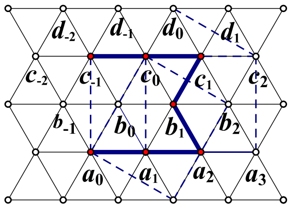

Fig. 9. $b_1c_1 \in E(P)$.

Assume without loss of generality that $a_0a_1 \in E(P)$ and a_0 is an endvertex of P. The angle between any two incident edges is $\frac{2\pi}{3}$. Otherwise, suppose $a_1b_0 \in E(P)$, see Fig. 7. Since there exists a poidge in P containing a_0 and b_0, there must exist a linear subpath of length 2 containing a_0 or b_0, contradicting our assumption.

Suppose without loss of generality that $a_1b_1 \in E(P)$, see Fig. 8. If $b_1b_2 \in E(P)$, then, by the poidge-convexity of P, there exists a poidge in P containing a_0 and b_2, which implies that there exists a linear subpath of length 2 containing a_0 or b_2, a contradiction. So $b_1c_0 \in E(P)$. Since $a_0b_0 \notin E(P)$, by Lemma 1, $d_0 \notin V(P)$. Then $c_0c_{-1} \in E(P)$. Since $a_1b_0 \notin E(P)$, by Lemma 1, $d_{-2} \notin V(P)$. Suppose that $b_{-1} \in V(P)$. Due to the poidge-convexity of P, there exists a poidge in P containing a_0 and b_{-1}. Because $a_0b_{-1} \notin E(P)$, this poidge must include a linear subpath of length 2 containing a_0 or b_{-1}, a contradiction. Thus, c_{-1} is an endvertex of P, and it is clear that $P = a_0a_1b_1c_0c_{-1}$ is poidge-convex (Type 1).

Second, suppose the length of a maximal linear subpath of P is 2.

Assume without loss of generality that $a_0a_1a_2$ is a linear subpath of length 2, and b_1a_2 belongs to $E(P)$, by Lemma 3. According to Lemma 3, we have $b_2 \notin V(P)$.

Case 1. a_0 is an endvertex of P. If b_1 is also an endvertex, it is clear that $P = a_0a_1a_2b_1$ is a poidge-convex path, see Fig. 6 (Type 2). Now, suppose that b_1 is not an endvertex of P. Assume $b_2 \in V(P)$. Since $a_0b_0 \notin E(P)$ and $a_2b_2 \notin E(P)$, we have no poidge in P containing a_0 and b_2. Hence, $b_2 \notin V(P)$. Then $b_1c_1 \in E(P)$, or $b_1c_0 \in E(P)$, or $b_1b_0 \in E(P)$.

Subcase 1.1. $b_1c_1 \in E(P)$, see Fig. 9. Then at most one of the two edges c_1c_2 and c_1c_0 belongs to $E(P)$. Assuming $c_1c_2 \in E(P)$, there is no poidge in P containing a_1 and c_2, because $a_2a_3 \notin E(P)$. Hence, $c_1c_2 \notin E(P)$. Since a_0b_0, a_1b_0, a_1b_1, b_1c_0 are not in $E(P)$, according to Lemma 1 we conclude that d_0, d_{-2}, d_1, d_{-1} do not belong to $V(P)$. If P has an endvertex at c_1, then it includes no poidge containing b_1 and c_1. Hence, $c_1c_0 \in E(P)$. By the poidge-convexity of P, there exists a poidge in P containing b_1 and c_0. This yields $c_{-1}c_0 \in E(P)$. Since c_0b_0, a_1b_0, a_1b_1 and b_1c_0 are not in $E(P)$, we have no poidge in P containing b_1 and b_0. This implies that $b_0 \notin V(P)$. Thus, c_{-1} is an endvertex of P. Obviously, $P = a_0a_1a_2b_1c_1c_0c_{-1}$ is a poidge-convex path (Type 3).

Subcase 1.2. $b_1c_0 \in E(P)$, see Fig. 10. If c_0 is an endvertex of P, then it is easily seen that $P = a_0a_1a_2b_1c_0$ is a poidge-convex path (Type 4). Otherwise, since $a_0b_0 \notin E(P)$ and $a_2b_1c_0$ is a linear subpath of length 2, we have $d_0 \notin V(P)$, $c_0d_{-1} \notin E(P)$ and $c_0c_{-1} \notin E(P)$. Thus $c_0c_1 \in E(P)$ or $c_0b_0 \in E(P)$.

Subcase 1.2.1. $c_0c_1 \in E(P)$, see Fig. 10. Since a_0b_0, a_1b_1 are not in $E(P)$, by Lemma 1, d_0 and d_1 do not belong to $V(P)$. Assume $c_1c_2 \in E(P)$. From a_0b_0, a_2b_2, $c_{-1}c_0$, c_2b_3, $a_2a_3 \notin E(P)$, it follows that no poidge in P contains a_0 and c_2. Hence, $c_1c_2 \notin E(P)$. Since $b_2 \notin V(P)$ and $c_1b_1 \notin E(P)$, c_1 is an endvertex of P. It is clear that $P = a_0a_1a_2b_1c_0c_1$ is a poidge-convex path (Type 5).

Subcase 1.2.2. $c_0b_0 \in E(P)$, see Fig. 11. Next, we claim that b_0 is an endvertex of P. Suppose this is not the case. The existence of $a_2b_1c_0 \subset P$ prohibits $c_{-1} \notin V(P)$, by Lemma 3. Then $b_0b_{-1} \in E(P)$. Since there is a poidge in P containing a_0 and b_{-1}, $b_{-1}b_{-2} \in E(P)$.

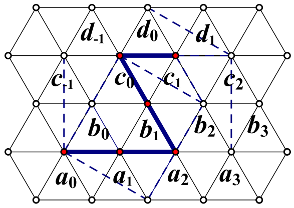

Fig. 10. $c_0c_1 \in E(P)$.

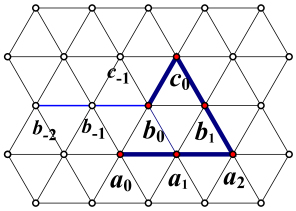

Fig. 11. $c_0b_0 \in E(P)$.

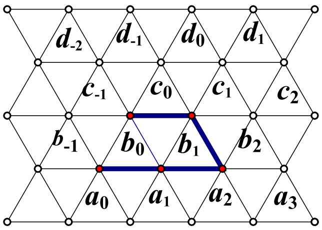

Fig. 12. $b_1b_0 \in E(P)$.

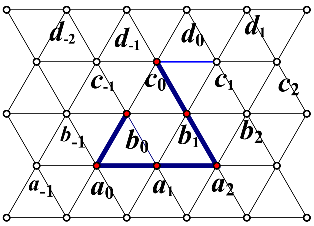

Fig. 13. $a_0b_0 \in E(P)$.

This contradicts Lemma 1. Hence, b_0 is an endvertex of P. It is easy to check that $P = a_0a_1a_2b_1c_0b_0$ is a poidge-convex path (Type 6).

Subcase 1.3. $b_1b_0 \in E(P)$, see Fig. 12. Since a_0b_0 and b_1c_0 are not in $E(P)$, according to Lemma 1 we conclude that d_0 and d_{-1} do not belong to $V(P)$. By Lemma 3, $a_0a_1a_2 \subset P$ yields $b_{-1} \notin V(P)$. Hence, P includes no poidge containing b_1 and b_0, which is not possible.

Case 2. a_0 is not an endvertex of P.

Subcase 2.1. $a_0b_0 \in E(P)$, see Fig. 13. Then $b_0b_1 \notin E(P)$. Since there exists a poidge in P containing b_0 and b_1, that poidge must be $a_0b_0c_0 \cup b_1$ or $a_2b_1c_0 \cup b_0$. Suppose without loss of generality that $b_1c_0 \in E(P)$ and $b_0c_0 \notin E(P)$. Then $d_0 \notin V(P)$. Since $a_2b_1c_0$ is a linear subpath of length 2, we have $c_0d_{-1} \notin E(P)$ and $d_{-1} \notin V(P)$. Next, we claim that c_0 is an endvertex of P. Suppose the claim is false. Since $a_2b_1c_0 \subset P$, by Lemma 3, $c_{-1} \notin V(P)$. Then $c_0c_1 \in E(P)$. Recall that $b_2 \notin V(P)$. Since d_0, b_2 and c_{-1} are not in $V(P)$ and $b_0a_1 \notin E(P)$, no poidge in P contains c_1 and b_0, a contradiction.

Thus, c_0 is an endvertex of P. From the analysis of the symmetric Subcase 1.2.2, it follows that b_0 is an endvertex of P, and $P = b_0a_0a_1a_2b_1c_0$ is poidge-convex (Type 6).

Subcase 2.2. $a_0e_1 \in E(P)$, see Fig. 14. If e_1 and b_1 are endvertices of P, then $P = e_1a_0a_1a_2b_1$, and P is poidge-convex (Type 7). Now we suppose that at least one of the vertices e_1, b_1 is not an endvertex of P. Assume without loss of generality that b_1 is not an endvertex of P.

Recall that $b_2 \notin V(P)$. Since $e_1 \in V(P)$ and $e_1a_1 \notin E(P)$, according to Lemma 1, $c_1 \notin V(P)$. Then $b_1c_0 \in E(P)$ or $b_1b_0 \in E(P)$.

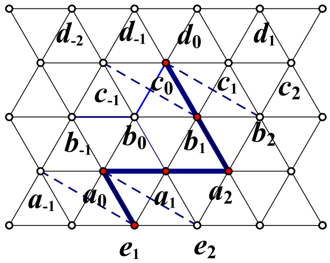

Fig. 14. $a_0e_1 \in E(P)$ and $b_1c_0 \in E(P)$.

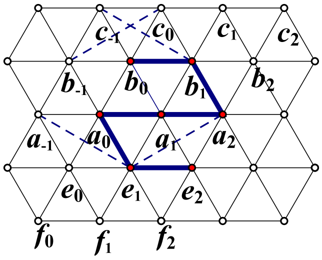

Fig. 15. $b_1b_0 \in E(P)$.

Subcase 2.2.1. If $b_1c_0 \in E(P)$, then $c_0d_{-1} \notin E(P)$, see Fig. 14. Due to $a_2b_1c_0 \subset P$, by Lemma 3, we have $c_{-1} \notin V(P)$. Since $a_0a_{-1} \notin E(P)$ and $a_2 \in V(P)$, from Lemma 1 we get $a_{-1} \notin V(P)$. Next, we claim that c_0 ia an endvertex of P. Suppose the claim were false. Then $c_0b_0 \in E(P)$ or $c_0c_1 \in E(P)$. The latter case is excluded by Lemma 1. If $c_0b_0 \in E(P)$, there is no poidge in P containing e_1 and b_0, like in the symmetric subcase 2.1, a contradiction. Hence, c_0 ia an endvertex of P. From the analysis of the symmetric Subcase 1.2.1, it follows that e_1 ia an endvertex of P. It is easily seen that $P = e_1a_0a_1a_2b_1c_0$ is a poidge-convex path (Type 5).

Subcase 2.2.2. $b_1b_0 \in E(P)$, see Fig. 15. Since $a_0a_1a_2 \subset P$, by Lemma 3, $b_{-1} \notin V(P)$. Assume $c_{-1} \in V(P)$ or $c_0 \in V(P)$. Due to b_1c_0, $a_1e_1 \notin E(P)$ and $b_{-1} \notin V(P)$, no poidge in P contains e_1, c_{-1} or e_1, c_0. Hence, c_0 and c_{-1} do not belong to $V(P)$. Thus, b_0 ia an endvertex of P.

If e_1 is also an endvertex of P, then $P = e_1a_0a_1a_2b_1b_0$, and it is easily checked that P is poidge-convex (Type 8), see Fig. 1. Suppose now that e_1 is not an endvertex of P. Since $b_1a_1 \notin E(P)$ and $b_1 \in V(P)$, by Lemma 1, we have $f_1 \notin V(P)$. On the other hand, due to $a_0a_1a_2 \subset P$ and $b_0 \in V(P)$, according to Lemma 3, we have $e_0 \notin V(P)$ and $e_1f_2 \notin E(P)$. Thus, $e_1e_2 \in E(P)$. Like in the case of b_0, we conclude that e_2 is an endvertex of P. Then $P = e_2e_1a_0a_1a_2b_1b_0$, and this path is poidge-convex (Type 9). □

3. Poidge-convex grid graphs

We investigate here the grid graphs in $\mathcal{T}$ with convex boundary, and characterize those which are poidge-convex.

Proposition 1. *Let L_1 and L_2 be two parallel lattice lines, and $a \in L_1$ be a lattice point. Then $\pi(a, L_2)$ is a lattice point if and only if $\rho(L_1, L_2)$ is even.*

Proposition 2. *Suppose that a, b are two vertices on the same lattice line. If L_a and L_b are two parallel distinct lattice lines containing a and b, respectively, then $\rho(L_a, L_b) = \delta(a, b)$.*

In the following results, G is a grid graph and C its boundary cycle.

Lemma 4. *Let G be a grid graph with* set(C) *convex. If G is poidge-convex and the angles at two vertices of* set(C) *are acute, then the vertices are neighbours* (*in* set(C)).

Proof. Suppose the lemma were false. Let a and b be two vertices of set(C) such that the angles at a and b are acute, and a and b are not neighbours. According to the convexity of set(C), there exist two parallel lattice lines $\overline{aa_1}$ and $\overline{bb_1}$, such that set$(C) \subseteq$ conv$\{a, a_1, b, b_1\}$ and $\overline{a_1b}$, $\overline{ab_1}$ are also two parallel lattice lines, as shown in Fig. 16. There exist two parallel lattice lines L_a and L_b, such that $L_a \cap \text{conv}\{a, a_1, b, b_1\} = \{a\}$, $L_b \cap \text{conv}\{a, a_1, b, b_1\} = \{b\}$; then conv$\{a, a_1, b, b_1\}$ is contained in the strip bounded by L_a and L_b. Thus, $\pi(a, L_b)$ and $\pi(b, L_a)$ are not vertices of G, or they equal b and a, respectively. On the other hand, since $\widehat{aa_1b}$ and $\widehat{ab_1b}$ are obtuse, $\pi(a, \overline{a_1b})$, $\pi(b, \overline{aa_1})$, $\pi(a, \overline{bb_1})$ and $\pi(b, \overline{ab_1})$ do not belong to $V(G)$. Thus, there is no poidge in G containing a and b, contrary to the poidge-convexity of G. □

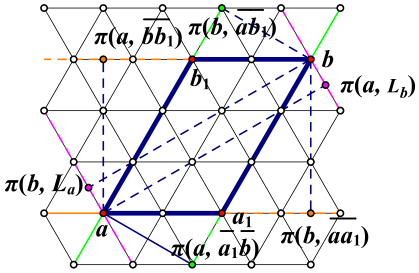

Fig. 16. $\text{conv}\{a, a_1, b, b_1\}$.

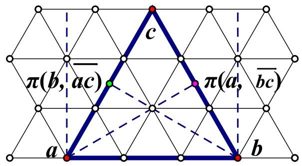

Fig. 17. $\text{set}(C) = ab \cup bc \cup ca$.

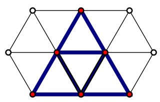

Fig. 18. The grid graph G.

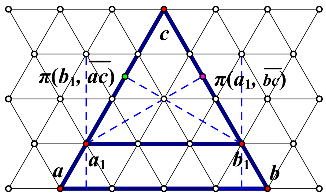

Fig. 19. $\text{set}(C) \subset \text{conv}\{a, b, c\}$.

Lemma 5. *Let G be a grid graph, such that* $\text{set}(C)$ *is a convex polygon having two neighbouring vertices a, b with acute angles. If G is poidge-convex, then $\delta(a, b)$ is even.*

Proof. On the contrary, suppose that $\delta(a, b)$ is odd. By Propositions 1 and 2, $\pi(a, \overline{bc})$, $\pi(b, \overline{ac})$ are not in $V(G)$, see Fig. 17. On the other hand, since the angles at a, b are acute, there exists no vertex $u \in V(G)$ such that $\angle uab = \frac{\pi}{2}$ or $\angle uba = \frac{\pi}{2}$. Therefore, no poidge in G contains a and b, and this is not possible. □

Theorem 2. *Suppose that G is a grid graph, such that* $\text{set}(C)$ *is a convex polygon having at least two vertices a, b with acute angles. Then G is poidge-convex if and only if* $\text{set}(C)$ *is an equilateral triangle with side length* 2 (see Fig. 18).

Proof. Let a, b be two vertices of $\text{set}(C)$, such that the angles at a, b are acute. We now claim that $\delta(a, b) = 2$. Suppose, contrary to our claim, that $\delta(a, b) \geq 3$.

According to Lemma 4, the vertices a, b are neighbours in $\text{set}(C)$; thus, there exists a lattice point c such that $\text{set}(C) \subset \text{conv}\{a, b, c\}$, see Fig. 19. Let $a_1 \in ac \cap V(C)$ and $b_1 \in bc \cap V(C)$ such that $\delta(a, a_1) = \delta(b, b_1) = 1$. Then $\angle ca_1b_1 < \frac{\pi}{2}$ and $\angle cb_1a_1 < \frac{\pi}{2}$. Thus, there exists no vertex $u \in V(G)$ such that $\angle ua_1b_1 = \frac{\pi}{2}$ or $\angle ub_1a_1 = \frac{\pi}{2}$. Since, by Lemma 5, $\delta(a, b)$ is even, $\delta(a_1, b_1)$ is odd. Then, by Propositions 1 and 2, $\pi(a_1, \overline{bc})$ and $\pi(b_1, \overline{ac})$ are not in $V(G)$. Therefore, no poidge in G contains a_1 and b_1, a contradiction. □

Lemma 6. *Let G be a grid graph such that* $\text{set}(C)$ *is a convex pentagon* $[abcde]$, *$\delta(a, e) \geq \delta(a, b)$ and the angle at a is acute. If $\delta(a, e)$ is odd, then G is not poidge-convex.*

Proof. We first suppose that $\delta(d, e) > 1$. Let $f \in ed \cap V(G)$ such that $\delta(e, f) = 2$. Assume that $\overline{aa'}$, $\overline{ee'}$ and $\overline{ff'}$ are three lattice lines parallel to cd, and $\overline{ff''}$ is a lattice line parallel to bc, where e', f' and f'' belong to $V(C)$, see Fig. 20.

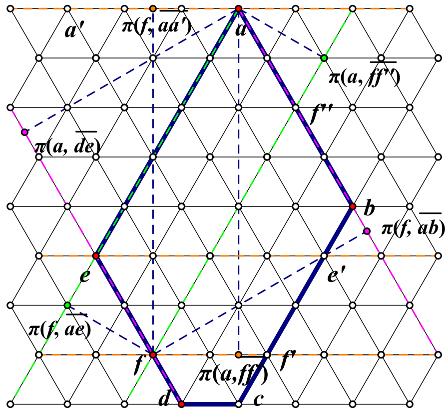

Fig. 20. $\delta(d,e) > 1$.

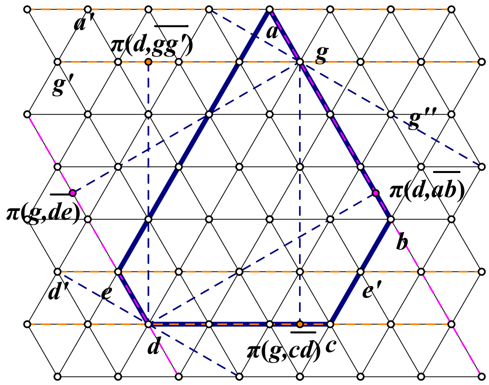

Fig. 21. $g \in ab \cap V(G)$ and $\delta(a,g) = 1$.

Since $\delta(a,e)$ is odd, $\rho(\overline{ab},\overline{de})$ is odd, according to Proposition 2. By Proposition 1, $\pi(f,\overline{ab})$ and $\pi(a,\overline{de})$ do not belong to $V(G)$. Because $\rho(\overline{aa'},\overline{ff'}) = \rho(\overline{aa'},\overline{ee'}) + \rho(\overline{ee'},\overline{ff'})$, $\rho(\overline{aa'},\overline{ff'})$ is odd. According to Proposition 1, $\pi(f,\overline{aa'})$ and $\pi(a,\overline{ff'})$ do not belong to $V(G)$. Since $\widehat{aef}$ and $\widehat{af''f}$ are obtuse, $\pi(f,\overline{ae})$ and $\pi(a,\overline{ff''})$ do not belong to $V(G)$. Thus, no poidge in G contains f and a.

Now, assume $\delta(d,e) = 1$.

Let $g \in ab \cap V(G)$ such that $\delta(a,g) = 1$. Assume that $\overline{gg'}$ is a lattice line parallel to cd, while $\overline{dd'}$ and $\overline{gg''}$ are orthogonal to $\overline{dg}$, see Fig. 21.

As before, $\rho(\overline{ab},\overline{de})$ is odd. By Proposition 1, $\pi(g,\overline{de})$ and $\pi(d,\overline{ab})$ do not belong to $V(G)$. Since $\delta(g,d) = \delta(a,e)$ is odd, $\rho(\overline{gg'},\overline{cd})$ is odd. According to Proposition 1, $\pi(d,\overline{gg'})$ and $\pi(g,\overline{cd})$ do not belong to $V(G)$. Since $\widehat{agd}$ is obtuse and $d(a,g) = 1$, $\overline{gg''} \cap (V(G) \setminus \{g\}) = \emptyset$. Since $\widehat{cde}$ is obtuse, $\overline{dd'} \cap (V(G) \setminus \{d\}) = \emptyset$. Thus, no poidge in G contains d and g. □

Lemma 7. *Let G be a grid graph and* set(C) *a convex pentagon* $[abcde]$*, such that $\delta(a,e) \geq \delta(a,b)$ and the angle at a is acute. If G is poidge-convex, then $1 \leq \delta(d,e) \leq 2$.*

Proof. Set $\delta(a,e) = n$. Since set(C) is a convex pentagon $abcde$, $1 \leq \delta(d,e)$. Now we claim that $\delta(d,e) \leq 2$. Suppose, contrary to our claim, that $\delta(d,e) \geq 3$. Let $p \in ed \cap V(G)$ such that $\delta(e,p) = 3$. Suppose $\overline{pq}$ is a lattice line parallel to cd such that $q \in V(G)$ and $\delta(p,q) = 1$, see Fig. 22. Since $\delta(a,e) = n$ is even by Lemma 6, and $\delta(e,p) = 3$ by Proposition 2, $\rho(\overline{aa'},\overline{pq})$ is odd. According to Proposition 1, $\pi(q,\overline{aa'})$ and $\pi(a,\overline{pq})$ are not vertices of (G). Because $\delta(p,q) = 1$, $\rho(\overline{ab},\overline{qq'}) = \rho(\overline{ab},\overline{ed}) - \rho(\overline{ed},\overline{qq'}) = n - 1$ is odd. According to Proposition 1, $\pi(a,\overline{qq'})$ and $\pi(q,\overline{ab})$ are not in $V(G)$. Since $\widehat{eab}$ is acute, $\pi(a,\overline{qq''}) \notin V(G)$. Since $\delta(e,q) = 3$ and $\delta(p,q) = 1$, $\rho(\overline{ae},\overline{qq''}) = 4$. Thus, $\pi(q,\overline{ae}) \notin V(G)$. Therefore, there is no poidge in G containing a and q, a contradiction. Hence, $\delta(d,e) \leq 2$. □

Lemma 8. *Let G be a grid graph such that* set(C) *is a convex pentagon* $[abcde]$*, $\delta(a,e) \geq \delta(a,b)$, and the angle at a is acute. If G is poidge-convex, then $\delta(a,b) = \delta(a,e)$.*

Proof. By Lemma 6, $\delta(a,e)$ is even. We first suppose that $\delta(a,b)$ is odd. Then $\delta(b,c) \geq 2$. Let $f \in bc \cap V(G)$ such that $\delta(b,f) = 2$. Assume that $\overline{aa'}$, $\overline{ff'}$ and $\overline{bb'}$ are three lattice lines parallel to cd, and $\overline{ff''}$ is a lattice line parallel to ab, where b', f' and f'' belong to $V(C)$, see Fig. 23.

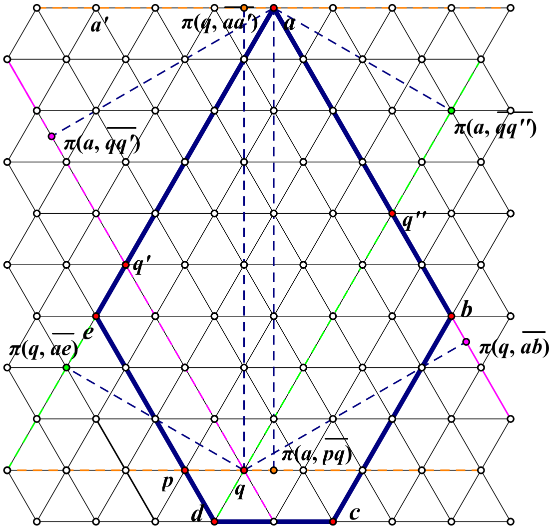

Fig. 22. $\delta(d,e)\geq 3$.

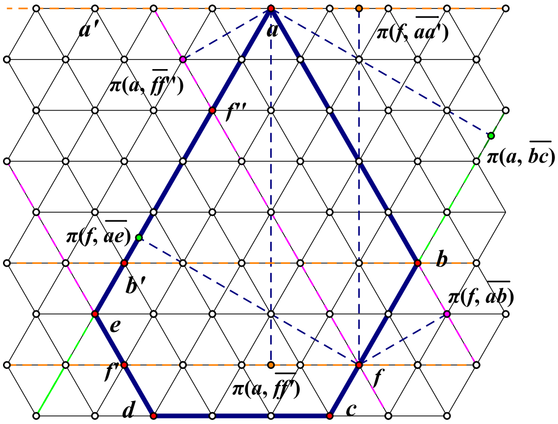

Fig. 23. $f\in bc\cap V(G)$ and $\delta(b,f)=2$.

By Proposition 1, $\pi(a,\overline{bc})$ and $\pi(f,\overline{ae})$ are not vertices of G. Moreover, $\rho(\overline{aa'},\overline{ff'})=\rho(\overline{aa'},\overline{bb'})+\rho(\overline{bb'},\overline{ff'})=\delta(a,b)=2$ is odd. Again by Proposition 1, $\pi(a,\overline{ff'})$ and $\pi(f,\overline{aa'})$ do not belong to $V(G)$. Since $\widehat{eab}$ is acute and $\widehat{abc}$ obtuse, $\pi(f,\overline{ab})$, $\pi(a,\overline{ff''})\notin V(G)$. Thus, there is no poidge in G containing f and a. Therefore, $\delta(a,b)$ is even.

Next we suppose that $\delta(a,b)<\delta(a,e)-2$. Since $\delta(a,b)$ and $\delta(a,e)$ are even, $\delta(a,b)\leq\delta(a,e)-4$. Let $g\in ed\cap V(G)$ and $h\in ab\cap V(G)$ such that $\delta(e,g)=1$ and $\delta(a,h)=2$. Assume that $\overline{gm}$, $\overline{ee'}$ and $\overline{hk}$ are three lattice lines parallel to cd, where $m\in V(C)$ and $k\in gk\cap V(G)$, see Fig. 24. Suppose that $\overline{kk'}$ and $\overline{mm'}$ are two lattice lines parallel to ab, where m' and k' belong to $V(C)$.

Since $\delta(a,b)$ is even and $\delta(e,g)=1$, $\rho(\overline{bc},\overline{gk})=\rho(\overline{bc},\overline{ae})-\rho(\overline{ae},\overline{gk})=\rho(\overline{bc},\overline{ae})-1$ is odd. According to Proposition 1, $\pi(m,\overline{gk})$ and $\pi(k,\overline{bc})$ are not in $V(G)$. By Proposition 2, since $\delta(a,e)$ is even, $\rho(\overline{hk},\overline{mg})=\rho(\overline{aa'},\overline{ee'})-\rho(\overline{aa'},\overline{hk})+\rho(\overline{ee'},\overline{mg})=\rho(\overline{aa'},\overline{ee'})-1$ is odd. Again by Proposition 1, $\pi(m,\overline{hk})$ and $\pi(k,\overline{mg})$ do not belong to $V(G)$. Due to $\angle abc>\frac{\pi}{2}$ and $k'\in ab$, we have $\angle kk'm>\frac{\pi}{2}$ and $\pi(m,\overline{kk'})\notin V(G)$. Since $\delta(a,b)\leq\delta(a,e)-4$, $\delta(m,b)\geq 4+1=5$ and $\rho(\overline{mm'},\overline{kk'})\geq 4$. Thus, $\pi(k,\overline{mm'})\notin V(G)$. Therefore, there is no poidge in G containing m and k, a contradiction. Hence, $\delta(a,b)\geq\delta(a,e)-2$.

Now we claim that $\delta(a,b)\neq\delta(a,e)-2$. Suppose, contrary to our claim, that $\delta(a,b)=\delta(a,e)-2$. According to Lemma 7, $\delta(d,e)=1$ or $\delta(d,e)=2$.

Case 1. If $\delta(d,e)=1$, then $\delta(b,c)=3$. Let $p\in cd\cap V(G)$ such that $\delta(c,p)=1$. Assume that $\overline{pp'}$ is a lattice line parallel to ab, and $\overline{pp''}$ is a lattice line parallel to ae, see Fig. 25.

Since $\delta(a,e)$ is even and $\delta(d,e)=1$, $\rho(\overline{aa'},\overline{cd})=\rho(\overline{aa'},\overline{ee'})+\rho(\overline{ee'},\overline{cd})=\rho(\overline{aa'},\overline{ee'})+1$ is odd. According to Proposition 1, $\pi(a,\overline{cd})$ and $\pi(p,\overline{aa'})$ are not in $V(G)$. Because $\delta(a,b)$ is even and $\delta(c,p)=1$, $\rho(\overline{pp''},\overline{ae})=\rho(\overline{bc},\overline{ae})-\rho(\overline{bc},\overline{pp''})=\rho(\overline{bc},\overline{ae})-1$ is odd. Again by Proposition 1, $\pi(a,\overline{pp''})$ and $\pi(p,\overline{ae})$ do not belong to $V(G)$. Due to $\angle bae<\frac{\pi}{2}$, $\pi(a,\overline{pp'})$ does not belong to $V(G)$. Because $\angle bae>\frac{\pi}{2}$ and $\delta(c,p)=1$, $\pi(p,\overline{ab})$ is not a vertex of G. Therefore, there is no poidge in G containing a and p, a contradiction.

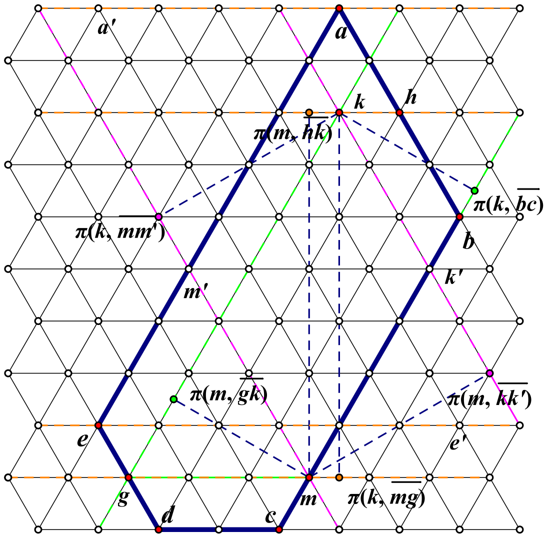

Fig. 24. $\delta(a,b) \le \delta(a,e) - 4$.

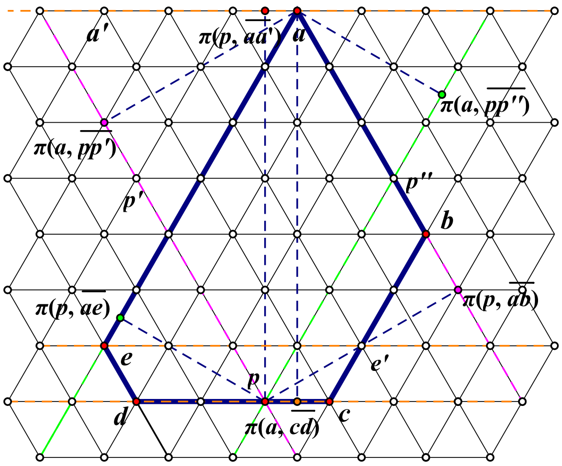

Fig. 25. Case 1.

Case 2. If $\delta(d,e)=2$, then $\delta(b,c)=4$. Since $\delta(d,e)=2$ and $\delta(a,b)=\delta(a,e)-2$, $\delta(c,d)\ge 2$. Let $s\in ab\cap V(G)$ and $t\in cd\cap V(G)$ such that $\delta(a,s)=1$ and $\delta(c,t)=2$. Assume that $\overline{ss'}$ is a lattice line parallel to cd, $\overline{tt'}$ is a lattice line parallel to ab, and $\overline{ss''}$ and $\overline{tt''}$ are lattice lines parallel to bc, see Fig. 26.

Since $\delta(a,s)=1$, $\rho(\overline{aa'},\overline{ss'})=1$ and $\rho(\overline{ss''},\overline{ae})=1$. Since $\delta(d,e)=2$ and $\delta(a,e)$ is even, $\rho(\overline{ss'},\overline{cd})=\rho(\overline{aa'},\overline{ee'})+\rho(\overline{ee'},\overline{cd})-\rho(\overline{aa'},\overline{ss'})=\rho(\overline{aa'},\overline{ee'})+1$ is odd. According to Proposition 1, $\pi(s,\overline{cd})$ and $\pi(t,\overline{ss'})$ are not in $V(G)$. Since $\delta(c,t)=2$ and $\delta(a,b)$ is even, $\rho(\overline{ss''},\overline{tt''})=\rho(\overline{bc},\overline{ae})-\rho(\overline{bc},\overline{tt''})-\rho(\overline{ss''},\overline{ae})=\rho(\overline{bc},\overline{ae})-3$ is odd. Again by Proposition 1, $\pi(s,\overline{tt''})$ and $\pi(t,\overline{ss''})$ do not belong to $V(G)$. Because $\widehat{bae}$ is acute and $\delta(a,s)=1$, $\pi(s,\overline{tt'})\notin V(G)$. Since $\widehat{bcd}$ is obtuse and $\delta(c,t)=2$, $\pi(t,\overline{ab})\notin V(G)$. Thus, there is no poidge in G containing s and t, a contradiction.

Hence, $\delta(a,b)=\delta(a,e)$. □

Theorem 3. *Let G be a grid graph such that* set(C) *is a convex pentagon* $[abcde]$ *and the angle at a is acute. Then G is poidge-convex if and only if* $\delta(a,b)=\delta(a,e)$ *is even and* $1\le\delta(d,e)\le 2$.

Proof. Suppose without loss of generality that $\delta(a,e)\ge\delta(a,b)$. According to Lemma 6, Lemma 7 and Lemma 8, the necessity of the condition in the statement is obvious. Now we prove its sufficiency.

Let x,y belong to $V(G)$. We prove the existence of a poidge in G containing both x and y.

Case 1. There exists a lattice line L containing x and y.

Subcase 1.1. L parallel to cd. Let $L\cap(ab\cup bc)=\{u\}$ and $L\cap(ae\cup ed)=\{v\}$. If $u=b$, then $L\cap \text{set}(C)=\{b,e\}$ and be is a linear path in G containing x and y. Since $\delta(a,e)$ is even and the angles $\widehat{bae}$ and $\widehat{abe}$ are acute, $\pi(e,\overline{ab})\in V(G)$. Thus, $\{\pi(e,\overline{ab})\}\cup be$ is a

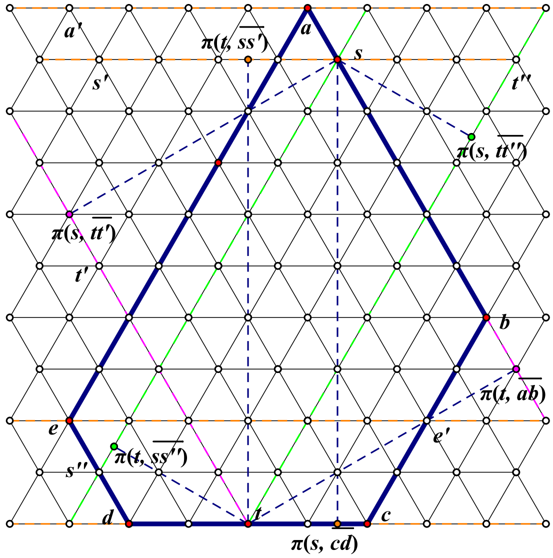

Fig. 26. Case 2.

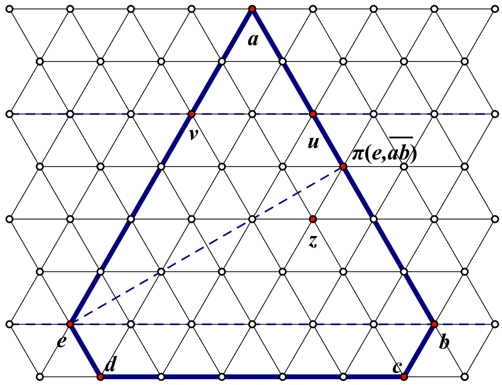

Fig. 27. Subcase 1.1.

poidge in G containing x and y, see Fig. 27. In case $u \neq b$, let P be the linear path between u and v. It is easily seen that there exists $z \in V(G)$ such that $\delta(u,z) = 2$ and $\angle zuv = \frac{\pi}{2}$. Thus, $\{z\} \cup P$ is a poidge in G containing x and y, see Fig. 27.

Subcase 1.2. L parallel to ab. Let $L \cap (bc \cup cd) = \{u\}$ and $L \cap ae = \{v\}$; then, $u, v \in V(G)$. Assume that P is the linear path between u and v.

If $u \in bc$, then $\{\pi(u, \overline{ae})\} \cup P$ is a poidge in G containing x and y, because $\rho(\overline{bc}, \overline{ae})$ is even, which implies $\pi(u, \overline{ae}) \in V(G)$, see Fig. 28.

If $u \in (cd \setminus \{c\})$, then there exists a vertex $z \in V(G)$ such that $\delta(u,z) = 2$ and $\angle zuv = \frac{\pi}{2}$. Thus, $\{z\} \cup P$ is a poidge in G containing x and y, see Fig. 29.

Subcase 1.3. L parallel to ae. This is symmetric to Subcase 1.2.

Case 2. There exist no lattice line containing x and y. Then there will exist two distinct parallel lattice lines L_1 and L_2 such that $x \in L_1$, $y \in L_2$ and $\rho(L_1, L_2)$ is even. Let $L_i \cap \text{set}(C) = \{u_i, v_i\}$, then $u_i, v_i \in V(G)$, where $i = 1, 2$. Suppose without loss of generality that $\delta(u_1, v_1) \leq \delta(u_2, v_2)$.

Subcase 2.1. If $\text{set}(C) \cap (L_1 \cup L_2)$ is a trapezoid or a hexagon, then $\pi(u_1v_1, L_2) \subset u_2v_2$ (see Fig. 30). Since $\rho(L_1, L_2)$ is even, $\pi(x, L_2) \in V(G)$, by Proposition 1. Let $z = \pi(x, L_2)$, P be the linear path between z and u_2, and Q is the linear path between z and v_2, then $y \in P$ or $y \in Q$. Therefore, $\{x\} \cup P$ or $\{x\} \cup Q$ is a poidge in G containing x and y.

Subcase 2.2. If $\text{set}(C) \cap (L_1 \cup L_2)$ is a parallelogram, then $\delta(b,c) = \delta(d,e) = 2$, and $ab \subset (L_1 \cup L_2)$ or $cd \subset (L_1 \cup L_2)$. Let $\overline{aa'}$ be a lattice line parallel to cd.

Subcase 2.2.1. If $ab \subset (L_1 \cup L_2)$, then $c \in (L_1 \cup L_2)$. Assume without loss of generality that $ab \subset L_1$, $c \in L_2$ and $L_2 \cap ae = c'$ (see Fig. 31).

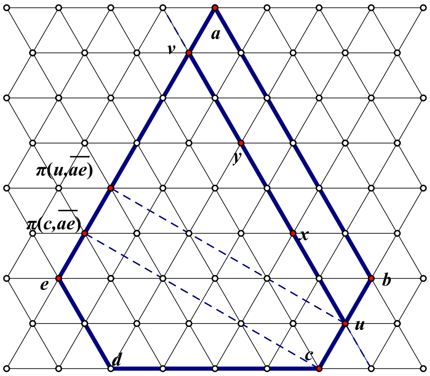

Fig. 28. $u \in bc$.

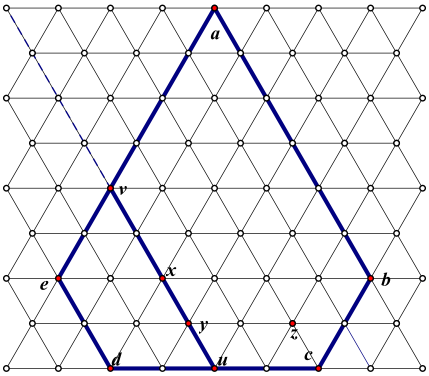

Fig. 29. $u \in (cd \setminus \{c\})$.

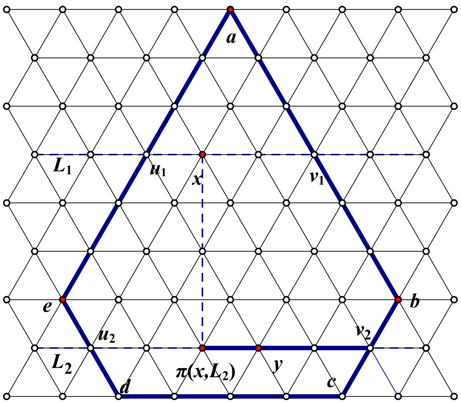

Fig. 30. Subcase 2.1.

If $x = a$ and $y = c$, then $\pi(x, L_1) \notin V(G)$ and $\pi(y, L_2) \notin V(G)$. Since $\delta(a, b)$ and $\delta(b, c)$ are even, $\rho(\overline{aa'}, \overline{cd}) = \rho(\overline{aa'}, \overline{be}) + \rho(\overline{be}, \overline{cd})$ is even. According to Proposition 1, $\pi(a, \overline{cd}) \in V(G)$. Let P_1 be the linear path between $\pi(a, \overline{cd})$ and c, then $\{a\} \cup P_1$ is a poidge in G containing x and y.

If $x \neq a$, then $\pi(x, L_2) \in V(G)$. Let P_2 be the linear path between $\pi(x, L_2)$ and c, and P_3 be the linear path between $\pi(x, L_2)$ and c', then $y \in P_3$ or $y \in P_4$. Therefore, $\{x\} \cup P_3$ or $\{x\} \cup P_4$ is a poidge in G containing x and y.

If $y \neq c$, then $\pi(y, L_1) \in V(G)$. Let P_5 be the linear path between $\pi(y, L_1)$ and a, and P_6 be the linear path between $\pi(y, L_1)$ and b, then $x \in P_5$ or $x \in P_6$. Therefore, $\{y\} \cup P_5$ or $\{y\} \cup P_6$ is a poidge in G containing x and y.

Subcase 2.2.2. If $ae \subset (L_1 \cup L_2)$, then $d \in (L_1 \cup L_2)$. From the analysis of the symmetric Subcase 2.2.1, it follows that there exists a poidge in G containing x and y. □

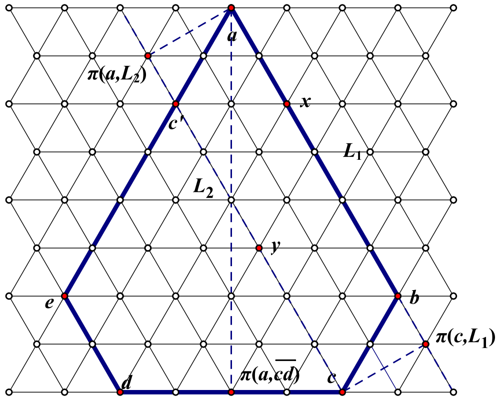

Fig. 31. Subcase 2.2.1.

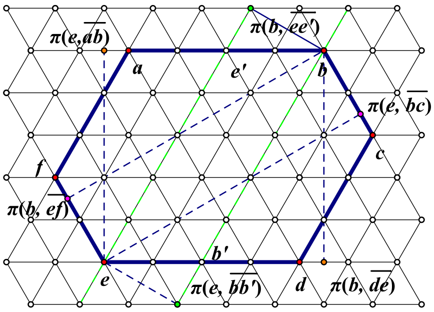

Fig. 32. $\rho(\overline{ab},\overline{de})$ and $\rho(\overline{bc},\overline{ef})$ are odd.

Lemma 9. *Let G be a grid graph and* $\text{set}(C)$ *be a convex hexagon* $[abcdef]$*. If G is poidge-convex, then at most one of the three numbers* $\rho(\overline{ab},\overline{de})$, $\rho(\overline{bc},\overline{ef})$ *and* $\rho(\overline{cd},\overline{fa})$ *is odd.*

Proof. Suppose the assertion of the lemma is false. Assume without loss of generality that $\rho(\overline{ab},\overline{de})$ and $\rho(\overline{bc},\overline{ef})$ are odd and $\rho(\overline{ab},\overline{de}) \le \rho(\overline{bc},\overline{ef})$. Since $\rho(\overline{ab},\overline{de})$ is odd, $\pi(b,\overline{de})$ and $\pi(e,\overline{ab})$ are not vertices of G, by Proposition 1. On the other hand, $\rho(\overline{bc},\overline{ef})$ is odd implies $\pi(b,\overline{ef})$ and $\pi(e,\overline{bc})$ do not belong to $V(G)$.

If $\rho(\overline{ab},\overline{de}) = \rho(\overline{bc},\overline{ef})$, then there exists a lattice line containing b and e. Because all of the four angles $\widehat{abe}$, $\widehat{cbe}$, $\widehat{deb}$ and $\widehat{feb}$ are acute, there exists no vertex of G, say u, such that $\angle ube = \frac{\pi}{2}$ or $\angle ueb = \frac{\pi}{2}$. Thus, there is no poidge in G containing b and e, a contradiction.

If $\rho(\overline{ab},\overline{de}) < \rho(\overline{bc},\overline{ef})$, then there exist two distinct lattice lines parallel to cd, say $\overline{bb'}$ and $\overline{ee'}$, such that $b \in \overline{bb'}$ and $e \in \overline{ee'}$, where b' and e' belong to $V(C)$, see Fig. 32. Since the angles $\widehat{baf}$ and $\widehat{cde}$ are obtuse and the lattices lines $\overline{bb'}$ and $\overline{ee'}$ are parallel to cd, $\pi(b,\overline{ee'})$ and $\pi(e,\overline{bb'})$ are not in $V(G)$. Thus, there is no poidge in G contains b and e, a contradiction. □

Theorem 4. *Let G be a grid graph with* $\text{set}(C)$ *a convex hexagon* $[abcdef]$ *such that* $\rho(\overline{ab},\overline{de})$ *is odd. If G is poidge-convex, then* $\rho(\overline{ab},\overline{de}) - \rho(\overline{bc},\overline{ef}) = \pm 1$ *and* $\rho(\overline{ab},\overline{de}) - \rho(\overline{cd},\overline{fa}) = \pm 1$.

Proof. Since $\rho(\overline{ab},\overline{de})$ is odd, both $\rho(\overline{bc},\overline{ef})$ and $\rho(\overline{cd},\overline{fa})$ are even, according to Lemma 9. Let $d_1, d_2 \in V(C)$ be the two neighbours of d and $e_1, e_2 \in V(C)$ the two neighbours of e, such that $d_1, e_1 \in de$. Suppose that $\overline{d_1 d_1'}$ is a lattice line parallel to cd and $\overline{e_1 e_1'}$ a lattice line parallel to bc. By Proposition 1, since $\rho(\overline{ab},\overline{de})$ is odd, $\pi(a,\overline{de})$, $\pi(b,\overline{de})$, $\pi(d_1,\overline{ab})$ and $\pi(e_1,\overline{ab})$ are not in $V(G)$, see Fig. 33.

Since $\delta(d,d_1) = 1$, $\rho(\overline{d_1 d_1'},\overline{fa}) = \rho(\overline{cd},\overline{fa}) - 1$ is odd, by Proposition 2. According to Proposition 1, $\pi(a,\overline{d_1 d_1'})$ and $\pi(d_1,\overline{fa})$ are not in $V(G)$. On the other hand, $\overline{d_1 d_2}$ is orthogonal to $\overline{bc}$ and $\overline{d_1 d_2} \cap V(G) = \{d_1, d_2\}$. By the poidge-convexity of G, there exists a poidge in G containing a and d_1, which means that $\overline{ad_1}$ or $\overline{ad_2}$ must be a lattice line. Similarly, $\overline{be_1}$ or $\overline{be_2}$ must be a lattice line.

If $\overline{ad_1}$ is a lattice line, then $\rho(\overline{d_1 d_1'},\overline{fa}) = \rho(\overline{ab},\overline{de})$ and $\rho(\overline{ab},\overline{de}) - \rho(\overline{cd},\overline{fa}) = -1$. Otherwise, $\overline{ad_2}$ is a lattice line, which implies that $\rho(\overline{cd},\overline{fa}) = \rho(\overline{ab},\overline{d_2 e_2})$ and $\rho(\overline{ab},\overline{de}) - \rho(\overline{cd},\overline{fa}) = 1$.

Simmetrically, if $\overline{be_1}$ is a lattice line, then $\rho(\overline{ab},\overline{de}) - \rho(\overline{bc},\overline{ef}) = -1$, and if $\overline{be_2}$ is a lattice line, then $\rho(\overline{ab},\overline{de}) - \rho(\overline{bc},\overline{ef}) = 1$. □

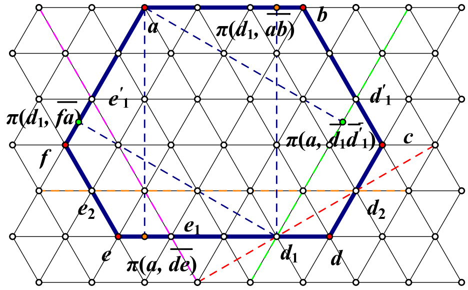

Fig. 33. $\rho(\overline{ab}, \overline{de})$ is odd.

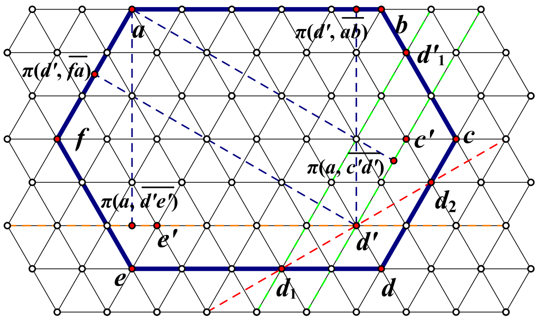

Fig. 34. Theorem 5.

Theorem 5. *Let G be a grid graph such that* $\mathrm{set}(C)$ *is a convex hexagon* $[abcdef]$ *and* $\rho(\overline{ab}, \overline{de})$, $\rho(\overline{bc}, \overline{ef})$ *and* $\rho(\overline{cd}, \overline{fa})$ *are all even. If G is poidge-convex, then any pair of these numbers are either equal or consecutive even numbers.*

Proof. Let $c', d', e' \in V(G) \setminus V(C)$ such that $\delta(c, c') = \delta(d, d') = \delta(e, e') = 1$. Then $\overline{c'd'}$ and $\overline{d'e'}$ will be two lattice lines parallel to cd and de, respectively, see Fig. 34.

Let $d_1 \in de$ and $d_2 \in cd$ such that $\delta(d, d_1) = \delta(d, d_2) = 2$. Now we show that the poidge-convexity of G implies that one of the three lines $\overline{ad}$, $\overline{ad_1}$ and $\overline{ad_2}$ is a lattice line. Since $\delta(d, d') = 1$ and $\rho(\overline{ab}, \overline{de})$ is even, $\rho(\overline{ab}, \overline{d'e'}) = \rho(\overline{ab}, \overline{de}) - 1$ is odd. It follows that $\pi(a, \overline{d'e'})$ and $\pi(d', \overline{ab})$ do not belong to $V(G)$. By a similar argument, $\delta(d, d') = 1$ and $\rho(\overline{cd}, \overline{fa})$ even imply that $\pi(a, \overline{c'd'})$ and $\pi(d', \overline{fa})$ are not in $V(G)$. Obviously, $\overline{d_1 d_2}$ is orthogonal to bc and $\overline{d_1 d_2} \cap V(G) = \{d_1, d', d_2\}$. By the poidge-convexity of G, there is a poidge in G containing a and d', which means that one of the three lines $\overline{ad}$, $\overline{ad_1}$ and $\overline{ad_2}$ must be a lattice line.

Suppose $\overline{d_1 d_1'}$ is a lattice line parallel to cd. If $\overline{ad_1}$ is a lattice line, then $\rho(\overline{ab}, \overline{de}) = \rho(\overline{d_1 d_1'}, \overline{fa})$. Because $\delta(d, d_1) = 2$, $\rho(\overline{cd}, \overline{fa}) = \rho(\overline{ab}, \overline{de}) + 2$. If $\overline{ad}$ is a lattice line, then $\rho(\overline{ab}, \overline{de}) = \rho(\overline{cd}, \overline{fa})$. Otherwise, $\overline{ad_2}$ is a lattice line and $\rho(\overline{cd}, \overline{fa}) = \rho(\overline{ab}, \overline{de}) - 2$.

Similar arguments apply to b and e', f and c', respectively; thus, if G is poidge-convex, then $|\rho(\overline{ab}, \overline{de}) - \rho(\overline{bc}, \overline{ef})| \le 2$ and $|\rho(\overline{bc}, \overline{fe}) - \rho(\overline{cd}, \overline{fa})| \le 2$. □

We arrive at the following characterization.

Theorem 6. *Let G be a grid graph such that* $\mathrm{set}(C)$ *is a convex hexagon, and p, q, r are the intrinsic distances of its opposite sides. G is poidge-convex if and only if at most one of the numbers p, q, r is odd, and the difference between any two of them is at most 2.*

Proof. For the “only if” implication, combine Theorems 4 and 5. The verification of the “if” implication is a routine matter. □

To summarize the results of this section, a grid graph with boundary cycle C is poidge-convex, if and only if $\mathrm{set}(C)$ is an equilateral triangle with side length 2 or a pentagon, as described in Theorem 3, or a hexagon, as described in Theorem 6.

Data availability

No data was used for the research described in the article.

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