# Introduction to Topological Groups

Dikran Dikranjan

To the memory of Ivan Prodanov (1935 – 1985)

**Abstract**

These notes provide a brief introduction to topological groups with a special emphasis on Pontryagin-van Kampen's duality theorem for locally compact abelian groups. We give a completely self-contained elementary proof of the theorem following the line from [57, 67]. According to the classical tradition, the structure theory of the locally compact abelian groups is built parallelly.

## 1 Introduction

Let $\mathcal{L}$ denote the category of locally compact abelian groups and continuous homomorphisms and let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the unit circle group. For $G \in \mathcal{L}$ denote by $\widehat{G}$ the group of continuous homomorphisms (*characters*) $G \to \mathbb{T}$ equipped with the compact-open topology. Then the assignment

$$G \mapsto \widehat{G}$$

is a contravariant endofunctor $\widehat{\ }: \mathcal{L} \to \mathcal{L}$. The celebrated Pontryagin-van Kampen duality theorem ([122]) says that this functor is, up to natural equivalence, an involution i.e., $\widehat{\widehat{G}} \cong G$ and this isomorphism is "well behaved" (see Theorem 12.5.4 for more detail). Moreover, this functor sends compact groups to discrete ones and viceversa, i.e., it defines a duality between the subcategory $\mathcal{C}$ of compact abelian groups and the subcategory $\mathcal{D}$ of discrete abelian groups. This allows for a very efficient and fruitful tool for the study of compact abelian groups, reducing many problems related to topological properties of these group to the related problems concerning algebraic properties in the category of discrete groups. The reader is advised to give a look at the Mackey's beautiful survey [114] for the connection of charactres and Pontryagin-van Kampen duality to number theory, physics and elsewhere. This duality inspired a huge amount of related research also in category theory, a brief comment on a specific categorical aspect (uniqueness and representability) can be found in §8.1 of the Appendix.

The aim of these notes is to provide a self-contained proof of this remarkable duality theorem, providing all necessary steps, including basic background on topological groups and the structure theory of locally compact abelian groups. Peter-Weyl's theorem asserting that the continuous characters of the compact abelian groups separate the points of the groups (see Theorem 11.2.1) is certainly the most important tool in proving the duality theorem. The Peter-Weyl's theorem is valid for arbitrary compact groups, but then continuous characters must be replaced by finite-dimensional unitary representations and the usual proof of the theorem in this general case involves Haar integration. In the case of abelian groups the irreducible ones turn out the be one-dimensional, i.e., continuous characters. We prefer here a different approach. Namely, Peter-Weyl's theorem in the abelian case can be obtained as an immediate corollary of a theorem of Følner (Theorem 10.3.5) whose elementary proof uses nothing beyond elementary properties of the finite abelian groups, a local version of the Stone-Weierstraß approximation theorem proved in §2 and the Stone-Čech compactification of discrete spaces. As another application of Følner's theorem we describe the precompact groups (i.e., the subgroups of the compact groups) as having a topology generated by continuous characters. As a third application of Følner's theorem one can obtain the existence of the Haar integral on locally compact abelian groups for free (see Theorem 11.4.21, the proof follows [57, §2.4, Theorem 2.4.5]).

The notes are organized as follows. In Section 2 we recall basic results and notions on abelian groups and general topology, which will be used in the rest of the paper. Section 3 contains background on topological groups, starting from scratch. Various ways of introducing a group topology are considered (§3.2), of which the prominent one is by means of characters (§3.2.3). In §4.3 we recall the construction of Protasov and Zelenyuk [131] of topologies arising from a given sequence that is required to be convergent to 0.

In §4.1 we discuss separation axioms and metrizability of topological groups. Connectedness and related properties in topological groups are discussed in §4.2.

In §5 the Markov's problems on the existence of non-discrete Hausdorff group topologies is discussed. In §5.1 we introduce two topologies, the Markov topology and the Zariski topology, that allow for an easier understanding of

Markov's problems. In §5.2 we describe the Markov topology of the infinite permutation groups, while §5.3 contains the first two examples of non-topologizable groups, given by Shelah and Ol′shanskii, respectively. The problems arising in extension of group topologies are the topic of §5.4. Several cardinal invariants (weight, character and density character) are introduced in §6.1, whereas §6.2 discuses completeness and completions. Further general information on topological groups can be found in the monographs or surveys [4, 36, 37, 38, 57, 106, 119, 122].

Section 7 is dedicated to specific properties of the (locally) compact groups used essentially in these notes. The most important property of locally compact group we recall in §7.3 is the open mapping theorem. §7.4 is dedicated to the minimal and the totally minimal groups, which need not be locally compact , yet satisfy the open mapping theorem.

In §8 we recall (with complete proofs) the structure of the closed subgroups of $\mathbb{R}^n$ as well as the description of the closure of an arbitrary subgroup of $\mathbb{R}^n$. These groups play an important role in the whole theory of locally compact abelian groups.

Section 9 starts with §9.1 dedicated to big (large) and small subsets of abstract groups. In §9.2.1 we give an internal description of the precompact groups using the notion of a big set of a group and we show that these are precisely the subgroups of the compact groups. Moreover, we define a precompact group $G^+$ that "best approximates" $G$. Its completion $bG$, the Bohr compactification of $G$, is the compact group that "best approximates" $G$. Here we introduce almost periodic functions and briefly comment their connection to the Bohr compactification of $G$. In §9.2.2 we establish the precompactness of the topologies generated by characters. In §9.3 we recall (without proofs) some relevant notions in the non-abelian case, as Haar integral, unitary representation, etc., that play a prominent role in the general theory of topological groups, but are not used in this exposition. The Haar integral in locally compact abelian groups is built in §11.4.3.

In §10 prepares all ingredients for the proof of Følner's theorem (see Theorem 10.3.5). This proof, follows the line of [57]. An important feature of the proof is the crucial idea, due to Prodanov, to eliminate all discontinuous characters in the uniform approximation of continuous functions via linear combinations of characters obtained by means of Stone-Weierstraß approximation theorem. This step is ensured by Prodanov's lemma 10.3.1, which has many other relevant applications towards independence of characters and the construction of the Haar intergral for LCA groups. The last subsection contains the final stage of the proof of Følner's theorem.

In Section 11 gives various applications of Følner's theorem. The first one is a description of the precompact group topologies of the abelian groups (§10.1). The main application of Følner's theorem is an immediate proof of Peter-Weyl's theorem (in §11.2). To the structure of the compactly generated locally compact abelian groups is dedicated §11.3. Applications of these structure theorems are given in §11.4.3. In §11.3 we provide useful information on the dual of a locally compact abelian group, to be used in Section 12. §11.4 is dedicated to the almost periodic functions of the abelian group. As another application of Følner's theorem we give a proof of Bohr - von Neumann's theorem describing the almost periodic functions as uniform limits of linear combinations of characters. Among other things, we obtain as a by-product of Prodanov's approach an easy construction of the Haar integral for almost periodic functions on abelian groups, in particular for all continuous functions on a compact abelian group (§11.4.2). In §11.4.3 we build a Haar integral on arbitrary locally compact abelian groups, using the construction from §11.4.2 in the compact case. In §11.5 we consider a precompact version of the construction form §4.3 of topologies making a fixed sequence converge to 0.

Section 12 is dedicated to Pontryagin-van Kampen duality. In §§12.1-12.3 we construct all tools for proving the duality theorem 12.5.4. More specifically, §§12.1 and 12.2 contain various properties of the dual groups that allow for an easier computation of the dual in many cases. Using further the properties of the dual, we see in §12.3 that many specific groups $G$ satisfy the duality theorem, i.e., $G \cong \widehat{\widehat{G}}$. In §12.4 we stress the fact that the isomorphism $G \cong \widehat{\widehat{G}}$ is *natural* by studying in detail the natural transformation $\omega_G : G \to \widehat{\widehat{G}}$ connecting the group with its bidual. It is shown in several steps that $\omega_G$ is an isomorphism, considering larger and larger classes of locally compact abelian groups $G$ where the duality theorem holds (elementary locally compact abelian groups, compact abelian groups, discrete abelian groups, compactly generated locally compact abelian groups). The last step uses the fact that the duality functor is exact, this permits us to use all previous steps in the general case. As an immediate application of the duality theorem we obtain the main structure theorem for the locally compact abelian groups, a complete description of the monothetic compact groups, the torsion compact abelian groups, the connected compact abelian groups with dense torsion subgroup, etc.

In the Appendix we dedicate some time to several topics that are not discussed in the main body of the notes: uniqueness of the duality, dualities for non-abelian or non-locally compact-groups, pseudocompact groups and finally, some connection to the topological properties of compact group and dynamical systems.

A large number of exercises is given in the text to ease the understanding of the basic properties of group topologies and the various aspects of the duality theorem.

These notes are born out of three courses in the framework of the PhD programs at the Department of Mathematics at Milan University, the Department of Geometry and Topology at the Complutense University of Madrid and Department of Mathematics at Nanjing Normal University held in 2006, 2007 and 2016, respectively. Among the participants there were various groups, interested in different fields. To partially satisfy the interest of the audience I included various parts that can be eventually skipped, at least during the first reading. For example, the reader who is not interested in non-abelian groups can skip §§3.2.4, the entire §5 and take all groups abelian in §§3,4, 6 and 7 (conversely, the reader interested in non-abelian groups or rings may dedicate more time to §§3.2.4, 5 and consider the non-abelian case also in the first half of §7.2, see the footnote at the beginning of §7.2). For the category theorists §§7.3, 8, 9.2–9.3 may have less interest, compared to §§3-6, 7.2, 9.1, 10.1-10.4 and 11.1-11.2. Finally, those interested to get as fast as possible to the proof of the duality theorem can skip §§3.2.3, 3.2.4 and 4.3-6.2 (in particular, the route §§8–10 is possible for the reader with sufficient knowledge of topological groups).

Several favorable circumstances helped in creating these notes. My sincere thanks go to my colleagues E. Martín-Peinador, M. J. Chasco, M. G. Bianchi, Wei He, L. Außenhofer, X. Domíngues, M. Bruguera, S. Trevijano, and E. Pacifici who made these courses possible. The younger participants of the course motivated me with their constant activity and challenging questions. I thank them for their interst and patience. Thanks are due to George Bergman, who kindly pointed out an error in the proof of Theorem 4.3.4.

This notes are dedicated to the memory of my teacher, Professor Ivan Prodanov, whose original contributions to Pontryagin-van Kampen duality and its applications can hardly by overestimated. The line adopted here in the proof of Pontryagin-van Kampen duality theorem follows his approach from [127, 128] and [57] (see also the recent [67]).

The course of Topological groups (Topologia 2) started in the academic year 1998/99. The lectures notes (in Italian) of that course, merged to a preliminary much shorter version of §§10–12, kindly prepared by Anna Giordano Bruno in 2005, became the backbone of these notes around May/June 2007, available on-line on `https://users.dimi.uniud.it/~dikran.dikranjan/ITG.pdf`. Since then, they were periodically up-dated and some more material was gradually added over the years.

Udine, February 26, 2016

Dikran Dikranjan

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**Notation and terminology**

We denote by $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{N}_+$ respectively the set of primes, the set of natural numbers and the set of positive integers. The symbol $\mathfrak{c}$ stands for the cardinality of the continuum. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ will denote the integers, the rationals, the reals and the complex numbers, respectively.

The quotient $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a compact divisible abelian group, topologically isomorphic to the unitary circle $\mathbb{S}$ (i.e., the subgroup of all $z \in \mathbb{C}$ with $|z| = 1$). For $\mathbb{S}$ we use the multiplicative notation, while for $\mathbb{T}$ we use the additive notation.

For a topological group $G$ we denote by $c(G)$ the connected component of the identity $e_G$ in $G$. If $c(G)$ is trivial, the group $G$ is said to be *totally disconnected*. If $M$ is a subset of $G$ then $\langle M\rangle$ is the smallest subgroup of $G$ containing $M$ and $\overline{M}$ is the closure of $M$ in $G$. The symbol $w(G)$ stands for the weight of $G$. Moreover $\widetilde{G}$ stands for the completion of a Hausdorff topological abelian group $G$ (see §6.2).

# 2 Background on abstract groups, topological spaces and category theory

## 2.1 Background on groups

Generally a group $G$ will be written multiplicatively and the neutral element will be denoted by $e_G$, simply $e$ or 1 when there is no danger of confusion. For abelian groups we use additive notation, consequently 0 will denote the neutral element in such a case.

For a subset $A, A_1, A_2, \ldots, A_n$ of a group $G$ we write

$$A^{-1} = \{a^{-1} : a \in A\}, \quad \text{and} \quad A_1A_2\ldots A_n = \{a_1\ldots a_n : a_i \in A_i, i = 1, 2, \ldots, n\} \tag{$*$}$$

and we write $A^n$ for $A_1A_2\ldots A_n$ if all $A_i = A$. Moreover, for $A \subseteq G$ we denote by $c_G(A)$ the *centralizer* of $A$, i.e., the subgroup $\{x \in G : xa = ax \text{ for every } a \in A\}$.

For a family $\{G_i : i \in I\}$ of groups we denote by $\prod_{i\in I} G_i$ the *direct product* $G$ of the groups $G_i$. The underlying set of $G$ is the Cartesian product $\prod_{i\in I} G_i$ and the operation is defined coordinatewise. For $x = (x_i) \in \prod_{i\in I} G_i$ the *support of* $x$ is the set $\{i \in I : x_i \neq e_{G_i}\}$. The *direct sum* $\bigoplus_{i\in I} G_i$ is the subgroup of $\prod_{i\in I} G_i$ consisting of all elements of finite support. If all $G_i$ are isomorphic to the same group $G$ and $|I| = \alpha$, we write $\bigoplus_\alpha G$ (or $G^{(\alpha)}$, or $\bigoplus_I G$) for the direct sum $\bigoplus_{i\in I} G_i$.

Clearly, the counterpart of (*) will be $-A$ and $A_1 + A_2 + \ldots + A_n$ (and $nA$ for $A^n$).

A standard reference for abelian groups is the monograph [80]. We give here only those facts or definitions that appear very frequently in the sequel.

For abelian groups $G, H$ we denote by $\mathrm{Hom}\,(G, H)$ the group of all homomorphisms from $G$ to $H$ where the operation is defined pointwise. The group $\mathrm{Hom}\,(G, \mathbb{T})$ will be written additively. Sometimes the multiplicative form $G^* = \mathrm{Hom}\,(G, \mathbb{S}) \cong \mathrm{Hom}\,(G, \mathbb{T})$ will be used as well, when necessary (e.g., concerning easier computation in $\mathbb{C}$, etc.). We call the elements of $\mathrm{Hom}\,(G, \mathbb{T}) \cong \mathrm{Hom}\,(G, \mathbb{S})$ *characters*.

### 2.1.1 Torsion groups and torsion-free groups

For $m \in \mathbb{N}_+$, we use $\mathbb{Z}_m$ or $\mathbb{Z}(m)$ for the finite cyclic group of order $m$. Let $G$ be an abelian group and $m \in \mathbb{N}_+$ let

$$G[m] = \{x \in G : mx = 0\} \quad \text{and} \quad mG = \{mx : x \in G\}.$$

Then the torsion elements of $G$ form a subgroup of $G$ denoted by $t(G)$. The increasing union $\bigcup_n G[p^n]$ is a subgroup of $G$ that we denote by $t_p(G)$ and call *p-primary component* of $G$. It is not hard to check that $t(G) = \bigoplus_{p\in\mathbb{P}} t_p(G)$.

For an abelian group $G$ and a prime number $p$ the subgroup $G[p]$ is a vector space over the finite field $\mathbb{Z}/p\mathbb{Z}$. We denote by $r_p(G)$ its dimension over $\mathbb{Z}/p\mathbb{Z}$ and call it *p-rank* of $G$. The *socle* of $G$ is the subgroup $Soc(G) = \bigoplus_{p\in\mathbb{P}} G[p]$. Note that the non-zero elements of $Soc(G)$ are precisely the elements of square-free order of $G$.

Let us start with the structure theorem for finitely generated abelian groups.

**Theorem 2.1.1.** *If $G$ is a finitely generated abelian group, then $G$ is a finite direct product of cyclic groups. Moreover, if $G$ has $m$ generators, then every subgroup of $G$ is finitely generated as well and has at most $m$ generators.*

**Definition 2.1.2.** An abelian group $G$ is

(a) *torsion* if $t(G) = G$ (a *p-group*, for a prime $p$, if $t_p(G) = G$);

(b) *torsion-free* if $t(G) = 0$;

(c) *bounded* if $mG = 0$ for some $m > 0$.

**Example 2.1.3.** (a) The groups $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are torsion-free. The class of torsion-free groups is stable under taking direct products and subgroups.

(b) The groups $\mathbb{Z}_m$ and $\mathbb{Q}/\mathbb{Z}$ are torsion. The class of torsion groups is stable under taking direct sums, subgroups and quotients.

(c) Let $m_1, m_2, \ldots, m_k > 1$ be naturals and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be cardinal numbers. Then the group $\bigoplus_{i=1}^k \mathbb{Z}_{m_i}^{(\alpha_i)}$ is bounded. According to a theorem of Prüfer every bounded abelian group has this form [80]. This generalizes the Frobenius-Stickelberger theorem about the structure of the finite abelian groups (see Theorem 2.1.1).

### 2.1.2 Free abelian groups

A subset $X$ of an abelian group $G$ is *independent*, if $\sum_{i=1}^n k_i x_i = 0$ with $k_i \in \mathbb{Z}$ and distinct elements $x_i$ of $X$, $i = 1, 2, \ldots, n$, imply $k_1 = k_2 = \ldots = k_n = 0$. The maximum size of an independent subset of $G$ is called *free-rank* of $G$ and denoted by $r_0(G)$ (see Exercise 2.1.5 for the correctness of this definition).

The next pair of exercises takes care of the correctness of the definition of $r_0(G)$.

**Exercise 2.1.4.** *For a torsion-free abelian group $H$ prove that:*

*(a) there exists a linear space $D(H)$ over the field $\mathbb{Q}$ containing $H$ as a subgroup and such that $D(H)/H$ is torsion;*

*(b) for $H$ and $D(H)$ as in (a) prove that a subset $X$ in $H$ is independent (resp., maximal independent) iff it is linearly independent (resp., a base) in the $\mathbb{Q}$-vector space $D(H)$;*

*(c) conclude from (b) that all maximal independent subsets of $H$ have the same size (namely,* $\dim_{\mathbb{Q}}(H)$*).*

Hint. (a) Consider the relation $\sim$ in $X = H \times \mathbb{N}_+$ defined by $(h, n) \sim (h', n')$ precisely when $n'h = nh'$. Then the quotient set $D(H) = X/\sim$ carries a binary operation defined by $(h, n) + (h', n') = (n'h + nh')/nn'$. Show that $D(H)$ is the desired linear space.

**Exercise 2.1.5.** *For an abelian group $G$ and the canonical homomorphism $f : G \to G/t(G)$ prove that:*

*(a) if $X$ is a subset of $G$, then $X$ is independent iff $f(X)$ is independent;*

*(b) conclude from (a) and Exercise 2.1.4 that all maximal independent subsets of any abelian group have the same size.*

*(c) $r_0(G) = r_0(G/t(G))$ for every abelian group $G$.*

An abelian group $G$ is *free* , if $G$ has an independent set of generators $X$. In such a case $G \cong \bigoplus_{|X|} \mathbb{Z}$, the isomorphism defined by $G \ni g \mapsto (k_x)_{x \in X}$, where $g = \sum_{x \in X} k_x x$ (note that only finitely many $k_x \neq 0$, so effectively $(k_x)_{x \in X} \in \bigoplus_{|X|} \mathbb{Z}$).

**Lemma 2.1.6.** *An abelian group $G$ is free iff $G$ has a set of generators $X$ such that every map $f : X \to H$ to an abelian group $H$ can be extended to a homomorphism $\bar{f} : G \to H$.*

*Proof.* Let $F = \bigoplus_X \mathbb{Z}$ be the free group of $|X|$-many generators and let $e_x$ denote the generator of the $x$-th copy of $\mathbb{Z}$ in $F$. The set $S = \{e_x : x \in X\}$ generates $F$. Every map $f : S \to G$ to an abelian group $G$ extends to a homomorphism $\bar{f} : F \to G$ by letting $\bar{f}(\sum_{i=1}^n k_i e_{x_i}) = \sum_{i=1}^n k_i f(e_{x_i})$.

Now assume that the group $G$ has a set of generators $X$ with the above property. To prove that $G$ is free, it suffices to show that $X$ is independent. As above, let $F = \bigoplus_X \mathbb{Z}$ be the free group of $|X|$-many generators, where $e_x$ denotes the generator of the $x$-th copy of $\mathbb{Z}$ in $F$, so that $S = \{e_x : x \in X\}$ is an independent set of generators of $F$. Define $f : X \to F$ by $f(x) = e_x$ and let $\bar{f} : G \to H$ be its extension. Since $f(X) = S$ is independent in $F$, we deduce that $X$ is independent as well. □

We collect here some useful properties of the free abelian groups.

**Lemma 2.1.7.** *(a) Every abelian group is (isomorphic to) a quotient group of a free group.*

*(b) If $G$ is an abelian group such that for a subgroup $H$ of $G$ the quotient group $G/H$ is free, then $H$ is a direct summand of $G$.*

*(c) A subgroup of a free abelian group is free.*

*Proof.* (a) follows from Lemma 2.1.6. To prove (b), fix an independent set of generators $X$ of $G/H$ and let $q : G \to G/H$ be the quotient homomorphism. For every $x \in X$ pick an element $s(x) \in G$ such that $q(s(x)) = x$. Let $f : G/H \to f$ be the homomorphism extending $s$. Then $q \circ f = id_{G/H}$ as $q \circ s = id_X$. This implies that $H \cap f(G/H) = \{0\}$ and $H + f(G/H) = G$. Hence, $G = H \oplus f(G/H)$.

For a proof of (c) see [80]. □

### 2.1.3 Divisible abelian groups

**Definition 2.1.8.** An abelian group $G$ is said to be

(a) *divisible* if $G = mG$ for every $m > 0$;

(b) *p-divisible*, for $p \in \mathbb{P}$, if $G = mG$.

As $nG \cap mG = mnG$ whenever $n, m$ are co-prime, it is clear that a group is divisible if and only if it is $p$-divisible for every prime $p$.

**Example 2.1.9.** (a) The groups $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{T}$ are divisible.

(b) For $p \in \mathbb{P}$ we denote by $\mathbb{Z}(p^\infty)$ the *Prüfer group*, namely the $p$-primary component of the torsion group $\mathbb{Q}/\mathbb{Z}$ (so that $\mathbb{Z}(p^\infty)$ has generators $c_n = 1/p^n + \mathbb{Z}$, $n \in \mathbb{N}$). The group $\mathbb{Z}(p^\infty)$ is divisible.

(c) The class of divisible groups is stable under taking direct products, direct sums and quotients. In particular, every abelian group has a maximal divisible subgroup $div(G)$.

(d) [80] Every divisible group $G$ has the form $(\bigoplus_{r_0(G)} \mathbb{Q}) \oplus (\bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(r_p(G))})$.

If $X$ is a set, a set $Y$ of functions of $X$ to a set $Z$ *separates the points* of $X$ if for every $x, y \in X$ with $x \neq y$, there exists $f \in Y$ such that $f(x) \neq f(y)$. Now we see that the characters separate the points of a discrete abelian groups.

**Theorem 2.1.10.** *Let $G$ be an abelian group, $H$ a subgroup of $G$ and $D$ a divisible abelian group. Then for every homomorphism $f : H \to D$ there exists a homomorphism $\overline{f} : G \to D$ such that $\overline{f}\restriction_H = f$.*

*If $a \in G \setminus H$ and $D$ contains elements of arbitrary finite order, then $\overline{f}$ can be chosen such that $\overline{f}(a) \neq 0$.*

*Proof.* Let $H'$ be a subgroup of $G$ such that $H' \supseteq H$ and suppose that $g : H' \to D$ is such that $g\restriction_H = f$. We prove that for every $x \in G$, defining $N = H' + \langle x\rangle$, there exists $\overline{g} : N \to D$ such that $\overline{g}\restriction_{H'} = g$. There are two cases.

If $\langle x\rangle \cap H' = \{0\}$, then define $\overline{g}(h + kx) = g(h)$ for every $h \in H'$ and $k \in \mathbb{Z}$. Then $\overline{g}$ is a homomorphism. This definition is correct because every element of $N$ can be represented in a unique way as $h+kx$, where $h \in H'$ and $k \in \mathbb{Z}$.

If $C = \langle x\rangle \cap H' \neq \{0\}$, then $C$ is cyclic, being a subgroup of a cyclic group. So $C = \langle lx\rangle$ for some $l \in \mathbb{Z}$. In particular, $lx \in H'$ and we can consider the element $a = g(lx) \in D$. Since $D$ is divisible, there exists $y \in D$ such that $ly = a$. Now define $\overline{g} : N \to D$ putting $\overline{g}(h + kx) = g(h) + kx$ for every $h + kx \in N$, where $h \in H'$ and $k \in \mathbb{Z}$. To see that this definition is correct, suppose that $h+kx = h'+k'x$ for $h, h' \in H'$ and $k, k' \in \mathbb{Z}$. Then $h-h' = k'x-kx = (k'-k)x \in C$. So $k - k' = sl$ for some $s \in \mathbb{Z}$. Since $g : H' \to D$ is a homomorphism and $lx \in H'$, we have

$$g(h) - g(h') = g(h - h') = g(s(lx)) = sg(lx) = sa = sly = (k' - k)y = k'y - ky.$$

Thus, from $g(h) - g(h') = k'y - ky$ we conclude that $g(h) + ky = g(h') + k'y$. Therefore $\overline{g}$ is correctly defined. Moreover $\overline{g}$ is a homomorphism and extends $g$.

Let $\mathcal{M}$ be the family of all pairs $(H_i, f_i)$, where $H_i$ is a subgroup of $G$ containing $H$ and $f_i : H_i \to D$ is a homomorphism extending $f : H \to D$. For $(H_i, f_i), (H_j, f_j) \in \mathcal{M}$ let $(H_i, f_i) \leq (H_j, f_j)$ if $H_i \leq H_j$ and $f_j$ extends $f_i$. In this way $(\mathcal{M}, \leq)$ is partially ordered. Let $\{(H_i, f_i)\}_{i\in I}$ a totally ordered subset of $(\mathcal{M}, \leq)$. Then $H_0 = \bigcup_{i\in I} H_i$ is a subgroup of $G$ and $f_0 : H_0 \to D$ defined by $f_0(x) = f_i(x)$ whenever $x \in H_i$, is a homomorphism that extends $f_i$ for every $i \in I$. This proves that $(\mathcal{M}, \leq)$ is inductive and so we can apply Zorn's lemma to find a maximal element $(H_{\max}, f_{\max})$ of $(\mathcal{M}, \leq)$. Using the first part of the proof, we can conclude that $H_{\max} = G$.

Suppose now that $D$ contains elements of arbitrary finite order. If $a \in G \setminus H$, we can extend $f$ to $H + \langle a\rangle$ defining it as in the first part of the proof. If $\langle a\rangle \cap H = \{0\}$ then $\overline{f}(h + ka) = f(h) + ky$ for every $k \in \mathbb{Z}$, where $y \in D \setminus \{0\}$. If $\langle a\rangle \cap H \neq \{0\}$, since $D$ contains elements of arbitrary order, we can choose $y \in D$ such that $\overline{f}(h + ka) = f(h) + ky$ with $y \neq 0$. In both cases $\overline{f}(a) = y \neq 0$. □

**Corollary 2.1.11.** *Let $G$ be an abelian group and $H$ a subgroup of $G$. If $\chi \in \mathrm{Hom}\,(H, \mathbb{T})$ and $a \in G \setminus H$, then $\chi$ can be extended to $\overline{\chi} \in \mathrm{Hom}\,(G, \mathbb{T})$, with $\overline{\chi}(a) \neq 0$.*

*Proof.* In order to apply Theorem 2.1.10 it suffices to note that $\mathbb{T}$ has elements of arbitrary finite order. □

**Corollary 2.1.12.** *If $G$ is an abelian group, then* $\mathrm{Hom}\,(G,\mathbb{T})$ *separates the points of $G$.*

*Proof.* If $x \neq y$ in $G$, then $a = x - y \neq 0$ so there exists $\chi \in Hom\,(G,\mathbb{T})$ with $\chi(a) \neq 0$, i.e., $\chi(x) \neq \chi(y)$. □

**Corollary 2.1.13.** *If $G$ is an abelian group and $D$ a divisible subgroup of $G$, then there exists a subgroup $B$ of $G$ such that $G = D \times B$. Moreover, if a subgroup $H$ of $G$ satisfies $H \cap D = \{0\}$, then subgroup $B$ can be chosen to contain $H$.*

*Proof.* Since the first assertion can be obtained from the second one with $H = \{0\}$, let us prove directly the second assertion.

Since $H \cap D = \{0\}$, we can define a homomorphism $f : D + H \to D$ by $f(x + h) = x$ for every $x \in D$ and $h \in H$. By Theorem 2.1.10 we can extend $f$ to $\overline{f} : G \to G$. Then put $B = \ker \overline{f}$ and observe that $H \subseteq B$, $G = D + B$ and $D \cap B = \{0\}$; consequently $G \cong D \times B$. □

Call a subgroup $H$ of a not necessarily abelian group $G$ *essential*, if every non-trivial normal subgroup of $G$ non-trivially meets $H$.

**Exercise 2.1.14.** *Prove that for every abelian group $G$ there exists a divisible abelian group $D(G)$ containing $G$ as an essential subgroup. If $D'(G)$ is another group with the same properties, then there exists an isomorphism $i : D(G) \to D'(G)$ such that $i \restriction_G) id_G$.*

The divisible group $D(G)$ defined above is called *divisible hull* of $G$. When $G$ is torsion-free, then $D(G)$ is torsion-free as well, so it is a $\mathbb{Q}$-linear space (it coincides with the group $D(G)$ built in Exercise 2.1.4).

**Exercise 2.1.15.** *Prove that:*

*(a)* $D(\mathbb{Z}_p) = \mathbb{Z}(p^\infty)$ *for every prime $p$.*

*(b)* $G = D(Soc(G))$ *for every torsion divisible group $G$.*

### 2.1.4 Reduced abelian groups

**Definition 2.1.16.** An abelian group $G$ is *reduced* if the only divisible subgroup of $G$ is the trivial one.

**Example 2.1.17.** It is easy to see that every free abelian group is reduced. Moreover, every bounded torsion group is reduced as well. Finally, every proper subgroup of $\mathbb{Q}$ is reduced.

**Exercise 2.1.18.** *Prove that*

*(a) subgroups, as well as direct products of reduced groups are reduced.*

*(b) every abelian group is a quotient of a reduced group.*

(Hint. (a) is easy, for (b) use Fact 2.1.7 and Example 2.1.17.)

A group $G$ is said to be *residually finite*, if $G$ us isomorphic to a subgroup of a direct product of finite groups. The *Ulm subgroup* $G^1$ of an abelian group $G$ is defined by $G^1 := \bigcap_{n=1}^{\infty} nG$.

**Exercise 2.1.19.** *Prove that*

*(a) a group $G$ is residually finite if and only if the intersection of all normal subgroups of $G$ of finite index is trivial;*

*(b) every residually finite abelian group is reduced;*

*(c) an abelian group $G$ is residually finite if and only if $G^1 = \{0\}$;*

*(d) a torsion-free abelian group is reduced if and only if $G^1 = \{0\}$.*

Item (d) of the above exercise allows us to conclude that for torsion-free abelian groups the notions "reduced" and "residually finite" coincide. This fails to be true for torsion abelian groups.

Now we obtain as a consequence of Corollary 2.1.13 the following important factorization theorem for arbitrary abelian groups.

**Theorem 2.1.20.** *Every abelian group $G$ can be written as $G = div(G) \times R$, where $R$ is a reduced subgroup of $G$.*

*Proof.* By Corollary 2.1.13 there exists a subgroup $R$ of $G$ such that $G = div(G) \times R$. To conclude that $R$ is reduced it suffices to apply he definition of $div(G)$. □

In particular, this theorem implies that every abelian $p$-group $G$ can be written as $G = (\bigoplus_\kappa \mathbb{Z}(p^\infty)) \times R$, where $\kappa = r_p(div(G))$ and $R$ is a reduced $p$-group.

Clearly, direct sums of cyclic groups is a reduced group. The following notion is important in the study of reduced $p$-groups, as allows to "approximate" them appropriately via direct sums of cyclic subgroups.

**Definition 2.1.21.** Let $G$ be a $p$-group. Then a *basic subgroup* of $G$ is a subgroup $B$ of $G$ with the following properties:

(a) $B$ is a direct sum of cyclic subgroups;

(b) $B$ is pure (i.e., $p^nG \cap B = p^nB$ for every $n \in \mathbb{N}$),

(c) $G/B$ is divisible.

It can be proved that every abelian $p$-group admits a basic subgroup ([80]).

**Example 2.1.22.** If an abelian $p$-group has a bounded basic subgroup $B$, then $B$ splits off as a direct summand, so $G = B \oplus D$, where $D \cong G/B$ is divisible. (Indeed, if $p^nB = 0$, then by (b) we get $p^nG \cap B = 0$. On the other hand, by (c), $G = p^nB + B$, so this sum is direct.)

In partcular, if $r_p(G) < \infty$ and $G$ is infinite, then $G$ contains a copy of the group $\mathbb{Z}(p^\infty)$. Indeed, fix a basic subgroup $B$ of $G$. Then $r_p(B) \leq r_p(G)$ is finite, so $B$ is bounded (actually, finite). Hence $G = B \oplus D$ with $D \cong \mathbb{Z}(p^\infty)^k$ with $k \leq r_p(G)$. Since $B$ is finite, necessarily $k > 0$, so $G$ contains a copy of the group $\mathbb{Z}(p^\infty)$.

The ring of endomorphisms of the group $\mathbb{Z}(p^\infty)$ will be denoted by $\mathbb{J}_p$, it is isomorphic to the inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ of the finite rings $\mathbb{Z}/p^n\mathbb{Z}$, known also as the ring of *p-adic integers*. The field of quotients of $\mathbb{J}_p$ is called *field of p-adic numbers*) and will be denoted by $\mathbb{Q}_p$. Sometimes we shall consider only the underlying groups of these rings and will simply speak of "the group $p$-adic integers", or "the group $p$-adic numbers".

### 2.1.5 Extensions of abelian groups

**Definition 2.1.23.** Let $A$ and $C$ be abelian groups. An abelian group $B$ is said to be an *extension of $A$ by $C$* if $B$ has a subgroup $A' \cong A$ such that $B/A' \cong C$.

In such a case, if $i : A \to B$ is the injective homomorphism with $i(A) = A'$ and $B/A' \cong C$, we shall briefly denote this by the diagram

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ q\ } C \longrightarrow 0, \tag{1}$$

where $q$ is the composition of the canonical homomorphism $B \to B/i(A)$ and the isomorphism $B/i(A) \cong C$. More generally, we shall refer to (1), as well as to any pair of group homomorphisms $i : A \to B$ and $q : B \to C$ with $\ker q = i(A)$, $\ker i = 0$ and $Cokerq = 0$, speaking of a *short exact sequence*.

**Example 2.1.24.** A typical extension of a group $A$ by a group $C$ is the direct sum $B = A \oplus D$. This extension we call *trivial extension*.

(a) There may exist non-trivial extenisons, e.g. $\mathbb{Z}$ is a non-trivial extension of $\mathbb{Z}$ and $\mathbb{Z}_2$.

(b) In some cases only trivial extensions are available of $A$ by $C$ (e.g., for $A = \mathbb{Z}_2$ and $C = \mathbb{Z}_3$, for more examples see Exercise 2.1.33).

A property $\mathfrak{G}$ of abelian groups is called *stable under extension* (or, *three space property*), if every group $B$ that is an extension of groups both having $\mathfrak{G}$, necessarily has $\mathfrak{G}$.

**Exercise 2.1.25.** *Prove that the following properties of the abelian groups are stable under extension:*
*(a) torsion;*
*(b) torsion-free;*
*(c) divisible;*
*(d) reduced;*
*(e) p-torsion;*
*(f) having no non-trivial p-torsion elements.*

**Lemma 2.1.26.** *Assume that the two horizontal rows are exact in the following commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & G_1 \arrow[r, "i_1"] \arrow[d, "f_1"'] & G \arrow[r, "q_1"] \arrow[d, "f"] & G_2 \arrow[r] \arrow[d, "f_2"] & 0 \\ 0 \arrow[r] & H_1 \arrow[r, "i_2"'] & H \arrow[r, "q_2"'] & H_2 \arrow[r] & 0 \end{tikzcd} \qquad (*)$$

*If the homomorphism $f_1$and $f_2$ and surjective (injective), then also $f$ is surjective (injective).*

*Proof.* Assume that both $f_1$ and $f_2$ are surjective. Since the second row is exact, we have $\ker q_2 = Imi_2$. We prove now that

$$\ker q_2 = Imi_2 \leq f(G).$$

Indeed, if $x \in H_1$, then $i_2(x) \in i_2(f_1(G_1)) = f(i_1(G_1)) \leq f(G)$ by the surjectivity of $f_1$. This proves (*). Now to check the surjectivity of $f$ pick $y \in H$. Then $q_2(y) \in f_2(q_1(G)) = q_2(f(G))$ by the surjectivity of $f_2$ and $q_1$. Therefore, $q_2(x) = q_2(z)$ for some $z \in f(G)$. This yields $x \in z + \ker q_2 \leq f(G) + f(G) = f(G)$, by (*).

Now assume that both $f_1$ and $f_2$ are injective. To prove that $f$ is injective, assume that $f(x) = 0$ for some $x \in G$. Then $0 = q_2(f(x)) = f_2(q_1(x)))0$. By the injectivity of $f_2$ we deduce that $q_1(x) = 0$. Hence, $x \in i_1(G_1)$. Let $x = i_1(y)$ for some $y \in G_1$. So $f(x) = f(i_1(y)) = i_2(f_1(y)) = 0$. As both $i_2$ and $f_1$ are injective, we deduce that $y = 0$. Therefore, $x = 0$. $\square$

Let us find a description of an extension $B$ of given groups $A$ and $B$. Suppose for simplicity that $A$ is a subgroup of $B$ and $C = B/A$. Let $q : B \to C = B/A$ be the canonical map. Since it is surjective one can fix a *section* $s : C \to B$ (namely a map such that $q(s(c)) = c$ for all $c \in C$) with $s(0) = 0$. For $b \in B$ the element $r(b) = b - s(q(b))$ belongs to $A$. This defines a map $r : B \to A$ such that $r\restriction_A = id_A$. Therefore, every element $b \in B$ is uniquely described by the pair $(q(b), r(b)) \in C \times A$ by $b = s(q(b)) + r(b)$. Hence, every element $b \in B$ can be encoded in a unique way by a pair $(c, a) \in C \times A$. If $s$ is a homomorphism, the image $s(C)$ is a subgroup of $B$ and $B \cong s(C) \times A$ splits. From now on we consider the case when $s$ is not a homomorphism. Then, for $c, c' \in C$, the element $f(c, c') := s(c) + s(c') - s(c+c') \in B$ need not be zero, but certainly belongs to $A$, as $q$ is a homomorphism. This defines a function $f : C \times C \to A$ uniquely determined by the extension $B$ and the choice of the section $s$.

The commutativity and the associativity of the operation in $B$ yield:

$$f(c, c') = f(c', c) \text{ and } f(c, c') + f(c + c', c'') = f(c, c' + c'') + f(c', c'') \qquad (1)$$

for all $c, c', c'' \in C$. As the section $s$ satisfies $s(0) = 0$, one has also

$$f(c, 0) = f(0, c) = 0 \text{ for all } c \in C. \qquad (2)$$

A function $f : C \times C \to A$ satisfying (1) and (2) is called a *factor set* (on $C$ to $A$).

The proof of the next proposition is left to the reader.

**Proposition 2.1.27.** *Every factor set $f$ on $C$ to $A$ gives rise to an extension $B$ of $A$ by $C$ defined in the following way. The support of the group is $B = C \times A$, with operation*

$$(c, a) + (c', a') = (c + c', a + a' + f(c, c')) \textit{ for } c, c' \in C, a, a' \in A$$

*and subgroup $A' = \{0\} \times A \cong A$, such that $B/A' \cong C$. Letting $s(c) = (c, 0)$ defines a section $s : C \to B$ giving rise to exactly the initial factor set $f$.*

Let us note that the subset $C' = C \times \{0\} \cong C = s(C)$ *need not be* a subgroup of $B$.

This proposition shows that one can obtain a description of the extension of a given pair of groups $A, C$ by studying the factor sets $f$ on $C$ to $A$. The trivial extension is determined by the identically zero function $f$, if the section $s(c) = c$ is chosen. More precisely one has:

**Example 2.1.28.** It is easy to see that every section $s : C \to B$ with $s(0) = 0$ of the trivial extension $B = C \oplus A$ is defined by $s(c) = c + h(c)$ for $c \in C$, where $h : C \to A$ is map with $h(0) = 0$ (here we identify $C, A$ with subgroups of $C \oplus A$). The factor set associated to this section is obtained, for $c, c' \in C$, by

$$f(c, c') = h(c) + h(c') - h(c + c'). \qquad (1)$$

If $s_1, s_2 : C \to B$ are two sections of the same extension $B$ of $A$ by $C$, then for every $c \in C$ one has $h(c) := s_1(c) - s_2(c) \in A$, i.e., one gets a function $h : C \to A$, such that $s_1 - s_2 = h$. One can see that the factor set $f_1$ and $f_2$ corresponding to $s_1$ and $s_2$ satisfy:

$$f_1(c, c') - f_2(c, c') = h(c) + h(c') - h(c + c'). \tag{2}$$

This motivates the following definition:

**Definition 2.1.29.** Call two factor sets $f_1, f_2 : C \times C \to B$ *equivalent* if (2) holds for some map $h : C \to A$.

**Definition 2.1.30.** Call two extensions $B_1, B_2$ of $A$ by $C$ *equivalent* if there exists a homomorphism $\xi : B_1 \to B_2$ so that the following diagram, where both horizontal rows describe the respective extension,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_1"] \arrow[d, "id_A"'] & B_1 \arrow[r, "q_1"] \arrow[d, "\xi"] & C \arrow[r] \arrow[d, "id_A"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_2"'] & B_2 \arrow[r, "q_2"'] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative.

We leave the proof of the next theorem toe the reader.

**Theorem 2.1.31.** *In the above notation:*

*(a) the homomorphism $\xi$ in the above definition is necessarily an isomorphisms, provided it exists;*

*(b) the extensions $B_1$ and $B_2$ are equivalent iff the factor sets $f_1, f_2$ are equivalent.*

The above theorem gives a description of the set $Ext(C, A)$ of all equivalence classes of extensions of $A$ by $C$ establishing a bijection with the set of all equivalence classes of factor sets. One can prove that $Ext(C, A)$ carries a structure of abelian group (see [112]). We provide a different argument below, using the bijection between $Ext(C, A)$ and the equivalence classes of factor sets from Theorem 2.1.31.

For the reader who is not familiar with cohomology we recall briefly the definition of the cohomology group $H^2(C, A)$ that is nothing else but the set of equivalence classes of factor sets. Since $H^2(C, A)$ carries a natural structure of an abelian group, this provides a group structure also on $Ext(C, A)$ via the bijection from Theorem 2.1.31.

For $n > 0$, let $K^n(C, A)$ denote the set if all maps $C^n \to A$, the elements of $K^n(C, A)$ are named *n-cochains*. Define the *co-boundary operator* $d_n : K^n(C, A) \to K^{n+1}(C, A)$ by

$$d_n f(c_0, c_1, \ldots, c_n) = f(c_1, \ldots, c_n) - f(c_0 + c_1, c_2, \ldots, c_n) + f(c_0, c_1 + c_2, \ldots, c_n) + (-1)^{n+1} f(c_0, c_2, \ldots, c_{n-1}).$$

Then $d_{n+1} \circ d_n = 0$ for all $n$, so that $\ker d_n$ contains Im $d_{n-1}$. Call the elements of $\ker d_n$ *n-cocycles* and the elements of Im $d_{n-1}$ *n-coboundaries*. Let $H^n(C, A) = \ker d_n/\text{Im } d_{n-1}$, the *n-the cohomology group of $C$ with coefficients on $A$*.

**Theorem 2.1.32.** *$Ext(C, A)$ is isomorphic to the cohomology group $H^2(C, A)$.*

*Proof.* We have to prove that:

(a) $f \in K^2(C, A)$ is a 2-cocycle precisely when $f$ satisfies the equation (1);

(b) two cocylces $f_1$ and $f_2$ give rise to the same extension iff (†) holds.

Indeed, each extension $B$ is determined by its factor set $f$, that is a map $C^2 \to A$, i.e., an element of the group $K^2(C, A)$ of all 2-cochains (these are the maps $C \times C \to A$). Let $d_2 : K^2(C, A) \to K^3(C, A)$ be the co-boundary operator. Then the equation (1), written as

$$d_2 f(c, c', c'') = f(c', c'') - f(c + c', c'') + f(c, c' + c'') - f(c, c') = 0$$

wintesses that $d_2 f = 0$, i.e., $f$ is a cocycle in $K^2(C, A)$. Finally, by Exercise 2.1.31 two cocylces $f_1$ and $f_2$ give rise to the same extension iff (†) holds. Since $h(c) + h(c') - h(c + c') = d_1(c, c')$ for $h \in K^1(C, A)$ and the coboundary map $d_1 : K^1(C, A) \to K^2(C, A)$, (†) says that $f_1 - f_2$ is a coboundary in $K^2(C, A)$. Therefore, these two cocycles give rise to the same element in $H^2(C, A)$. □

In the sequel, using the fact that $Ext(C, A)$ is a group, we write $Ext(C, A) = 0$ to say that there are only trivial extensions of $A$ by $C$.

**Exercise 2.1.33.** *Prove that $Ext(C, A) = 0$ in the following cases:*

*(a) $A$ is divisible;*

*(b) $C$ is free.*

*(c) both $A$ and $C$ are torsion and for every $p$ either $r_p(A) = 0$ or $r_p(C) = 0$;*

*(d*) (Theorem of Prüfer) $C$ is torsion free and $A$ has finite exponent.*

Hint. For (a) use Exercise 2.1.13, for (b) – Exercise 2.1.6. For (c) deduce first that every extension $B$ of $A$ by $C$ is torsion and then argue using the hypothesis on $r_p(A)$ and $r_p(C)$. A proof of (d*) can be found in [80].

## 2.2 Background on topological spaces

For the sake of completeness we recall here some frequently used notions and notations from topology.

### 2.2.1 Basic definitions

We start with the definition of a filter and a topology.

**Definition 2.2.1.** Let $X$ be a set. A family $\mathcal{F}$ of non-empty subsets of $X$ is said

(a) to have the *finite intersection property*, if $F_1 \cap F_2 \cap \ldots \cap F_n \neq \emptyset$ for any $n$-tuple $F_1, F_2, \ldots, F_n \in \mathcal{F}$, $n > 1$.

(b) to be a *filterbase* if for $A, B \in \mathcal{F}$ there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$;

(c) A filterbase $\mathcal{F}$ is called a *filter* if $F \subseteq F'$ and $F \in \mathcal{F}$ yield $F' \in \mathcal{F}$;

(d) A filter $\mathcal{F}$ is called an *ultrafilter* if $\mathcal{F} \subseteq \mathcal{F}'$ for some filter $\mathcal{F}'$ yields $\mathcal{F}' = \mathcal{F}$.

Clearly, every filter is a filterbase, while every filterbase has the finite intersection property. If $\mathcal{F}$ has the finite intersection property, then the family $\mathcal{F}^*$ of all finite intersection $F_1 \cap F_2 \cap \ldots \cap F_n \neq \emptyset$, with $F_1, F_2, \ldots, F_n \in \mathcal{F}$, is a filter-base.

For a set $X$ a subfamily $\mathfrak{B}$ of $\mathcal{P}(X)$ is called a *$\sigma$-algebra* on $X$ if $X \in \mathfrak{B}$ and $\mathfrak{B}$ is closed under taking complements and countable unions.

**Exercise 2.2.2.** *Let $f : X \to Y$ be a map. Prove that*

*(a) if $\mathcal{F}$ is a filter on $X$, then $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$ is a filter-base in $Y$;*

*(b) if $f$ is surjective[1] and $\mathcal{F}$ is a filter on $Y$, then $f^{-1}(\mathcal{F}) = \{f^{-1}(F) : F \in \mathcal{F}\}$ is a filter-base in $X$.*

**Exercise 2.2.3.** *Let $X$ be a non-empty set. Prove that every filter $\mathcal{F}$ on $X$ is contained in some ultrafilter.*

(Hint. Apply Zorn's lemma to the ordered by inclusion set of all filters of $X$ containing $\mathcal{F}$.)

**Definition 2.2.4.** Let $X$ be a set. A family $\tau$ of subsets of $X$ is called a *topology* on $X$ if

($c_1$) $X, \emptyset \in \tau$,

($c_2$) $\{U_1, \ldots, U_n\} \subseteq \tau \Rightarrow U_1 \cap \ldots \cap U_n \in \tau$,

($c_3$) $\{U_i : i \in I\} \subseteq \tau \Rightarrow \bigcup_{i \in I} U_i \in \tau$.

The pair $(X, \tau)$ is called a *topological space* and the members of $\tau$ are called *open sets*, the complement of an open set is called *closed*. A set that is simultaneously closed and open is called *clopen*. For $x \in X$ a *neighborhood* of $x$ is any subset of $X$ containing an open set $U \ni x$. The neighborhoods of a point $x$ form a filter $\mathcal{V}(x)$ in $X$. We say that a filter $\mathcal{F}$ on $X$ *converges* to $x \in X$ when $\mathcal{V}(x) \subseteq \mathcal{F}$. We also say that $x$ is a *limit point* of $\mathcal{F}$. In case every member of $\mathcal{F}$ meets every neighborhood of $x$ we say that $x$ is an *adherent point* of $\mathcal{F}$ and we write $x \in ad\,\mathcal{F}$.

**Exercise 2.2.5.** *Prove that if $x$ is an adherent point of an ultrafilter $\mathcal{U}$, then $x$ is also a limit point of $\mathcal{U}$.*

[1] or more generally, $\mathcal{F} \cup \{f(X)\}$ has the finite intersection property.

For a subset $M$ of a space $X$ we denote by $\overline{M}$ is the *closure* of $M$ in $X$, namely the set of all points $x \in X$ such that every $U \in \mathcal{V}(x)$ meets $M$. (Obviously, $M$ is closed iff $\overline{M} = M$.) The set $M$ is called *dense* if $\overline{M} = X$. A topological space $X$ is *separable*, if $X$ has a dense countable subset.

For a subset $M$ of a space $X$ we denote by $\mathrm{Int}\,(M)$ the *interior* of $M$ in $X$, namely the set of all points $x \in M$ such that every $x \in U \subseteq M$ for some $U \in \tau$. (Obviously, $M$ is open iff $\mathrm{Int}\,(M) = M$.) An open set is said to be *regular open* if it coincides with the interior of its closure.

**Exercise 2.2.6.** *For a subset $M$ of a space $X$ prove that*

*(a) $\overline{M}$ is the smallest closed subset of $X$ containing $M$.*

*(b) Int$(M)$ is the largest open subset of $X$ contained in $M$.*

Let $(X,\tau)$ be a topological space and let $Y$ be a subset of $X$. Then $Y$ becomes a topological space when endowed with the *topology induced* by $X$, namely $\tau \restriction_Y = \{Y \cap U : U \in \tau\}$.

For a topological space $(X,\tau)$ a family $\emptyset \notin \mathcal{B} \subseteq \tau$ is a *base* of a the space $X$, if for every $x \in X$ and for every $x \in U \in \tau$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. The symbol $w(X)$ stands for the *weight* of $X$, i.e., $w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } X\}$. We say that a family $\emptyset \notin \mathcal{B} \subseteq \tau$ is a *prebase* of a the space $X$, if the the family $\mathcal{B}^*$ of all finite non-empty intersections of members of $\mathcal{B}$ is a base of $X$.

**Examples:** (1) For every set $X$ the *discrete* topology has as open sets all subsets of $X$; the *indiscrete* topology has as open sets only the sets $X$ and $\emptyset$.

(2) The canonical topology attached to the Euclidean space $\mathbb{R}^n$ $(n \geq 1)$ is defined by the collection of sets $U$ such that, if $x \in U$, then $\{y \in \mathbb{R}^n : \|y - x\| < r\} \subseteq U$ for some $r > 0$.

More general examples can be obtained as follows. We need to recall first the definition of a *pseudometric* on a set $X$. This is a map $d : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$ one has:

**(1)** $d(x,x) = 0$;

**(2)** $d(x,y) = d(y,x)$;

**(3)** $d(x,z) \leq d(x,y) + d(y,z)$.

In case $d(x,y) = 0$ always implies $x = y$, the function $d$ is called a *metric*. A set $X$ provided with a metric $d$ is called a *metric space* and we usually denote a metric space by $(X,d)$. For a point $x \in X$ and $\varepsilon > 0$ the set $B_\varepsilon(x) = \{y \in X : d(y,x) < \varepsilon\}$ is called the *open disk* (*open ball*) with center $x$ and radius $\varepsilon$.

**Example 2.2.7.** Let $(X,d)$ be a metric space. The family $\mathcal{B}$ of all open disks $\{B_\varepsilon(x) : x \in X, \varepsilon > 0\}$ is a base of a topology $\tau_d$ on $X$ called the *metric topology* of $(X,d)$.

For a topological space $(X,\tau)$ denote by $\mathcal{B}(X)$ the $\sigma$-algebra generated by $\tau \subseteq \mathcal{B}(X)$. The members of $\mathcal{B}(X)$ are called *Borel sets*. Some of the Borel sets of $X$ have special names:

**Definition 2.2.8.** A subset $A$ of a topologcal space $X$ is said to be $G_\delta$-*dense*, if every non-empty $G_\delta$ subset of $X$ meets $A$.

The set $\mathcal{T}(X)$ of all topologies on a given set $X$ is ordered by inclusion. For two topologies $\tau_1 \subseteq \tau_2$ on $X$ we write sometimes $\tau_1 \leq \tau_2$ and say that $\tau_1$ is *coarser* than $\tau_2$, while $\tau_2$ is *finer* than $\tau_1$.

Let $\{\tau_i : i \in I\}$ be a family of topologies on a set $X$. Then the intersection $\bigcap_{i\in I} \tau_i$ is a topology on $X$ and it coincides with the *infimum* $\tau = \inf_{i\in I} \tau_i$ of the family $\{\tau_i : i \in I\}$, i.e., it is the finest topology on $X$ contained in every $\tau_i$, $i \in I$.

On the other hand, the *supremum* $\tau = \sup_{i\in I} \tau_i$ is the topology on $X$ with base $\bigcup_{i\in I} \tau_i$, i.e., a basic neighborhood of a point $x \in X$ is formed by the family of all finite intersection $U_1 \cap U_2 \cap \ldots \cap U_n$, where $U_k \in \mathcal{V}_{\tau_{i_k}}(x)$, for $k = 1, 2, \ldots, n$. This is the smallest topology on $X$ that contains all topologies $\tau_i$, $i \in I$. In this way $(\mathcal{T}(X), \inf, \sup)$ becomes a complete lattice with top element the discrete topology and bottom element the indiscrete topology.

### 2.2.2 Separation axioms and other properties of the topological spaces

Now we recall the so called separation axioms for topological spaces:

**Definition 2.2.9.** A a topological space $X$ is

(a) a $T_0$-*space*, if for every pair of distinct points $x, y \in X$ there exists an open set $U$ such that either $x \in U, y \notin U$, or $y \in U, x \notin U$;

(b) a $T_1$-*space*, if for every pair of distinct points $x, y \in X$ there exist open sets $U$ and $V$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$ (or, equivalently, every singleton of $X$ is closed);

(c) a $T_2$-*space* (or, a *Hausdorff space*) , if for every pair of distinct points $x, y \in X$ there exist disjoint open sets $U$, $V$ such that $x \in U$ and $y \in V$.

Moreover,

(d) a $T_0$ space $X$ is called a $T_3$-*space* (or, a *regular space*), if for every $x \in X$ and every open set $x \in U$ in $X$ there exists a an open set $V$ such that $x \in V \subseteq \overline{V} \subseteq U$;

(e) a $T_0$ space $X$ is called a $T_{3.5}$-*space* (or, a *Tychonov space*), if for every $x \in X$ and every open set $x \in U$ in $X$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in X$ and $y \notin U$;

(f) a $T_0$ space $X$ is called a $T_4$-*space* (or, a *normal space*), if for every pair of closed disjoint sets $F, G$ in $X$ there exists a pair of open disjoint sets $U, V$ in $X$ such that $F \subseteq U$ and $G \subseteq V$.

The following implications hold true between these properties

$$T_0 \longleftarrow T_1 \longleftarrow T_2 \longleftarrow T_3 \longleftarrow T_{3.5} \longleftarrow T_4.$$

While the first four implications are more or less easy to see, the last implication $T_4 \to T_{3.5}$ requires the following deep fact:

**Theorem 2.2.10.** (Urysohn Lemma) *Let $X$ be a normal space. Then for every pair of closed disjoint sets $F, G$ in $X$ there esists a continuous function $f : X \to [0, 1]$ such that $f(F) = 1$ and $f(G) = 0$.*

A $T_0$ topological space $X$ having a base of clopen sets is called *zero-dimensional*, denoted by dim $X = 0$. Obviosuly, zero-dimensional spaces are $T_{3.5}$. All these properties (beyond $T_4$) are preserved by taking subspaces.

For the sake of completeness we recall here some frequently used properties of topological spaces. Most of them are related to compactness. A family $\mathcal{U} = \{U_i : i \in I\}$ of non-empty open sets is an *open cover* of $X$ if $X = \bigcup_{i \in I} U_i$. A subfamily $\{U_i : i \in J\}$, $J \subseteq I$, is a *subcover* of $\mathcal{U}$ if $X = \bigcup_{i \in J} U_i$.

**Definition 2.2.11.** *A topological space $X$ is*

- compact *if for every open cover of $X$ there exists a finite subcover;*
- countably compact *if for every countable open cover of $X$ there exists a finite subcover;*
- Lindelöff *if for every open cover of $X$ there exists a countable subcover;*
- pseudocompact *if every continuous function $X \to \mathbb{R}$ is bounded;*
- locally compact *if every point of $X$ has compact neighborhood in $X$;*
- $\sigma$-compact *if $X$ is the union of countably many compact subsets;*
- hemicompact *if $X$ is $\sigma$-compact and has a countable family of compact subsets such that every compact set of $X$ is contained in one of them;*
- Baire space*, if any countable intersection of dense open sets of $X$ is still dense*[2]*;*
- of first category*, if $X = \bigcup_{n=1}^{\infty} A_n$ and every $A_n$ is a closed subset of $X$ with empty interior;*
- of second category*, if $X$ is not of first category;*
- connected *if every proper open subset of $X$ with open complement is empty.*

**Example 2.2.12.** Let $B$ be a subset of $\mathbb{R}^n$ equipped with the usual metric topology. Then $B$ is compact iff $B$ is closed and bounded (i.e., $B$ has finite diameter).

[2]if any countable intersection of dense $G_\delta$-sets of $X$ is still a dense $G_\delta$-set

Obviously, a space is compact iff it is both Lindelöff and countably compact. Compact spaces are locally compact and $\sigma$-compact.

Compactness-like properties "improve" separation properties in the following sense:

**Theorem 2.2.13.** *(a) Every Hausdorff compact space is normal.*

*(b) Every regular Lindelöff space is normal.*

*(c) Every Hausdorff locally compact space is Tychonov.*

It follows from item (a) of Theorem 2.2.13 that every subspace of a compact Hausdorff space is necessarily a Tychonov space. According to Tychonov's embedding theorem every Tychonov space $X$ is a subspace of a compact space $K$, so taking the closure $Y$ of $X$ in $K$ one obtains also a compact space $Y$ containing $X$ as a dense subspace, i.e., a *compactification* of $X$.

### 2.2.3 Relations between the various generalizations of compactness

In the sequel we show the other relations between these properties (see the diagram below for all implications between them).

**Lemma 2.2.14.** *If $X$ is a $\sigma$-compact space, then $X$ is a Lindelöff space.*

*Proof.* Let $X = \bigcup_{\alpha\in I} U_\alpha$. Since $X$ is $\sigma$-compact, $X = \bigcup_{n=1}^\infty K_n$ where each $K_n$ is a compact subset of $X$. Thus for every $n \in \mathbb{N}_+$ there exists a finite subset $F_n$ of $I$ such that $K_n \subseteq \bigcup_{n\in F_n} U_n$. Now $I_0 = \bigcup_{n=1}^\infty F_n$ is a countable subset of $I$ and $K_n \subseteq \bigcup_{\alpha\in I_0} U_\alpha$ for every $N \in \mathbb{N}_+$ yields $X = \bigcup_{\alpha\in I_0} U_\alpha$. □

**Lemma 2.2.15.** *If $X$ is a dense countably compact subspace of a regular space $Y$, then $X$ is $G_\delta$-dense in $Y$.*

*Proof.* Let $O$ be a non-empty $G_\delta$ subset of $Y$. Then there exists $y \in O$ and $O = \bigcap_{n=1}^\infty U_n$, where each $U_n$ is open in $Y$. By the regularity of $Y$ we can find for each $n$ an open set $V_n$ of $Y$ such that $y \in V_n \subseteq \overline{V}_n \subseteq U_n$. Since $W_n = U_1 \cap \ldots \cap U_n \neq \emptyset$ is open, $X \cap W_n \neq \emptyset$. Therefore $F_n = X \cap \overline{V}_1 \cap \ldots \cap \overline{V}_n \neq \emptyset$ for each $n$. Since $X$ is countably compact, also

$$\bigcap_n F_n = X \cap \bigcap_n \overline{V}_n \subseteq X \cap \bigcap_n U_n = X \cap O$$

is non-empty. □

Let us start with a criterion for (countable) compactness.

**Lemma 2.2.16.** *Let $X$ be a topological space.*

*(a) $X$ is (countably) compact iff every (countable) family of closed sets with the finite intersection property has a non-empty intersection.*

*(b) $X$ is compact iff every ultrafilter of $X$ is convergent.*

*Proof.* For the proof (a) note that every family $\mathcal{F}$ of closed sets with the finite intersection property having empty intersection corresponds to an open cover of $X$ without finite subcovers (simply take the complement of the members of $\mathcal{F}$).

(b) Follows from (a) and Exercise 2.2.5. □

**Theorem 2.2.17.** *[Arhangel′skij] If $X$ is a countable compact space, then $X$ is metrizable. In particular, a countably infinite compact space has a non-trivial convergent sequences.*

*Proof.* Obviously, $X$ is normal. On the other hand, $X$ has countable pseudocharacter, so $X$ is first countable (being compact). Hence, $X$ is second countable, as well. By Urysohn metrization theorem, $X$ is metrizable. □

Here comes a criterion for pseudocompactness.

**Theorem 2.2.18.** *Let $X$ be a Tychonov space. Prove that the following are equivalent:*
*(a) $X$ is pseudocompat;*
*(b) every locally finite family of non-empty open sets is finite;*
*(c) for every chain of non-empty open sets $V_1 \supseteq V_2 \supseteq \ldots$ in $Y$ with*

$$\overline{V}_n \subseteq V_{n-1} \text{ for every } n > 1 \qquad (*)$$

*one has $\bigcap_n V_n \neq \emptyset$.*

*Proof.* (a) → (b) Assume that $\{V_n n \in \mathbb{N}\}$ is an infinite locally finite family of non-empty open sets. Fix a point $x_n \in V_n$ for every $n \in \mathbb{N}$. Since $X$ is Tychonov, there exists a continuous function $f_n : X \to [0, 1]$ such that $f_n$ vanishes on $X \setminus V_n$ and $f_n(x_n) = 1$. Define a function $f : X \to \mathbb{R}$ by $f(x) = \sum_n nf_n(x)$, for $x \in X$. Since the family $\{V_n n \in \mathbb{N}\}$ is locally finite and each $f_n$ is continuous, $f$ is continuous as well. Obviously, $f$ is unbounded, as $f(x_n) = n$, for $n \in \mathbb{N}$. This contradicts the pseudocompactness of $X$.

(b) → (c) is obvious.

(c) → (a) Assume that $f : X \to \mathbb{R}$ is an unbounded continuous function. Then for every $n \in \mathbb{N}$ the open set $V_n = f^{-1}(\mathbb{R} \setminus [-n, n])$ is non-empty and obviously (*) and $\bigcap_n V_n = \emptyset$ hold, a contradiction. □

**Remark 2.2.19.** Using the above criterion, one can prove the obvious counterpart of Lemma 2.2.15 for Tychonov spaces: *If $X$ is a dense pseudocompact subspace of a Tychonov space $Y$, then $X$ is $G_\delta$-dense in $Y$* For a proof argue as in the proof of Lemma 2.2.15, starting with a non-empty $G_\delta$ subset $O$ of $Y$, presented as the intersection of a chain of open sets $V_1 \supseteq V_2 \supseteq \dots$ in $Y$ with (*). Then $U_n = X \cap V_n$ is an open set of $X$ with $\overline{U}_n = \overline{V}_n$ in view of the density of $X$ in $Y$. So $\overline{U}_n^X = X \cap \overline{U}_n = X \cap \overline{V}_n \subseteq X \cap V_{n-1} = U_{n-1}$. So, $\bigcap_n \overline{U}_n^X = \bigcap_n \overline{U}_n \neq \emptyset$, by Theorem 2.2.18. Therefore,

$$X \cap O = X \cap \bigcap_n V_n = \bigcap_n U_n = \bigcap_n \overline{U}_n \neq \emptyset.$$

**Exercise 2.2.20.** *If a Tichonov space $X$ is a $G_\delta$-dense subspace of $\beta X$, then $X$ is pseudocompact.*

In the next exercise we resume 2.2.19 and 2.2.20:

**Exercise 2.2.21.** *A Tichonov space $X$ is pseudocompact if and only if $X$ is $G_\delta$-dense in $\beta X$.*

A Baire space $X$ is of second category. Indeed, assume that $X = \bigcup_{n=1}^\infty A_n$ such that every $A_n$ is closed with empty interior. Then the sets $D_n = X \setminus A_n$ are open and dense in $X$. Then $\bigcap_{n=1}^\infty D_n$ is dense, in particular non-empty, so $X \neq \bigcup_{n=1}^\infty A_n$, a contradicton.

According to the Baire category theorem complete metric spaces are Baire. Now we prove that also locally compact spaces are Baire spaces.

**Theorem 2.2.22.** *A Hausdorff locally compact space $X$ is a Baire space.*

*Proof.* Suppose that the sets $D_n$ are open and dense in $X$. We show that $\bigcap_{n=1}^\infty D_n$ is dense. To this end fix an arbitrary open set $V \neq \emptyset$. According to Theorem 2.2.13, a Hausdorff locally compact space is regular. Hence there exists an open set $U_0 \neq \emptyset$ with $\overline{U}_0$ compact and $\overline{U}_0 \subseteq V$. Since $D_1$ is dense, $U_0 \cap D_1 \neq \emptyset$. Pick $x_1 \in U_0 \cap D_1$ and an open set $U_1 \ni x_1$ in $X$ with $\overline{U}_1$ compact and $\overline{U}_1 \subseteq U_0 \cap D_1$ . Proceeding in this way, for every $n \in \mathbb{N}_+$ we can find an open set $U_n \neq \emptyset$ in $G$ with $\overline{U}_n$ compact and $\overline{U}_n \subseteq U_{n-1} \cap D_n$. By the compactness of every $\overline{U}_n$ there exists a point $x \in \bigcap_{n=1}^\infty \overline{U}_n$. Obviously, $x \in V \cap \bigcap_{n=1}^\infty D_n$. □

The above proof works also in the case of complete meric spaces, but the neighborhoodd $U_n$ must be chose each time with $diam\, B_n \leq 1/n$. Then Cantor's theorem (for complete metric spaces) guarantees $\bigcap_{n=1}^\infty \overline{U}_n \neq \emptyset$.

**Exercise 2.2.23.** *Let $Y$ be a Baire space and $X$ be a $G_\delta$-dense subspace of $Y$. Then $X$ is a Baire space as well.*

**Theorem 2.2.24.** *Every countably compact Tychonov space is a Baire space.*

*Proof.* Let $X$ be a countably compact Tychonov space. Take any compactification $Y$ of $X$. Then $Y$ is a Baire space by Theorem 2.2.22. Since $Y$ is regular (Theorem 2.2.13), every non-empty $G_\delta$ subset of $Y$ meets $X$ by Lemma 2.2.15. Now Exercise 2.2.23 applies. □

According to Exercise 2.2.21), a Tichonov space $X$ is pseudocompact iff $X$ is $G_\delta$-dense in $\beta X$. Combining with Exercise 2.2.23, we conclude that pseudocompact spaces are Baire.

In the next diagram we collect all implications between the properties we have discussed so far.

$$\begin{tikzcd}
\text{Lindelöf} & \sigma\text{-compact} \arrow[l] & \text{hemicompact} \arrow[l] & \text{compact} \arrow[l] \arrow[r] \arrow[d] \arrow[dlll, leftrightarrow] & \text{loc.compact} \arrow[r] & \text{Baire} \arrow[r] & 2^d \text{ categ.} \\
\text{Lindelöf+count.compact} \arrow[u] \arrow[rrr] & & & \text{count.compact} \arrow[rr] \arrow[urr] & & \text{pseudocompact} \arrow[u] &
\end{tikzcd}$$

The metric countably compact spaces are compact.

**Exercise 2.2.25.** *Locally compact $\sigma$-compact spaces $X$ are hemicompact.*

(Hint. Use the definitions. If $X$ is Hausdorff, then $X$ is Tychonov by Theorem 2.2.13 (c), so one can consider the one-point compactification of $X$.)

Most of these properties are preserved by taking closed subspaces:

**Lemma 2.2.26.** *If $X$ is a closed subspace of a space $Y$, then $X$ is compact (resp., Lindelöff, countably compact, $\sigma$-compact, locally compact) whenever $Y$ has the same property.*

Now we discuss preservation of properties under unions.

**Lemma 2.2.27.** *Let $X$ be a topological space and assume $X = \bigcup_{i\in I} X_i$, where $X_i$ are subspaces of $X$.*

- *If $I$ is finite and each $X_i$ is (countably) compact, then $X$ is (countably) compact.*
- *If $I$ is countable and each $X_i$ is $\sigma$-compact (resp., Lindelöff), then $X$ has the same property.*
- *If $\bigcap_{i\in I} X_i \neq \emptyset$ and each $X_i$ is connected, then $X$ is connected.*

For every topological space $X$ and $x \in X$ there is a largest connected subset $x \in C_x \subseteq X$, called *connected component of* $x$ in $X$. It is always a closed subset of $X$ and $X = \bigcup_{x\in X} C_x$ is a partition of $X$. The space $X$ is called *totally disconnected* if all connected components are singletons. Obviously, zero-dimensional spaces are totally disconnected (as every point is an intersection of clopen sets). Both properties are preserved by taking subspaces.

In a topological space $X$ the *quasi-component* of a point $x \in X$ is the intersection of all clopen sets of $X$ containing $x$.

**Lemma 2.2.28.** (Shura-Bura) *In a compact space $X$ the quasi-components and the connected components coincide.*

**Theorem 2.2.29.** (Vedenissov) *Every totally disconnected locally compact space is zero-dimensional.*

### 2.2.4 Properties of the continuous maps

Here we recall properties of maps:

**Definition 2.2.30.** For a map $f : (X, \tau) \to (Y, \tau')$ between topological spaces and a point $x \in X$ we say that:

(a) $f$ is *continuous* at $x$ if for every neighborhood $U$ of $f(x)$ in $Y$ there exists a neighborhood $V$ of $x$ in $X$ such that $f(V) \subseteq U$;

(b) $f$ is *open* at $x \in X$ if for every neighborhood $V$ of $x$ in $X$ there exists a neighborhood $U$ of $f(x)$ in $Y$ such that $f(V) \supseteq U$;

(c) $f$ is continuous (resp., open) if $f$ is continuous (resp., open) at every point $x \in X$;

(d) $f$ is *closed* if the subset $f(A)$ of $Y$ is closed for every closed subset $A \subseteq X$;

(e) $f$ is *perfect* if $f$ is closed and $f^{-1}(y)$ is compact for all $y \in Y$;

(f) $f$ is a *homeomorphism* if $f$ is continuous, open and bijective.

In item (a) and (b) one can limit the test to only basic neighborhoods. A topological space $X$ is *homogeneous*, if for every pair of points $x, y \in X$ there exists a homeomorphism $f : X \to X$ such that $f(x) = y$.

Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Consider the Cartesian product $X = \prod_{i\in I} X_i$ with its canonical projections $p_i : X \to X_i$, $i \in I$. Then $X$ usually carries the *product topology* (or *Tichonov topology*), having as a base $\mathcal{B}$ the family $\bigcap\{p_i^{-1}(U_i) : (\forall\, i \in J)U_i$ open in $X_i\}$, where $J$ runs over the finite subsets of $I$. When $X$ is equipped with this topology the projections $p_i$ are both open and continuous.

Some basic properties relating spaces to continuous maps are collected in the next lemma:

**Lemma 2.2.31.** *If $f : X \to Y$ is a continuous surjective map, then $Y$ is compact (resp., Lindelöff, countably compact, $\sigma$-compact, connected) whenever $X$ has the same property.*

A partially ordered set $(A, \leq)$ is *directed* if for every $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A subset $B$ of $A$ is *cofinal*, if for every $\alpha \in A$ there exists $\beta \in B$ with $\alpha \leq \beta$.

A *net* in a topological space $X$ is a map from a directed set $A$ to $X$. We write $x_\alpha$ for the image of $\alpha \in A$ so that the net can be written in the form $N = \{x_\alpha\}_{\alpha\in A}$. A *subnet* of a net $N$ is $S = \{x_\beta\}_{\beta\in B}$ such that $B$ is a cofinal subset of $A$.

A net $\{x_\alpha\}_{\alpha\in A}$ in $X$ *converges* to $x \in X$ if for every neighborhood $U$ of $x$ in $X$ there exists $\beta \in A$ such that $\alpha \in A$ and $\beta \leq \alpha$ implies $\alpha \in U$.

**Lemma 2.2.32.** *Let $X$ be a topological space.*

*(a) If $Z$ is a subset of $X$, then $x \in \overline{Z}$ if and only if there exists a net in $Z$ converging to $x$.*

*(b) $X$ is compact if and only if every net in $X$ has a convergent subnet.*

*(c) A map $f : X \to Y$ (where $Y$ is a topological space) is continuous if and only if $f(x_\alpha) \to f(x)$ in $Y$ for every net $\{x_\alpha\}_{\alpha \in A}$ in $X$ with $x_\alpha \to x$.*

*(d) The space $X$ is Hausdorff if and only if every net in $X$ converges to at most one point in $X$.*

By $\beta X$ we denote the *Čech-Stone compactification* of a topological Tychonov space $X$, that is the compact space $\beta X$ together with the dense immersion $i : X \to \beta X$, such that for every function $f : X \to [0,1]$ there exists $f^\beta : \beta X \to [0,1]$ which extends $f$ (this is equivalent to ask that every function of $X$ to a compact space $Y$ can be extended to $\beta X$). Here $\beta X$ will be used mainly for a discrete space $X$.

The next theorem shows that many of the properties of the topological spaces are preserved under products. As far as compactness is concerned, this is known as *Tichonov Theorem*:

**Theorem 2.2.33.** *Let $X = \prod_{i \in I} X_i$. Then*

*(a) $X$ is compact (resp., connected, totally disconnected, zero-dimensional, $T_0$, $T_1$, $T_2$, $T_3$, $T_{3.5}$) iff every space $X_i$ has the same property.*

*(b) if $I$ is finite, the same holds for local compactness and $\sigma$-compactness.*

Let us mention here that countable compactness as well as Lindelöff property are not stable even under finite products.

For a set $X$ we denote by $B(X)$ ($B^r(X)$, $B^+(X)$) the algebra of all bounded complex-valued (resp., real-valued, non-negative real-valued) functions on $X$. If $X$ is also a topological space, we denote by $C(X)$ ($C_0(X)$) the space of all continuous complex-valued functions on $X$ (with compact support, i.e., functions vanishing out of a compact subset of $X$). Moreover, we let $C_0^r(X) = C_0(X) \cap B^r(X)$ and $C_0^+(X) = C_0(X) \cap B^+(X)$. Note, that $C_0(X) \subseteq B(X)$

Let $X$ be a topological space. If $f \in C(X)$ let

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

**Theorem 2.2.34** (Stone-Weierstraß theorem)**.** *Let $X$ be a compact topological space. A $\mathbb{C}$-subalgebra $\mathcal{A}$ of $C(X)$ containing all constants and closed under conjugation is dense in $C(X)$ for the norm $\| \ \|_\infty$ if and only if $\mathcal{A}$ separates the points of $X$.*

We shall need in the sequel the following local form of Stone-Weierstraß theorem.

**Corollary 2.2.35.** *Let $X$ be a compact topological space and $f \in C(X)$. Then $f$ can be uniformly approximated by a $\mathbb{C}$-subalgebra $\mathcal{A}$ of $C(X)$ containing all constants and closed under the complex conjugation if and only if $\mathcal{A}$ separates the points of $X$ separated by $f \in C(X)$.*

*Proof.* Denote by $G : X \to \mathbb{C}^{\mathcal{A}}$ the diagonal map of the family $\{g : g \in \mathcal{A}\}$. Then $Y = G(X)$ is a compact subspace of $\mathbb{C}^{\mathcal{A}}$ and by the compactness of $X$, its subspace topology coincides with the quotient topology of the map $G : X \to Y$. The equivalence relation $\sim$ in $X$ determined by this quotient is as follows: $x \sim y$ for $x, y \in X$ by if and only if $G(x) = G(y)$ (if and only if $g(x) = g(y)$ for every $g \in \mathcal{A}$). Clearly, every continuous function $h : X \to \mathbb{C}$, such that $h(x) = h(y)$ for every pair $x, y$ with $x \sim y$, can be factorized as $h = \overline{h} \circ q$, where $\overline{h} \in C(Y)$. In particular, this holds true for all $g \in \mathcal{A}$ and for $f$ (for the latter case this follows from our hypothesis). Let $\overline{\mathcal{A}}$ be the $\mathbb{C}$-subalgebra $\{\overline{h} : h \in \overline{\mathcal{A}}\}$ of $\mathcal{C}(Y, \mathbb{C})$. It is closed under the complex conjugation and contains all constants. Moreover, it separates the points of $Y$. (If $y \neq y'$ in $Y$ with $y = G(x)$, $y' = G(x')$, $x, x' \in X$, then $x \not\sim x'$. So there exists $h \in \mathcal{A}$ with $\overline{h}(y) = h(x) \neq h(x') = \overline{h}(y')$. Hence $\overline{\mathcal{A}}$ separates the points of $Y$.) Hence we can apply Stone - Weierstraß theorem 2.2.34 to $Y$ and $\overline{\mathcal{A}}$ to deduce that we can uniformly approximate the function $\overline{f}$ by functions of $\overline{\mathcal{A}}$. This produces uniform approximation of the function $f$ by functions of $\mathcal{A}$. $\square$

## 2.3 Background on categories and functors

**Definition 2.3.1.** A *category* $\mathcal{X}$ consists of

- a class $\mathcal{O}b(\mathcal{X})$ whose elements $X$ are called *objects* of the category;

- a class $\mathrm{Hom}\,(\mathcal{X})$ whose elements are sets $\mathrm{Hom}\,(X_1, X_2)$, where $(X_1, X_2)$ varies among the ordered pairs of object of the category, the elements $\phi : X_1 \to X_2$ (written shortly as $\phi$) of $\mathrm{Hom}\,(X_1, X_2)$ are called *morphisms with domain $X_1$ and codomain $X_2$*;

- an associative composition law

$$\circ : \mathrm{Hom}\,(X_2, X_3) \times \mathrm{Hom}\,(X_1, X_2) \to \mathrm{Hom}\,(X_1, X_3),$$

  for every ordered triple $(X_1, X_2, X_3)$ of objects of the category, that associates to every pair of morphisms $(\phi, \psi)$ from $\mathrm{Hom}\,(X_2, X_3) \times \mathrm{Hom}\,(X_1, X_2)$, a morphism $\phi \circ \psi \in \mathrm{Hom}\,(X_1, X_3)$ called composition of $\phi$ and $\psi$.

The following conditions must be satisfied:

1. the sets $\mathrm{Hom}\,(X, X')$ and $\mathrm{Hom}\,(Y, Y')$ are disjoint if the pairs of objects $(X, X')$ and $(Y, Y')$ do not coincide;

2. for every object $X$ there exists a morphism $1_X : X \to X$ in $\mathrm{Hom}\,(X, X)$ such that $1_X \circ \alpha = \alpha$ e $\beta \circ 1_X = \beta$ for every pair of morphisms $\alpha \in \mathrm{Hom}\,(X', X)$ and $\beta \in \mathrm{Hom}\,(X, X')$.

**Example 2.3.2.** In the sequel we make use of the following categories:

- **Set** – sets and maps,
- $\mathbf{Vect}_K$ – vector spaces over a field $K$ and linear maps,
- **Grp** – groups and group homomorphisms,
- **AbGrp** – abelian groups and group homomorphisms,
- **Rng** – rings and ring homomorphisms,
- $\mathbf{Rng}_1$ – unitary rings and homomorphisms of unitary rings,
- **Top** – topological spaces and continuous maps,

A morphism $f : X \to Y$ in a category $\mathcal{A}$ is an *isomorphism*, if there exists a morphism $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$.

Consider two categories $\mathcal{A}$ and $\mathcal{B}$. A *covariant [contravariant] functor* $F : \mathcal{A} \to \mathcal{B}$ assigns to each object $A \in \mathcal{A}$ an object $FA \in \mathcal{B}$ and to each morphism $f : A \to A'$ in $\mathcal{A}$ a morphism $Ff : FA \to FA'$ $[Ff : FA' \to FA]$ such that $Fid_A = id_{FA}$ and $F(g \circ f) = Fg \circ Ff$ $[F(g \circ f) = Ff \circ Fg]$ for every morphism $f : A \to A'$ and $g : A' \to A''$ in $\mathcal{A}$.

If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are functors, one can define a functor $G \circ F : \mathcal{A} \to \mathcal{C}$ by letting $(G \circ F)A = G(FA)$ for every object $A$ in $\mathcal{A}$ and $(G \circ F)f = G(Ff)$ for every arrow $f$ in $\mathcal{A}$. It is easy to see that functor $G \circ F$ is covariant whenever both functors are simultaneously covariant or contravariant. If one of them is covariant and the other contravariant, then the functor $G \circ F$ is contravariant.

A functor $T : \mathcal{A} \to \mathcal{B}$ defines a map

$$\mathrm{Hom}\,(X, X') \to \mathrm{Hom}\,(T(X), T(X'))$$

for every pair of objects of the category $\mathcal{A}$. We say that $F$ is *faithful* if these maps are injective, *full* if they are surjective.

**Example 2.3.3.** A category $\mathcal{A}$ is called *concrete* if it admits a faithful functor $U : \mathcal{A} \to \mathbf{Set}$ (in such a case the functor is called *forgetful*). All examples above are concrete categories.

**Exercise 2.3.4.** *Build forgetful functors* $\mathbf{Vect}_K \to \mathbf{AbGrp}$ *and* $\mathbf{Rng} \to \mathbf{AbGrp}$.

Let $F, F' : \mathcal{A} \to \mathcal{B}$ be covariant functors. A *natural transformation* $\gamma$ from $F$ to $F'$ assigns to each $A \in \mathcal{A}$ a morphism $\gamma_A : FA \to F'A$ such that for every morphism $f : A \to A'$ in $\mathcal{A}$ the following diagram is commutative

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\gamma_A"'] & FA' \arrow[d, "\gamma_{A'}"] \\ F'A \arrow[r, "F'f"'] & F'A' \end{tikzcd}$$

A *natural equivalence* is a natural transformation $\gamma$ such that each $\gamma_A$ is an isomorphism.

# 3 General properties of topological groups

## 3.1 Definition of a topological group

Let us start with the following fundamental concept:

**Definition 3.1.1.** Let $G$ be a group.

- A topology $\tau$ on $G$ is said to be a *group topology* if the map $f : G \times G \to G$ defined by $f(x, y) = xy^{-1}$ is continuous.
- A *topological group* is a pair $(G, \tau)$ of a group $G$ and a group topology $\tau$ on $G$.

If $\tau$ is Hausdorff (resp., compact, locally compact, connected, etc.), then the topological group $(G, \tau)$ is called Hausdorff (resp., compact, locally compact, connected, etc.). Analogously, if $G$ is cyclic (resp., abelian, nilpotent, etc.) the topological group $(G, \tau)$ is called cyclic (resp. abelian, nilpotent, etc.). Obviously, a topology $\tau$ on a group $G$ is a group topology iff the maps

$$\mu : G \times G \to G \text{ and } \iota : G \to G$$

defined by $\mu(x, y) = xy$ and $\iota(x) = x^{-1}$ are continuous when $G \times G$ carries the product topology.

Here are some examples, starting with two trivial ones: for every group $G$ the discrete topology and the indiscrete topology on $G$ are group topologies. Non-trivial examples of a topological group are provided by the additive group $\mathbb{R}$ of the reals and by the multiplicative group $\mathbb{S}$ of the complex numbers $z$ with $|z| = 1$, equipped both with their usual topology. This extends to all powers $\mathbb{R}^n$ and $\mathbb{S}^n$. These are abelian topological groups. For every $n$ the linear group $GL_n(\mathbb{R})$ equipped with the topology induced by $\mathbb{R}^{n^2}$ is a non-abelian topological group. The groups $\mathbb{R}^n$ and $GL_n(\mathbb{R})$ are locally compact, while $\mathbb{S}$ is compact.

**Example 3.1.2.** For every prime $p$ the group $\mathbb{J}_p$ of $p$-adic integers carries the topology induced by $\prod_{n=1}^{\infty} \mathbb{Z}(p^n)$, when we consider it as the inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$. The same topology can be obtained also when we consider $\mathbb{J}_p$ as the ring of all endomorphims of the group $\mathbb{Z}(p^\infty)$. Now $\mathbb{J}_p$ embeds into the product $\mathbb{Z}(p^\infty)^{\mathbb{Z}(p^\infty)}$ carrying the product topology, while $\mathbb{Z}(p^\infty)$ is discrete. We leave to the reader the verification that this is a compact group topology on $\mathbb{J}_p$. Basic open neighborhoods of 0 in this topology are the subgroups $p^n\mathbb{J}_p$ of $(\mathbb{J}_p, +)$ (actually, these are ideals of the ring $\mathbb{J}_p$) for $n \in \mathbb{N}$. The field $\mathbb{Q}_p$ becomes a locally compact group by declaring $\mathbb{J}_p$ open in $\mathbb{Q}_p$ (i.e., an element $x \in \mathbb{Q}_p$ has as typical neighborhoods the cosets $x + p^n\mathbb{J}_p$, $n \in \mathbb{N}$).

Other examples of group topologies will be given in §3.3.

If $G$ is a topological group written multiplicatively and $a \in G$, then the *left translation* $x \overset{_a t}{\mapsto} ax$, the *right translation* $x \overset{t_a}{\mapsto} xa$, as well as the *internal automorphism* $x \mapsto axa^{-1}$ are homeomorphisms. Consequently, the group $G$ is discrete iff the point $e_G$ is isolated, i.e., the singleton $\{e_G\}$ is open. In the sequel $aM$ will denote the image of a subset $M \subseteq G$ under the (left) translation $x \mapsto ax$, i.e., $aM := \{am : m \in M\}$. This notation will be extended also to families of subsets of $G$, in particular, for every filter $\mathcal{F}$ we denote by $a\mathcal{F}$ the filter $\{aF : F \in \mathcal{F}\}$.

Making use of the homeomorphisms $x \mapsto ax$ one can prove:

**Exercise 3.1.3.** *Every topological group is a homogeneous topological space.*

**Example 3.1.4.** For every $n$ the group $GL_n(\mathbb{C})$ equipped with the topology induced from $\mathbb{C}^{n^2}$, is a topological group. Indeed, the known formulas for multiplication and inversion of matrices immediately show that both operations are continuous.

For a topological group $G$ and $g \in G$ we denote by $\mathcal{V}_{G,\tau}(g)$ the filter of all neighborhoods of the element $g$ of $G$. When no confusion is possible, we shall write briefly also $\mathcal{V}_G(g)$, $\mathcal{V}_\tau(g)$ or even $\mathcal{V}(g)$. Among these filters the filter $\mathcal{V}_{G,\tau}(e_G)$, obtained for the neutral element $g = 1$, plays a central role. It is useful to note that for every $a \in G$ the filter $\mathcal{V}_G(a)$ coincides with $a\mathcal{V}_G(e_G) = \mathcal{V}_G(e_G)a$. More precisely, we have the following:

**Theorem 3.1.5.** *Let $G$ be a group and let $\mathcal{V}(e_G)$ be the filter of all neighborhoods of $e_G$ in some group topology $\tau$ on $G$. Then:*

**(a)** *for every $U \in \mathcal{V}(e_G)$ there exists $V \in \mathcal{V}(e_G)$ with $V \cdot V \subseteq U$;*

**(b)** *for every $U \in \mathcal{V}(e_G)$ there exists $V \in \mathcal{V}(e_G)$ with $V^{-1} \subseteq U$;*

**(c)** *for every $U \in \mathcal{V}(e_G)$ and for every $a \in G$ there exists $V \in \mathcal{V}(e_G)$ with $aVa^{-1} \subseteq U$.*

*Conversely, if $\mathcal{V}$ is a filter on $G$ satisfying (a), (b) and (c), then there exists a unique group topology $\tau$ on $G$ such that $\mathcal{V}$ coincides with the filter of all $\tau$-neighborhoods of $e_G$ in $G$.*

*Proof.* To prove (a) it suffices to apply the definition of the continuity of the multiplication $\mu : G \times G \to G$ at $(e_G, e_G) \in G \times G$. Analogously, for (b) use the continuity of the map $\iota : G \to G$ at $e_G \in G$. For item (c) use the continuity of the internal automorphism $x \mapsto axa^{-1}$ at $e_G \in G$.

Let $\mathcal{V}$ be a filter on $G$ satisfying all conditions (a), (b) and (c). Let us see first that every $U \in \mathcal{V}$ contains $e_G$. In fact, take $W \in \mathcal{V}$ with $W \cdot W \subseteq U$ and choose $V \in \mathcal{V}(e_G)$ with $V \subseteq W$ and $V^{-1} \subseteq W$. Then $e_G \in V \cdot V^{-1} \subseteq U$.

Now define a topology $\tau$ on $G$ whose open sets $O$ are defined by the following property:

$$\tau := \{O \subseteq G : (\forall a \in O)(\exists U \in \mathcal{V}) \text{ such that } aU \subseteq O\}.$$

Let us check that that $\tau$ is a topology on $G$. Obviously $\emptyset, G \in \tau$ and $\tau$ is stable under arbitrary unions. Assume $O_1, O_2 \in \tau$. If $O_1 \cap O_2 = \emptyset$ then $O_1 \cap O_2 \in \tau$. Otherwise pick an element $x \in O_1 \cap O_2$ and find $V_i \in \mathcal{V}$ such that $xO_i \subseteq O_1$ for $i = 1, 2$. Then $V := V_1 \cap V_2 \in \mathcal{V}$ and $xV \subseteq O_1 \cap O_2$.

Let us see now that for every $g \in G$ the filter $g\mathcal{V}$ coincides with the filter $\mathcal{V}_{(G,\tau)}(g)$ of all $\tau$-neighborhoods of $g$ in $(G, \tau)$. The inclusion $g\mathcal{V} \supseteq \mathcal{V}_{(G,\tau)}(g)$ is obvious. Assume $U \in \mathcal{V}$. To see that $gU \in \mathcal{V}_{(G,\tau)}(g)$ we have to find a $\tau$-open $O \subseteq gU$ that contains $g$. Let $O := \{h \in gU : (\exists W \in \mathcal{V})\ hW \subseteq gU\}$. Obviously $g \in O$. To see that $O \in \tau$ pick $x \in O$. Then there exists $W \in \mathcal{V}$ with $xW \subseteq gU$. Let $V \in \mathcal{V}$ with $V \cdot V \subseteq W$, then $xV \subseteq O$ since $xvV \subseteq gU$ for every $v \in V$.

We have seen that $\tau$ is a topology on $G$ such that the $\tau$-neighborhoods of any $x \in G$ are given by the filter $x\mathcal{V}$. It remains to see that $\tau$ is a group topology. To this end we have to prove that the map $f$ defined by $(x, y) \mapsto xy^{-1}$ is continuous. Fix $x, y$ and pick a $U \in \mathcal{V}$. By (c) there exists a $W \in \mathcal{V}$ with $Wy^{-1} \subseteq y^{-1}U$. Now choose $V \in \mathcal{V}$ with $V \cdot V^{-1} \subseteq W$. Then $O = xV \times yV$ is a neighborhood of $(x, y)$ in $G \times G$ and $f(O) \subseteq xV \cdot V^{-1}y^{-1} \subseteq xWy^{-1} \subseteq xy^{-1}U$. □

In the above theorem one can take instead of a filter $\mathcal{B}$ also a filter base. For example, some authors prefer the characterize the base $\mathcal{B}$ of $\mathcal{V}(e_G)$ formed by the open neighborhoods. In such a case one has to add the (a)–(c) also the following property

(d) $(\forall U \in \mathcal{B})(\forall x \in U)(\exists V \in \mathcal{B}) Vx \subseteq U$.

A neighborhood $U \in \mathcal{V}(e_G)$ is *symmetric*, if $U = U^{-1}$. Obviously, for every $U \in \mathcal{V}(e_G)$ the intersection $U \cap U^{-1} \in \mathcal{V}(e_G)$ is a symmetric neighborhood, hence every neighborhood of $e_G$ contains a symmetric one.

Let $G$ and $H$ be topological groups and let $f : G \to H$ be a homomorphism. If $f$ is simultaneously an isomorphism and a homeomorphism, then $f$ is called a *topological isomorphism*.

**Remark 3.1.6.** Due to the homogeneity of topological groups, a homomorphism $f : G \to H$ is continuous iff it is continuous at $1_G$, i.e., if for every $U \in \mathcal{V}_H(1_H)$ there exists $V \in \mathcal{V}_G(1_G)$ such that $f(V) \subseteq U$.

The group topology $\tau$ built in Theorem 3.1.5 starting from a filter $\mathcal{V}$ on $G$ with the properties (a)-(c) was defined by letting the neighborhood filter at $g \in G$ to be the filter $g\mathcal{V} = \mathcal{V}g$. The coincidence of these two filters is ensured /actually, equivalent to) property (c). In case (c) fails, one obtains two topologies $\tau_r$ and $\tau_l$ on $G$ (having as neighborhood base at $g \in G$ the filter $\mathcal{V}g$ and $g\mathcal{V}$, respectively), such that for the pair $(G, \tau_r)$ all right translations $x \mapsto xg$ are continuous (actually, homeomorphisms), and all left translations $x \mapsto gx$ are continuous for the pair $(G, \tau_r)$. Pairs with this property are called *right topological groups* (*left topological groups*, respectively). A pair $(G, \tau)$ that is simultaneously a left and a right topological group is called a *semi-topological group*. The filter of neighborhoods of the neutral element of a semi-topological group satisfies (c), and every filter with the property (c) on a group $G$ gives rise to a semi-topological group topology $\tau$ on $G$, defined as above.

The right topological groups were introduced by Namioka [117] and largely used since then. Robert Ellis, in his two fundamental papers [74, 75] proved that locally compact semitopological groups are topological. Later this was generalized by A. Bouziad to Čech-complete groups.

Along with right topological groups, left topological groups and semi-topological groups, one can find in the literature also the following weak versions of the notion of a topological group. A pair $(G, \tau)$ is called a:

(a) *quasitopological group* – when the multiplication map is separately continuous in both variables and the inverse map is continuous (i.e., it is a semitopological group such that the inverse is continuous);

(b) *paratopological group* – when the multiplication map is jointly continuous in both variables

The filter of neighborhoods of the neutral element of a quasitopological (paratopological) group is characterized with the properties (b) and (c) ((a) and (c), resp.).

For paratopological groups, having some compact-like properties (as pseudocompactness, etc.), one has analogues of Ellis' theorem. More details on this issue can be found in the survey [140] and the monograph [3].

## 3.2 Comparing group topologies

The family $\mathfrak{L}(G)$ of all group topologies on a group $G$ is naturally ordered by inclusion, having as a top element the discrete topology of $G$ and as bottom element the indiscrete topology of $G$. Due to Theorem 3.1.5 one can describe the poset $\mathfrak{L}(G)$ by the poset $\mathfrak{F}(G)$ of all filters $\mathfrak{V}$ on $G$ satisfying conditions (a)-(c) form the theorem. The poset $\mathfrak{F}(G)$ is ordered again by inclusion (of filters) and very often we simply study $\mathfrak{F}(G)$ in place of the more complicated poset $\mathfrak{L}(G)$.

If $\{\tau_i : i \in I\}$ is a family of group topologies on a group $G$, then their supremum $\tau = \sup_{i\in I} \tau_i$, taken in the larger lattice $\mathcal{T}(G)$, is a group topology on $G$ with a base of neighborhoods of $e_G$ formed by the family of all finite intersection $U_1 \cap U_2 \cap \ldots \cap U_n$, where $U_k \in \mathcal{V}_{\tau_{i_k}}(e_G)$ for $k = 1, 2, \ldots, n$ and the $n$-tuple $i_1, i_2, \ldots, i_n$ runs over all finite subsets of $I$. Since the poset $\mathfrak{L}(G)$ has a bottom element, this proves that $\mathfrak{L}(G)$ is a complete lattice. Nevertheless, $\mathfrak{L}(G)$ is not a sublattice of the complete lattice $\mathcal{T}(G)$. Indeed, the intersection of group topologies need not be a group topology (examples will be given below).

**Exercise 3.2.1.** *If $G$ is an abelian group and $\tau_1, \tau_2 \in \mathfrak{L}(G)$, prove that the family $\{U_1 + U_2 : U_i \in \mathcal{V}_{(G,\tau_i)}(0), i = 1, 2\}$ is a base of the filter $\mathcal{V}_{(G,\inf\{\tau_1,\tau_2\})}(0)$.*

**Exercise 3.2.2.** *If $(a_n)$ is a sequence in $G$ such that $a_n \to e_G$ for every member $\tau_i$ of a family $\{\tau_i : i \in I\}$ of group topologies on a group $G$, then $a_n \to e_G$ also for the supremum $\sup_{i\in I} \tau_i$.*

## 3.3 Examples of group topologies

Now we give several series of examples of group topologies, introducing them by means of the filter $\mathcal{V}(e_G)$ of neighborhoods of $e_G$ as explained above. However, in all cases we avoid to treat the whole filter $\mathcal{V}(1)$ and we prefer to deal with an essential part of it, namely a base. Let us recall the precise definition of a base of neighborhoods.

**Definition 3.3.1.** Let $G$ be a topological group. A family $\mathcal{B} \subseteq \mathcal{V}(e_G)$ is said to be a *base of neighborhoods of $e_G$* (or briefly, a *base at 1*) if for every $U \in \mathcal{V}(e_G)$ there exists a $V \in \mathcal{B}$ contained in $U$ (such a family will necessarily be a filterbase).

### 3.3.1 Linear topologies

Let $\mathcal{V} = \{N_i : i \in I\}$ be a filter base consisting of normal subgroups of a group $G$. Then $\mathcal{V}$ satisfies (a)–(c), hence generates a group topology on $G$ having as basic neighborhoods of a point $g \in G$ the family of cosets $\{gN_i : i \in I\}$. Group topologies of this type will be called *linear topologies*. Let us see now various examples of linear topologies.

**Example 3.3.2.** Let $G$ be a group and let $p$ be a prime:

- the *pro-finite* topology, with $\{N_i : i \in I\}$ all normal subgroups of finite index of $G$;
- the *pro-p-finite* topology, with $\{N_i : i \in I\}$ all normal subgroups of $G$ of finite index that is a power of $p$;
- the *p-adic* topology, with $I = \mathbb{N}$ and for $n \in \mathbb{N}$, $N_n$ is the subgroup (necessarily normal) of $G$ generated by all powers $\{g^{p^n} : g \in G\}$.
- the *natural* topology (or *$\mathbb{Z}$-topology*), with $I = \mathbb{N}$ and for $n \in \mathbb{N}$, $N_n$ is the subgroup (necessarily normal) of $G$ generated by all powers $\{g^n : g \in G\}$.
- the *pro-countable* topology, with $\{N_i : i \in I\}$ all normal subgroups of at most countable index $[G : N_i]$.

When $G$ is an abelian group, then the basic subgroup $N_n$ defining the $p$-adic topology of $G$ has the form $N_n = p^n G$. Analogously, the basic subgroup $N_n$ defining the natural topology of $G$ has the form $N_n = nG$.

**Exercise 3.3.3.** *Let $G$ be a group. Prove that*

*(a) the profinite topology of $G$ is discrete (resp., indiscrete) iff $G$ is finite ($G$ has no subgroups of finite index); in case $G$ is abelian, the profinite topology of $G$ is indiscrete iff $G$ is divisible;*

*(b) the pro-p-finite topology of $G$ is discrete (resp., indiscrete) iff $G$ is a finite $p$-group (resp., $G$ has no subgroups of index power of $p$);*

*(c) the $p$-adic topology of $G$ is discrete (resp., indiscrete) iff $G$ is a $p$-group of finite exponent ($G$ is $p$-divisible);*

*(d) the natural topology of $G$ is discrete (resp., indiscrete) iff $G$ is a group of finite exponent ($G$ is divisible);*

*(e) the pro-countable topology of $G$ is discrete (resp., indiscrete) iff $G$ is countable ($G$ has no subgroups of finite index); in case $G$ is abelian, the pro-countable topology of $G$ is indiscrete iff $G$ is trivial;*

*(f) if $m$ and $k$ are co-prime integers, then $mG \cap kG = mkG$; hence the natural topology of $G$ coincides with the supremum of all $p$-adic topologies of $G$.*

**Lemma 3.3.4.** *Let $f : G \to H$ be a homomorphism of groups. Then $f$ is continuous when both groups are equipped with their profinite (resp., pro-$p$-finite, $p$-adic, natural, pro-countable) topology.*

*Proof.* Let $N$ be a subgroup of finite index of $H$. Then obviously $f^{-1}(N)$ is a subgroup of finite index of $G$. The other cases are similar. □

This lemma shows that the above mentioned topologies have a "natural" origin, whatever this may mean. Here comes a definition that makes this idea more precise.

**Definition 3.3.5.** Assume that every abelian group $G$ is equipped with a group topology $\tau_G$ such that every group homomorphism $f : (G, \tau_G) \to (H, \tau_H)$ is continuous. Then we say that the class of topologies $\{\tau_G : G \in \mathbf{AbGrp}\}$ is a *functorial topology.*

The next simple construction belongs to Taimanov. Now neighborhoods of $e_G$ are subgroups, that are not necessarily normal.

**Exercise 3.3.6.** *Let $G$ be a group with trivial center. Then $G$ can be considered as a subgroup of Aut $(G)$ making use of the internal automorphisms. Identify Aut $(G)$ with a subgroup of the power $G^G$ and equip Aut $(G)$ with the group topology $\tau$ induced by the product topology of $G^G$, where $G$ carries the discrete topology. Prove that:*

- *the filter of all $\tau$-neighborhoods of $e_G$ has as base the family of centralizers $\{c_G(F)\}$, where $F$ runs over all finite subsets of $G$;*
- *$\tau$ is Hausdorff;*
- *$\tau$ is discrete iff there exists a finite subset of $G$ with trivial centralizer.*

Furstenberg used the natural topology $\nu$ of $\mathbb{Z}$ (see Example 3.3.2) to find a new proof of the infinitude of prime numbers.

**Exercise 3.3.7.** *Prove that there are infinitely many primes in $\mathbb{Z}$ using the natural topology $\nu$ of $\mathbb{Z}$.*

(Hint. If $p_1, p_2, \dots, p_n$ were the only primes, then consider the union of the open subgroups $p_1\mathbb{Z}, \dots, p_n\mathbb{Z}$ of $(\mathbb{Z}, \nu)$ and use the fact that every integer $n \neq 0, \pm 1$ has a prime divisor, so belongs to the closed set $F = \bigcup_{i=1}^n p_i\mathbb{Z}$. Therefore the set $\{0, \pm 1\} = \mathbb{Z} \setminus F$ is open, so must contain a non-zero subgroup $m\mathbb{Z}$, a contradiction.)

### 3.3.2 Topologies generated by characters

Let $(G, +)$ be an abelian group. A *character* of $G$ is a homomorphism $\chi : G \to \mathbb{S}$. For a character $\chi$ and $\delta > 0$ let $U_G(\chi; \delta) := \{x \in G : |\mathrm{Arg}\ (\chi(x))| < \delta\}$.

**Example 3.3.8.** (a) For a fixed character $\chi$ the family $\mathcal{B} = \{U_G(\chi; \frac{\pi}{n+1}) : n \in \mathbb{N}\}$, where the argument Arg $(z)$ of a complex number $z$ is taken in $(-\pi, \pi]$, is a filter base satisfying conditions (a) – (c) of Theorem 3.1.5. We denote by $\mathcal{T}_\chi$ the group topology on $G$ generated by $\mathcal{B}$. Then $\chi : (G, \mathcal{T}_\chi) \to \mathbb{S}$ is continuous, so $\ker \chi$ is a closed subgroup of $(G, \mathcal{T}_\chi)$ contained in $U_G(\chi; \delta)$ for every $\delta > 0$. On the other hand, every subgroup of $G$ contained in $U_G(\chi; \pi/2)$ is contained in $\ker \chi$ as well (since $\mathbb{S}_+ = \{z \in \mathbb{S} : Re\ z \geq 0\}$ contains no non-trivial subgroups).

(b) With $G$ and $\chi$ as above, consider $n \in \mathbb{Z}$. Then $\mathcal{T}_{\chi^n} \subseteq \mathcal{T}_\chi$, where the character $\chi^n : G \to \mathbb{T}$ is defined by $(\chi^n)(x) := (\chi(x))^n$. Obviously, $\mathcal{T}_{\chi^{-1}} = \mathcal{T}_\chi$. One can show that for $\chi, \xi \in \mathbb{Z}^*$ with $\ker \chi = \ker \xi = 0$ the equality $\mathcal{T}_\chi = \mathcal{T}_\xi$ holds true if and only if $\xi = \chi^{\pm 1}$.

For characters $\chi_i$, $i = 1, \dots, n$, of $G$ and $\delta > 0$ let

$$U_G(\chi_1, \dots, \chi_n; \delta) := \{x \in G : |\mathrm{Arg}\ (\chi_i(x))| < \delta, i = 1, \dots, n\}, \tag{1}$$

One can describe (1) alternatively, using the target group $\mathbb{T}$ instead of $\mathbb{S}$. In such a case characters $\xi_i : G \to \mathbb{T}$ must be used and the inequality $|\mathrm{Arg}\ (\chi_i(x))| < \delta$ must be replaced by $\|\xi_i(x)\| < \delta/2\pi$, where for $z = r + \mathbb{Z} \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ one has $\|z\| = \|r + \mathbb{Z}\| = d(r, \mathbb{Z}) = \min\{(d(r, m) : m \in \mathbb{Z}\}$ and $d$ is the usual metric in $\mathbb{R}$.

**Example 3.3.9.** Let $G$ be an abelian group and let $H$ be a family of characters of $G$. Then the family

$$\{U_G(\chi_1, \dots, \chi_n; \delta) : \delta > 0, \chi_i \in H, i = 1, \dots, n\}$$

is a filter base satisfying the conditions (a)–(c) of Theorem 3.1.5, hence it gives rise to a group topology $\mathcal{T}_H$ on $G$. Since $U_G(\chi_1, \dots, \chi_n; \delta) = \bigcap_{i=1}^n U_G(\chi_i; \delta)$, $\mathcal{T}_H$ coincides with the supremum $\sup\{\mathcal{T}_\chi : \chi \in H\}$.

(a) The assignment $H \mapsto \mathcal{T}_H$ is monotone, i.e., if $H \subseteq H'$ then $\mathcal{T}_H \subseteq \mathcal{T}_{H'}$.

(b) By item (b) of Example 3.3.8, $\mathcal{T}_{\langle\chi\rangle} = \mathcal{T}_\chi$. This suggests the equation $\mathcal{T}_{\langle H\rangle} = \mathcal{T}_H$ for every family $H$. Indeed, the inclusion $\supseteq$ follows from monotonicity. Let $\chi_1, \chi_2 \in H$. Then one can easily show that $U_G(\chi_1\chi_2; \delta) \subseteq U_G(\chi_1, \chi_2; \delta/2)$. Thus $\mathcal{T}_{\chi_1\chi_2} \subseteq \mathcal{T}_H$. Along with item (b) of Example 3.3.8 this proves that $\mathcal{T}_{\langle H\rangle} \subseteq \mathcal{T}_H$.

We refer to the group topology $\mathcal{T}_H$ as *topology generated by the characters* of $H$. Due to the equation $\mathcal{T}_{\langle H\rangle} = \mathcal{T}_H$, it is worth studying the topologies $\mathcal{T}_H$ when $H$ is a *subgroup* of $G^*$. The topology $\mathcal{T}_{G^*}$, generated by all characters of $G$, is called *Bohr topology* of $G$ and the topological group $(G, \mathcal{T}_{G^*})$ will often be written shortly as $G^\#$.

**Lemma 3.3.10.** *Let $f : G \to H$ be a homomorphism of abelian groups. Then $f$ is continuous when both groups are equipped with their Bohr topology (i.e., $f : G^\# \to H^\#$ is continuous).*

*Proof.* Let $\chi_1, \dots, \chi_n \in H^*$ and $\delta > 0$. Then $f^{-1}(U_H(\chi_1, \dots, \chi_n; \delta)) = U_G(\chi_1 \circ f, \dots, \chi_n \circ f; \delta)$ is a neighborhood of 0 in $G^\#$. $\square$

For an abelian group $G$ some of the linear topologies on $G$ are also generated by appropriate families of characters.

**Proposition 3.3.11.** *The profinite topology of an abelian group $G$ is contained in the Bohr topology of $G$.*

*Proof.* If $H$ is a subgroup of $G$ of finite index, then $G/H$ is finite, so has the form $C_1 \times \ldots \times C_n$, where each $C_n$ is a finite cyclic group. Let $q : G \to G/H$ be the quotient map, let $p_i : C_1 \times \ldots \times C_n \to C_i$ be the $i$-th projection, let $q_i = p_i \circ q : G \to C_i$. and let $H_i = \ker q_i$. Then $G/H_i \cong C_i$. Moreover, we can identify each $C_i$ with the unique cyclic subgroup of $\mathbb{T}$ of order $m_i = |C_i|$, so that we can consider $q_i : G \to C_i \hookrightarrow \mathbb{T}$ as a character of $G$. Then $H_i = \ker q_i = U_G(q_i; 1/2m_i) \in \mathcal{T}_{G^*}$. To end the proof note that $H = \bigcap_{i=1}^n H_i \in \mathcal{T}_{G^*}$. $\square$

Call a character $\chi : G \to \mathbb{T}$ *torsion* if there exists $n > 0$ such that $\chi$ vanishes on the subgroup $nG := \{nx : x \in G\}$. Equivalently, the character $\chi$ is a torsion element of the group $G^*$, i.e., if $o(\chi) = n$, then $\chi^n$ coincides with the trivial character. This occurs precisely when the subgroup $\chi(G)$ of $\mathbb{S}$ is finite cyclic. Therefore, $G^*$ is torsion-free when $G$ is divisible.

**Lemma 3.3.12.** *If $H$ is a family of characters of an abelian group $G$, then the topology $\mathcal{T}_H$ is contained in the pro-finite topology of $G$ iff every character of $H$ is torsion.*

*Proof.* Note that for a torsion character $\chi$ the basic neighborhood $U_G(\chi; \pi/2)$ contains a closed subgroup $\ker\chi$ of finite index (as $\chi(G) \cong G/\ker\chi$ is finite). Hence $\ker\chi$ is open, so a neighborhood of 0 in the pro-finite topology. Therefore, $\mathcal{T}_H$ is contained in the pro-finite topology of $G$.

Now assume that $\mathcal{T}_H$ is contained in the pro-finite topology of $G$. Then for any $\chi \in H$ the basic $\mathcal{T}_\chi$-neighborhood $U_G(\chi; \pi/2)$ must contain a finite-index subgroup $N$ of $G$. Then $N \subseteq \ker\chi$ by Example 3.3.9 (b). Thus $\ker\chi$ has finite index, consequently $\chi$ is torsion. $\square$

**Exercise 3.3.13.** *Let $G$ be an abelian group.*

1. *Give an example of a group $G$ where profinite topology of $G$ and the Bohr topology of $G$ differ.*
2. *Let $H$ be the family of all torsion characters $\chi$ of $G$. Prove that the topology $\mathcal{T}_H$ coincides with the pro-finite topology on $G$.*
3. *Let $H$ be the family of all characters $\chi$ of $G$ such that the subgroup $\chi(G)$ is finite and contained in the subgroup $\mathbb{Z}(p^\infty)$ of $\mathbb{T}$. Prove that the topology $\mathcal{T}_H$ coincides with the pro-$p$-finite topology on $G$.*

(Hint. 2. The above lemma implies that $\mathcal{T}_H$ is contained in the pro-finite topology on $G$. For the proof of the other inclusion it remains to argue as in the proof of Proposition 3.3.11 and observe that the characters appearing there are torsion.)

**Theorem 3.3.14.** *The Bohr topology of an abelian group $G$ coincides with the profinite topology of $G$ iff $G$ is bounded torsion.*

*Proof.* If $G$ is bounded torsion, of exponent $m$, then every character of $G$ is torsion, so Lemma 3.3.12 applies. Assume that the Bohr topology of $G$ coincides with its profinite topology. According to Lemma 3.3.12 $G^*$ is torsion. This immediately implies that $G$ is torsion. If $r_p(G) \neq 0$ for infinitely many primes, then we find a subgroup $G_1$ of $G$ isomorphic to $\bigoplus_{n=1}^{\infty} \mathbb{Z}_{p_n}$, where $p_n$ are distinct primes. Then take an embedding $j : G_1 \to \mathbb{S}$ and extend $j$ to a character of the whole group $G$. It cannot be torsion, a contradiction. If only finitely many $r_p(G) > 0$, then at least one of the primary components $t_p(G)$ is infinite. If $r_p(G) < \infty$, then $t_p(G)$ contains a copy of the group $\mathbb{Z}(p^\infty)$ by Example 2.1.22. Now take an embedding $j : \mathbb{Z}(p^\infty) \to \mathbb{S}$ and extend $j$ to a character of the whole group $G$. It cannot be torsion, a contradiction. If $r_p(G)$ is infinite and $t_p(G)$ contains no copies of the group $\mathbb{Z}(p^\infty)$, then either $t_p(G)$ is bounded, or there exists a subgroup of $t_p(G)$ isomorphic to $L = \bigoplus_n \mathbb{Z}(p^n)$. It is easy to build a surjective homomorphism $h : L \to \mathbb{Z}(p^\infty) \subseteq \mathbb{S}$. Now extend $h$ to a character $h_1 : G \to \mathbb{S}$. Obviously, $h_1$ is not torsion, a contradiction. □

**Exercise 3.3.15.** *The Bohr topology of an abelian group $G$ coincides with its pro-$p$-finite topology of $G$ iff $G$ is a bounded $p$-group.*

### 3.3.3 Pseudonorms and invariant pseudometrics in a group

According to Markov a *pseudonorm* in a group $(G, \cdot)$ is a map $v : G \to \mathbb{R}$ such that for every $x, y \in G$:

**(1)** $v(e_G) = 0$;

**(2)** $v(x^{-1}) = v(x)$;

**(3)** $v(xy) \leq v(x) + v(y)$.

A pseudonorm with the additional property, $v(x) = 0$ iff $x = e_G$ is called a *norm*. Note that the values of a pseudonorm are necessarily *non-negative* reals, since

$$0 = v(e_G) = v(x^{-1}x) \leq v(x^{-1}) + v(x) = v(x) + v(x) = 2v(x)$$

for every $x \in G$.

The norms defined in a (real) vector space are obviously norms of the underlying abelian group (although they have a stronger property).

Every pseudonorm $v$ generates a pseudometric $d_v$ on $G$ defined by $d_v(x, y) := v(x^{-1}y)$. This pseudometric is *left invariant* in the sense that $d_v(ax, ay) = d_v(x, y)$ for every $a, x, y \in G$. Conversely, every left invariant pseudometric on $G$ gives rise to a pseudonorm of $G$ defined by $v_d(x) = d(x, e_G)$. Obviously, this pseudonorm generates the original left invariant pseudometric $d$ (i.e., $d = d_{v_d}$). This defines a bijective correspondence between pseudonorms $v$ and left invariant pseudometrics $d_v$.

Clearly $d_v$ is a metric iff $v$ is a norm. Denote by $\tau_v$ the topology induced on $G$ by this pseudometric. A base of $\mathcal{V}_{\tau_v}(e_G)$ is given by the open balls $\{B_{1/n}(e_G) : n \in \mathbb{N}_+\}$.

**Example 3.3.16.** Let $\ell_2$ denote the set of all sequences $x = (x_n)$ of real numbers such that the series $\sum_{n=1}^{\infty} x_n^2$ converges. Then $\ell_2$ has a natural structure of vector space (induced by the product $\mathbb{R}^{\mathbb{N}} \supseteq \ell_2$). Let $\|x\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}$. This defines a norm of the abelian group $(\ell_2, +)$, that provides an invariant metric on $\ell_2$ making it a metric space and a topological group.

In order to build metrics generating the topology of a given topological group $(G, \tau)$ we need the following lemma (for a proof see [102, 8.2], [119]). We say that a pseudometric $d$ on $G$ is *continuous* if the map $d : G \times G \to \mathbb{R}_+$ is continuous. This is equivalent to have the topology induced by the metric $d$ coarser than the topology $\tau$ (i.e., every open set with respect to the metric $d$ is $\tau$-open).

**Lemma 3.3.17.** *Let $G$ be a topological group and let*

$$U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n \supseteq \ldots \tag{2}$$

*be symmetric neighborhoods of 1 with $U_n^3 \subseteq U_{n-1}$ for every $n \in \mathbb{N}$. Then there exists a continuous left invariant pseudometric $d$ on $G$ such that $U_n \subseteq B_{1/n}(e_G) \subseteq U_{n-1}$ for every $n$.*

**Exercise 3.3.18.** *Prove that in the previous lemma $H = \bigcap_{n=1}^{\infty} U_n$ is a closed subgroup of $G$ with the property $H = \{x \in G : d(x, e_G) = 0\}$. In particular, $d$ is a metric iff $H = \{e_G\}$.*

**Remark 3.3.19.** (a) If the chain (2) has also the property $xU_nx^{-1} \subseteq U_{n-1}$ for every $x \in G$ and for every $n$, the subgroup $H$ is normal and $d$ defines a metric on the quotient group letting $\tilde{d}(xH, yH) := d(x, y)$. The metric $\tilde{d}$ induces the quotient topology on $G/H$ (see §3.4).

(b) Assume $U_0$ is a subgroup of $G$ and all $U_n = U_0$ in (2). Then this stationary chain satisfies the hypothesis of the lemma. The pseudometric $d$ is defined as follows $d(x, y) = 0$ if $xU_0 = yU_0$, otherwise $d(x, y) = 1$.

### 3.3.4 Function spaces as topological groups

The function spaces were the first instances of topological spaces. Since the target of the functions (the reals, the complex number field, etc.) has usually (at least) a topological group structure itself, the functions spaces have a very rich structure from both points of view (topology and algebra).

The norm $\|\|_\infty$ defined on the space $B(X)$ of all bounded complex-valued functions gives rise to an invariant metric whose metric topology is a group topology. This topology takes the name *uniform convergence topology*.

Let $X$ be a set and let $Y$ be a topological space. The set of all maps $X \to Y$, i.e., the Cartesian power $Y^X$, often carries also two weaker topologies. The *pointwise convergence topology*, has as a base all sets of the form $\{f \in Y^X : (\forall x \in F) f(x) \in U_x\}$, where $F$ is a finite subset of $X$ and $U_x$ $(x \in F)$ are non-empty open sets in $Y$. In case $Y$ is a topological group, this topology makes $Y^X$ a topological group.

The *compact-open topology* is defined on the set $Z$ of all $Y$-valued (continuous) functions $X \to Y$, where $X$ is a topological spaces and $(Y, d)$ is a metric space. It has as a base of the filter of neighborhoods of $f \in Z$ the family of sets $W(K, \varepsilon, f) = \{g \in Z : d(f(x), g(x)) < \varepsilon\}$, where $K \subseteq X$ is compact and $\varepsilon > 0$. In case $Y$ is a topological group, this topology makes $Z$ a topological group. A base of neighborhoods of the constant function $f = e_Y$ is given by the sets $W(K, \varepsilon) = \{g \in Z : d(g(x), e_Y) < \varepsilon\}$, where $K \subseteq X$ is compact and $\varepsilon > 0$. Since finite sets are compact, this topology is finer than the pointwise convergence topology.

These topologies have many applications in analysis and topological algebra. The compact-open topology will be used to define the Pontryagin dual $\widehat{X}$ of an abelian topological group $X$, with target group $Y = \mathbb{S}$.

Here comes a further specialization from algebra (module theory). Fix and $V, U$ vector spaces over a field $K$. Now consider on the space $Hom(V, U)$ of all linear maps $V \to U$ the so called *finite topology* having as typical neighborhoods of 0 all sets $W(F) = \{f \in Hom(V, U) : (\forall x \in F) f(x) = 0\}$, where $F$ runs over all finite subsets of $V$. It is easy to see that $W(F)$ is a linear subspace of $V$.

**Exercise 3.3.20.** *(a) Prove that the finite topology of $Hom(V, U)$ coincides with the pointwise convergence topology when $Hom(V, U)$ is considered as a subset of $U^V$ and $U$ carries the discrete topology.*

*(b) Prove that if $\dim U < \infty$, then $W(F)$ has finite co-dimension in $Hom(V, U)$ (i.e., $\dim Hom(V, U)/W(F) < \infty$). Conclude that in this case $Hom(V, U)$ is discrete iff $\dim V < \infty$ as well.*

The finite topology is especially useful when imposed on the dual $V^* = Hom(V, K)$ of the space $V$. Then the continuous linear functionals $V^* \to K$ of the dual space $V$ equipped with the finite topology (and $K$ discrete) form a subspace of the second dual $V^{**}$ that is canonically isomorphic to the original space $V$ via the usual evaluation map $V \to V^{**}$. This fact is known as *Lefschetz duality*.

### 3.3.5 Transformation groups

We shall start with the basic example, the permutation groups.

Let $X$ be an infinite set and let $G$ briefly denote the group $S(X)$ of all permutations of $X$. A very natural topology on $G$ is defined by taking as filter of neighborhoods of $1 = id_X$ the family of all subgroups of $G$ of the form

$$S_F = \{f \in G : (\forall x \in F)\ f(x) = x\},$$

where $F$ is a finite subset of $X$.

This topology can be described also as the topology induced by the natural embedding of $G$ into the Cartesian power $X^X$ equipped with the product topology, where $X$ has the discrete topology.

This topology is also the point-wise convergence topology on $G$. Namely, if $(f_i)_{i \in I}$ is a net in $G$, then $f_i$ converges to $f \in G$ precisely when for every $x \in X$ there exists an $i_0 \in I$ such that for all $i \geq i_0$ in $I$ one has $f_i(x) = f(x)$.

**Exercise 3.3.21.** *If $S_\omega(X)$ denotes the subset of all permutations of finite support in $S(X)$ prove that $S_\omega(X)$ is a dense normal subgroup of $G$.*

**Exercise 3.3.22.** *Prove that $S(X)$ has no proper closed normal subgroups.*

Let $(X, d)$ be a compact metric space. Then the group $\mathcal{H}omeo(X)$ of all homeomorphisms of $X$ admits a norm $v$ defined by $v(f) = \sup\{d(x, f(x)) + d(x, f^{-1}(x)) : x \in X\}$ for $f \in \mathcal{H}omeo(X)$. It defines an invariant metric in $\mathcal{H}omeo(X)$ that makes it a topological group.

## 3.4 Subgroups and direct products of topological groups

Let $G$ be a topological group and let $H$ be a subgroup of $G$. Then $H$ becomes a topological group when endowed with the topology induced by $G$. Sometimes we refer to this situation by saying that $H$ is a *topological subgroup* of $G$. It is easy to see that the filter $\mathcal{V}\restriction_H = \{H \cap V : V \in \mathcal{V}_\tau(e)\}$ in $H$ satisfies (a)–(c) from Theorem 3.1.5, it coincides precisely with the filter of neighborhoods of $e$ in the topological group $(H, \tau\restriction_H)$.

If $f : G \to f(G) \subseteq H$ is a topological isomorphism, where $f(G)$ carries the topology induced by $H$, then $f$ is called *topological group embedding*, or shortly *embedding*. In such a case $f(G)$ is a topological subgroup of $G$.

We start with two properties of the open subgroups.

**Proposition 3.4.1.** *Let $G$ be a topological group and let $H$ be a subgroup of $G$. Then:*

**(a)** *$H$ is open in $G$ iff $H$ has a non-empty interior;*

**(b)** *if $H$ is open, then $H$ is also closed;*

*Proof.* (a) Let $\emptyset \neq V \subseteq H$ be an open set and let $h_0 \in V$. Then $1 \in h_0^{-1}V \subseteq H = h_0^{-1}H$. Now $U = h_0^{-1}V$ is open, contains 1 and $h \in hU \subseteq H$ for every $h \in H$. Therefore $H$ is open.

(b) If $H$ is open then every coset $gH$ is open and consequently the complement $G \setminus H$ is open. So $H$ is closed. □

Let us see now how the *closure* $\overline{H}$ of a subset $H$ of a topological group $G$ can be computed.

**Lemma 3.4.2.** *Let $H$ be a subset of $G$. Then with $\mathcal{V} = \mathcal{V}(e_G)$ one has*

**(a)** $\overline{H} = \bigcap_{U\in\mathcal{V}} UH = \bigcap_{U\in\mathcal{V}} HU = \bigcap_{U,V\in\mathcal{V}} UHV$*;*

**(b)** *if $H$ is a subgroup of $G$, then $\overline{H}$ is a subgroup of $G$; if $H$ is a normal subgroup, then also $\overline{H}$ is a normal subgroup;*

**(c)** *$N = \overline{\{e_G\}}$ is a closed normal subgroup.*

**(d)** *for $x \in G$, one has $\overline{\{x\}} = xN = Nx$.*

*Proof.* (a) For $x \in G$ one has $x \notin \overline{H}$ iff there exists $U \in \mathcal{V}$ such that $xU \cap H = \emptyset = Ux \cap H$. Pick a symmetric $U$, i.e., $U = U^{-1}$. Then the latter property is equivalent to $x \notin UH \cup HU$. This proves $\overline{H} = \bigcap_{U\in\mathcal{V}} UH = \bigcap_{U\in\mathcal{V}} HU$. To prove the last equality in (a) note that the already established equalities yield

$$\bigcap_{U,V\in\mathcal{V}} UHV = \bigcap_{U\in\mathcal{V}} (\bigcap_{V\in\mathcal{V}} UHV) = \bigcap_{U\in\mathcal{V}} \overline{UH} \subseteq \bigcap_{U\in\mathcal{V}} U^2H = \bigcap_{W\in\mathcal{V}} WH = \overline{H}.$$

(b) Let $x, y \in \overline{H}$. According to (a), to verify $xy \in \overline{H}$ it suffices to see that $xy \in UHU$ for every $U \in \mathcal{V}$. This follows from $x \in UH$ and $y \in HU$ for every $U \in \mathcal{V}$. If $H$ is normal, then for every $a \in G$ and for $U \in \mathcal{V}$ there exists a symmetric $V \in \mathcal{V}$ with $aV \subseteq Ua$ and $Va^{-1} \subseteq a^{-1}U$. Now for every $x \in \overline{H}$ one has $x \in VHV^{-1}$, hence $axa^{-1} \in aVHV^{-1}a^{-1} \subseteq UaHa^{-1}U \subseteq UHU$. This proves $axa^{-1} \in \overline{H}$ according to (a).

(c) follows from (b) with $H = \{e_G\}$. □

**Corollary 3.4.3.** *If $A, B$ are non-empty subsets of a topological group, then $\overline{A} \cdot \overline{B} \subseteq \overline{AB}$. If one of the sets is a singleton, then $\overline{A} \cdot \overline{B} = \overline{AB}$.*

*Proof.* The inclusion follows from item (a) of the above lemma. (As $\overline{A} \cdot \overline{B} \subseteq UABU$ for every $U \in \mathcal{V}$.) In case $B = \{b\}$ is a singleton, $AB = Ab = t_b(A)$. Since $t_b$ is a homeomorphism, one has

$$\overline{A \cdot B} = \overline{Ab} = \overline{t_b(A)} = t_b(\overline{A}) = \overline{A}b \subseteq \overline{A} \cdot \overline{B}.$$

This proves the missing inlcusion. □

Clearly $\overline{A} \cdot \overline{B}$ is dense in $\overline{AB}$, as it contains the dense subset $AB$ of $\overline{AB}$. Therefore, the equality $\overline{A} \cdot \overline{B} = \overline{AB}$ holds true precisely when $\overline{A} \cdot \overline{B}$ is closed. We shall give examples showing that this often fails even in the group $\mathbb{R}$. On the other hand, we shall see that the equality holds true when $B$ is compact.

**Proposition 3.4.4.** *Let $\{G_i : i \in I\}$ be a family of topological groups. Then the direct product $G = \prod_{i\in I} G_i$, equipped with the product topology, is a topological group.*

*Proof.* The filter $\mathcal{V}(e_G)$ of all neighborhoods of $e_G$ in the product topology of $G$ has a base of neighborhoods of the form $U_{j_1} \times \ldots \times U_{j_n} \times \prod_{i\in I\setminus J} G_i$, where $J = \{j_1, \ldots j_n\}$ varies among all finite subsets of $I$ and $U_j \in \mathcal{V}(e_{G_j})$ for all $j \in J$. Now it is easy to check that the filter $\mathcal{V}(e_G)$ satisfies the conditions (a) − (c) from Theorem 3.1.5. For an arbitrary element $a \in G$ one can easily check that $\mathcal{V}(a) = a\mathcal{V}(e_G) = \mathcal{V}(e_G)a$. Hence $G$ is a topological group. □

**Exercise 3.4.5.** *Let $G = G_1 \times G_2$. Identify $G_1$ and $G_2$ with the subgroups $G_1 \times \{e_2\}$ and $\{e_1\} \times G_2$, respectively, of $G$. For a group topology $\tau$ on $G$ denote by $\tau_i$ the topology induced on $G_i$ by $\tau$, $i = 1, 2$. Prove that $\tau$ is coarser than the product topology $\tau_1 \times \tau_2$ of $G$.*

(Hint. Let $W$ be a $\tau$-neighborhood of the identity of $G$. Find a $\tau$-neighborhood $V$ of the identity of $G$ such that $V^2 \subseteq W$. Now $V \cap G_i$ is a $\tau_i$-neighborhood of the identity of $G_i$ for $i = 1, 2$, hence

$$\tau_1 \times \tau_2 \ni U = (V \cap G_1) \times (V \cap G_2) = ((V \cap G_1) \times \{e_2\}).(\{e_1\} \times (V \cap G_2)) \subseteq V^2 \subseteq W,$$

therefore, $W$ is also a $\tau_1 \times \tau_2$-neighborhood of the identity of $G$.)

**Theorem 3.4.6.** *Let $G$ be an abelian group equipped with its Bohr topology and let $H$ be a subgroup of $G$. Then:*

*(a) $H$ is closed in $G$;*

*(b) the topological subgroup topology of $H$ coincides with its Bohr topology.*

*Proof.* (a) Consider the quotient $G/H$. Then for every non-zero element $y$ of $G/H$ there exists a character of $G/H$ that does not vanish at $y$ by Corollary 2.1.12. Thus for every $x \in G \setminus H$ there exists a character $\chi : G \to \mathbb{T}$ such that $\chi(H) = 0$ and $\chi(x) \neq 0$. Since $\ker \chi$ is closed in the Bohr topology of $G$ and contains $H$, we conclude that $x \notin \overline{H}$.

(b) The inclusion $j : H^\# \hookrightarrow G^\#$ is continuous by Lemma 3.3.10. To see that $j : H^\# \to j(H)$ is open take a basic neighborhood $U_H(\chi_1, \ldots, \chi_n; \delta)$ of 0 in $H^\#$, where $\chi_1, \ldots, \chi_n \in H^*$. By Theorem 2.1.10 each $\chi_i$ can be extended to some character $\xi_i \in G^*$, hence $U_H(\chi_1, \ldots, \chi_n; \delta) = H \cap U_G(\xi_1, \ldots, \xi_n; \delta)$ is open in $j(H)$. This proves that the topological subgroup topology of $H$ coincides with its Bohr topology. □

**Remark 3.4.7.** In Lemma 3.3.10 we saw that if $f : G \to H$ is a homomorphism of abelian groups, then $f$ is continuous when both groups are equipped with their profinite (resp., pro-$p$-finite, $p$-adic, natural, pro-countable) topology. The above theorem shows that $f$ is actually a topological embedding if $f$ is simply the embedding of a subgroup and one takes the Bohr topology in both groups. One can show that this fails for the profinite, pro-$p$-finite, $p$-adic or the natural topology (take for example $G = \mathbb{Q}$ and $H = \mathbb{Z}$).

## 3.5 Separation axioms

Making use of Lemma 3.4.2 we show now that for a topological group all separation axioms $T_0 - T_{3.5}$ are equivalent.

**Proposition 3.5.1.** *Every topological group is a regular topological space. Moreover, for a topological group $G$ the following are equivalent:*

**(a)** *$G$ is $T_0$.*

**(b)** $\overline{\{e_G\}} = \{e_G\}$.

**(c)** *$G$ is Hausdorff;*

**(d)** *$G$ is $T_3$ (where $T_3$ stands for "regular and $T_1$").*

*Proof.* To prove the first statement it sufficies to check the regularity axiom at $e_G$. Let $U \in \mathcal{V}$. Pick a $V \in \mathcal{V}$ such that $V^2 \subseteq U$. Then $\overline{V} \subseteq V^2 \subseteq U$ by Lemma 3.4.2. This property proves the implication (b) $\to$ (d). Indeed, to see that (b) $\to$ (d) it suffices to deduce from (b) that $G$ is a $T_1$ space. This follows from the fact that all singletons $\{g\}$ of $G$ are closed, as $\{e_G\}$ closed.

On the other hand, obviously (d) $\to$ (c) $\to$ (b). Therefore, the properties (b), (c) and (d) are equivalent and obviously imply (a).

It remains to prove the implication (a) $\to$ (b). Let $N = \overline{\{x\}}$ and assume for a contradiction that there exists an element $x \in N$, $x \neq e_G$. Then $\overline{\{x\}} = xN = N$ according to Lemma 3.4.2 (d). Hence, $e_G \in \overline{\{x\}}$. This contradicts our assumption that $G$ is $T_0$. □

Let us see now that every $T_0$ topological group is also a Tychonov space.

**Theorem 3.5.2.** *Every Hausdorff topological group is a Tychonov space.*

*Proof.* Let $F$ be a closed set with $e_G \notin F$. Then we can find a chain (2) of open neighborhoods $(U_n)$ of $e_G$ as in Lemma 3.3.17 such that $F \cap U_0 = \emptyset$. Let $d$ be the pseudometric defined in Lemma 3.3.17 and let $f_F(x) = d(x, F)$ be the distance function from $F$. This function is continuous in the topology induced by the pseudometric. By the continuity of $d$ it will be continuous also with respect to the topology of $G$. It suffices to note now that $f_F(F) = 0$, while $f_F(1) = 1$. This proves that the space $G$ is Tychonov, as the pseudometric is left invariant, so the separation of a generic point $a \in G$ from a closed set $F$ that does not contain $a$ can be obtained by translating $a$ to the origin and applying the above argument. □

**Exercise 3.5.3.** *Let $G$ be a group. Prove that the following are equivalent:*

*(a) the profinite topology of $G$ is Hausdorff*

*(b) $G$ is residually finite.*

*If $G$ is abelian, then they are equivalent also to*

*(d) the natural topology of $G$ is Hausdorff.*

*(e) $G^1 = \{0\}$.*

(Hint. Prove that for an abelian group $G$ the subgroup $G^1$ coincides with the intersection of all subgroups of finite index of $G$.)

The above exercise allows one to obtain a purely topological proof of the fact that a reduced abelian group $G$ has no non-trivial divisible subgroups. Indeed, if $D$ is a divisible subgroup of $G$, then equip $G$ and $D$ with their profinite topologies. This will make the inclusion $j : D \hookrightarrow G$ continuous. Since $D$ is indiscrete (so $j(G)$ is an indiscrete subgroup of $G$), while $G$ is Hausdorff, this yealds that $D = j(G)$ is trivial.

## 3.6 Quotients of topological groups

Let $G$ be a topological group and $H$ a normal subgroup of $G$. Consider the quotient $G/H$ with the *quotient topology*, namely the finest topology on $G/H$ that makes the canonical projection $q : G \to G/H$ continuous. Since we have a group topology on $G$, the quotient topology consists of all sets $q(U)$, where $U$ runs over the family of all open sets of $G$ (as $q^{-1}(q(U))$ is open in $G$ in such a case). In particular, one can prove the following important properties of the quotient topology.

**Lemma 3.6.1.** *Let $G$ be a topological group, let $H$ be a normal subgroup of $G$ and let $G/H$ be equipped with the quotient topology. Then*

**(a)** *the canonical projection $q : G \to G/H$ is open.*

**(b)** *If $f : G/H \to G_1$ is a homomorphism to a topological group $G_1$, then $f$ is continuous iff $f \circ q$ is continuous.*

*Proof.* (a) Let $U \neq \emptyset$ be an open set in $G$. Then $q^{-1}(q(U)) = HU = \bigcup_{h \in H} hU$ is open, since each $hU$ is open. Therefore, $q(U)$ is open in $G/H$.

(b) If $f$ is continuous, then the composition $f \circ q$ is obviously continuous. Assume now that $f \circ q$ is continuous. Let $W$ be an open set in $G_1$. Then $(f \circ q)^{-1}(W) = q^{-1}(f^{-1}(W))$ is open in $G$. Then $f^{-1}(W)$ is open in $G/H$. Therefore, $f$ is continuous. □

The next theorem is due to Frobenius.

**Theorem 3.6.2.** *If $G$ and $H$ are topological groups, $f : G \to H$ is a continuous surjective homomorphism and $q : G \to G/\ker f$ is the canonical homomorphism, then the unique homomorphism $f_1 : G/\ker f \to H$, such that $f = f_1 \circ q$, is a continuous isomorphism. Moreover, $f_1$ is a topological isomorphism iff $f$ is open.*

*Proof.* Follows immediately from the definitions of quotient topology and open map and Lemma 3.6.1. □

As a first application of Theorem 3.6.2 we show that the quotient is invariant under isomorphism in the following sense:

**Corollary 3.6.3.** *Let $G$ and $H$ be topological groups and let $f : G \to H$ be a topological isomorphism. Then for every normal subgroup $N$ of $G$ the quotient $H/f(N)$ is isomorphic to $G/N$.*

*Proof.* Obviously $q(N)$ is a normal subgroup of $H$ and the surjective quotient homomorphism $q : H \to H/q(N)$ is continuous and open by Lemma 3.6.1. Therefore, the composition $h = q \circ f : G \to H/f(N)$ is a surjective continuous and open homomorphism with $\ker h = N$. Therefore, $H/f(N)$ is isomorphic to $G/N$ by Theorem 3.6.2. □

One can order continuous surjective homomorphisms with a common domain $G$ saying that $f : G \to H$ is projectivelly bigger than $f' : G \to H'$ when there exists continuous homomorphism $\iota : H \to H'$ such that $f' = \iota \circ f$. In the next proproposition we show, roughly speaking, that the projective order between continuous surjective open homomorphisms with the same domain corresponds to the order by inclusion of their kernels.

**Proposition 3.6.4.** *Let $G, H_1$ and $H_2$ be topological abelian groups and let $\chi_i : G \to H_i$, $i = 1, 2$, be continuous surjective open homomorphisms. Then there exists a continuous homomorphism $\iota : H_1 \to H_2$ such that $\chi_2 = \iota \circ \chi_1$ iff* $\ker \chi_1 \leq \ker \chi_2$. *If* $\ker \chi_1 = \ker \chi_2$ *then $\iota$ will be a topological isomorphism.*

*Proof.* The necessity is obvious. So assume that $\ker \chi_1 \leq \ker \chi_2$ holds. By the homomorphism theorem applied to $\chi_i$ there exists a topological isomorphisms$j_i : G/\ker \chi_i \to H_i$ such that $\chi_i = j_i \circ q_i$, where $q_i : G \to G/\ker \chi_i$ is the canonical homomorphism for $i = 1, 2$. As $\ker \chi_1 \leq \ker \chi_2$ we get a continuous homomorphism $t$ that makes commutative the following diagram

$$\begin{tikzcd}
& & G \arrow[dll, "\chi_1"'] \arrow[dl, "q_1"] \arrow[dr, "q_2"'] \arrow[drr, "\chi_2"] & & \\
H_1 \arrow[rrrr, bend right, "\iota"'] & G/\ker \chi_1 \arrow[l, "j_1"] \arrow[rr, dashed, "t"] & & G/\ker \chi_2 \arrow[r, "j_2"] & H_2
\end{tikzcd}$$

Obviously $\iota = j_2 \circ t \circ j_1^{-1}$ works. If $\ker \chi_1 = \ker \chi_2$, then $t$ is a topological isomorphism, hence $\iota$ will be a topological isomorphism as well. □

Independently on its simplicity, Theorem 3.6.2 is very important since it produces topological isomorphisms as in the above proof. Openness of the map $f$ is its main ingredient, so from now on we shall be interested in providing conditions that ensure openness (see also §4.1).

**Lemma 3.6.5.** *Let $X, Y$ be topological spaces and let $\varphi : X \to Y$ be a continuous open map. Then for every subspace $P$ of $Y$ with $P \cap \varphi(X) \neq \emptyset$ the restriction $\psi : H_1 \to P$ of the map $\varphi$ to the subspace $H_1 = \varphi^{-1}(P)$ is open.*

*Proof.* To see that $\psi$ is open choose a point $x \in H_1$ and a neighborhood $U$ of $x$ in $H_1$. Then there exists a neighborhood $W$ of $x$ in $X$ such that $U = H_1 \cap W$. To see that $\psi(U)$ is a neighborhood of $\psi(x)$ in $P$ it suffices to note that if $\varphi(w) \in P$ for $w \in W$, then $w \in H_1$, hence $w \in H_1 \cap W = U$. Therefore $\varphi(W) \cap P \subseteq \varphi(U) = \psi(U)$. □

We shall apply this lemma when $X = G$ and $Y = H$ are topological group and $\varphi = q : G \to H$ is a continuous open homomorphism. Then the restriction $q^{-1}(P) \to P$ of $q$ is open for every subgroup $P$ of $H$. Nevertheless, even in the particular case when $q$ is surjective, the restriction $H_1 \to \varphi(H_1)$ of $q$ to an arbitrary closed subgroup $H_1$ of $G$ need not be open as the following example shows.

**Example 3.6.6.** Let $G = \mathbb{T}$ and $N = \langle 1/2 \rangle$ be the 2-element cyclic subgroup of $G$. Then the quotient map $q : G \to G/N$ is a continuous open homomorphism. Let now $H_1 = \mathbb{Z}(3^\infty)$ be the Prüfer subgroup. The restriction $q' : H_1 \to q(H_1)$ of $q$ is a continuous isomorphism. To see that $q'$ is not open it suffices to notice that the sequence $x_n = \sum_{k=1}^{n} 1/3^k$ in $H_1$ is not convergent (as it converges to the point $1/2 \in \mathbb{T}$ that does not belong to $H_1$). On the other hand, $q'(x_n) \to 0$ in $q(H_1)$ since every neighborhood $W$ of 0 in $q'(H_1)$ has the form $U \cap H_1$, where $U = q^{-1}(V)$ and $V$ is a neighborhood of 0 in $\mathbb{T}$. Hence $U$ is an open set of $\mathbb{T}$ containing 0 and $1/2$. Hence $q'(x_n) \in U$ for all sufficiently large $n$, thus $q'(x_n) \to 0$ in $q(H_1)$.

In the next theorem we see some isomorphisms related to the quotient groups.

**Theorem 3.6.7.** *Let $G$ be a topological group, let $N$ be a normal subgroup of $G$ and let $p : G \to G/N$ be the canonical homomorphism.*

**(a)** *If $H$ is a subgroup of $G$, then the homomorphism $p_1 : HN/N \to p(H)$, defined by $p_1(xN) = p(x)$, is a topological isomorphism.*

**(b)** *If $H$ is a closed normal subgroup of $G$ with $N \subseteq H$, then $p(H) = H/N$ is a closed normal subgroup of $G/N$ and the map $j : G/H \to (G/N)/(H/N)$, defined by $j(xH) = (xN).(H/N)$, is a topological isomorphism.*

**(c)** *If $H$ is a subgroup of $G$, then the map $s : H/H \cap N \to (HN)/N$, defined by $s(x(H \cap N)) = xN$, is a continuous isomorphism. It is a topological isomorphism iff the restriction $p\restriction_H : H \to (HN)/N$ is open.*

*(Both in (a) and (b) the quotient groups are equipped with the quotient topology.)*

*Proof.* (a) As $HN = p^{-1}(p(H))$ we can apply Lemma 3.6.5 and conclude that the restriction $p' : HN \to p(H)$ of $p$ is an open map. Now Theorem 3.6.2 appplies to $p'$.

(b) Since $H = HN$, item (a) implies that the induced topology of $p(H)$ coincides with the quotient topology of $H/N$. Hence we can identify $H/N$ with the topological subgroup $p(H)$ of $G/N$. Since $H = HN$, the set $(G/N) \setminus p(HN) = p(G \setminus HN)$ is open, hence $p(H)$ is closed. Finally note that the composition $f : G \to (G/N)/(H/N)$ of $p$ with the canonical homomorphism $G/N \to (G/N)/(H/N)$ is open, being the latter open. Applying to the open homomorphism $f$ with $\ker f = H$ Theorem 3.6.2 we can conclude that $j$ is a topological isomorphism.

(c) To the continuous surjective homomorphism $p\restriction_H : H \to (HN)/N$ apply Theorem 3.6.2 to find a continuous isomorphism $s : H/H \cap N \to (HN)/N$, that is necessarily defined by $s(x(H \cap N)) = xN$. By the same theorem, $s$ is a topological isomorphism iff the map $p\restriction_H : H \to (HN)/N$ is open. $\square$

Example 3.6.6 shows that the continuous isomorphism $s : H/H \cap N \to (HN)/N$ need not be open (take $G = \mathbb{T}$, $H = \mathbb{Z}(3^\infty)$ and $N = \langle 1/2 \rangle$, so that $H \cap N = \{0\}$ and $s : H \to (H+N)/N = q(H)$ is not open).

**Example 3.6.8.** Let us see that for every $m$ the quotient group $\mathbb{T}/\mathbb{Z}_m$ is isomorphic to $\mathbb{T}$ itself. To this end consider the subgroup $H = \langle 1/m \rangle$ of $\mathbb{R}$ containing $N = \mathbb{Z}$. Then $H/N \cong \mathbb{Z}_m$ so that $\mathbb{T}/\mathbb{Z}_m = (\mathbb{R}/\mathbb{Z})/(H/\mathbb{Z}) \cong \mathbb{R}/H$. Hence it remains to note that $\mathbb{R}/H \cong \mathbb{R}/\mathbb{Z} = \mathbb{T}$, as the automorphism $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x/m$ takes $\mathbb{Z}$ to $H$.

Here we consider the counterpart of Remark 3.4.7 for quotient groups:

**Proposition 3.6.9.** *If $f : G \to H$ is a surjective homomorphism of abelian groups, then $f$ is (continuous and) open when both groups are equipped with their profinite (resp., pro-p-finite, p-adic, natural, pro-countable) topology.*

*Proof.* For the profinite and for the pro-countable topology use that fact that for a subgroup $N$ of $G$ one has a surjective homomorphism $f_1 : G/N \to H/f(N)$ induced by $f$. The remaining cases are trivial. $\square$

**Remark 3.6.10.** We shall see in the sequel that if $G$ is an abelian group equipped with its Bohr topology and $H$ is a subgroup of $G$, then the quotient topology of $G/H$ coincides with the Bohr topology of $G/H$. Moreover, $G$ has no convergent sequences [57, §3.4].

## 3.7 The Hausdorff reflection of a topological group

For a topological group $(G, \tau)$ denote by $\mathrm{core}(G, \tau)$, or simply by $\mathrm{core}(G)$, when the topology is clear, the intersection $\bigcap \mathcal{V}_{(G,\tau)}(e_G)$.

**Proposition 3.7.1.** *Let $(G, \tau)$ be a topological group.*

*(a) $\mathrm{core}(G, \tau)$ is a closed normal subgroup of $G$ and coincides with the closure of $\{e_G\}$.*

*(b) The quotient group $\mathfrak{h}G := G/\mathrm{core}(G, \tau)$, equipped with the quotient topology, is a Hausdorff group.*

*(c) If $f : (G, \tau) \to H$ is a continuous homomorphism to a Hausdorff group $H$, then there exists a unique continuous homomorphism $f_1 : \mathfrak{h}G \to H$ such that $f_1 \circ q = f$, where $q : G \to \mathfrak{h}G$ is the canonical homomorphism.*

The Hausdorff quotient group $\mathfrak{h}G$ associated to $(G, \tau)$ is its best approximation of $(G, \tau)$ by Hausdorff groups, as item (c) of the above proposition shows.

Let us see that the assignment $G \mapsto \mathfrak{h}G$ defines a functor from the category of all topological group to the subcategory of Hausdorff groups:

**Proposition 3.7.2.** *If $f : (G, \tau) \to (H, \sigma)$ is a continuous homomorphism of topological groups, then the map $\mathfrak{h}f : \mathfrak{h}G \to \mathfrak{h}H$ defined by $\mathfrak{h}f(x\mathrm{core}(G)) := f(x)\mathrm{core}(H)$ is a continuous homomorphism commuting with the canonical maps $q_G : G \to \mathfrak{h}G$ and $q_H : H \to \mathfrak{h}H$. If $f$ is an embedding, then so is $\mathfrak{h}f$.*

*Proof.* Since $f(e_G) = e_H$, item (a) of the abobe proposition implies that $f(\mathrm{core}(G)) \leq \mathrm{core}(H)$. This proves the correctness of the definition of $\mathfrak{h}f$ and the commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle q_G}\downarrow & & \downarrow{\scriptstyle q_H} \\ \mathfrak{h}G & \xrightarrow[\mathfrak{h}f]{} & \mathfrak{h}H \end{array}$$

The continuity of $\mathfrak{h}f$ easily follows from the continuity of $f$, as $\mathfrak{h}G$ and $\mathfrak{h}H$ carry the quotient topologies.

Now assume that $f$ is an embedding. For simplicity we assume that $G$ is simply a topological subgroup of $H$ and $f$ is the inclusion. Then obvisouly $\mathrm{core}(G) = \mathrm{core}(H) \cap G$, so that $\mathfrak{h}f$ is injective. It remains to prove that $\mathfrak{h}f : \mathfrak{h}G \to q(G) = \mathfrak{h}f(\mathfrak{h}G)$ is open, when $q(G)$ carries the topologies induced by $\mathfrak{h}H$. Let us see first that $h := q \restriction_G$ is open. To this end take a neighborhood $W$ of $e_H$ then $q(W \cap H)$ is a generaic neighborhood of $e$ in $q(G)$ equipped with the quotient topology. Pick a symmetric neighborhood $W_1$ of $e_H$ with $W_1^2 \subseteq W$. Then

$$q(W_1) \cap q(H) \subseteq q(W \cap H) \tag{3}$$

and this will prove that $q(W \cap H)$ isa neighborhood of $e$ in $q(G)$ equipped with the topology induced by $\mathfrak{h}H$. To prove (3) pick $w \in W_1$ such that $q(w) \in q(W_1) \cap q(H)$, i.e., $q(w) \in q(H)$. Then there exists $h \in H$ such that $w = hy$, where $y \in \ker q = \mathrm{core}(H)$. As $\mathrm{core}(H) \subseteq W_1$, this implies that $h \in W_1^2 \subseteq W$, so $h \in W \cap H$. Therefore, $q(w) = q(h) \in q(W \cap H)$ and (3) is proved. □

In the next remark we collect further properties of the reflection functor $\mathfrak{h}$ and the map $q_G$:

**Remark 3.7.3.** We shall see in the sequel that:
(a) a group $G$ is (locally) compact iff $\mathfrak{h}G$ is locally compact.
(b) if $H$ is a closed (oepn) subgroup of $G$, then so is $q : G(H)$ in $\mathfrak{h}G$.
(c) If a continuous homomorphism $f : (G, \tau) \to (H, \sigma)$ of topological groups is open (closed), then so if $\mathfrak{h}f$.

## 3.8 Initial and final topologies

Let $G$ be a group and let $\{K_i : i \in I\}$ be a family of topological groups. For a given family $\mathcal{F}$ of group homomorphisms $f_i : G \to K_i$ one defines the *initial topology* of the family $\mathcal{F}$ as the coarsest group topology that makes continuous all the homomorphisms $f_i \in \mathcal{F}$. Namely, the group topology on $G$ obtained by taking as a filter-base of neighborhoods at $1_G$ all finite intersections $\bigcap_{i=1}^n f_i^{-1}(U_i)$, where $U_i \in \mathcal{V}_{K_i}(1_{K_i})$, $n \in \mathbb{N}$.

There are many instances of initial topologies:

**Example 3.8.1.**

1. For a topological group $K$ and a subgroup $G$ of $K$, the induced topology of $G$ is the initial topology of the inclusion map $G \hookrightarrow K$.
2. For a family $\{K_i : i \in I\}$ of topological groups, the product topology of $G = \prod_{i \in I} K_i$ is the initial topology of the family of the projections $p_i$.
3. Let $G$ be a group, let $\{K_i : i \in I\}$ be the family of all finite quotient groups $G/N_i$ of $G$ equipped with the discrete topology and let $f_i : G \to K_i$ be the canonical homomorphism for $i \in I$. Then the pro-finite topology of $G$ coincides with the initial topology of the family $(f_i)$.
4. For a fixed prime $p$ the pro-$p$-topology of a group $G$ can be described in a similar manner as the pro-finite topology, using the finite quotients $G/N_i$ of $G$ that are $p$-groups. The $p$-adic topology of $G$ is obtained if instead of the all finite quotient of $G$ that are $p$-groups, one takes all quotients of $G$ of finite exponent, that is a power of $p$.
5. To obtain the natural topology of a group $G$ as the initial topology in the above sense, one has to make recourse to all quotients of $G$ of finite exponent.
6. The co-countable topology of a group $G$ can be obtained as the initial topology in the above sense, if one takes all countable quotients of $G$.
7. For a family $H$ of characters $f_i : G \to \mathbb{T}$, the initial topology of the family $H$ coincides with the topology $\mathcal{T}_H$ defined in §3.3.2.

Now we define an inverse system of topological groups and inverse limit of such a system.

**Definition 3.8.2.** Let $(I, \leq)$ be a directed set.

(a) An *inverse system* of topological groups, indexed by $(I, \leq)$, is a family of $\{G_i : i \in I\}$ topological groups and continuous homomorphisms $\nu_{ij} : G_j \to G_i$ for every pair $i \leq j$ in $I$, such that for every triple $i \leq j \leq k$ in $I$ one has $\nu_{ij} \circ \nu_{jk} = \nu_{ik}$.

(b) An *inverse limit*, of an inverse system as in (a) is a topological group $G$ and a family of continuous homomorphisms $p_i : G \to G_i$ satisfying $p_i = \nu_{ij} \circ p_j$ for every pair $i \leq j$ in $I$, such that for every topological group $H$ and every family of continuous homomorphisms $q_i : H \to G_i$ satisfying $q_i = \nu_{ij} \circ q_j$ for every pair $i \leq j$ in $I$ there exists a unique continuous homomorphism $f : H \to G$ such that $q_i = p_i \circ f$ for every $i \in I$.

We denote by $\lim\limits_{\leftarrow} G_i$ the inverse limit determined in item (b).

**Exercise 3.8.3.** *(a) Prove that the inverse limit $\lim\limits_{\leftarrow} G_i$ is uniquely determined up to isomorphism.*

*(b) Let $\{G_i : i \in I\}$ and $\nu_{ij} : G_j \to G_i$ be as in (a) of the above definition. In the product $H = \prod_{i\in I} G_i$ consider the subgroup $G = \{x = (x_i) \in H : \nu_{ij}(x_j) = x_i$ whenever $i \leq j\}$ and denote by $p_i$ the restriction to $G$ of the canonical projection $H \to G_i$. Prove that*

*($b_1$) $G \cong \lim\limits_{\leftarrow} G_i$, i.e., the group $G$ along with the family of projections $p_i$ is an inverse limit of $\{G_i : i \in I\}$ and $\nu_{ij} : G_j \to G_i$.*

*($b_2$) the group $G$ has the initial topology of the family of all projections $G \to G_i$.*

**Exercise 3.8.4.** *For $G$ and $f_i : G \to K_i$ as above, let $\tau_i$ denote the initial topology of the single homomorphism $f_i \in \mathcal{F}$. Then the initial topology of the family $\mathcal{F}$ coincides with $\sup_{i\in I} \tau_i$.*

**Exercise 3.8.5.** *For $G$ and $f_i : G \to K_i$ as above, a homomorphism $h : H \to G$ is continuous w.r.t. the initial topology of the family $\mathcal{F}$ on $G$ iff all compostions $f_i \circ h : H \to K_i$ are continuous.*

**Lemma 3.8.6.** *If for $G$ and $f_i : G \to K_i$ as above the homomorphisms $f_i \in \mathcal{F}$ separate the points of $G$, then the initial topology of the family $\mathcal{F}$ coincides with the topology induced on $G$ by the injective diagonal map $\Delta_{\mathcal{F}} : G \to \prod_{i\in I} K_i$ of the family $\mathcal{F}$, defined by $\Delta_{\mathcal{F}}(x) = (f_i(x)) \in \prod_{i\in I} K_i$.*

**Exercise 3.8.7.** *Let $G$ be a group and let $\mathcal{T} = \{\tau_i : i \in I\}$ be a family of group topologies on $G$. Then $\sup\{\tau_i : i \in I\}$ coincides with the initial topology of the family $\mathcal{F}$ of all maps $id_G : G \to (G, \tau_i)$ and also with the topology induced on $G$ by the diagonal map $\Delta_{\mathcal{F}} : G \to \prod_{i\in I} G = G^I$ of the family $\mathcal{F}$, i.e., $(G, \inf\{\tau_i : i \in I\})$ is topologically isomorphic to the diagonal subgroup $\Delta = \{x = (x_i) \in G^I : x_i = x_j$ for all $i, j \in I\}$ of $\prod_{i\in I}(G_i, \tau_i)$.*

Let $G$ be a group and let $\{K_i : i \in I\}$ be a family of topological groups. For a given family $\mathcal{F}$ of group homomorphisms $f_i : K_i \to G$ one defines the *final topology* of the family $\mathcal{F}$ as the finest group topology on $G$ that makes continuous all the homomorphisms $f_i \in \mathcal{F}$. The prominent example in this direction is the quotient topology of a quotient group $G = K/N$ of a topological group $K$. It coincides with the final topology of the quotient homomorphism $q : K \to G$.

**Exercise 3.8.8.** *For $G$ and $f_i : K_i \to G$ as above, a homomorphism $h : G \to H$ is continuous w.r.t. the final topology of the family $\mathcal{F}$ on $G$ iff all compostions $h \circ f_i : K_i \to H$ are continous.*

**Exercise 3.8.9.** *(a) If $V, U$ are vector spaces over a field $K$, then the finite toplogy of $Hom(V, U)$ is the initial topology of all linear maps $f : V \to U$, when $U$ is equipped with the discrete topology.*

*(b) If $U = K = \mathbb{Z}_p$ is a finite field, then the finite topology of $V^*$ coincides with the pro-finite topology of $V^*$.*

# 4 Closed subgroups, metrizability and connectedness

## 4.1 Closed subgroups

**Theorem 4.1.1.** *If $G$ is a Hausdorff topological group containing a dense abelian group, then $G$ is abelian.*

*Proof.* Let $H$ be the dense abelian subgroup of $H$. Take $x, y \in H$. Then $x = \lim_i h_i$ and $y = \lim_i g_i$, where $h : i, g : i \in H$. It is easy to see that $[x, y] = \lim_i [h_i, g_i] = e_G$ as $H$ is abelian. Then $[x, y] = e_G$ by the uniqueness of the limit in Hausdorff groups (see Lemma 2.2.32(d)). □

**Exercise 4.1.2.** *Let $G$ be a Hausdorff topological group. Prove that the centralizer of an element $g \in G$ is a closed subgroup. In particular, the center $Z(G)$ is a closed subgroup of $G$.*

**Exercise 4.1.3.** *If $G$ is a Hausdorff topological group containing a dense nilpotent group, then $G$ is nilpotent.*

Next we see that the discrete subgroups of the Hausdorff groups are always closed.

**Proposition 4.1.4.** *Let $G$ be a topological group and let $H$ be a subgroup of $G$. If $H$ is discrete and $G$ is $T_1$, then $H$ is closed.*

*Proof.* Since $H$ is discrete there exists $U \in \mathcal{V}(e_G)$ with $U \cap H = \{e_G\}$. Choose $V \in \mathcal{V}(e_G)$ with $V^{-1} \cdot V \subseteq U$. Then $|xV \cap H| \leq 1$ for every $x \in G$, as $h_1 = xv_1 \in xV \cap H$ and $h_2 = xv_2 \in xV \cap H$ give $h_1^{-1} h_2 \in V^{-1} \cdot V \cap H = \{e_G\}$, hence $h_1 = h_2$. Therefore, if $x \notin H$ one can find a neighborhood $W \subseteq xV$ of $x$ with $W \cap H = \emptyset$, i.e., $x \notin \overline{H}$. Indeed, if $xV \cap H = \emptyset$, just take $W = xV$. In case $xV \cap H = \{h\}$ for some $h \in H$, one has $h \neq x$ as $x \notin H$. Then $W = xV \setminus \{h\}$ is the desired neighborhood of $x$. □

**Example 4.1.5.** (a) Let $H$ be a Hausdorff non-trivial group and let $G = H \times N$, where $N$ is an indiscrete non-trivial group. If $H$ is discrete, then $H \times \{e_G\}$ is a discrete dense (hence, non-closed) subgroup of $G$.

(b) In general, if $H$ is a Hausdorff subgroup of a topological group $G$, then $H \cap N = \{e_G\}$ and $\overline{H} = H \cdot N$, where $N = \overline{\{e_G\}}$. Hence, one can identify $H \cdot \overline{\{e_G\}}$ with the Cartesian product $H \times N$, in an obvious way. If moreover $H$ is discrete, then the Cartesian product carries the product topology, where $H$ is discrete and $N$ is indiscrete (argue as in the proof of Proposition 4.1.4).

**Exercise 4.1.6.** *Prove that for every infinite set $X$ and every group topology on the permutation group $S(X)$ the subgroups $S_x = \{f \in S(X) : f(x) = x\}$, $x \in X$, are either closed or dense. (Hint. Prove that $S_x$ is a maximal subgroup of $S(X)$, see Fact 5.2.3.)*

Now we relate proprieties of the quotient $G/H$ to those of the subgroup $H$ of $G$.

**Lemma 4.1.7.** *Let $G$ be a topological group and let $H$ be a normal subgroup of $G$. Then:*

*(1) the quotient $G/H$ is discrete if and only if $H$ is open;*

*(2) the quotient $G/H$ is Hausdorff if and only if $H$ is closed.*

*Proof.* let $q : G \to G/H$ denote the quotient homomorphism. (1) If $G/H$ is discrete, then $H = q^{-1}(1_{G/H})$ is open since the singleton $\{1_{G/H}\}$ is open. Conversely, if $H$ is open, then $\{1_{G/H}\} = q(H)$ is open since the map $q$ is open.

(2) If $G/H$ is Hausdorff, then $H = q^{-1}(1_{G/H})$ is closed since the singleton $\{1_{G/H}\}$ is closed. Conversely, if $H$ is closed, then $\{1_{G/H}\} = q(H)$ is closed by Theorem 3.6.7. □

**Lemma 4.1.8.** *Let $(G, \tau)$ be a topological group and let $N$ denote the closure of $\{e_G\}$. Then:*

*(1) $N$ is an indiscrete closed normal subgroup of $G$ and the quotient $G/N$ is Hausdorff,*

*(2) $\tau$ coincides with the initial topology of $G$ w.r.t. the quotient map $G \to G/N$;*

*(3) every continuous homomorphism $f : G \to H$, where $H$ is a Hausdorff group, factorizes through the quotient $G \to G/N$.*

*Proof.* (1) Since $N$ is contained in every neighborhood of 1, closed normal subgroup $N$ of $G$ is indiscrete. The last assertion follows from item (2) of the previous lemma.

(2) Let $V \in \mathcal{V}(e_G)$ be open. Then $N \subseteq V$. Fix arbitrarily $v \in V$. Then there exits $U \in \mathcal{V}(1)$ such that $xU \subseteq V$. Since $N \subseteq U \subseteq V$, we conclude that $xN \subseteq V$. This proves that $VN \subseteq V$. On the other hand, $V \subseteq VN$, hence $V = VN = q^{-1}(q(N))$. Hence $\tau$ coincides with the initial topology w.r.t. the quotient map $G \to G/N$.

(3) Let $L = \ker f$. Then $L$ is a closed normal subgroup of $G$, so $L \geq N$. By Frobenius theorem there exists a continuous injective homomorphism $f_1 : G/L \to H$, such that $f = f_1 \circ \pi$, where $\pi : G \to G/L$ is the quotient homomorphism. By $L \geq N$ there exists a homomorphism $h : G/N \to G/L$ such that $\pi = h \circ q$. Moreover, $h$ is continuous by the continuity of $\pi = h \circ q$. Now the composition $\eta = f_1 \circ h : G/N \to H$ provides the desired factorization $f = \eta \circ q$. □

This lemma shows that the properties of $G$ are easily determined from the corresponding properties of the Hausdorff quotient $G/N$. This is why, it is not restrictive to work mainly with Hausdorff groups. Therefore, most often the topological groups in the sequel will be assumed to be Hausdorff.

The next example shows that the closed subgroups of $\mathbb{R}$ have a very simple description. The closed subgroups of $\mathbb{R}^n$ will be described in §8.1.

**Example 4.1.9.** For a proper closed subgroup $H$ of $\mathbb{R}$ the following properties are equivalent:

**(a)** $H$ is cyclic;

**(b)** $H$ is discrete;

**(c)** $H$ is closed.

Indeed, it is easy to see that cyclic subgroups of $\mathbb{R}$ are discrete, so (a) $\to$ (b). By Proposition 4.1.4 (b) $\to$ (c).

To prove (c) $\to$ (a) assume that $H$ is a proper closed non-trivial subgroup of $\mathbb{R}$. Let $h_0$ be the greatest lower bound of the set $\{h \in H : h > 0\}$. Assume that $h_0 = 0$. Then for every $\varepsilon > 0$ there exists $h \in (0, \varepsilon) \cap H$. Therefore, $H$ hits every open interval of $\mathbb{R}$ of lenght $\leq 2\varepsilon$. This proves that $H$ is dense in $\mathbb{R}$, a contradiction. Therefore, $h_0 > 0$. Now it is easy to see that that $H = \langle h_0 \rangle$. Indeed, for a positive $h \in H$ pick the greatest integer $m$ such that $mh_0 \leq h < (m+1)h_0$. Then $0 \leq h - mh_0 < h_0$ and $h - mh_0 \in H$. Hence $h - mh_0 = 0$. Therefore, $h \in \langle h_0 \rangle$.

Consequently, a subgroup of $\mathbb{R}$ is dense iff it is not cyclic. This gives easy examples of closed subgroups $H_1, H_1$ of $\mathbb{R}$ such that $H_1 + H_2$ is not closed, actually it is dense in $\mathbb{R}$. Indeed, suhc $H_1, H_2$ are necessarily cyclic. Take $H_1 = \mathbb{Z}$ and $H_2$ any cyclic subgroup generated by an irrational number. Then $H_1 + H_2$ is not cyclic, so by what we proved above, it cannot be dense. In fact, it is *dense* in $\mathbb{R}$.

We shall see now that a cyclic subgroup need not be closed in general (compare with Example 4.1.9). A topological group $G$ is *monothetic* if there exists $x \in G$ with $\langle x \rangle$ dense in $G$.

**Exercise 4.1.10.** *Prove that:*

*(a) a Hausdorff monothetic group is necessarily abelian.*

*(b) $\mathbb{T}$ is monothetic.*

*Is $\mathbb{T}^2$ monothetic? What about $\mathbb{T}^{\mathbb{N}}$?*

(Hint. (a) Apply Theorem 4.1.1. (b) By Example 4.1.9 the subgroup $N = \mathbb{Z} + \langle a \rangle$ of $\mathbb{R}$ is dense whenever $a \in \mathbb{R}$ is rational. Then the image $a + \mathbb{Z}$ of $a$ in $\mathbb{T}$ generates a dense subgroup of $\mathbb{T}$. The questions have positive answer, see §8.1.)

Let $G$ be an abelian group and let $H$ be a family of characters of $G$. Then the characters of $H$ separate the points of $G$ iff for every $x \in G$, $x \neq 0$, there exists a character $\chi \in H$ with $\chi(x) \neq 1$.

**Exercise 4.1.11.** *Let $G$ be an abelian group and let $H$ be a family of characters of $G$. Prove that the topology $\mathcal{T}_H$ is Hausdorff iff the characters of $H$ separate the points of $G$.*

**Proposition 4.1.12.** *Let $G$ be an infinite abelian group and let $H = Hom(G, \mathbb{S})$. Then the following holds true:*

**(a)** *the characters of $H$ separate the points of $G$,*

**(b)** *the Bohr topology $\mathcal{T}_H$ is Hausdorff and non-discrete.*

*Proof.* (a) This is Corollary 2.1.12.

(b) According to Exercise 4.1.11 item (a) implies that the topology $\mathcal{T}_H$ is Hausdorff. Suppose, for a contradiction, that $\mathcal{T}_H$ is discrete. Then there exist $\chi_i \in H, i = 1, \ldots, n$ and $\delta > 0$ such that $U(\chi_1, \ldots, \chi_n; \delta) = \{0\}$. In particular, $H = \bigcap_{i=1}^n \ker \chi_i = \{0\}$. Hence the diagonal homomorphism $f = \chi_1 \times \ldots \times \chi_n : G \to \mathbb{S}^n$ is injective and $f(G) \cong G$ is an infinite discrete subgroup of $\mathbb{S}^n$. According to Proposition 3.4.1 $f(G)$ is closed in $\mathbb{S}^n$ and consequently, compact. The compact discrete spaces are finite, a contradiction. □

**Example 4.1.13.** (a) Countable Hausdorff topological groups are normal, since a regular Lindelöff space is normal (Theorem 2.2.13).

(b) The situation is not so clear for uncountable ones. Contrary to what we proved in Theorem 3.5.2 Hausdorff topological groups need not be normal as topological spaces (see Exercise 4.1.14). A nice "uniform" counter-example to this was given by Trigos: for every uncountable group $G$ the topological group $G^{\#}$ is not normal as a topological space.

**Exercise 4.1.14.** $^*$ *The group $\mathbb{Z}^{\aleph_1}$ equipped with the Tychonov topology (where $\mathbb{Z}$ is discrete) is not a normal space [102].*

### 4.1.1 Metrizability of topological groups

**Theorem 4.1.15.** (Birkhoff-Kakutani) *A topological group is metrizable iff it has a countable base of neighborhoods of $e_G$.*

*Proof.* The necessity is obvious as every point $x$ in a metric space has a countable base of neighborhoods. Suppose now that $G$ has countable base of neighborhoods of $e_G$. Then one can build a chain (2) of neighborhoods of $e_G$ as in Lemma 3.3.17 that form a base of $\mathcal{V}(e_G)$, in particular, $\bigcap_{n=1}^\infty U_n = \{e_G\}$. Then the pseudometric produced by the lemma is a metric that induces the topology of the group $G$ because of the inclusions $U_n \subseteq B_{1/n} \subseteq U_{n-1}$. □

**Exercise 4.1.16.** *Prove that subgroups, countable products and quotients of metrizable topological groups are metrizable.*

**Theorem 4.1.17.** *Prove that every Hausdorff topological abelian group admits a continuous isomorphism into a product of metrizable abelian groups.*

*Proof.* For $x \in G$, $x \neq 0$ choose an open neighborhood $U$ of 0 with $x \in U$. Build a sequence $\{U_n\}$ of symmetric open neighborhoods of 0 with $U_0 \subseteq U$ and $U_n + U_n \subseteq U_{n-1}$. Then $H_U = \bigcap_{n=1}^{\infty} U_n$ is a closed subgroup of $G$ .Let $\tau_U$ be the group topology on the quotient $G/H_U$ having as a local base at 0 the countable family $\{f_U(U_n)\}$, where $f_U : F \to G/H_U$ is the canonical homomorphism. According to Birkhoff-Kakutani's theorem, $(G/H, \tau_U)$ is metrizable. Now take the product of all groups $(G/H, \tau_U)$. To conclude observe that the diagonal map of the family $f_U$ into the product of all groups $(G/H, \tau_U)$ is continuous and injective. □

This theorem fails for non-abelian groups. Indeed, for an uncountable set $X$ the permutation group $S(X)$, equipped with the topology described in §3.3.5, admits no non-trivial continuous homomorphism to a metrizable abelian group.

**Exercise 4.1.18.** *Let $G$ be an abelian group and let $H$ be a countable set of characters of $G$. Prove that $\mathcal{T}_H$ is metrizable.*

## 4.2 Connectedness in topological groups

For a topological group $G$ we denote by $c(G)$ the connected component of $e_G$ and we call it briefly *connected component of* $G$.

Before proving some basic facts about the connected component, we need an elementary property of the connected sets in a topological groups.

**Lemma 4.2.1.** *Let $G$ be a topological group.*

*(a) If $C_1, C_2, \dots, C_n$ are connected sets in $G$, then also $C_1C_2 \dots C_n$ is connected.*

*(b) If $C$ is a connected set in $G$, then the set $C^{-1}$ as well as the subgroup generated by $C$ are connected.*

*Proof.* (a) Let us conisder the case $n = 2$, the general case easily follows from this one by induction. The subset $C_1 \times C_2$ of $G \times G$ is connected. Now the map $\mu : G \times G \to G$ defined $\mu(x, y) = xy$ is continuous and $\mu(C_1 \times C_2) = C_1C_2$. So by Lemma 2.2.31 also $C_1C_2$ is connected.

(a) For the first part it suffices to note that $C^{-1}$ is a continuous image of $C$ under the continuous map $x \mapsto x^{-1}$. Since $C$ is connected, by Lemma 2.2.31 we conclude that $C^{-1}$ is connected as well.

To prove the second assertion consider the set $C_1 = CC^{-1}$. It is connected by the previous lemma and obviously $e_G \in C_1$. Moreover, $C_1^2 \supseteq C \cup C^{-1}$. It remains to note now that the subgroup generated by $C_1$ coincides with the subgroup generated by $C$. Since the former is the union of all sets $C_1^n$, $n \in \mathbb{N}$ and each set $C_1^n$ is connected by item (a), we are done. □

**Proposition 4.2.2.** *The connected component $c(G)$ a topological group $G$ is a closed normal subgroup of $G$. The connected component of an element $x \in G$ is simply the coset $xc(G) = c(G)x$.*

*Proof.* To prove that $c(G)$ is stable under multiplication it suffices to note that $c(G)c(G)$ is still connected (applying item (a) of the above lemma) and contains $e_G$, so must be contained in the connected component $c(G)$. Similarly, an application of item (b) implies that $c(G)$ is stable also w.r.t. the operation $x \mapsto x^{-1}$, so $c(G)$ is a subgroup of $G$. Moreover, for every $a \in G$ the image $ac(G)a^{-1}$ under the conjugation is connected and contains 1, so must be contained in the connected component $c(G)$. So $c(G)$ is stable also under conjugation. Therefore $c(G)$ is a normal subgroup. The fact that $c(G)$ is closed is well known.

To prove the last assertion it suffices to recall that the maps $y \mapsto xy$ and $y \mapsto yx$ are homeomorphisms. □

Our next aim is to see that the quotient $G/c(G)$ is totally disconnected. We need first to see that connectedness and total disconnectedness are properties stable under extension:

**Proposition 4.2.3.** *Let $G$ be a topological group and let $N$ be a closed normal subgroup of $G$.*

*(a) If both $N$ and $G/N$ are connected, then also $G$ is connected.*

*(b) If both $N$ and $G/N$ are totally disconnected, then also $G$ is totally disconnected.*

*Proof.* Let $q : G \to G/N$ be the canonical homomorphism.

(a) Let $A \neq \emptyset$ be a clopen set of $G$. As every coset $aN$ is connected, one has either $aN \subseteq A$ or $aN \cap A = \emptyset$. Hence, $A = q^{-1}(q(A))$. This implies that $q(A)$ is a non-empty clopen set of the connected group $G/N$. Thus $q(A) = G/N$. Consequently $A = G$.

(b) Assume $C$ is a connected set in $G$. Then $q(C)$ is a connected set of $G/N$, so by our hypothesis, $q(C)$ is a singleton. This means that $C$ is contained in some coset $xN$. Since $xN$ is totally disconnected as well, we conclude that $C$ is a singleton. This proves that $G$ is totally disconnected. □

**Lemma 4.2.4.** *If $G$ is a topological group, then the group $G/c(G)$ is totally disconnected.*

*Proof.* Let $q : G \to G/c(G)$ be the canonical homomorphism and let $H$ be the inverse image of $c(G/c(G))$ under $q$. Now apply Proposition 4.2.3 to the group $H$ and the quotient group $H/c(G) \cong c(G/c(G))$ to conclude that $H$ is connected. Since it contains $c(G)$, we have $H = c(G)$. Hence $G/c(G)$ is totally disconnected. □

For a topological group $G$ denote by $a(G)$ the set of points $x \in G$ connected to $e_G$ by an arc, i.e., a continuous map $f : [0,1] \to G$ such that $f(0) = e_G$ and $f(1) = x$. We call arc the image $f([0,1])$ in $G$ and arc component the subset $a(G)$. Obviously, all points of the image belong to $a(G)$.

**Exercise 4.2.5.** *(a) If $G, H$ are topological groups, then $a(G \times H) = a(G) \times a(H)$.*

*(b) If $f : G \to H$ is a continuous map of topological groups with $f(e_G) = e_H$, then $f(a(G)) \subseteq a(H)$.*

*(c) Let $G$ be an abelian topological group and let $l(G)$ be the set of elements $x \in G$ such that there exists a continuous homomorphism $f : \mathbb{R} \to G$ with $f(1) = x$. Check that $l(G)$ is a subgroup of $G$ contained in $a(G)$. If $G$ is also locally compact, $a(G) = l(G)$.*

*(d) Can (a) be extended to arbitrary products?*

The following theorem can be proved in analogy with Proposition 4.2

**Proposition 4.2.6.** *For a topological group $G$ the arc component $a(G)$ of $G$ is a normal subgroup of $G$.*

*Proof.* Use the previous exercise and the continuity of the multiplication map $G\times G \to G$ to show that $a(G)a(G) \subseteq a(G)$. Analogously, using the continuity of the inverse $x \mapsto x^{-1}$, prove that $a(G)^{-1} \subseteq a(G)$. This proves that $a(G)$ is a subgroup of $G$. To show that $a(G)$ is stable under conjugation, use again item (b) of the above exercise and the continuity of conjugation. □

In general, $a(G)$ need not be closed in $G$. Actually, for every compact connected group $G$ the subgroup $a(G)$ is dense in $G$.

For a topological group $G$ denote by $Q(G)$ the quasi-component of the neutral element $e_G$ of $G$ (i.e., the intersection of all clopen sets of $G$ containing $e_G$) and call it *quasi-component* of $G$.

**Proposition 4.2.7.** *For a topological group $G$ the quasi-component $Q(G)$ is a closed normal subgroup of $G$. The quasi-component of $x \in G$ coincides with the coset $xQ(G) = Q(G)x$.*

*Proof.* Let $x, y \in Q(G)$. To prove that $xy \in Q(G)$ we need to verify that $xy \in O$ for every clopen set $O$ containing $e_G$. Let $O$ be such a set, then $x, y \in O$. Obviously $Oy^{-1}$ is a clopen set containing 1, hence $x \in Oy^{-1}$. This implies $xy \in O$. Hence $Q(G)$ is stable under multiplication. For every clopen set $O$ containing 1 the set $O^{-1}$ has the same propriety, hence $Q(G)$ is stable also w.r.t. the operation $a \mapsto a^{-1}$. This implies that $Q(G)$ is a subgroup. Moreover, for every $a \in G$ and for every clopen set $O$ containing 1 also its image $aOa^{-1}$ under the conjugation is a clopen set containing $e_G$. So $Q(G)$ is stable also under conjugation. Therefore $Q(G)$ is a normal subgroup. Finally, as an intersection of closen sets, $Q(G)$ is closed. □

**Remark 4.2.8.** It follows from Lemma 2.2.28 that $c(G) = Q(G)$ for every compact topological group $G$. Actually, this remains true also in the case of locally compact groups $G$ (cf. 7.4.5) as well as countably compact groups [32].

In the next remark we discuss zero-dimensionality.

**Remark 4.2.9.** (a) It follows immediately from Proposition 3.4.1 that every topological group with linear topology is zero-dimensional; in particular, totally disconnected.

(b) Every countable Hausdorff topological group $G$ is zero-dimensional (this is true for regular topological spaces as well, but not for Hausdorff ones). Indeed, using the Tychonov separation axiom, for every $U \in \mathcal{V}(e_G)$ we can separate $e_G$ from the complement of $U$ by a continuous functions $f : G \to [0,1]$ such that $f(e_G) = 0$ and $f(G \setminus U) = 0$. The subset $X = f(G)$ of $[0,1]$ is countable, hence there exists $a \in [0,1] \setminus f(G)$. Then $W = (a,1] \cap f(G)$ is a clopen subset of $f(G)$. Therefore $f^{-1}(W) \subseteq U$ is a clopen set containing $e_G$. Hence $G$ has a base of clopen sets.

We shall see in the sequel that for locally compact groups or compact groups the implication from item (a) can be inverted (see Theorem 7.4.1). On the other hand, the next example shows that local compactness is essential.

**Example 4.2.10.** The group $\mathbb{Q}/\mathbb{Z}$ is zero-dimensional but has no proper open subgroups.

**Exercise 4.2.11.** *Let $G$ be a connected group and let $H$ be a Hausdorff topological group. Prove that*

*(a) if $h : G \to H$ a continuous homomorphism, then $h$ is trivial whenever* $\ker h$ *has non-empty interior;*

*(b) if $G, H$ are abelian and $f_1, f_2 : G \to H$ are continuous homomorphisms that coincide on some neighborhood of 0 in $G$, then $f_1 = f_2$.*

Hint. (a) Use that fact that $\ker f$ is an open subgroup of $G$, so must coincide with $G$. (b) Apply (a) to the homomorphism $h = f_1 - f_2 : G \to H$.

**Exercise 4.2.12.** *If $n \in \mathbb{N}$ and $f_1, f_2 : \mathbb{R}^n \to H$ are continuous homomorphisms to some Hausdorff topological group $H$ that coincide on some neighborhood of 0 in $\mathbb{R}^n$, then $f_1 = f_2$.*

## 4.3 Group topologies determined by sequences

Let $G$ be an abelian group and let $(a_n)$ be a sequence in $G$. The question of the existence of a Hausdorff group topology that makes the sequence $(a_n)$ converge to 0 is not only a mere curiosity. Indeed, assume that some Hausdorff group topology $\tau$ makes the sequence $(p_n)$ of all primes converge to zero. Then $p_n \to 0$ would yield $p_n - p_{n+1} \to 0$ in $\tau$, so this sequence cannot contain infinitely many entries equal to 2. This would provide a very easy negative solution to the celebrated problem of the infinitude of twin primes (actually this argument would show that the shortest distance between two consecutive primes converges to $\infty$).

**Definition 4.3.1.** [132] A one-to-one sequence $A = \{a_n\}_n$ in an abelian group $G$ is called a *T-sequence* is there exists a Hausdorff group topology on $G$ such that $a_n \to 0$.

We shall see below that the sequence $(p_n)$ of all primes is not a $T$-sequence in the group $\mathbb{Z}$ (see Exercise 9.2.17). So the above mentioned possibility to resolve the problem of the infinitude of twin primes does not work.

Let $(a_n)$ be a T-sequence in an abelian group $G$. Hence the family $\{\tau_i : i \in I\}$ of Hausdorff group topologies on the group $G$ such that $a_n \to 0$ in $\tau_i$ is non-empty. Let $\tau = \sup_{i\in I} \tau_i$, then by Exercise 3.2.2 $a_n \to 0$ in $\tau$ as well. Clearly, this is the finest group topology in which $a_n$ converges to 0. This is why we denote it by $\tau_A$ or $\tau_{(a_n)}$. Since we consider only sequences without repetition, the convergence to zero $a_n \to 0$ depends only on the set $A = \{a_n\}_n$, it does not depend on the enumeration of the sequence.

Before discussing the topology $\tau_{(a_n)}$ and how $T$-sequences can be described in general we consider a couple of examples:

**Example 4.3.2.** (a) Let us see that the sequences $(n^2)$ and $(n^3)$ are not a $T$-sequence in $\mathbb{Z}$. Indeed, suppose for a contradiction that some Hausdorff group topology $\tau$ on $\mathbb{Z}$ makes $n^2$ converge to 0. Then $(n+1)^2$ converges to 0 as well. Taking the difference we conclude that $2n+1$ converges to 0 as well. Since obviously also $2n+3$ converges to 0, we conclude, after substraction, that the constant sequence 2 converges to 0. This is a contradiction, since $\tau$ is Hausdorff. We leave the case $(n^3)$ as an exercise to the reader.

(b) A similar argument proves that the sequence $P_d(n)$, where $P_d(x) \in \mathbb{Z}[x]$ is a fixed polynomial with deg$P_d = d > 0$, is not a $T$-sequence in $\mathbb{Z}$.

Protasov and Zelenyuk [131] established a number of nice properties of the finest group topology $\tau_{(a_n)}$ on $G$ that makes $(a_n)$ converge to 0.

For an abelian group $G$ and subsets $A_1, \ldots, A_n \ldots$ of $G$ we denote by $A_1 + \ldots + A_n$ the set of all sums $g = g_1 + \ldots + g_n$, where $g_i \in A_i$ for every $i = 1, \ldots, n$. Let

$$A_1 + \ldots + A_n + \ldots = \bigcup_{n=1}^{\infty} A_1 + \ldots + A_n.$$

If $A = \{a_n\}_n$ is a sequence in $G$, for $m \in \mathbb{N}$ denote by $A_m^*$ the "tail" $\{a_m, a_{m+1}, \ldots\}$ and let $A_m = \{0\} \cup A_m^* \cup -A_m^*$ For $k \in \mathbb{N}$ let $A(k, m) = A_m + \ldots + A_m$ ($k$ times).

**Remark 4.3.3.** The existence of a finest group topology $\tau_A$ on an abelian group $G$ that makes an arbitrary given sequence $A = \{a_n\}_n$ in $G$ converge to 0 is easy to prove as far as we are not interested on imposing the Hausdorff axiom. Indeed, as $a_n$ converges to 0 in the indiscrete topology, $\tau_A$ is simply the supremum of all group topologies $\tau$ on $G$ such that $a_n$ converges to 0 in $\tau$. This gives no idea on how this topology looks like. One can easily describe it as follows.

Let $m_1, \ldots, m_n, \ldots$ be a sequence of natural numbers. Denote by $A(m_1, \ldots, m_n)$ the set $A_{m_1} + \ldots + A_{m_n}$ and let

$$A(m_1, \ldots, m_n, \ldots) = A_{m_1} + \ldots + A_{m_n} + \ldots = \bigcup_n A(m_1, \ldots, m_n)$$

Then the family $\mathcal{B}_A$ of all sets $A(m_1,\ldots,m_n,\ldots)$, when $m_1,\ldots,m_n,\ldots$ vary in $\mathbb{N}^{\mathbb{N}}$, is a filter base, satisfying the axioms of group topology. The group topology $\tau$ defined in this way satisfies the required conditions. Indeed, obviously $a_n \to 0$ in $(G,\tau)$ and $\tau$ contains any other group topology with this property. Consequently, $\tau = \tau_A$.

Note that

$$A(k,m) \subseteq A(m_1,\ldots,m_n,\ldots), \tag{1}$$

for every $k \in \mathbb{N}$, where $m = \max\{m_1,\ldots,m_k\}$. The sets $A(k,m)$, for $k,m \in \mathbb{N}$, form a filter base, but the filter they generate need not be the filter of neighborhoods of 0 in a group topology. The utility of this family becomes clear now.

**Theorem 4.3.4.** *A sequence $A = \{a_n\}_n$ in an abelian group $G$ is a T-sequence iff*

$$\bigcap_{m=1}^{\infty} A(k,m) = 0 \text{ for every } k \in \mathbb{N}. \tag{2}$$

*Proof.* Obviously the sequence $A = \{a_n\}_n$ is a T-sequence iff the topology $\tau_A$ is Hausdorff. Clearly, $\tau_A$ is Hausdorff iff

$$\bigcap_{m_1,\ldots,m_n,\ldots}^{\infty} A(m_1,\ldots,m_n,\ldots) = 0. \tag{$*$}$$

If $\tau_A$ is Hausdorff, then (2) holds by (1). It remains to see that (2) implies (*). We prove first that

$$\bigcap_{i=1}^{\infty}(A(m_1,\ldots,m_k)+A_i) = A(m_1,\ldots,m_k) \tag{$**$}$$

for every $k \geq 0$ and every sequence $m_1,\ldots,m_k$, where we agree to let $A(m_1,\ldots,m_k) = 0$ when $k = 0$ (i.e., the sum is empty). We argue by induction on $k$. The case $k = 0$ follows directly from (2) with $k = 1$.

Now assume that $k > 0$ and (**) is true for $k-1$ and all sequences $m_1,\ldots,m_{k-1}$. Take $g \in \bigcap_{i=1}^{\infty} A(m_1,\ldots,m_k)+A_i$. Then for every $j = 1,2,\ldots,k$ and every $i \in \mathbb{N}$ one can find $b_j(i) \in A_{m_j}$ and $a(i) \in A_i$ such that $g = b_1(i)+\ldots+b_k(i)+a(i)$. If there exists some $j = 1,2,\ldots,k$, such that $b_j(i) = 0$ for infinitely many $i$, then $g \in \sum_{\nu\neq j} A_{m_\nu} + A_i$ for infinitely many $i$, so

$$g \in \bigcap_{i=1}^{\infty}\left(\sum_{\nu\neq j} A_{m_\nu} + A_i\right) = \sum_{\nu\neq j} A_{m_\nu} \subseteq A(m_1,\ldots,m_k)$$

by our inductive hypothesis. Hence we may assume that $b_j(i) \neq 0$ for all $i > i_0$ and for all $j = 1,\ldots,k$. Then for all $i > i_0$ and for all $j = 1,\ldots,k$ there exists $m_j(i) \geq m_j$ so that $b_j(i) = \pm a_{m_j(i)} \in A_{m_j}$. If $\lim_i m_j(i) = \infty$ for all $j = 1,\ldots,k$, then $g \in A(k+1,i)$ for infinitely many $i$, so $g \in \bigcap_{i=1}^{\infty} A(k+1,i) = 0$ by (2). Hence $g = 0 \in A(m_1,\ldots,m_n)$. If there exists some $j = 1,\ldots,k$ such that $m_j(i) = l$ for some $l \geq m_j$ and infinitely many $i$, then $g^* = g \mp a_l \in \sum_{\nu\neq j} A_{m_\nu} + A_i$ for infinitely many $i$, so

$$g^* \in \bigcap_{i=1}^{\infty}\left(\sum_{\nu\neq j} A_{m_\nu} + A_i\right) = \sum_{\nu\neq j} A_{m_\nu}$$

by our inductive hypothesis. Therefore

$$g = a_l + g^* \in a_l + \sum_{\nu\neq j} A_{m_\nu} \subseteq A(m_1,\ldots,m_n).$$

This proves (**)

To prove (*) assume that $g \in G$ is non-zero. Then using our assumption (2) and (**) it is easy to build inductively a sequences $(m_n)$ such that $g \notin A(m_1,\ldots,m_n)$ for every $n$, i.e., $g \notin A(m_1,\ldots,m_n,\ldots)$. □

Since every infinite abelian group $G$ admits a non-discrete metrizable group topology, there exist non-trivial (i.e., having all members non-zero) $T$-sequences.

A notion similar to $T$-sequence, but defined with respect to only topologies induced by characters, will be given in §11.5. From many points of view it turns out to be easier to deal with than $T$-sequence. In particular, we shall see easy sufficient condition for a sequence of integers to be a $T$-sequence.

We give without proof the following technical lemma that will be useful in §11.5.

**Lemma 4.3.5.** [132] *For every $T$-sequence $A = \{a_n\}$ in $\mathbb{Z}$ there exists a sequence $\{b_n\}$ in $\mathbb{Z}$ such that for every choice of the sequence $(e_n)$, where $e_n \in \{0,1\}$, the sequence $q_n$ defined by $q_{2n} = b_n + e_n$ and $q_{2n-1} = a_n$, is a $T$-sequence.*

**Exercise 4.3.6.** *(a)* Prove that there exists a $T$-sequence $(a_n)$ in $\mathbb{Z}$ with $\lim_n \frac{a_{n+1}}{a_n} = 1$ [132] (see also Example 11.5.4).*

*(b)* Every sequence $(a_n)$ in $\mathbb{Z}$ with $\lim_n \frac{a_{n+1}}{a_n} = +\infty$ is a $T$-sequence [132, 10] (see Theorem 11.5.3).*

*(c)* Every sequence $(a_n)$ in $\mathbb{Z}$ such that $\lim_n \frac{a_{n+1}}{a_n} \in \mathbb{R}$ is transcendental is a $T$-sequence [132].*

# 5 Markov's problems

## 5.1 The Zariski topology and the Markov topology

Let $G$ be a Hausdorff topological group, $a \in G$ and $n \in \mathbb{N}$. Then the set $\{x \in G : x^n = a\}$ is obviously closed in $G$. This simple fact motivated the following notions due to Markov [115].

A subset $S$ of a group $G$ is called:

(a) *elementary algebraic* if there exist an integer $n > 0$, $a_1, \ldots, a_n \in G$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ such that

$$S = \{x \in G : x^{\varepsilon_1} a_1 x^{\varepsilon_2} a_2 \ldots a_{n-1} x^{\varepsilon_n} = a_n\},$$

(b) *algebraic* if $S$ is an intersection of finite unions of elementary algebraic subsets,

(c) *unconditionally closed* if $S$ is closed in *every* Hausdorff group topology of $G$.

Since the family of all finite unions of elementary algebraic subsets is closed under finite unions and contains all finite sets, it is a base of closed sets of some $T_1$ topology $\mathfrak{Z}_G$ on $G$, called the *Zariski topology*[3]. Clearly, the $\mathfrak{Z}_G$-closed sets are precisely the algebraic sets in $G$.

Analogously, the family of all unconditionally closed subsets of $G$ coincides with the family of closed subsets of a $T_1$ topology $\mathfrak{M}_G$ on $G$, namely the infimum (taken in the lattice of all topologies on $G$) of all Hausdorff group topologies on $G$. We call $\mathfrak{M}_G$ the *Markov topology* of $G$. Note that $(G, \mathfrak{Z}_G)$ and $(G, \mathfrak{M}_G)$ are quasi-topological groups, i.e., the inversion and translations are continuous. Nevertheless, when $G$ is abelian $(G, \mathfrak{Z}_G)$ and $(G, \mathfrak{M}_G)$ are not group topologies unless they are discrete.

Since an elementary algebraic set of $G$ must be closed in every Hausdorff group topology on $G$, one always has $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$. In 1944 Markov [115] asked if the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ holds for every group $G$. He himself showed that the answer is positive in case $G$ is countable [115]. Moreover, in the same manuscript Markov attributes to Perel'man the fact that $\mathfrak{Z}_G = \mathfrak{M}_G$ for every Abelian group $G$ (a proof has never appeared in print until [61]). An example of a group $G$ with $\mathfrak{Z}_G \neq \mathfrak{M}_G$ was given by Gerchard Hesse [101].

**Exercise 5.1.1.** *Show that if $(G, \cdot)$ is an abelian group, then every elementary algebraic set of $G$ has the form $\{x \in G : x^n = a\}$, $a \in G$.*

## 5.2 The Markov topology of the symmetric group

Let $X$ be an infinite set. In the sequel we denote by $\tau_X$ the pointwise convergence topology of the infinite symmetric group $S(X)$ defined in §3.3.5. It turns out that the Markov topology of $S(X)$ coincides with $\tau_X$:

**Theorem 5.2.1.** *Then Markov topology on $S(X)$ coincides with the topology $\tau_X$ of pointwise convergence of $S(X)$.*

This theorem follows immediately from the following old result due to Gaughan.

**Theorem 5.2.2.** *([57]) Every Hausdorff group topology of the infinite permutation group $S(X)$ contains the topology $\tau_X$.*

The proof of this theorem follows more or less the line of the proof exposed in [57, §7.1] with several simplifications. The final stage of the proof is preceded by a number of claims (and their corollaries) and two facts about *purely algebraic* properties of the group $S(X)$ (5.2.3 and 5.2.6). The claims and their corollaries are given with complete proofs. To give an idea about the proofs of the two algebraic facts, we prove the first and a part of the second one.

We say for a subset $A$ of $S(X)$ that $A$ is $m$-transitive for some positive integer $m$ if for every $Y \subseteq X$ of size at most $m$ and every injection $f : Y \to X$ there exists $a \in A$ that extends $f$. [4] The leading idea is that a transitive subset $A$ of $S(X)$ is placed "generically" in $S(X)$, whereas a non-tranisitve one is a subset of some subgroup of $S(X)$ that is a

[3] Some authors call it also the *verbal topology* [23], we prefer here *Zariski topology* coined by most authors [13].

[4] Note that a countable subset $H$ of $S(X)$ cannot be transitive unless $X$ itself is countable.

direct product $S(Y) \times S(X \setminus Y)$. (Here and in the sequel, for a subset $Y$ of $X$ we tacitly identify the group $S(Y)$ with the subgroup of $S(X)$ consisting of all permutations of $S(X)$ that are identical on $X \setminus Y$.)

The first fact concerns the stabilizers $S_x = S_{\{x\}} = \{f \in S(X) : f(x) = x\}$ of points $x \in X$. They consitute a prebase of the filter of neighborhoods of $id_X$ in $\tau_X$.

**Fact 5.2.3.** *For every $x \in X$ the subgroup $S_x$ of $S(X)$ is maximal.*

*Proof.* Assume $H$ is a subgroup of $S(X)$ properly containing $S_x$. To show that $H = S(X)$ take any $f \in S(X)$. If $y = f(x)$ coincides with $x$, then $f \in S_x \subseteq H$ and we are done. Assume $y \neq x$. Get $h \in H \setminus S_x$. Then $z = h(x) \neq x$, so $x \notin \{z, y\}$. There exists $g \in S(X)$ such that $g(x) = x, g(y) = z$ and $g(z) = y$. Then $g \in S_x \subseteq H$ and $f(x) = g(h(x)) = y$, so $h^{-1}g^{-1}f(x) = x$ and $h^{-1}g^{-1}f \in S_x \cap G \subseteq H$. So $f \in ghH = H$. □

**Claim 5.2.4.** *Let $T$ be a Hausdorff group topology on $S(X)$. If a subgroups of $S(X)$ of the form $S_x$ is $T$-closed, then it is also $T$-open.*

*Proof.* As $S_x$ is $T$-closed, for every fixed $y \neq x$ the set $V_y = \{f \in S(X) : f(x) \neq y\}$ is $T$-open and contains 1. So there exists a symmetric neighborhood $W$ of 1 in $T$ such that $W.W \subseteq V_y$. By the definition of $V_y$ this gives $Wx \cap Wy = \emptyset$. Then either $|X \setminus Wx| = |X|$ or $|X \setminus Wy| = |X|$. Suppose this occurs with $x$, i.e., $|X \setminus Wx| = |X|$. Then one can find a permutation $f \in S(X)$ that sends $Wx \setminus \{x\}$ to the complement of $Wx$ and $f(x) = x$. Such an $f$ satisfies:

$$fWf^{-1} \cap W \subseteq S_x$$

as $fWf^{-1}(x)$ meets $Wx$ precisely in the singleton $\{x\}$ by the choice of $f$. This proves that $S_x$ is $T$-open.

Analogous argument works for $S_y$ when $|X \setminus Wy| = |X|$. □

**Corollary 5.2.5.** *If $T$ be a Hausdorff group topology on $S(X)$ that does not contain $\tau_X$, then all subgroups of $S(X)$ of the form $S_x$ are $T$-dense.*

*Proof.* Since the subgroups $S_x$ of $S(X)$ form a prebase of the filter of neighborhoods of $id_X$ in $S(X)$, out hypothesis implies that some subgroup $S_x$ is not $T$-open. By Claim 5.2.4 $S_x$ is not $T$-closed either. By Fact 5.2.3 $S_x$ is $T$-dense. Since all subgroups of the form $S_y$ are conjugated, this implies that stabilizers $S_y$ are $T$-dense. □

This was the first step in the proof. The next step will be establishing that $S_{x,y}$ are never dense in any Hausdorff group topology on $S(X)$ (Corollary 5.2.9).

In the sequel we need the subgroup $\widetilde{S}_{x,y} := S_{x,y} \times S(\{x, y\})$ of $S(X)$ that contains $S_{x,y}$ as a subgroup of index 2. Note that $\widetilde{S}_{x,y}$ is precisely the subgroup of all permutations in $S(X)$ that leave the doubleton $\{x, y\}$ set-wise invariant.

**Fact 5.2.6.** *For any doubleton $x, y$ in $X$ the following holds true:*

*(a) the subgroup $\widetilde{S}_{x,y}$ of $S(X)$ is maximal;*

*(b) every proper subgroup of $S(X)$ properly containing $S_{x,y}$ coincides with one of the subgroups $S_x, S_y$ or $\widetilde{S}_{x,y}$.*

*Proof.* (a) This is [57, Lemmas 7.1.4].

(b) Assume $H$ is a subgroup of $S(X)$ properly containing $S_{x,y}$. Assume that $H$ does not coincide with $S_x, S_y$. We aim to show that $H = \widetilde{S}_{x,y}$, i.e., $(xy) \in H$.

Since $S_{x,y}$ is a maximal subgroup of $S_x$ by Fact 5.2.3 that $H \cap S_x = S_{x,y}$. Analogously, $H \cap S_y = S_{x,y}$. Take $f \in H \subseteq S_{x,y}$. Then $f \notin S_x$ and $f \notin S_y$. Let $z = f(x)$ and $t = f(y)$. Then $z \neq x$ and $t \neq y$. Consider the following three cases:

1. $\{z, t\} = \{x, y\}$. This is possible precisely when $z = y$ and $t = x$. Then $(xy)f \in S_{x,y} \subseteq H$. Thus $(xy) \in H$.

2. $\{z, t\} \cap \{x, y\} = \emptyset$. Then $(zt) \in S_{x,y} \subseteq H$, so $(xy) = f^{-1}(zt)f \in H$.

3. $\{z, t\} \cap \{x, y\} = \{z\} = \{y\}$ (so $x \neq t$). Let $u = f^{-1}(x)$ and consider first the case when $u \neq t$. Then $(ut) \in S_{xy}$, so $g = f(ut) \in H$. Then $h = (xyt)^{-1}g = (tyx)g \in S_{xy}$, so $(xyt) \in H$. If $u = t$, then $h = (xyt)^{-1}f = (tyx)f \in S_{xy}$, so we again have $(xyt) \in H$. Choose an arbitrary $v \in X \setminus \{x, y, t\}$. Then $(tv) \in S_{xy}$. Since

$$f_1 = (xt)(yv) = (xyt)(tv)(xyt)(tv) \in H,$$

we have an element $f_1 \in H$ with $f_1(x) = t \notin \{x, y\}$ and $f_1(y) \notin \{x, y\}$. Applying the argument from case 2 we conclude that $(xy) \in H$. □

**Claim 5.2.7.** *Let $T$ be a Hausdorff group topology on $S(X)$, then there exists a $T$-nbd of 1 that is not 2-transitive.*

*Proof.* Assume for a contradiction that all $T$-neighborhoods of $id_X$ that are 2-transitive. Fix distinct $u, v, w \in X$. We show now that the 3-cycle $(u, v, w) \in V$ for every arbitrarily fixed $T$-neighborhood of $id_X$. Indeed, choose a symmetric $T$-neighborhood $W$ of $id_X$ such that $W^2 \subseteq V$. Let $f$ be the transposition $(uv)$. Then $U = fWf \cap W \in T$ is a neighborhood of 1 and $fUf = U$. Since $U$ is 2-transitive there exists $g \in U$ such that $g(u) = u$ and $g(v) = w$. Then $(u, v, w) = gfg^{-1}f \in W \cdot (fUf) \subseteq W^2 \subseteq V$. □

**Claim 5.2.8.** *Let $T$ be a group topology on $S(X)$. Then*

*(a) every $T$-nbd $V$ of $id_X$ in $S(X)$ is transitive iff every stabilizer $S_x$ is $T$-dense;*

*(b) every $T$-nbd $V$ of $id_X$ in $S(X)$ is $m$-transitive iff every stabilizer $S_F$ with $|F| \leq m$ is $T$-dense.*

*Proof.* Assume that some (hence all) $S_z$ is $T$-dense in $S(X)$. To prove that $V$ is transitive consider a pair $x, y \in X$. Let $t = (xy)$. By the $T$-density of $S_x$ the $T$-nbd $t^{-1}V$ of $t^{-1}$ meets $S_x$, i.e., for some $v \in V$ one has $t^{-1}v \in S_x$. Then $v \in tS_x$ obviously satisfies $vx = y$.

A similar argument proves that transitivity of each T-nbd of 1 entails that every stabilizer $S_x$ is $T$-dense.

(b) The proof in the case $m > 1$ is similar. □

What we really need further on (in particular, in the next corollary) is that the density of the stabilizers $S_{x,y}$ imply that every $T$-nbd $V$ of $id_X$ in $S(X)$ is 2-transitive.

**Corollary 5.2.9.** *Let $T$ be a Hausdorff group topology on $S(X)$. Then $S_{x,y}$ is $T$-dense for no pair $x, y$ in $X$.*

*Proof.* Follows from claims 5.2.7 and 5.2.8 □

**Proof of Theorem 5.2.2.** Assume for a contradiction that $T$ is a Hausdorff group topology on $S(X)$ that does not contain $\tau_X$. Then by corollaries 5.2.5 and 5.2.9 all subgroups of the form $S_x$ are $T$-dense and no subgroup of the form $S_{x,y}$ is $T$-dense. Now fix a pair $x, y \in X$ and let $G_{x,y}$ denote the $T$-closure of $S_{x,y}$. Then $G_{x,y}$ is a proper subgroup of $S(X)$ containing $S_{x,y}$. Since $S_x$ is dense, $G_{x,y}$ cannot contain $S_x$, so $S_x \cap G_{x,y}$ is a proper subgroup of $S_x$ containing $S_{x,y}$. By Claim 5.2.3 applied to $S_x = S(X \setminus \{x\})$ and its subgroup $S_{x,y}$ (the stabilizer of $y$ in $S_x$), we conclude that $S_{x,y}$ is a maximal subgroup of $S_x$. Therefore, $S_x \cap G_{x,y} = S_{x,y}$. This shows that $S_{x,y}$ is a $T$-closed subgroup of $S_x$. By Claim 5.2.4 applied to $S_x = S(X \setminus \{x\})$ and its subgroup $S_{x,y}$, we conclude that $S_{x,y}$ is a $T$-open subgroup of $S_x$. Since $S_x$ is dense in $S(X)$, we can claim that $G_{x,y}$ is a $T$-open subgroup of $S(X)$. Since $S_x$ is a proper dense subgroup of $S(X)$, it is clear that $S_x$ cannot contain $G_{x,y}$. Analogously, $S_y$ cannot contain $G_{x,y}$ either. So $G_{x,y} \neq S_{x,y}$ is a proper subgroup of $S(X)$ containing $S_{x,y}$ that does not coincide with $S_x$ or $S_y$. Therefore $G_{x,y} = \widetilde{S}_{x,y}$ by Fact 5.2.6. This proves that $\widetilde{S}_{x,y}$ is $T$-open. Since all subgroups of the form $\widetilde{S}_{x,y}$ are pairwise conjugated, we can claim that all subgroups $\widetilde{S}_{x,y}$ is $T$-open.

Now we can see that the stabilizers $S_F$ with $|F| > 2$ are T-open, as

$$S_F = \bigcap \{\widetilde{S}_{x,y} : x, y \in F, x \neq y\}.$$

This proves that all basic neighborhoods $S_F$ of 1 in $\tau_X$ are $T$-open. In particular, also the subgroups $S_x$ are $T$-open, contrary to our hypothesis.

## 5.3 Existence of Hausdorff group topologies

According to Proposition 4.1.12 every infinite abelian group admits a non-discrete Hausdorff group topology, for example the Bohr topology. This gives immediately the following

**Corollary 5.3.1.** *Every group with infinite center admits a non-discrete Hausdorff group topology.*

*Proof.* The center $Z(G)$ of the group $G$ has a non-discrete Hausdorff group topology $\tau$ by the above remark. Now consider the family $\mathcal{B}$ of all sets of the form $aU$, where $a \in G$ and $U$ is a non-empty $\tau$-subset of $Z(G)$. It is easy to see that it is a base of a non-discrete Hausdorff group topology on $G$. □

In 1946 Markov set the problem of the existence of a (countably) infinite group $G$ that admits no Hausdorff group topology beyond the discrete one. Let us call such a group a *Markov group*. Obviously, $G$ is a Markov group precisely when $\mathfrak{M}_G$ is discrete. A Markov group must have finite center by Corollary 5.3.1.

According to Proposition 3.5.1, the closure of the neutral element of every topological group is always a normal subgroup of $G$. Therefore, a simple topological group is either Hausdorff, or indiscrete. So a simple Markov group $G$ admits only two group topologies, the discrete and the indiscrete ones.

The equality $\mathfrak{Z}_G = \mathfrak{M}_G$ established by Markov in the countable case was intended to help in finding a countably infinite Markov group $G$. Indeed, a countable group $G$ is Markov precisely when $\mathfrak{Z}_G$ is discrete. Nevertheless, Markov failed in building a countable group $G$ with discrete Zariski topology; this was done much later, in 1980, by Ol′shanskii [118] who made use of the so called *Adian groups* $A = \mathcal{A}(m,n)$ (constructed by Adian to negatively resolve the famous 1902 Burnside problem on finitely generated groups of finite exponent). Let us sketch here Ol′shanskii's elegant short proof.

**Example 5.3.2.** [118] Let $m$ and $n$ be odd integers $\geq 665$, and let $A = \mathcal{A}(m,n)$ be Adian's group having the following properties

(a) $A$ is generated by $n$-elements;

(b) $A$ is torsion-free;

(c) the center $C$ of $A$ is infinite cyclic.

(d) the quotient $A/C$ is infinite, of exponent $m$, i.e., $y^m \in C$ for every $y \in A$.[5]

By (a) the group $A$ is countable. Denote by $C^m$ the subgroup $\{c^m : c \in C\}$ of $A$. Let us see that (b), (c) and (d) jointly imply that the Zariski topology of the infinite quotient $G = A/C^m$ is discrete (so $G$ is a countably infinite Markov group). Let $d$ be a generator of $C$. Then for every $x \in A\setminus C$ one has $x^m \in C\setminus C^m$. Indeed, if $x^m = d^{ms}$, then $(xd^{-s})^m = e_A$ for some $s \in \mathbb{Z}$, so $xd^{-s} = e_A$ and $x \in C$ by (b). Hence

$$\text{for every } u \in G\setminus\{e_G\} \text{ there exists } a \in C\setminus C^m, \text{ such that either } u = a \text{ or } u^m = a. \tag{3}$$

As $|C/C^m| = m$, every $u \in G\setminus\{e_G\}$ is a solution of some of the $2(m-1)$ equations in (3). Thus, $G\setminus\{e_G\}$ is closed in the Zariski topology $\mathfrak{Z}_G$ of $G$. Therefore, $\mathfrak{Z}_G$ is discrete.

Now we recall an example, due to Shelah [137], of an uncountable group which is non-topologizable. It appeared about a year or two earlier than the ZFC-example of Ol′shanskii exposed above.

**Example 5.3.3.** [137] Under the assumption of CH there exists a group $G$ of size $\omega_1$ satisfying the following conditions (a) (with $m = 10000$) and (b) (with $n = 2$):

(a) there exists $m \in \mathbb{N}$ such that $A^m = G$ for every subset $A$ of $G$ with $|A| = |G|$;

(b) for every subgroup $H$ of $G$ with $|H| < |G|$ there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in G$ such that the intersection $\bigcap_{i=1}^n x_i^{-1} H x_i$ is finite.

Let us see that $G$ is a Markov group (i.e., $\mathfrak{M}_G$ is discrete)[6]. Assume $\mathcal{T}$ be a Hausdorff group topology on $G$. There exists a $\mathcal{T}$-neighbourhood $V$ of $e_G$ with $V \neq G$. Choose a $\mathcal{T}$-neighbourhood $W$ of $e_G$ with $W^m \subseteq V$. Now $V \neq G$ and (a) yield $|W| < |G|$. Let $H = \langle W \rangle$. Then $|H| = |W| \cdot \omega < |G|$. By (b) the intersection $O = \bigcap_{i=1}^n x_i^{-1} H x_i$ is finite for some $n \in \mathbb{N}$ and elements $x_1, \ldots, x_n \in G$. Since each $x_i^{-1} H x_i$ is a $\mathcal{T}$-neighbourhood of $e_G$, this proves that $e_G \in O \in \mathcal{T}$. Since $\mathcal{T}$ is Hausdorff, it follows that $\{e_G\}$ is $\mathcal{T}$-open, and therefore $\mathcal{T}$ is discrete.

One can see that even the weaker form of (a) (with $m$ depending on $A \in [G]^{|G|}$), yields that every proper subgroup of $G$ has size $< |G|$. In the case $|G| = \omega_1$, the groups with this property are known as *Kurosh groups* (in particular, this is a *Jonsson semigroup* of size $\omega_1$, i.e., an uncountable semigroup whose proper subsemigroups are countable).

Finally, this remarkable construction from [137] furnished also the first consistent example to a third open problem. Namely, a closer look at the above argument shows that the group $G$ is simple. As $G$ has no maximal subgroups, it shows also that taking Frattini subgroup[7] "does not commute" with taking finite direct products (indeed, $Fratt(G) = G$, while $Fratt(G \times G) = \Delta_G$ the "diagonal" subgroup of $G \times G$).

[5] i.e., the finitely generated infinite quotient $A/C$ negatively resolves Burnside's problem.

[6] Hesse [101] showed that the use of $CH$ in Shelah's construction of a Markov group of size $\omega_1$ can be avoided.

[7] the Frattini subgroup of a group $G$ is the intersection of all maximal subgroups of $G$.

## 5.4 Extension of group topologies

The problem of the existence of (Hausdorff non-discrete) group topologies can be considered also as a problem of extension of (Hausdorff non-discrete) group topologies.

The theory of extension of topological spaces is well understood. If a subset $Y$ of a set $X$ carries a topology $\tau$, then it is easy to extend $\tau$ to a topology $\tau^*$ on $X$ such that $(Y,\tau)$ is a subspace of $(X,\tau^*)$. The easiest way to do it is to consider $X = Y \cup (X \setminus Y)$ as a partition of the new space $(X,\tau^*)$ into clopen sets and define the topology of $X \setminus Y$ arbitrarily. Usually, one prefers to define the extension topology $\tau^*$ on $X$ in such a way to have $Y$ dense in $X$. In such a case the extensions of a given space $(Y,\tau)$ can be described by means of appropriate families of *open filters* of $Y$ (i.e., filters on $Y$ having a base of $\tau$-open sets).

The counterpart of this problem for groups and group topologies is much more complicated because of the presence of group structure. Indeed, let $H$ be a subgroup of a group $G$ and assume that $\tau$ is a group topology of $H$. Now one has to build a group topology $\tau^*$ on $G$ such that $(H,\tau)$ is a topological subgroup of $(G,\tau^*)$. The first idea to extend $\tau$ is to imitate the first case of extension considered above by declaring the subgroup $H$ a $\tau^*$-open topological subgroup of the new topological group $(G,\tau^*)$. Let us note that this would immediately determine the topology $\tau^*$ in a unique way. Indeed, every coset $gH$ of $H$ must carry the topology transported from $H$ to $gH$ by the translation $x \mapsto gx$, i.e., the $\tau^*$-open subsets of $gH$ must have the form $gU$, where $U$ is an open subset of $(H,\tau)$. In other words, the family $\{gU : \emptyset \neq U \in \tau\}$ is a base of $\tau^*$. This idea has worked in the proof of Corollary 5.3.1 where $H$ was the center of $G$. Indeed, this idea works in the following more general case.

**Lemma 5.4.1.** *Let $H$ be a subgroup of a group $G$ such that $G = Hc_G(H)$. Then for every group topology $\tau$ on $H$ the above described topology $\tau^*$ is a group topology of $G$ such that $(H,\tau)$ is an open topological subgroup of $(G,\tau^*)$.*

*Proof.* The first two axioms on the neighborhood base are easy to check. For the third one pick a basic $\tau^*$-neighborhood $U$ of 1 in $G$. Since $H$ is $\tau^*$-open, we can assume wlog that $U \subseteq H$, so $U$ is a $\tau$-neighborhood of 1. Let $x \in G$. We have to produce a $\tau^*$-neighborhood $V$ of 1 in $G$ such that $x^{-1}Vx \subseteq U$. By our hypothesis there exist $h \in H, z \in c_G(H)$, such that $x = hz$. Since $\tau$ is a group topology on $H$ there exist $V \in \mathcal{V}_{H,\tau}(1)$ such that $h^{-1}Vh \subseteq U$. Then

$$x^{-1}Vx = z^{-1}h^{-1}Vhz \subseteq z^{-1}Uz = U$$

as $z \in c_G(H)$. This proves that $\tau^*$ is a group topology of $G$ . □

Clearly, the condition $G = Hc_G(H)$ is satisfied when $H$ is a central subgroup of $G$. It is satisfied also when $H$ is a direct summand of $G$. On the other hand, subgroups $H$ satisfying $G = Hc_G(H)$ are normal.

Two questions are in order here:

- is the condition $G = Hc_G(H)$ really necessary for the extension problems;
- is it possible to definite the extension $\tau^*$ in a different way in order to have *always* the possibility to extend a group topology?

Our next theorem shows that the difficulty of the extension problem are not hidden in the special features of the extension $\tau^*$.

**Theorem 5.4.2.** *Let $H$ be a normal subgroup of the group $G$ and let $\tau$ be a group topology on $H$. Then the following are equivalent:*

*(a) the extension $\tau^*$ is a group topology on $G$;*

*(b) $\tau$ can be extended to a group topology of $G$;*

*(c) for every $x \in G$ the automorphism of $H$ induced by the conjugation by $x$ is $\tau$-continuous.*

*Proof.* The implication (a) → (b) is obvious, while the implication (b) → (c) follows from the fact that the conjugations are continuous in any topological group. To prove the implication (c) → (a) assume now that all automorphisms of $N$ induced by the conjugation by elements of $G$ are $\tau$-continuous. Take the filter of all neighborhoods of 1 in $(H,\tau^*)$ as a base of neighborhoods of 1 in the group topology $\tau^*$ of $G$. This works since the only axiom to check is to find for every $x \in G$ and every $\tau^*$-nbd $U$ of 1 a $\tau^*$-neighborhood $V$ of 1 such that $V^x := x^{-1}Vx \subseteq U$. Since we can choose $U,V$ contained in $H$, this immediately follows from our assumption of $\tau$-continuity of the restrictions to $H$ of the conjugations in $G$. □

Now we give an example showing that the extension problem cannot be resolved for certain triples $G, H, \tau$ of a group $G$, its subgroup $H$ and a group topology $\tau$ on $H$.

**Example 5.4.3.** In order to produce an example when the extension is not possible we need to produce a triple $G, H, \tau$ such that at least some conjugation by an element of $G$ is not $\tau$-continuous when considered as an automorphism of $H$. The best tool to face this issue is the use of semi-direct products.

Let us recall that for groups $K$, $H$ and a group homomorphism $\theta : K \to Aut(H)$ one defines the semi-direct product $G = H \rtimes_\theta K$, where we shall identify $H$ with the subgroup $H \times \{1_K\}$ of $G$ and $K$ with the subgroup $\{1_H\} \times K$. In such a case, the conjugation in $G$ by an element $k$ of $K$ restricted to $H$ is precisely the automorphism $\theta(k)$ of $H$. Now consider a group topology $\tau$ on $H$. According to Theorem 5.4.2 $\tau$ can be extended to a group topology of $G$ iff for every $k \in K$ the automorphism $\theta(k)$ of $H$ is $\tau$-continuous. (Indeed, every element $x \in G$ has the form $x = hk$, where $h \in H$ and $k \in K$; hence it remains to note that the conjugation by $x$ is composition of the (continuous) conjugation by $h$ and the conjugation by $k$. )

In order to produce the required example of a triple $G, H, \tau$ such that $\tau$ cannot be extended to $G$ it suffices to find a group $K$ and a group homomorphism $\theta : K \to Aut(H)$ such that at least one of the automorphisms $\theta(k)$ of $H$ is $\tau$-discontinuous. Of course, one can simplify the construction by taking the cyclic group $K_1 = \langle k \rangle$ instead of the whole group $K$, where $k \in K$ is chosen such that the automorphisms $\theta(k)$ of $H$ is $\tau$-discontinuous. A further simplification can be arranged by taking $k$ in such a way that the automorphism $f = \theta(k)$ of $H$ is also an involition, i.e., $f^2 = id_H$. Then $H$ will be an index two subgroup of $G$.

Here is an example of a topological abelian group $(H, \tau)$ admitting a $\tau$-discontuous involition $f$. Then the triple $G, H, \tau$ such that $\tau$ cannot be extended to $G$ is obtained by simply taking $G = H \rtimes \langle f \rangle$, where the involition $f$ acts on $H$. Take as $(H, \tau)$ the torus group $\mathbb{T}$ with the usual topology. Then $\mathbb{T}$ is algebraically isomorphic to $(\mathbb{Q}/\mathbb{Z}) \oplus_{\mathfrak{c}} \bigoplus \mathbb{Q}$, so $\mathbb{T}$ has $2^{\mathfrak{c}}$ many involutions. Of these only the involutions $\pm id_{\mathbb{T}}$ of $\mathbb{T}$ are continuous (see Lemma 12.2.1).

Let us conclude now with a series of examples when the extension problem has always a positive solution.

**Example 5.4.4.** Let $p$ be a prime number. If the group of $p$-adic integers $N = \mathbb{Z}_p$ is a normal subgroup of some group $G$, then the $p$-adic topology of $N$ can be extended to a group topology on $G$. Indeed, it suffices to note that if $\xi : N \to N$ is an automorphism of $N$, then $\xi(p^n N) = p^n N$. Since the subgroups $p^n N$ define the topology of $N$, this proves that every automorphism of $N$ is continuous. Now Theorem 5.4.2 applies.

Clearly, the $p$-adic integers can be replaced by any topological group $N$ such that every automorphism of $N$ is continuous (e.g., products of the form $\prod_p \mathbb{Z}_p^{k_p} \times F_p$, where $k_p < \omega$ and $F_p$ is a finite abelian $p$-group).

**Exercise 5.4.5.** *Let $H$ be a discrete subgroup of a topological group $G$. Then $\overline{H}$ is isomorphic to the semi-direct product of $H$ and $\overline{\{e_G\}}$, carrying the product topology, where $H$ is discrete and $\overline{\{e_G\}}$ is indiscrete.*

# 6 Cardinal invariants and completeness

## 6.1 Cardinal invariants of topological groups

The *cardinal invariants* of the topological groups are cardinal numbers, say $\rho(G)$, associated to every topological group $G$ such that if $G$ is topologically isomorphic to the topological group $H$, then $\rho(G) = \rho(H)$. For example, the size $|G|$ is the simplest cardinal invariant of a topological group, it does not depend on the topology of $G$. Here we shall be interested in measuring the minimum size of a base (of neighborhoods of $e_G$) in a topological group $H$, as well as other cardinal functions related to $H$. The related cardinal invariants defined below are the *weight* $w(G)$, the *character* $\chi(G)$ and the *density character* $d(G)$.

It is important to relate the bases (of neighborhoods of 1) in $H$ to those of a subgroup $G$ of $H$.

**Exercise 6.1.1.** *If $G$ is a subgroup of a topological group $H$ and if $\mathcal{B}$ is a base (of neighborhoods of $e_G$) in $H$ then a base (of neighborhoods of 1) in $G$ is given by $\{U \cap G : U \in \mathcal{B}\}$.*

Now we consider the case when $G$ is a *dense* subgroup of $H$.

**Lemma 6.1.2.** *If $G$ is a dense subgroup of a topological group $H$ and $\mathcal{B}$ is a base of neighborhoods of $e_G$ in $G$, then $\{\overline{U}^H : U \in \mathcal{B}\}$ is a base of neighborhoods of $e$ in $H$.*

*Proof.* Since the topological group $H$ is regular, the closed neighborhoods form a base at $e_G$ in $H$. Hence for a neighborhood $V \ni e_G$ in $H$ one can find another neighborhood $V_0 \ni e_G$ such that $\overline{V_0} \subseteq V$. Since $G \cap V_0$ is a neighborhood of $e$ in $G$, there exists $U \in \mathcal{B}$ such that $U \subseteq G \cap V_0$. There exists also an open neighborhood $W$ of $e$ in $H$ such that $U = W \cap G$. Obviously, one can choose $W \subseteq V_0$. Hence $\overline{U}^H = \overline{W}$ as $G$ is dense in $H$ and $W$ is open in $H$. Thus $\overline{U}^H = \overline{W} \subseteq \overline{V}_0 \subseteq V$ is a neighborhood of $e$ in $H$. $\square$

**Lemma 6.1.3.** *Let $G$ be a dense subgroup of a topological group $H$ and let $\mathcal{B}$ be a base of symmetric neighborhoods of $e_G$ in $H$. Then $\{gU : U \in \mathcal{B}, g \in G\}$ is a base of the topology of $H$.*

*Proof.* Let $x \in H$ and let $x \in O$ be an open set. Then there exists a $U \in \mathcal{B}$ with $xU^2 \subseteq O$. Pick a $g \in G \cap xU$. Then $x^{-1}g \in U$, so $g^{-1}x \in U = U^{-1}$. So
$$x \in gU = xx^{-1}gU \subseteq xU^2 \subseteq O.$$
□

For a topological group $G$ set $d(G) = \min\{|X| : X \text{ is dense in } G\}$,
$$w(G) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } G\} \text{ and } \chi(G) = \min\{|\mathcal{B}| : \mathcal{B} \text{ a base of neighborhoods of } e_G \text{ in } G\}.$$

**Lemma 6.1.4.** *Let $G$ be a topological group. Then:*

*(a) $d(G) \leq w(G) \leq 2^{d(G)}$;*

*(b) $|G| \leq 2^{w(G)}$ if $G$ is Hausdorff.*

*Proof.* (a) To see that $d(G) \leq w(G)$ choose a base $\mathcal{B}$ of size $w(G)$ and for every $U \in \mathcal{B}$ pick a point $d_U \in U$. Then the set $D = \{d_U : U \in \mathcal{B}\}$ is dense in $G$ and $|D| \leq w(G)$.

To prove $w(G) \leq 2^{d(G)}$ note that $G$ is regular, hence every open base $\mathcal{B}$ on $G$ contains a base $\mathcal{B}_r$ of the same size consisting of regular open set. Let $\mathcal{B}$ be a base of $G$ of regular open sets and let $D$ be a dense subgroup of $G$ of size $d(G)$. If $U, V \in \mathcal{B}$, with $U \cap D = V \cap D$, then $\overline{U} = \overline{U \cap D} = \overline{V \cap D} = \overline{V}$. Being $U$ and $V$ regular open, the equality $\overline{U} = \overline{V}$ implies $U = V$. Hence the map $U \mapsto U \cap D$ from $\mathcal{B}$ to the power set $P(D)$ is injective. Therefore $w(G) \leq 2^{d(G)}$.

(b) To every point $x \in G$ assign the set $O_x = \{U \in \mathcal{B} : x \in U\}$. Then the axiom $T_2$ guarantees that the map $x \mapsto O_x$ from $G$ to the power set $P(\mathcal{B})$ is injective. Therefore, $|G| \leq 2^{w(G)}$. □

**Remark 6.1.5.** Two observations related to item (b) of the above lemma are in order here.

- The equality in item (b) can be attained (see Theorem 8.3.10).
- One cannot remove Hausdorffness in item (b) (any large indiscrete group provides a counter-example). This dependence on separation axioms is due to the presence of the size of the group in (b). We see in the next exercise that the Hausdorff axiom is not relevant as far as the other cardinal invariants are involved.

**Lemma 6.1.6.** *$w(G) = \chi(G) \cdot d(G)$ for every topological group $G$.*

*Proof.* The inequality $w(G) \geq \chi(G)$ is obvious. The inequality $w(G) \geq d(G)$ has already been proved in Lemma 12.4.20 (a). This proves the inequality $w(G) \geq \chi(G) \cdot d(G)$.

To prove the inequality $w(G) \leq \chi(G) \cdot d(G)$ pick a dense subgroup $D$ of $G$ of size $d(G)$ and a base $\mathcal{B}$ of symmetric open sets of $\mathcal{V}(e_G)$ with $|\mathcal{B}| = \chi(G)$ and apply Lemma 6.1.3. □

**Lemma 6.1.7.** *Let $H$ be a subgroup of a topological group $G$. Then:*

*(a) $w(H) \leq w(G)$ and $\chi(H) \leq \chi(G)$;*

*(b) if $H$ is dense in $G$, then $w(G) = w(H)$, $\chi(G) = \chi(H)$ and $d(G) \leq d(H)$.*

*Proof.* (a) Follows from Exercise 6.1.1, as $|\{U \cap G : U \in \mathcal{B}\}| \leq |\mathcal{B}|$ for every base (of neighborhoods of $e_G$) in $G$.

(b) We prove first $\chi(G) = \chi(H)$. The inequality $\chi(G) \geq \chi(H)$ follows from item (a). To prove the opposite inequality fix a base $\mathcal{B}$ of neighborhoods of $e_G$ in $H$ with $|\mathcal{B}| = \chi(H)$. By Lemma 6.1.2, $\mathcal{B}^* = \{\overline{U}^H : U \in \mathcal{B}\}$ is a base of neighborhoods of $e$ in $G$. Since $|\mathcal{B}^*| \leq \mathcal{B} = \chi(H)$, this proves $\chi(G) \leq \chi(H)$.

The inequality $d(G) \leq d(H)$ follows from the fact that the dense subsets of $H$ are dense in $G$ as well.

The inequality $w(G) \geq w(H)$ follows from item (a). According to Lemma 12.4.20, $H$ has a dense subgroup $D$ with $|D| \leq w(H)$. By the above argument $\chi(G) = \chi(H)$. Now $w(G) = \chi(G)d(G) \leq \chi(H)d(H) = w(H)$, where the first and the last equality follow from Lemma 6.1.6, the inequality follows from $d(G) \leq d(H)$. □

**Lemma 6.1.8.** *If $f : G \to H$ is a continuous surjective homomotphism, then $d(H) \leq d(G)$. If $f$ is open, then also $w(H) \leq w(G)$ and $\chi(H) \leq \chi(G)$.*

*Proof.* If $D$ is a dense subset of $G$, then $f(D)$ is a dense subset of $H$ with $|f(D)| \leq |D|$. This proves the first assertion. The second assertion follws from the fact that if $\mathcal{B}$ is a base (of neigborhoods of $e_G$), then $\mathcal{B}_0 = \{f(B) : B \in \mathcal{B}\}$ is a base (of neigborhoods of $e_H$) with $|\mathcal{B}_0| \leq |\mathcal{B}|$. □

**Theorem 6.1.9.** *If $\{G_i : i \in I\}$ is a family of non-trivial topological groups and $G = \prod_{i \in I} G_i$, then:*

*(a) $|I| \cdot \sup\{d(G_i) : i \in I\} \geq d(G) \geq \sup\{d(G_i) : i \in I\}$,*

*(b)* $\chi(G) = |I| \cdot \sup\{\chi(G_i) : i \in I\}$ *and* $w(G) = |I| \cdot \max\{w(G_i) : i \in I\}$.

*Proof.* (a) follows from tha fact that if $D_i$ is a dense countable subgroup of $G_i$ for each $i \in I$, then $D = \bigoplus_{i\in I} D_i$ is a dense subgroup of $\prod_{i\in I} G_i$ with $|D| \leq |I| \cdot \max\{|D_i|\}$.

(b) . . . □

**Example 6.1.10.** (a) The groups $G$ with $d(G) \leq \aleph_0$ are precisely the separable groups. If $\{G_i : i \in I\}$ is a family of separable groups, then $d(\prod_{i\in I} G_i) \leq \max\{\aleph_0, |I|\}$. (Indeed, if $D_i$ is a dense countable subgroup of $G_i$ for each $i \in I$, then $D = \bigoplus_{i\in I} D_i$ is a dense subgroup of $\prod_{i\in I} G_i$ with $|D| \leq \max\{\aleph_0, |I|\}$.) A stronger, yet non-trivial, inequality holds for this type of products: $d(\prod_{i\in I} G_i) \leq \kappa$, whenever $|I| \leq 2^\kappa$ (in particular, $\prod_{i\in I} G_i$ is separable whenever $|I| \leq \mathfrak{c}$), but we are going to prove it in §. . . by means of Pontryagin duality.

(b) Obviously, $\chi(G) \leq w(G)$. According to Birkhoff-Kakutani theorem, $\chi(G)$ is countable for a Hausdorff group $G$ precisely when $G$ is metrizable. Hence, every Hausdorff group of countable weight is metrizable.

(c) Let $\{G_i : i \in I\}$ be an infinite family of non-trivial metrizable Hausdorff groups. Then $G = \prod_{i\in I} G_i$ has $\chi(G) = |I|$. If $I$ is countable, then Exercise 6.1.10 applies. In the general case, for every $i \in I$ let $\mathcal{B}_i$ be a countable base of the filter $\mathcal{V}_{G_i}(1)$. Then for any finite subset $J \subseteq I$ and for $U_i \in \mathcal{B}_i$ when $i \in J$, let $W_J$ be the neighborhood $\prod_{i\in J} U_i \times \prod_{i\in I\setminus J} G_i$ of 1 in $G$. Then the family $(W_J)$, when $J$ runs over the family of finite subsets of $I$ and $U_i \in \mathcal{B}_i$ for $i \in J$, has size at most $|I|$ and forms a base of $\mathcal{V}_G(1)$. On the other hand, since every neighborhood $O$ in $\mathcal{V}_{G_i}(1)$ contains a subgroup $H_J := \prod_{i\in J}\{1_i\} \times \prod_{i\in I\setminus J} G_i \subseteq W_J$ for some finite subset $J \subseteq I$, it is clear that less than $|I|$ neighborhoods cannot give trivial intersection. Hence every base of the filter $\mathcal{V}_G(1)$ has size at least $|I|$. This proves $\chi(G) = |I|$.

**Example 6.1.11.** Let $\{G_i : i \in I\}$ be an infinite family of non-trivial Hausdorff groups of countable weight. Then $G = \prod_{i\in I} G_i$ satisfies $w(G) = \chi(G) = |I|$. Indeed, $\chi(G) = |I|$ was already proved in Example 6.1.10. In view of $w(G) = \chi(G) \cdot d(G)$, it remains to note that $d(G) \leq |I|$ in virtue of item (a) of Example 6.1.10.

**Exercise 6.1.12.** *Let $G$ be a topological group. Prove that:*

*(a)* $w(G) = w(G/\overline{\{e_G\}})$, $\chi(G) = \chi(G/\overline{\{e_G\}})$ *and* $d(G) = d(G/\overline{\{e_G\}})$*;*

*(b)* $d(U) = d(G)$ *for every non-empty open set* $U$*, if* $G$ *is Lindelöff;*

*(c) if $G$ is Hausdorff, then $\chi(G)$ is finite iff $G$ is discrete; in such a case $\chi(G) = 1$.*

*(d) if $G$ is Hausdorff, then $w(G)$ is finite iff $G$ is finite iff $d(G)$ is finite.*

**Exercise 6.1.13.** $w(G, \mathcal{T}_H) \leq |H|$.

(Hint. Since $(G, \mathcal{T}_H)$ is a topological subgroup of $\mathbb{T}^H$, one has $w(G, \mathcal{T}_H) \leq w(\mathbb{T}^H) = |H|$ by Example 6.1.11.)

We shall see in the sequel that $\chi(\mathcal{T}_H) = w(\mathcal{T}_H) = |H|$.

**Exercise 6.1.14.** *Show that $w(G)$, $\chi(G)$ and $d(G)$ are cardinal invariants in the sense explained above, i.e., if $G \cong H$, then $w(G) = w(H)$, $\chi(G) = \chi(H)$ and $d(G) = d(H)$.*

## 6.2 Completeness and completion

A net $\{g_\alpha\}_{\alpha\in A}$ in a topological group $G$ is a *Cauchy net* if for every neighborhood $U$ of $e_G$ in $G$ there exists $\alpha_0 \in A$ such that $g_\alpha^{-1} g_\beta \in U$ and $g_\beta g_\alpha^{-1} \in U$ for every $\alpha, \beta > \alpha_0$.

**Remark 6.2.1.** It is easy to see that a if $H$ is a subgroup of a topological group $G$, then a net $\{h_\alpha\}_{\alpha\in A}$ in $H$ is Cauchy iff it is a Cauchy net in $G$. In other words, this is an intrinsic property of the net and it does not depend on the topological group where the net is considered. (Consequently, a net $\{h_\alpha\}_{\alpha\in A}$ is Cauchy in $H$ iff it is a Cauchy net of the subgroup $\langle h_\alpha : \alpha \in A\rangle$ of $H$.)

By item (a) of the next exercise, a net $(g_\alpha)$ in a topological group $G$ is a Cauchy net whenever it converges in some larger topological group $H$ containing $G$ asa topological subgroup. Our aim in this subsection will be to see (see Theorem 6.2.3) that all Cauchy nets in $G$ arise in this way; moreover, there is a group $H$ witnessing this simultaneously for all Cauchy nest in $G$ and $G$ is dense in $H$.

**Exercise 6.2.2.** *(a) Let $G$ be a dense subgroup of a topological group $H$. If $(g_\alpha)$ is a net in $G$ that converges to some element $h \in H$, then $(g_\alpha)$ is a Cauchy net.*

*(b) Let $f : G \to H$ be a continuous homomorphism. If $\{g_\alpha\}_{\alpha\in A}$ is a (left, right) Cauchy net in $G$, then $\{f(g_\alpha)\}_{\alpha\in A}$ is a (left, right) Cauchy net in $H$.*

By the previous exercise, the convergent nets are Cauchy nets. A topological group $G$ is *complete (in the sense of Raĭkov)* if every Cauchy net in $G$ converges in $G$. We omit the tedious proof of the next theorem.

**Theorem 6.2.3.** *For every topological Hausdorff group $G$ there exists a complete topological group $\widetilde{G}$ and a topological embedding $i : G \to \widetilde{G}$ such that $i(G)$ is dense in $\widetilde{G}$.*

The completion $\widetilde{G}$ has an important universal property:

**Theorem 6.2.4.** *If $G$ is a topological Hausdorff group $G$ and $f : G \to H$ is a continuous homomorphism, where $H$ is a complete topological group, then there is a unique continuous homomorphism $\widetilde{f} : \widetilde{G} \to H$ with $f = \widetilde{f} \circ i$.*

*Proof.* Let $g \in \widetilde{G}$. Then there exists a net $\{g_\alpha\}_{\alpha\in A}$ in $G$ such that $g = \lim g_\alpha$. Then $\{g_\alpha\}_{\alpha\in A}$ is a Cauchy net, hence $\{f(g_\alpha)\}_{\alpha\in A}$ is a Cauchy net in $H$. By the completeness of $H$, it must be convergent. Put $\widetilde{f}(x) = \lim f(g_\alpha)$. One can prove that this limit does not depend on the choice of the net $\{g_\alpha\}_{\alpha\in A}$ with $g = \lim g_\alpha$ and $\widetilde{f}$ is continuous. This also shows the uniqueness of the extension $\widetilde{f}$. □

From this theorem one can deduce that every Hausdorff topological abelian group has a unique, up to topological isomorphisms, *(Raĭkov-)completion* $(\widetilde{G}, i)$ and we can assume that $G$ is a dense subgroup of $\widetilde{G}$.

**Definition 6.2.5.** *A net $\{g_\alpha\}_{\alpha\in A}$ in $G$ is a* left [resp., right] Cauchy net *if for every neighborhood $U$ of $e_G$ in $G$ there exists $\alpha_0 \in A$ such that $g_\alpha^{-1} g_\beta \in U$ [resp., $g_\beta g_\alpha^{-1} \in U$] for every $\alpha, \beta > \alpha_0$.*

Clearly, a net is Cauchy iff it is both left and right Cauchy.

**Lemma 6.2.6.** *Let $G$ be a Hausdorff topological group. Every left (resp., right) Cauchy net in $G$ with a convergent subnet is convergent.*

*Proof.* Let $\{g_\alpha\}_{\alpha\in A}$ be a left Cauchy net in $G$ and let $\{g_\beta\}_{\beta\in B}$ be a subnet convergent to $x \in G$, where $B$ is a cofinal subset of $A$. Let $U$ be a neighborhood of $e_G$ in $G$ and $V$ a symmetric neighborhood of $e_G$ in $G$ such that $VV \subseteq U$. Since $g_\beta \to x$, there exists $\beta_0 \in B$ such that $g_\beta \in xV$ for every $\beta > \beta_0$. On the other hand, there exists $\alpha_0 \in A$ such that $\alpha_0 \geq \beta_0$ and $g_\alpha^{-1} g_\gamma \in V$ for every $\alpha, \gamma > \alpha_0$. With $\gamma = \beta_0$ we have $g_\alpha \in xVV \subseteq xU$ for every $\alpha > \alpha_0$, that is $g_\alpha \to x$. □

**Proposition 6.2.7.** *A Hausdorff topological group $G$ is complete iff for every embedding $j : G \hookrightarrow H$ into a Hausdorff topological group $H$ the subgroup $j(G)$ of $H$ is closed.*

*Proof.* Assume that there exists an embedding $j : G \hookrightarrow H$ into a Hausdorff topological group $H$ such that $j(H)$ is not a closed subgroup of $H$. Then there exists a net $y_\alpha$ in $j(G)$ converging to some element $h \in H$ that does not belong to $j(G)$. By Remark 6.2.1, $(y_\alpha)$ is a Cauchy net in $j(G)$. Since it converges to $h \notin j(G)$ in $H$ and $H$ is a Hausdorff group, we conclude that this net does not converge in $j(G)$. Since $j : G \to j(G)$ is a topological group isomorphism, this provides a non-convergent Cauchy net in $G$. Hence $G$ is not complete. Now assume that $G$ is not complete and consider the dense inclusion $j : G \hookrightarrow \widetilde{G}$. Since $G = j(G)$ is a proper dense subgroup of $\widetilde{G}$, we conclude that $j(G)$ is not closed in $\widetilde{G}$. □

A topological group $G$ is *complete in the sense of Weil* if every left Cauchy net converges in $G$.

Every Weil-complete group is also complete, but the converse does not hold in general. It is possible to define the Weil-completion of a Hausdorff topological group in analogy with the Raĭkov-completion.

**Exercise 6.2.8.** *Prove that if a Hausdorff topological group $G$ admits a Weil-completion, then in $G$ the left Cauchy and the right Cauchy nets coincide.*

**Exercise 6.2.9.** *Let $X$ be an infinite set and let $G = S(X)$ equipped with the topology described in §3.3.5. Prove that:*

*(a) a net $\{f_\alpha\}_{\alpha\in A}$ in $G$ is left Cauchy iff there exists a (not necessarily bijective) map $f : X \to X$ so that $f_\alpha \to f$ in $X^X$, prove that such an $f$ must necessarily be injective;*

*(b) a net $\{f_\alpha\}_{\alpha\in A}$ in $G$ is right Cauchy iff there exists a (not necessarily bijective) map $g : X \to X$ so that $f_\alpha^{-1} \to g$ in $X^X$;*

*(c) the group $S(X)$ admits no Weil-completion.*

*(d) $S(X)$ is Raĭkov-complete.*

(Hint. (c) Build a left Cauchy net in $S(X)$ that is not right Cauchy and use items (a) and (b), as well as the previous exercise.) (d) Use items (a) and (b).)

**Exercise 6.2.10.** *(a) Let $G$ be a linearly topologized group and let $\{N_i : i \in I\}$ be its system of neighborhoods of $e_G$ consisting of open normal subgroups. Then the completion of $G$ is isomorphic to the inverse limit $\varprojlim G/N_i$ of the discrete quotients $G/N_i$.*

*(b) Show that the completion in (a) is compact iff all $N_i$ have finite index in $G$.*

*(c) Let $p$ be a prime number. Prove that the completion of $\mathbb{Z}$ equipped with the $p$-adic topology (see Example 3.3.2) is the compact group $\mathbb{J}_p$ of $p$-adic integers.*

*(d) Prove that the completion of $\mathbb{Z}$ equipped with the natural topology (see Example 3.3.2) is isomorphic to $\prod_p \mathbb{J}_p$.*

**Exercise 6.2.11.** *Let $p$ be a prime number. Prove that:*

*(a) $\mathbb{Z}$ admits a finest group topology $\tau$ such that $p^n$ converges to $0$ in $\tau$ (this is $\tau_{(p^n)}$ in the notation of §3.4);*

*(b)* [132, 131] $(\mathbb{Z}, \tau)$ is complete;*

*(c) conclude that $\tau$ is not metrizable.*

**Exercise 6.2.12.** *Let $V, U$ be linear spaces over a field $K$. Prove that the group $Hom(V, U)$, equipped with the finite topology, is complete.*

The proof of Theorem 6.2.3 becomes particularly involved when $G$ is not metrizable. This is why many authors prefer to avoid the Cauchy nest for the construction of the completion. This can be done by means of the following notions. A filter $\mathcal{F}$ on a Hausdorff topological group $G$ is called *Cauchy*, if for every $U \in \mathcal{V}_G(e_G)$ there exists $g \in G$ such that $gU \in \mathcal{F}$ and $Ug \in \mathcal{F}$.

**Exercise 6.2.13.** *Let $G$ be a Hausdorff topological group. Prove that:*

*(a) a filter $\mathcal{F}$ on $G$ Cauchy iff the filter $\mathcal{F}^{-1} = \{F^{-1} : F \in \mathcal{F}\}$ is Cauchy;*

*(b) if $\mathcal{F}$ is a Cauchy filter on $G$ and $x_F \in F$ for every $F \in \mathcal{F}$, then the net $\{x_F : F \in \mathcal{F}\}$ is a Cauchy net (here $\mathcal{F}$ is considered as a directed partially ordered set w.r.t. inclusion);*

*(c) if $\{x_i : i \in I\}$ is a Cauchy net in $G$ and $F_i = \{x_j : j \in I, j \geq i\}$, then the family $\{F_i : i \in I\}$ is a filter base of a Cauchy filter on $G$;*

*(d) $G$ is complete iff every Cauchy filter in $G$ converges.*

*(e) if $G$ is a topological subgroup of a topological group $H$ and $h \in \overline{G}$, then $\mathcal{F} = \{G \cap U : U \in \mathcal{V}_H(h)\}$ is an open Cauchy filter in $G$.*

The Raĭkov completion of a Hausdorff topological group $G$ is built by appropriately using the open Cauchy filters of $G$.

A weaker form of completeness was proposed in [65, 66]: a topological group $G$ is said to be *sequentially complete*topological group!sequentially complete, if every Cauchy sequence $(g_n)$ in $G$ is convergent. It is easy to realize that $G$ is sequentially complete if and only if $G$ is sequentially closed in its Raĭkov completion. Countably compact groups are sequentially complete, although they need not be complete in general. Plenty of results on this remarkable class can be found in [35, 65, 66].

# 7 Compactness and local compactness in topological groups

## 7.1 Examples

Clearly, a topological group $G$ is locally compact if there exists a compact neighborhood of $e_G$ in $G$ (compare with Definition 2.2.11). We shall assume without explicitly mentioning it, that all locally compact groups are Hausdorff.

Obviously, the group $\mathbb{T}$ is compact, so as an immediate consequence of Tychonov's theorem of compactness of products we obtain the following generic example of a compact abelian group:

**Example 7.1.1.** Every power $\mathbb{T}^I$ of $\mathbb{T}$, as well as every closed subgroup of $\mathbb{T}^I$, is compact. It will become clear in the sequel that this is the most general instance of a compact abelian group. Namely, *every compact abelian group is isomorphic to a closed subgroup of a power of* $\mathbb{T}$ (see Corollary 11.2.2).

The above example will help us to produce another important one.

**Example 7.1.2.** Let us see that for every abelian group $G$ the group $\mathrm{Hom}\,(G,\mathbb{S})$ is closed in the product $\mathbb{S}^G$, hence $G^* \cong \mathrm{Hom}\,(G,\mathbb{S})$ is compact. Indeed, consider the projections $\pi_x : \mathbb{S}^G \to \mathbb{S}$ for every $x \in G$ and the following equalities

$$\begin{aligned} G^* &= \bigcap_{h,g\in G} \{f \in \mathbb{S}^G : f(h+g) = f(h)f(g)\} = \bigcap_{h,g\in G} \{f \in \mathbb{S}^G : \pi_{h+g}(f) = \pi_h(f)\pi_g(f)\} \\ &= \bigcap_{h,g\in G} \{f \in \mathbb{S}^G : (\pi_{h+g}^{-1}\pi_h\pi_g)(f) = 1\} = \bigcap_{h,g\in G} \ker(\pi_{h+g}^{-1}\pi_h\pi_g). \end{aligned}$$

Since $\pi_x$ is continuous for every $x \in G$ and $\{1\}$ is closed in $\mathbb{S}$, then all $\ker(\pi_{h+g}^{-1}\pi_h\pi_g)$ are closed; so $\mathrm{Hom}\,(G,\mathbb{S})$ is closed too. Since $\mathbb{S}^G$ is compact by 7.1.1, this yields that $\mathrm{Hom}\,(G,\mathbb{S})$ is compact too.

It will become clear with the duality theorem 12.4.7 that this example is the most general one. Namely, *every compact abelian group $K$ is topologically isomorphic to some compact abelian group of the form $G^*$.*

The next lemma contains a well known useful fact – the existence of a "diagonal subnet".

**Lemma 7.1.3.** *Let $G$ be an abelian group and let $N = \{\chi_\alpha\}_\alpha$ be a net in $G^*$. Then there exist $\chi \in G^*$ and a subnet $S = \{\chi_{\alpha_\beta}\}_\beta$ of $N$ such that $\chi_{\alpha_\beta}(x) \to \chi(x)$ for every $x \in G$.*

*Proof.* By Tychonov's theorem, the group $\mathbb{T}^G$ endowed with the product topology is compact. Then there exist $\chi \in \mathbb{T}^G$ and a subnet $S = (\chi_{\alpha_\beta})$ of $N$ that converges to $\chi$. Therefore $\chi_{\alpha_\beta}(x) \to \chi(x)$ for every $x \in G$ and $\chi \in G^*$, because $G^*$ is closed in $\mathbb{T}^G$ by 7.1.2. □

An example of a non-abelian compact groups can be obtained as a topological subgroup of the full linear group $GL_n(\mathbb{C})$ considered in Example 3.1.4.

**Example 7.1.4.** The set $U(n)$ of all $n \times n$ unitary matrices over $\mathbb{C}$ is a subgroup of $GL_n(\mathbb{C})$. Moreover, since $U(n)$ is a closed and bounded subset of $\mathbb{C}^{n^2}$, we conclude with Example 2.2.12 that $U(n)$ is compact. It is easy to see that $U(1) \cong \mathbb{T}$ is precisely the unit circle group.

Clearly, $\mathbb{U} = \prod_{n=1}^{\infty} U(n)$ is still compact as well as all powers $\mathbb{U}^I$ and closed subgroups of $\mathbb{U}^I$. It is a remarkable fact of the theory of topological groups that *every compact group is isomorphic to a closed subgroup of a power of* $\mathbb{U}$ (Corollary 9.3.3).

Here we collect examples of locally compact groups.

**Example 7.1.5.** Obviously, every discrete group is locally compact.

(a) For every $n \in \mathbb{N}$ the group $\mathbb{R}^n$ is locally compact.

(b) If $G$ is a topological group having an open compact subgroup $K$, then $G$ is locally compact.

(c) Since finite products preserve local compactness (see Theorem 2.2.33(b)), it follows from (a) and (b) that every group of the form $\mathbb{R}^n \times G$, where $G$ has an open compact subgroup $K$, is necessarily locally compact. We shall prove below that every locally compact abelian group has this form.

**Example 7.1.6.** The group $\ell_2$ (see Example 7.1.6) is not locally compact. Indeed, it suffices to note that the closed unit disk is not compact (the sequence $(e_n)$ of the vectors of the canonical base has no adherence point).

## 7.2 Specific properties of (local) compactness

In this subsection we shall see the impact of local compactness in various directions (the open mapping theorem, properties related to connectedness, etc.).

**Lemma 7.2.1.** *Let $G$ be a topological group and let $C$ and $K$ be closed subsets of $G$:*

*(a) if $K$ is compact, then both $CK$ and $KC$ are closed;*

*(b) if both $C$ and $K$ are compact, then $CK$ and $KC$ are compact;*

*(c) if $K$ is contained in an open subset $U$ of $G$, then there exists an open neighborhood $V$ of $e_G$ such that $KV \subseteq U$.*

*Proof.* (a) Let $\{x_\alpha\}_{\alpha\in A}$ be a net in $CK$ such that $x_\alpha \to x_0 \in G$. It is sufficient to show that $x_0 \in CK$. For every $\alpha \in A$ we have $x_\alpha = y_\alpha z_\alpha$, where $y_\alpha \in C$ and $z_\alpha \in K$. Since $K$ is compact, then there exist $z_0 \in K$ and a subnet $\{z_{\alpha_\beta}\}_{\beta\in B}$ such that $z_{\alpha_\beta} \to z_0$. Thus $(x_{\alpha_\beta}, z_{\alpha_\beta})_{\beta\in B}$ is a net in $G\times G$ which converges to $(x_0, z_0)$. Therefore $y_{\alpha_\beta} = x_{\alpha_\beta} z_{\alpha_\beta}^{-1}$ converges to $x_0 z_0^{-1}$ because the function $(x,y)\mapsto xy^{-1}$ is continuous. Since $y_{\alpha_\beta}\in C$ for every $\beta\in B$ and $C$ is closed, $x_0z_0^{-1}\in C$. Now $x_0 = (x_0z_0^{-1})z_0 \in CK$. Analogously it is possible to prove that $KC$ is closed.

(b) The product $C\times K$ is compact by the Tychonov theorem and the function $(x,y)\mapsto xy$ is continuous and maps $C\times K$ onto $CK$. Thus $CK$ is compact.

(c) Let $C = G\setminus U$. Then $C$ is a closed subset of $G$ disjoint with $K$. Therefore, for the compact subset $K^{-1}$ of $G$ one has $1\notin K^{-1}C$. By (a) $K^{-1}C$ is closed, so there exists a symmetric neighborhood $V$ of 1 that misses $K^{-1}C$. Then $KV$ misses $C$ and consequently $KV$ is contained in $U$. □

**Exercise 7.2.2.** *(i) Prove item (c) of Lemma 7.2.1 directly, withouit making any recourse to item (a).*

*(ii) Deduce item (a) of Lemma 7.2.1 from item (c).*

(Hint. (i) If $U$ is an open set containing $K$, then for each $x\in K$ there exists an open $V_x\in\mathcal{V}(e_G)$ such that $xV_x^2\subseteq U$, so $\bigcup_{x\in K} xV_x$ covers the compact set $K$. Hence there exist $x_1,\dots,x_n\in K$ such that $K\subseteq\bigcup_{k=1}^n x_kV_{x_k}$. Let $V=\bigcap_{k=1}^n V_k$. Then $KV\subseteq U$, since for $x\in K$ there exists $k$ with $x\in x_kV$, so that $xV\subseteq x_kVV_k\subseteq x_kV_k^2\subseteq U$. (ii) Argue as in the proof of (c): if $x\in G$ and $x\notin KC$, then for the compact subset $K^{-1}$ of $G$ one has $K^{-1}x\cap C=\emptyset$, so the compact set $K^{-1}x$ is contained in the open subset $U=G\setminus C$ of $G$. So by (c) there exists an open neighborhood $V$ of $e_G$ such that $K^{-1}xV\subseteq U$. Hence $K^{-1}xV\cap C=\emptyset$ and consequently $xV\cap KC=\emptyset$. This proves that $KC$ is closed.)

Compactness of $K$ cannot be omitted in item (a) of Lemma 7.2.1. Indeed, $K=\mathbb{Z}$ and $C=\langle\sqrt{2}\rangle$ are closed subgroups of $G=\mathbb{R}$ but the subgroup $K+C$ of $\mathbb{R}$ is dense (see Exercie 4.1.9 or Proposition 8.3.9).

The canonical projection $\pi : G\to G/K$ from a topological group $G$ onto its quotient $G/K$ is always open. Now we see that it is also closed if $K$ is compact.

**Lemma 7.2.3.** *Let $G$ be a topological group and $K$ a compact normal subgroup of $G$. Then the canonical projection $\pi : G\to G/K$ is closed.*

*Proof.* Let $C$ be a closed subset of $G$. Then $CK$ is closed by Lemma 7.2.1 and so $U=G\setminus CK$ is open. For every $x\notin CK$, that is $\pi(x)\notin\pi(C)$, $\pi(U)$ is an open neighborhood of $\pi(x)$ such that $\pi(U)\cap\pi(C)$ is empty. So $\pi(C)$ is closed. □

**Lemma 7.2.4.** *Let $G$ be a topological group and let $H$ be a closed normal subgroup of $G$.*

*(1) If $G$ is compact, then $G/H$ is compact.*

*(2) If $H$ and $G/H$ are compact, then $G$ is compact.*

*Proof.* (1) is obvious.

(2) Let $\mathcal{F}=\{F_\alpha : \alpha\in A\}$ be a family of closed sets of $G$ with the finite intersection property. If $\pi : G\to G/H$ is the canonical projection, $\pi(\mathcal{F})$ is a family of closed subsets with the finite intersection property in $G/H$ by Lemma 7.2.3. By the compactness of $G/H$ there exists $\pi(x)\in\pi(F_\alpha)$ for every $\alpha\in A$. So $F_\alpha^* := F_\alpha\cap xH\neq\emptyset$ for every $\alpha\in A$. This gives rise to a family $\{F_\alpha^* : \alpha\}$ of closed sets of the compact set $xH$ with the finite intersection property. Thus $\bigcap_{\alpha\in A} F_\alpha^*\neq\emptyset$. So the intersection of all $F_\alpha$ is non-empty as well. □

Another proof of item (2) of the above lemma can be obtained using Lemma 7.2.3 which says that the canonical projection $\pi : G\to G/H$ is a perfect map when $H$ is compact. Applying the well known fact that inverse images of compact sets under a perfect map are compact to $G=\pi^{-1}(G/H)$, we conclude that $G$ is compact whenever $H$ and $G/H$ are compact.

**Lemma 7.2.5.** *Let $G$ be a locally compact group, $H$ be a closed normal subgroup of $G$ and $\pi : G\to G/H$ be the canonical projection. Then:*

*(a) $G/H$ is locally compact too;*

*(b) If $C$ is a compact subset of $G/H$, then there exists a compact subset $K$ of $G$ such that $\pi(K)=C$.*

*Proof.* Let $U$ be an open neighborhood of $e_G$ in $G$ with compact closure. Consider the open neighborhood $\pi(U)$ of $e_{G/H}$ in $G/H$. Then $\pi(\overline{U})\subseteq\overline{\pi(U)}$ by the continuity of $\pi$. Now $\pi(\overline{U})$ is compact in $G/H$, which is Hausdorff, and so $\pi(\overline{U})$ is closed. Since $\pi(U)$ is dense in $\pi(\overline{U})$, we have $\overline{\pi(U)}=\overline{\pi(\overline{U})}=\pi(\overline{U})$. So $G/H$ is locally compact.

(b) Let $U$ be an open neighborhood of $e_G$ in $G$ with compact closure. Then $\{\pi(sU) : s\in G\}$ is an open covering of $G/H$. Since $C$ is compact, a finite subfamily $\{\pi(s_iU) : i=1,\dots,m\}$ covers $C$. Then we can take $K=(s_1U\cup\dots\cup s_mU)\cap\pi^{-1}(C)$. □

**Lemma 7.2.6.** *A locally compact group is Weil-complete.*

*Proof.* Let $U$ be a neighborhood of $e_G$ in $G$ with compact closure and let $\{g_\alpha\}_{\alpha\in A}$ be a left Cauchy net in $G$. Then there exists $\alpha_0 \in A$ such that $g_\alpha^{-1} g_\beta \in U$ for every $\alpha, \beta \geq \alpha_0$. In particular, $g_\beta \in g_{\alpha_0} U$ for every $\beta > \alpha_0$. By the compactness of $g_{\alpha_0}\overline{U}$, we can conclude that there exists a convergent subnet $\{g_\beta\}_{\beta\in B}$ (for some cofinal $B \subseteq A$) such that $g_\beta \to g \in G$. Then also $g_\alpha$ converges to $g$ by Lemma 6.2.6. □

**Lemma 7.2.7.** *A locally compact countable group is discrete.*

*Proof.* By the Baire category theorem 2.2.22 $G$ is of second category. Since $G = \{g_1, \dots, g_n, \dots\} = \bigcup_{n=1}^\infty \{g_n\}$, there exists $n \in \mathbb{N}_+$ such that $\operatorname{Int}\{g_n\}$ is not empty and so $\{g_n\}$ is open. □

Now we introduce a special class of $\sigma$-compact groups that will play an essential role in determining the structure of the locally compact abelian groups.

**Definition 7.2.8.** *A group $G$ is* compactly generated *if there exists a compact subset $K$ of $G$ which generates $G$, that is $G = \langle K\rangle = \bigcup_{n=1}^\infty (K \cup K^{-1})^n$.*

**Lemma 7.2.9.** *If $G$ is a compactly generated group then $G$ is $\sigma$-compact.*

*Proof.* By the definition $G = \bigcup_{n=1}^\infty (K \cup K^{-1})^n$, where every $(K \cup K^{-1})^n$ is compact, since $K$ is compact. □

It should be emphasized that while $\sigma$-compactness is a purely topological property, being compactly generated involves essentially the algebraic structure of the group.

**Exercise 7.2.10.** *(a) Give examples of $\sigma$-compact groups that are not compactly generated.*

*(b) Show that every connected locally compact group is compactly generated.*

**Lemma 7.2.11.** *Let $G$ be a locally compact group.*

*(a) If $K$ a compact subset of $G$ and $U$ an open subset of $G$ such that $K \subseteq U$, then there exists an open neighborhood $V$ of $e_G$ in $G$ such that $(KV)\cup(VK) \subseteq U$ and $\overline{(KV)\cup(VK)}$ is compact.*

*(b) If $G$ is compactly generated, then there exists an open neighborhood $U$ of $e_G$ in $G$ such that $\overline{U}$ is compact and $U$ generates $G$.*

*Proof.* (a) By Lemma 7.2.1 (c) there exists an open neighborhood $V$ of $e_G$ in $G$ such that $(KV)\cup(VK) \subseteq U$. Since $G$ is locally compact, we can choose $V$ with compact closure. Thus $K\overline{V}$ is compact by Lemma 7.2.1. Since $KV \subseteq K\overline{V}$, then $\overline{KV} \subseteq K\overline{V}$ and so $\overline{KV}$ is compact. Analogously $\overline{VK}$ is compact, so $\overline{(KV)\cup(VK)} = \overline{KV} \cup \overline{VK}$ is compact.

(b) Let $K$ be a compact subset of $G$ such that $K$ generates $G$. So $K \cup \{e_G\}$ is compact and by (a) there exists an open neighborhood $U$ of $e_G$ in $G$ such that $U \supseteq K \cup \{e_G\}$ and $\overline{U}$ is compact. □

In the case of first countable topological groups Fujita and Shakmatov [83] have described the precise relationship between $\sigma$-compactness and the stronger property of being compactly generated.

**Theorem 7.2.12.** [83] *A metrizable topological group $G$ is compactly generated if and only if $G$ is $\sigma$-compact and, for every open subgroup $H$ of $G$, there exists a finite set $F \subseteq G$ such that $F \cup H$ algebraically generates $G$.*

This gives the following (for the definition of total boundedness see Definition 9.2.1):

**Corollary 7.2.13.** *A $\sigma$-compact metrizable group $G$ is compactly generated in each of the following cases:*

*(a) $G$ has no open subgroups*

*(b) the completion $\widetilde{G}$ is connected;*

*(c) $G$ is totally bounded.*

Moreover,

**Theorem 7.2.14.** [83] *A countable metrizable group is compactly generated iff it is algebraically generated by a sequence (possibly eventually constant) converging to its neutral element.*

Examples showing that the various conditions above cannot be omitted can be found in [83].

The question when will a topological group contain a compactly generated dense subgroup is considered in [84].

## 7.3 The open mapping theorem

Now we prove the open mapping theorem for locally compact topological groups.

**Theorem 7.3.1** (Open mapping theorem). *Let $G$ and $H$ be locally compact topological groups and let $h$ be a continuous homomorphism of $G$ onto $H$. If $G$ is $\sigma$-compact, then $h$ is open.*

*Proof.* Let $U$ be an open neighborhood of $e_G$ in $G$. There exists an open symmetric neighborhood $V$ of $e_G$ in $G$ such that $\overline{V}\,\overline{V} \subseteq U$ and $\overline{V}$ is compact. Since $G = \bigcup_{x\in G} xV$ and $G$ is Lindelöff by Lemma 2.2.14, we have $G = \bigcup_{n=1}^{\infty} x_nV$. Therefore $H = \bigcup_{n=1}^{\infty} h(x_n\overline{V})$, because $h$ is surjective. Put $y_n = h(x_n)$, hence $H = \bigcup_{n=1}^{\infty} y_n h(\overline{V})$ where each $h(\overline{V})$ is compact and so closed in $H$. Since $H$ is locally compact, Theorem 2.2.22 yields that there exists $n \in \mathbb{N}_+$ such that $\operatorname{Int} h(\overline{V})$ is not empty. So there exists a non-empty open subset $W$ of $H$ such that $W \subseteq h(\overline{V})$. If $w \in W$, then $w \in h(\overline{V})$ and so $w = h(v)$ for some $v \in \overline{V} = \overline{V}^{-1}$. Hence

$$e_G \in w^{-1}W \subseteq w^{-1}h(\overline{V}) = h(v^{-1})h(\overline{V}) \subseteq h(\overline{V}\,\overline{V}) \subseteq h(U)$$

and this implies that $h(U)$ is an open neighborhood of $e_G$ in $H$. □

The following immediate corollary is frequently used:

**Corollary 7.3.2.** *If $f : G \to H$ is a continuous surjective homomorphism of Hausdorff topological groups and $G$ is compact, then $f$ is open.*

**Exercise 7.3.3.** *Let $K$ be a compact torsion-free divisible abelian group. Then for every non-zero $r \in \mathbb{Q}$ the alfgebraic automorphism $\lambda_r$ of $K$, defined by setting $\lambda_r(x) = rx$ for every $x \in K$, is a topological isomorphism.*

(Hint. Write $r = n/m$. Note that the multiplication by $m$ is a continuous automorphism of $K$. By the compactness of $K$ and the open mapping theorem, it is a topological isomorphism. In particular, its inverse $x \mapsto \frac{1}{m}x$ is a topological isomorphism too. Since $n \neq 0$, the multiplication by $n$ is a topological isomorphism too. Being the composition of two topological isomorphisms, also $\lambda_r$ is a topological isomorphism.)

The topological groups satisfying the open mapping theorem are known also under the name *totally minimal*. More precisely, one has the following pair of concepts:

**Definition 7.3.4.** Let $G$ be a Hausdorff topological group.
(a) $G$ is said to be totally minimal, if every continuous homomorphism of $G$ onto a Hausdroff topological group $H$ is open.
(b) $G$ be *minimal*, if every continuous isomorphism of $G$ onto a Hausdorff topological group $H$ is open.

One can easily see that a Hausdorff topological group $G$ is totally minimal if and only if all Hausdorff quotients of $G$ are minimal. Therefore, compact groups are totally minimal (see Corollary 7.3.2).

The minimal groups were introduced simultaneously and independently in [138] and [72], where the first examples of non-compact minimal groupd can be found. The first examples of minimal non-totally-minimal groups can be found in [55], where the notion of totally minimal group was explicitly introduced. It was conjectured by Prodanov in 1971 that the minimal abelian groups are precompact. This was confirmed for totally minimal groups as well as some classes of minimal groups in [125, 126]. A positive solution in the general case was obtained by Prodanov and Stoyanov [126]. Answering a question of Choquet, Doïtchinov [72] showed that minimality (unlike compactness) is not preserved even under finite direct products. A complete description of the cases when minimality is preserved even under (arbitrary) direct products can be found in [31]. The surveys [34] and [35] contain various information on minimal groups.

The recent progress in this field is outlines in [50]

## 7.4 Compactness vs connectedness

Now we see that linearity and total disconnectedness of group topologies coincide for compact groups and for locally compact abelian groups.

**Theorem 7.4.1.** *[van Dantzig] Every locally compact totally disconnected group has a base of neighborhoods of $e$ consisting of open subgroups. In particular, a locally compact totally disconnected group that is either abelian or compact has linear topology.*

This can be derived from the following more precise result:

**Theorem 7.4.2.** *Let $G$ be a locally compact topological group and let $C = c(G)$. Then :*

**(a)** *$C$ coincides with the intersection of all open subgroups of $G$;*

**(b)** *if $G$ is totally disconnected, then every neighborhoodof $e_G$ contains an open subgroup of $G$.*

*If $G$ is compact, then the open subgroups in items (a) and (b) can be chosen normal.*

*Proof.* (a) follows from (b) as $G/C$ is totally disconnected hence the neutral element of $G/C$ is intersection of open (resp. open normal) subgroups of $G/C$. Now the intersection of the inverse images, w.r.t. the canonical homomorphism $G \to G/C$, of these subgroups coincides with $C$.

(b) Let $G$ be a locally compact totally disconnted group. By Vedenissov's Theorem $G$ has a base $\mathcal{O}$ of clopen symmetric compact neighborhoods of $e_G$. Let $U \in \mathcal{O}$. The

$$U = \overline{U} = \bigcap \{UV : V \in \mathcal{O}, \mathcal{V} \subseteq \mathcal{U}\}.$$

Then every set $U \cdot V$ is compact by Lemma 7.2.1, hence closed. Since $U$ is open and $U \supseteq \bigcap_{V \in \mathcal{O}} UV$, by the compactness of $UU = \overline{UU}$ we deduce that there exist $V_1, \dots, V_n \in \mathcal{O}$ such that $U \subseteq \bigcap_{k=1}^n UV_k$, so $U = \bigcap_{k=1}^n UV_k$. Then for $V := U \cap \bigcap_{k=1}^n V_k$ one has $UV = U$. This implies also $VV \subseteq U$, $VVV \subseteq U$ etc. Since $V$ is symmetric, the subgroup $H = \langle V \rangle$ is contained in $U$ as well. From $V \subseteq H$ one can deduce that $H$ is open (cf. 3.4.1). In case $G$ is compact, note that the heart $H_G = \bigcap_{x \in G} x^{-1}Hx$ of $H$ is an open normal subgroup as the number of all conjugates $x^{-1}Hx$ of $H$ is finite (being equal to $[G : N_G(H)] \leq [G : H] < \infty$). Hence $H_G$ is an open normal subgroup of $G$ contained in $H$, hence also in $U$. □

In general total disconnectedness is not preserved under taking quotients.

**Corollary 7.4.3.** *The quotient of a locally compact totally disconnected group is totally disconnected.*

*Proof.* Let $G$ be a locally compact totally disconnected group and let $N$ be a closed normal subgroup of $G$. It follows from the above theorem that $G$ has a linear topology. This yields that the quotient $G/N$ has a linear topology too. Thus $G/N$ is totally disconnected. □

**Corollary 7.4.4.** *The continuous homomorphic images of compact totally disconnected groups are totally disconnected.*

*Proof.* Follows from the above corollary and the open mapping theorem. □

According to Example 4.2.10 none of the items (a) and (b) of Theorem 7.4.2 remain true without the hypothesis "locally compact".

**Corollary 7.4.5.** *Let $G$ be a locally compact group. Then $Q(G) = c(G)$.*

*Proof.* By item (a) of the above theorema $C(G)$ is an intersection of open subgroups, that are clopen being open subgroups (cf. Proposizione 3.4.1). Hence $c(G)$ contains $Q(G)$ which in turn coincides with the intersection of *all* clopen sets of $G$ containing $e_G$. The inclusion $C(G) \subseteq Q(G)$ is always true. □

# 8 Properties of $\mathbb{R}^n$ and its subgroups

We saw in Exercise 4.2.12 that every continuous homomorphism $f : \mathbb{R}^n \to H$ is uniquely determined by its restriction to any (arbitrarily small) neighborhood of 0 in $\mathbb{R}^n$. Now we prove that every continuous map $f : U \to H$ defined only on a small neighborhood $U$ of 0 of $\mathbb{R}^n$ can be extended to a continuous homomorphism $f : \mathbb{R}^n \to H$ provided some natural additivity restraint is satisfied within that small neighborhood.

**Lemma 8.0.6.** *Let $n \in \mathbb{N}_+$, let $H$ be an abelian topological group and let $U, U_1$ be open symmetric neighborhoods of 0 in $\mathbb{R}^n$ with $U_1 + U_1 \subseteq U$. Then every map $f : U \to H$, such that $f(x + y) = f(x) + f(y)$ whenever $x, y \in U_1$, can be uniquely extended to a homomorphism $\bar{f} : \mathbb{R}^n \to H$. Moreover, $\bar{f}$ is continuous if and only if $f$ is continuous.*

*Proof.* Taking eventually smaller neighborhoods $U, U_1$ with $U_1 + U_1 \subseteq U$, we can assume without loss of generality that if $x \in U$, then also all $\frac{1}{n}x \in U$ for $n \in \mathbb{N}_+$, and similarly for $U_1$.

For $x \in \mathbb{R}^n$ there exists $n \in \mathbb{N}_+$ such that $\frac{1}{n}x \in U$. We put $\bar{f}(x) = nf(\frac{1}{n}x)$. To see that this definition does not depend on $n$ assume that $\frac{1}{m}x \in U$ as well and let $y = \frac{1}{mn}x \in U$. Then $ny = \frac{1}{m}x \in U$ and $my = \frac{1}{n}x \in U$. So

$$mf\left(\frac{1}{m}x\right) = mf(ny) = mnf(y) = nf(my) = nf\left(\frac{1}{n}x\right).$$

Next we prove now that $\bar{f}$ is a homomorphism. Take $x, y \in \mathbb{R}^n$. There exists an integer $n > 0$ such that $\frac{1}{n}x, \frac{1}{n}y \in U_1$ and so $\frac{1}{n}x + \frac{1}{n}y \in U$. By our hypothesis

$$f(x+y) = nf\left(\frac{1}{n}(x+y)\right) = nf\left(\frac{1}{n}x + \frac{1}{n}y\right) = nf\left(\frac{1}{n}x\right) + nf\left(\frac{1}{n}y\right) = f(x) + f(y).$$

Uniqueness of $\bar{f}$ follows from Exercise 4.2.12. (For a direct proof assume that $f' : \mathbb{R}^n \to H$ is another homomorphism extending of $f$. Then for every $x \in \mathbb{R}^n$ there exists $n \in \mathbb{N}_+$ such that $y = \frac{1}{n}x \in U$. So $f'(x) = f'(ny) = nf'(y) = nf(y) = \bar{f}(x)$.)

Since $\bar{f}$ is a homomorphism, it suffices to check its continuity at 0. This follows from our hypothesis that the map $f : U \to H$ is continuous map. $\square$

The next lemma will be used frequently in the sequel.

**Lemma 8.0.7.** *Let $H$ be an abelian topological group with a discrete subgroup $D$ and let $p : H \to H/D$ be the canonical map. Then for every continuous homomorphism $q : \mathbb{R}^n \to H/D$, $n \in \mathbb{N}_+$, there exists a continuous homomorphism $f : \mathbb{R}^n \to H$ such that $p \circ f = q$. Moreover, if $q$ is open, then $f$ can be chosen to be open.*

*Proof.* Let $W$ be a symmetric open neighborhood of 0 in $H$, such that $(W + W) \cap D = \{0\}$. Then the restriction $p \restriction_W$ is a one-to-one map from $W$ to $p(W)$. Moreover, both the bijection $p \restriction_W$ and its inverse $\xi : p(W) \to W$ are homeomorphisms. Pick a symmetric open neighborhood $W_1$ of 0 in $H$ such that $W_1 + W_1 \subseteq W$ and note that

$$\xi(x+y) = \xi(x) + \xi(y) \text{ whenever } x, y \in p(W_1). \tag{1}$$

Indeed, if $x = p(u), y = p(v)$ for $u, v \in W_1$, then $x + y = p(u) + p(v) = p(u + v)$, since $p$ is a homomorphism. Then $\xi(x + y) = u + v = \xi(x) + \xi(y)$, this porves (1).

Let $U = q^{-1}(p(W))$ and $U_1 = q^{-1}(p(W_1))$, so these are symmetric open neighborhoods of 0 in $\mathbb{R}^n$ with $U_1 + U_1 \subseteq U$.

Define the map $f : U \to H$ simply as the composition $\xi \circ q$. So $f : U \to H$ continuously maps $U_0$ onto the open subset $\xi(q(U))$ of $H$. Moreover, (1) yields that $f(x+y) = f(x)+f(y)$ whenever $x, y \in U_1$. Now Lemma 8.0.6 guarantees that the continuous map $f : U \to H$ can be extended to a continuous homomorphism $f : \mathbb{R}^n \to H$.

Now assume that $q$ is open. It suffices to show that the homomorphism $f$ defined above is open. To this end it suffices to check that for every neighborhood $U$ of 0 in $\mathbb{R}^n$ contained in $U_1$ also $f(U) \in \mathcal{V}_H(0)$. Since $\xi$ is a homeomorphism and $q(U) \in \mathcal{V}_{H/D}(0)$ is contained in $q(U_1) \subseteq W_1$, $\xi(q(U)) = f(U) \in \mathcal{V}_H(0)$. $\square$

## 8.1 The closed subgroups of $\mathbb{R}^n$

Our main goal here is to describe the closed subgroup of $\mathbb{R}^n$. In the next example we outline two important instances of such subgroups.

**Example 8.1.1.** Let $n, m \in \mathbb{N}_+$ and let $v_1, \ldots, v_m$ be linearly independent vectors in $\mathbb{R}^n$.

(a) The linear subspace $V = \mathbb{R}v_1 + \ldots + \mathbb{R}v_m \cong \mathbb{R}^m$ spanned by $v_1, \ldots, v_m$ is a closed subgroup of $\mathbb{R}^n$.

(b) The subgroup $D = \langle v_1 \rangle + \ldots + \langle v_m \rangle = \langle v_1, \ldots, v_m \rangle \cong \mathbb{Z}^m$ generated by $v_1, \ldots, v_m$ is a discrete (hence, closed) subgroup of $\mathbb{R}^n$. A subgroup $D$ of $\mathbb{R}^n$ of this form we call a *lattice* in $\mathbb{R}^n$. Clearly, a lattice $D$ in $\mathbb{R}^n$ is free with $r_0(D) = m$.

We prove that every closed subgroup of $\mathbb{R}^n$ is topologically isomorphic to a product $V \times D$ of subspace $V \cong \mathbb{R}^s$ and a lattice $D \cong \mathbb{Z}^m$, with $s, m \in \mathbb{N}$ and $s + m \leq n$. More precisely:

**Theorem 8.1.2.** *Let $n \in \mathbb{N}_+$ and let $H$ be a closed subgroup of $\mathbb{R}^n$. Then there exist $s + m \leq n$ linearly independent vectors $v_1, \ldots, v_s, v_{s+1}, \ldots, v_{s+m}$ such that $H = V + D$, where $V \cong \mathbb{R}^s$ is the vector subspace spanned by $v_1, \ldots, v_s$ and $D = \langle v_{s+1}, \ldots, v_{s+m} \rangle \cong \mathbb{Z}^m$.*

We give two proof of this theorem. The first one is relatively short and proceeds by induction. The second proof splits in several steps. Before starting the proofs, we note that the dichotomy imposed by Example 8.1.1 is reflected in the following *topological* dichotomy resulting from the theorem:

- the closed *connected* subgroups of $\mathbb{R}^n$ are always subspaces, isomorphic to some $\mathbb{R}^s$ with $s \leq n$;
- the closed *totally disconnected* subgroups $D$ of $\mathbb{R}^n$ are lattices in $\mathbb{R}^n$, so must be free and have free-rank $r_0(D) \leq n$; in particular they are discrete.

In the general case, for every closed subgroup $H$ of $\mathbb{R}^n$ the connected component $c(H)$ is open in $H$ and isomorphic to $\mathbb{R}^s$ for some $s \leq n$. Therefore, by the divisibility of $\mathbb{R}^s$ one can write $H = c(H) \times D$ for some discrete subgroup $D$ of $H$ (see Corollary 2.1.13). Necessarily $r_0(D) \leq n - s$ as $c(H) \cong \mathbb{R}^s$ contains a discrete subgroup $D_1$ of rank $s$, so that $D_1 \times D$ will be a discrete subgroup of $\mathbb{R}^n$.

The next lemma prepares the inductive step in the proof of Theorem 8.1.2.

**Lemma 8.1.3.** *If a closed subgroup $H$ of $\mathbb{R}^n$ and a one-dimentional subspace $L \cong \mathbb{R}$ of $\mathbb{R}^n$ satisfy $H \cap L \neq 0$, then the canonical map $p : \mathbb{R}^n \to \mathbb{R}^n/L$ sends $H$ to a closed subgroup $p(H)$ of $\mathbb{R}^n/L$.*

*Proof.* If $n = 1$, then $\mathbb{R}^n/L$ is trivial, so we are done. Assume $n > 1$. Consider the non-zero closed subgroup $H_1 = H \cap L$ of $L \cong \mathbb{R}$. If $H_1 = L$, i.e., if $L \subseteq H$, then the assertion follows from Theorem 3.6.7 (b). Now assume that $H_1 \neq L \cong \mathbb{R}$. Then $H_1 = \langle a \rangle$ is cyclic by Exercise 4.1.9. Making use of an appropriate linear automorphism $\alpha$ of $\mathbb{R}^n$ and replacing $H$ by $\alpha(H)$, we may assume without loss of generality that $L = \mathbb{R} \times \{0\}^{n-1}$ and $a = e_1$, i.e., $H_1 = \mathbb{Z} \times \{0\}^{n-1}$. Consider the canonical map $\pi : \mathbb{R}^n \to \mathbb{R}^n/H_1$. Since $H$ is a closed subgroup of $\mathbb{R}^n$ containing $H_1$, its image $\pi(H)$ is a closed subgroup of $\mathbb{R}^n/H_1 \cong \mathbb{T} \times \mathbb{R}^{n-1}$ by Theorem 3.6.7 (b). Next we observe that the projection $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the composition of $\pi$ and the canonical map $\rho : \mathbb{R}^n/H_1 \to \mathbb{R}^{n-1}$. Since $\ker \rho = L/H_1 \cong \mathbb{T}$ is compact and $\pi(H)$ is closed in $\mathbb{R}^n/H_1$, $p(H) = \rho(\pi(H))$ is a closed subgroup of $\mathbb{R}^{n-1}$ by Lemma 7.2.3. □

We shall see in §8.2 that "closed" can be replaced by "discrete" in this lemma.

**Example 8.1.4.** Let us see that the hypothesis $H \cap L \neq 0$ is relevant . Indeed, take the discrete (hence, closed) subgroup $H = \mathbb{Z}^2$ of $\mathbb{R}^2$ and the line $L = v\mathbb{R}$ in $\mathbb{R}^2$, where $v = (1, \sqrt{2})$. Then $L \cap H = 0$, while $\mathbb{R}^2/L \cong \mathbb{R}$, so by Exercise 4.1.9, the non-cyclic image $p(H) \cong \mathbb{Z}^2$ of $H$ in $\mathbb{R}$ is dense, so fails to be closed.

**First proof of of Theorem 8.1.2.** The case $n = 1$ is Exercise 4.1.9. Assume $n > 1$. If $H$ is a subspace of $\mathbb{R}^n$, then $H = V$ and we are done. Assume that $H$ is not a subspace. Then there exits a non-zero $h \in H$ such that the line $L = \mathbb{R}h$ through $h$ is not contained in $H$. Thus the closed non-zero subgroup $H_1 = H \cap L$ of $L \cong \mathbb{R}$ is proper, hence cyclic. Let $H_1 = \langle a \rangle$. By Lemma 8.1.3 the projection $p : \mathbb{R}^n \to \mathbb{R}^n/L \cong \mathbb{R}^{n-1}$ sends $H$ to a closed subgroup $p(H)$ of $\mathbb{R}^{n-1}$. By the inductive hypothesis there exist naturals $s, m$ with $s + m < n$ and $s + m$ linearly independent vectors $v'_1, \ldots, v'_s, v'_{s+1}, \ldots, v'_{s+m}$ of $\mathbb{R}^n/L \cong \mathbb{R}^{n-1}$ such that, with $V' = \mathbb{R}v'_1 + \ldots + \mathbb{R}v'_s \cong \mathbb{R}^s$ and $D' = \langle v'_{s+1}, \ldots, v'_{s+m} \rangle \cong \mathbb{Z}^m$, one has $p(H) = V' \times D'$. Since both $H$ and $p(H)$ are LCA and $H$ is also $\sigma$-compact (as a closed subgroup of $\mathbb{R}^n$), it follows from Theorem 7.3.1 that the continuous surjective homomorphism $p : H \to p(H)$ is open, i.e., $p(H) \cong H/H_1$. Since $H_1$ is discrete, we can apply Lemma 8.0.7 to obtain a continuous open homomorphism $f : V' \to H$ such that $p \circ f = j$ is the inclusion of $V' \cong \mathbb{R}^s$ in $p(H)$. Let $V = f(V')$, then $f : V' \to V$ will be a topological isomorphism. Let $v_i = f(v'_i)$ for $i = 1, 2, \ldots, s$. For every $j = 1, 2, \ldots, m$ find $v_{s+j} \in H$ such that $p(v_{s+j}) = v'_{s+j}$. Let $v_0 = a$. Since the projection $p : \mathbb{R}^n \to \mathbb{R}^n/L$ is $\mathbb{R}$-linear, the vectors $v_0, v_1, \ldots, v_s, v_{s+1}, \ldots, v_{s+m}$ are linearly independent, so $D = \langle v_0, v_{s+1}, \ldots, v_{s+m} \rangle$ is a lattice in $\mathbb{R}^n$. From $p(H) = V' \times D'$ and $\ker p = \langle a \rangle$, we deduce that $H = V \times D \cong \mathbb{R}^s \oplus \mathbb{Z}^{m+1}$. □

**Corollary 8.1.5.** *For every $n \in \mathbb{N}_+$ the only compact subgroup of $\mathbb{R}^n$ is the zero subgroup.*

*Proof.* Let $K$ be a compact subgroup of $\mathbb{R}^n$. So $K = V \times D$, where $V$ is a subspace of $\mathbb{R}^n$ and $D$ is a lattice in $\mathbb{R}^n$. The compactness of $K$ yield that both $V$ and $V$ are compact. Since $\mathbb{R}^s$ is compact only for $s = 0$ and $D \cong \mathbb{Z}^m$ is compact only for $m = 0$, we are done. □

## 8.2 A second proof of Theorem 8.1.2

This proof of Theorem 8.1.2 makes no recourse to induction, so from a certain point of view gives a better insight of the argument. By Exercise 4.1.9 every discrete subgroup of $\mathbb{R}$ is cyclic. The first part of this (second) proof consists in appropriately extending this property to discrete subgroups of $\mathbb{R}^n$ for every $n \in \mathbb{N}_+$ (see Proposition 8.2.2). The first step is to prove directly that the free-rank $r_0(H)$ of a discrete subgroup $H$ of $\mathbb{R}^n$ coincides with the dimension of the subspace of $\mathbb{R}^n$ generated by $H$.

**Lemma 8.2.1.** *Let $H$ be a discrete subgroup of $\mathbb{R}^n$. If the elements $v_1, \ldots, v_m$ of $H$ are $\mathbb{Q}$-linearly independent, then they are also $\mathbb{R}$-linearly indipendent.*

*Proof.* Let $V \cong \mathbb{R}^k$ be the subspace of $\mathbb{R}^n$ generated by $H$. We can assume wlog that $V = \mathbb{R}^n$, i.e., $k = n$. Hence we have to prove that the free-rank $m = r_0(H)$ of $H$ coincides with $n$. Obviously $m \geq n$. We need to prove that $m \leq n$. Let us fix $n$ $\mathbb{R}$-linearly independent vectors $v_1, \ldots, v_n$ in $H$. It is enough to see that for every $h \in H$ the vectors $v_1, \ldots, v_n, h$ are not $\mathbb{Q}$-linearly independent. This would imply $m \leq n$. Let us note first that we can assume wlog that $H \supseteq \mathbb{Z}^n$. Indeed, as $v_1, \ldots, v_n$ are $\mathbb{R}$-linearly independent, there exists a linear isomorphism $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ with

$\alpha(v_i) = e_i$ for $i = 1, 2, \ldots, n$, where $e_1, \ldots, e_n$ is the canonical base of $\mathbb{R}^n$. Clearly, $\alpha(H)$ is still a discrete subgroup of $\mathbb{R}^n$ and the vectors $v_1, \ldots, v_n, h$ are $\mathbb{Q}$-linearly independent iff the vectors $e_1 = \alpha(v_1), \ldots, e_n = \alpha(v_n), \alpha(h)$ are. The latter fact is equivalent to $\alpha(h) \notin \mathbb{Q}^n$. Therefore, arguing for a contradiction, assume for simplicity that $H \supseteq \mathbb{Z}^n$ and there exists $h = (h_1, \ldots, h_n) \in H$ such that

$$h \notin \mathbb{Q}^n. \tag{3}$$

By the discreteness of $H$ there exists an $\varepsilon > 0$ with $\max\{|h_i| : i = 1, 2, \ldots, n\} \geq \varepsilon$ for every $0 \neq h = (h_1, \ldots, h_n) \in H$. Represent the cube $C = [0, 1]^n$ as a finite union $\bigcup_i C_i$ of cubes $C_i$ of diameter $< \varepsilon$ (e.g., take them with faces parallel to the coordinate axes, although their precise position is completely irrelevant). For a real number $r$ denote by $\{r\}$ the unique number $0 \leq x < 1$ such that $r - x \in \mathbb{Z}$. Then $(\{mv_1\}, \ldots, \{mv_n\}) \neq (\{lh_1\}, \ldots, \{lh_n\})$ for every positive $l \neq m$, since otherwise, $(m - l)h \in \mathbb{Z}^n$ with $m - l \neq 0$ in contradiction with (3). Among the infinitely many points $a_m = (\{mh_1\}, \ldots, \{mh_n\}) \in C$ there exist two $a_m \neq a_l$ belonging to the same cube $C_i$. Hence, $|\{mh_j\} - \{lh_j\}| < \varepsilon$ for every $j = 1, 2, \ldots, n$. So there exists a $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n \subseteq H$, such that $0 \neq (m-l)h - z \in H$ and $|(m-l)h_j - z_j| < \varepsilon$ for every $j = 1, 2, \ldots, n$, this contradicts the choice of $\varepsilon$. □

The aim of the next step is to see that the discrete subgroups of $\mathbb{R}^n$ are free.

**Proposition 8.2.2.** *If $H$ is a discrete subgroup of $\mathbb{R}^n$, then $H$ is free and $r(H) \leq n$.*

*Proof.* In fact, let $m = r(H)$. By the definition of $r(H)$ there exist $m$ $\mathbb{Q}$-linearly independent vectors $v_1, \ldots, v_m$ of $H$. By the previous lemma the vectors $v_1, \ldots, v_m$ are also $\mathbb{R}$-linearly independent. Hence, $m \leq n$. Let $V \cong \mathbb{R}^m$ be the subspace of $\mathbb{R}^n$ generated by $v_1, \ldots, v_m$. Obviously, $H \subseteq V$, since $H$ is contained in the $\mathbb{Q}$-subspace of $\mathbb{R}^n$ generated by the free subgroup $F = \langle v_1, \ldots, v_m \rangle$ di $H$. Since $H$ is a discrete subgroup of $V$ too, we can argue with $V$ in place of $\mathbb{R}^n$. So, we can assume wlog that $m = n$ and $V = \mathbb{R}^n$. It suffices to see that $H/F$ is finite. Then $H$ will be finitely generated and torsion-free, hence $H$ must be free.

Since the vectors $v_1, \ldots, v_n$ are linearly independent on $\mathbb{R}$ we can assume wlog that $H \supseteq \mathbb{Z}^n$. In fact, let $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ be the linear isomorphism with $\alpha(v_i) = e_i$ for $i = 1, 2, \ldots, n$, where $e_1, \ldots, e_n$ is the canonical base of $\mathbb{R}^n$. Then $\alpha(H)$ is still a discrete subgroup of $\mathbb{R}^n$, $\mathbb{Z}^n = \alpha(F) \subseteq \alpha(H)$ and $H/F$ is finite iff $\alpha(H)/\alpha(F) \cong H/F$ is finite.

In the sequel we assume $H \supseteq \mathbb{Z}^n = F$ for the sake of simplicity. To check that $H/F$ is finite consider the canonical homomorphism $q : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$. According to Theorem 3.6.7, $q$ sends the closed subgroup $H$ onto a closed (hence compact) subgroup $q(H)$ of $\mathbb{T}^n$; moreover $H = q^{-1}(q(H))$, hence the restriction of $q$ to $H$ is open and $q(H)$ is discrete. Thus $q(H) \cong H/F$ is both compact and discrete, so $q(H)$ must be finite. □

The next lemma extends Lemma 8.1.3 to the case of discrete subgroups of $\mathbb{R}^n$.

**Lemma 8.2.3.** *If for a discrete subgroup $H$ of $\mathbb{R}^n$ and a one-dimentional subspace $L \cong \mathbb{R}$ of $\mathbb{R}^n$ one has $H \cap L \neq 0$, then the canonical map $p : \mathbb{R}^n \to \mathbb{R}^n/L$ sends $H$ to a discrete subgroup $p(H)$ of $\mathbb{R}^n/L$.*

*Proof.* If $n = 1$, then $L = \mathbb{R}^n$, so this case is trivial. Assume $n > 1$ in the sequel. Since $0 \neq H_1 = H \cap L$ is a discrete subgroup of $L \cong \mathbb{R}$, we conclude that $H = \langle a \rangle$ is cylic. Making use of an appropriate linear automorphism $\alpha$ of $\mathbb{R}^n$ and replacing $H$ by $\alpha(H)$, we may assume wlog that $L = \mathbb{R} \times \{0\}^{n-1}$ and $a = e_1$. Thus, $L \cap H = \mathbb{Z} \times \{0\}^{n-1}$. For $\varepsilon > 0$ let $B_\varepsilon(0) = (-\varepsilon, \varepsilon)^n$ and $U_\varepsilon = B_\varepsilon(0) + L$. Let us prove that for some $\varepsilon > 0$ also

$$U_\varepsilon \cap H = \mathbb{Z} \times \{0\}^{n-1}, \tag{4}$$

holds true. Assume for a contradiction that $U_{1/n} \cap H \not\subseteq L$ for every $n \in \mathbb{N}$ and pick $h(x_n, y_n) \in U_{1/n}(0) \cap H$ with $y_n \neq 0$. Since $\mathbb{Z} \times \{0\}^{n-1} \subseteq H$, we can assume without loss of generality that $0 \leq x_n < 1$ for every $n$. Then there exists a converging subsequences $x_{n_k} \to z$. Hence $h_{n_k} \to (z, 0) \in H$. Since $H$ is discrete, this sequence is eventually constant, so $y_{n_k} = 0$ for all sufficienlty large $k$, a contradiction. This proves that (4) holds true for some $\varepsilon > 0$. Let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection along $L$. Then $U_\varepsilon(0) = p^{-1}(p(B_\varepsilon(0)))$, so (4) implies that $p(B_\varepsilon(0)) \cap p(H) = (0)$ in $\mathbb{R}^{n-1}$. Thus the subgroup $p(H)$ of $\mathbb{R}^{n-1}$ is discrete. □

The next exercise provides a shorter alternative proof of the first part of Theorem 8.1.2 carried out in Proposition 8.2.2, namely the description of the discrete subgroups of $\mathbb{R}^n$.

**Exercise 8.2.4.** *Prove by induction on $n$ that for every discrete (so closed) subgroup $H$ of $\mathbb{R}^n$ there exist $m \leq n$ linearly independent vectors $v_1, \ldots, v_m$ such that $H = \langle v_1, \ldots, v_m \rangle \cong \mathbb{Z}^m$.*

*Proof.* The case $n = 1$ is Exercise 4.1.9. Assume $n > 1$. Pick any $0 \neq h \in H$ and let $L$ be the line $\mathbb{R}h$ in $\mathbb{R}^n$. Since $0 \neq H_1 = H \cap L$ is a discrete subgroup of $L \cong \mathbb{R}$, we conclude that $H = \langle a \rangle$ is cylic and we can apply Lemma 8.2.3 to claim that the image $p(H)$ of $H$ along the projection $p : \mathbb{R}^n \to \mathbb{R}^n/L \cong \mathbb{R}^{n-1}$ is a discrete subgroup of $\mathbb{R}^{n-1}$. Then our inductive hypothesis yields $p(H) = \langle v_2', \dots, v_m' \rangle$ for some linearly independent vectors $v_2', \dots, v_m'$ in $\mathbb{R}^n/L$. Pick $v_i \in \mathbb{R}^n$ such that $p(v_i) = v_i'$ for $i = 2, \dots, n$. Then with $v_1 = a$ we have the desired presentation

$$H = \langle v_1, \dots, v_m \rangle = \langle v_1 \rangle \oplus \langle v_2, \dots, v_m \rangle \cong \mathbb{Z} \oplus \mathbb{Z}^{m-1} \cong \mathbb{Z}^m.$$

□

Now we pass to the case of non-discrete closed subgroups of $\mathbb{R}^n$.

**Lemma 8.2.5.** *If $H$ is a closed non-discrete subgroup of $\mathbb{R}^n$, then $H$ contains a line through the origin.*

*Proof.* Consider the subset

$$M = \{u \in \mathbb{R}^n : \|u\| = 1 \text{ and } \exists \lambda \in (0, 1) \text{ with } \lambda u \in H\}$$

of the unitary sphere $S$ in $\mathbb{R}^n$. For $u \in S$ let $N_u = \{r \in \mathbb{R} : ru \in H\}$. Then $N_u$ is a closed subgroup of $\mathbb{R}$ and $H \cap \mathbb{R}u = N_u u$. Our aim will be to find some $u \in S$ such that the whole line $\mathbb{R}u$ is contained in $H$. This will allow us to use our inductive hypothesis. Since the proper closed subgroups of $\mathbb{R}$ are cyclic (see Exercise 4.1.9), it suffices to find some $u \in S$ such that $N_u$ is not cyclic.

**Case 1.** If $M = \{u_1, \dots, u_n\}$ is finite, then there exists an index $i$ such that $\lambda u_i \in H$ for infinitely many $\lambda \in (0, 1)$. Then the closed subgroup $N_{u_i}$ cannot be cyclic, so $H$ contains to line $\mathbb{R}u_i$ and we are done.

**Case 2.** Assume $M$ is infinite. By the assumption $H$ is not discrete there exists a sequence $u_n \in M$ such that the corresponding $\lambda_n$, with $\lambda_n u_n \in H$, converge to 0. By the compactness of $S$ there exists a limit point $u_0 \in S$ for the sequence $u_n \in M$. We can assume wlog that $u_n \to u_0$. Let $\varepsilon > 0$ and let $\Delta_\varepsilon$ be the open interval $(\varepsilon, 2\varepsilon)$. As $\lambda_n \to 0$, there exists $n_0$ such that $\lambda_n < \varepsilon$ for every $n \geq n_0$. Hence for every $n \geq n_0$ there exists an appropriate $k_n \in \mathbb{N}$ with $\eta_n = k_n \lambda_n \in \Delta_\varepsilon$. Taking again a subsequence we can assume wlog that there exists some $\xi_\varepsilon \in \overline{\Delta_\varepsilon}$ such that $\eta_n \to \xi_\varepsilon$. Hence $\xi_\varepsilon u_0 = \lim_n k_n \lambda_n u_0 \in H$. This argument shows that $N_{u_0}$ contains $\xi_\varepsilon \in \overline{\Delta_\varepsilon}$ with arbitrarily small $\varepsilon$. Therefore, $N_{u_0}$ cannot be cyclic. Hence $H$ contains the line $\mathbb{R}u_0$. □

Now we are in position to prove Theorem 8.1.2.

**Proof of Theorem 8.1.2.** If $H$ is a closed subgroup of $\mathbb{R}^n$ and $V_1, V_2$ are subspaces of $\mathbb{R}^n$ contained in $H$, then also the subspace $V_1 + V_2$ of $\mathbb{R}^n$ is contained in $H$. Therefore, $H$ contains a largest subspace $\lambda(H)$ of $\mathbb{R}^n$. Since $\lambda(H)$ is a closed subgroup of $\mathbb{R}^n$ contained in $H$, the projection $p : \mathbb{R}^n \to \mathbb{R}^n/\lambda(H) \cong \mathbb{R}^k$ (where $k = n - \dim \lambda(H)$) sends $H$ to a closed subgroup $p(H)$ of $\mathbb{R}^k$ by Theorem 3.6.7 (b). Moreover, $p(H)$ contains no lines $L$ since such a line $L$ would produce a subspace $p^{-1}(L)$ of $\mathbb{R}^n$ contained in $H$ and properly containing $\lambda(H)$. By the above lemma, $p(H)$ is discrete, i.e., $\lambda(H)$ is an open subgroup of $H$. Since $\lambda(H)$ is divisible, it splits, so $H = \lambda(H) \times H'$, where $H'$ is a discrete subgroup of $H$ (and of $\mathbb{R}^n$). By Proposition 8.2.2, $H' \cong \mathbb{Z}^m$. This proves Theorem 8.1.2.

## 8.3 Elementary LCA groups and Kronecker's theorem

**Definition 8.3.1.** An abelian topological group is

(a) *elementary compact* if it is topologically isomorphic to $\mathbb{T}^s \times F$, where $n$ is a positive integer and $F$ is a finite abelian group.

(b) *elementary locally compact* if it is topologically isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{T}^s \times F$, where $n, m, s$ are positive integers and $F$ is a finite abelian group.

Here we study properties of the elementary (locally) compact abelian groups. In particular, we see that the class of elementary locally compact abelian groups is closed under taking quotient, closed subgroups and finite products (see Theorem 8.1.2 and Corollary 8.3.3).

The next corollary describes the quotients of $\mathbb{R}^n$.

**Corollary 8.3.2.** *A quotient of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^k \times \mathbb{T}^m$, where $k + m \leq n$. In particular, a compact quotient of $\mathbb{R}^n$ is isomorphic to $\mathbb{T}^m$ for some $m \leq n$.*

*Proof.* Let $H$ be a closed subgroup of $\mathbb{R}^n$. Then $H = V + D$, where $V, D$ are as in Theorem 8.1.2. If $s = \dim V$ and $m = r_0(D)$, then $s + m \leq n$. Let $V_1$ be the linear subspace of $\mathbb{R}^n$ spanned by $D$. Pick a complementing subspace $V_2$ for the subspace $V + V_1$ and let $k = n - (s + m)$. Then $\mathbb{R}^n = V + V_1 + V_2$ is a factorization in direct product. Therefore $\mathbb{R}^n/H \cong (V_1/D) \times V_2$. Since $\dim V_1 = r_0(D) = m$, one has $V_1/D \cong \mathbb{T}^m$. Therefore, $\mathbb{R}^n/H \cong \mathbb{T}^m \times \mathbb{R}^k$. □

Now we prove that the closed subgroups of the finite-dimensional tori $\mathbb{T}^n$ are elementary compact abelian groups.

**Corollary 8.3.3.** *Let $C$ be a closed subgroup of $\mathbb{T}^n$. Then $C$ is isomorphic to $\mathbb{T}^s \times F$ where $s \leq n$ and $F$ is a finite abelian group.*

*Proof.* Let $q : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the canonical projection. If $C$ is a closed subgroup of $\mathbb{T}^n$, then $H = q^{-1}(C)$ is a closed subgroup of $\mathbb{R}^n$ that contains $\mathbb{Z}^n = \ker q$. Hence $H$ is a direct product $H = V + D$ with $V \cong \mathbb{R}^s$ and $D \cong \mathbb{Z}^m$, where $s$ and $m$ satisfy $s+m = n$ as $\mathbb{Z}^n \leq H$. Since the restriction of $q$ to $H$ is open by Theorem 3.6.7, we conclude that the restriction of $q$ to $V$ is open as far as $V$ is open in $H$. Hence $q\restriction_V : V \to q(V)$ is an open surjective homomorphism and the subgroup $q(V)$ is open in $C$. As $C$ is compact (as a closed subgroup of $\mathbb{T}^s \times F$), $q(V)$ has finite index in $C$. On the other hand, $q(V)$ is also divisible (being a quotient of the divisible group $V$), we can write $C = q(V) \times F$, where the subgroup $F$ is finite. On the other hand, as a compact quotient of $V \cong \mathbb{R}^s$ the group $q(V)$ is isomorphic to $\mathbb{T}^s$ by Corollary 8.3.2. Therefore, $C \cong \mathbb{T}^s \times F$. □

Using the above corollary one can prove that the closed subgroups and the quotients of the elementary compact abelian groups are still elementary compact abelian groups:

**Exercise 8.3.4.** *Prove that the class $\mathcal{EC}$ of elementary compact abelian groups is stable under taking closed subgroups, quotients and finite direct products.*

**Exercise 8.3.5.** *Prove that every elementary locally compact abelian groups is a quotient of an elementary locally compact abelian group of the form $\mathbb{R}^n \times \mathbb{Z}^m$.*

Our next aim is the description of the closure of an arbitrary subgroup of $\mathbb{R}^n$. To this end we shall exploit the scalar product $(x|y)$ of two vectors $x, y \in \mathbb{R}^n$.

Recall that every base $v_1, \dots, v_n$ di $\mathbb{R}^n$ admits a dual base $v_1', \dots, v_n'$ defined by the relations $(v_i|v_j') = \delta_{ij}$. For a subset $X$ of $\mathbb{R}^n$ define the *orthogonal subspace* $X^o$ setting

$$X^o := \{u \in \mathbb{R}^n : (\forall x \in X)(x|u) = 0\}.$$

If $X = \{v\} \neq \{0\}$ is a singleton, then $X^o$ is the hyperspace orthogonal to $v$, so in general $X^o$ is always a subspace, being an intersection of hyperspaces. If $V$ is a subspace of $V$, then $V^o$ is the orthogonal complement of $V$, so $\mathbb{R}^n = V \times V^o$.

For a subgroup $H$ of $\mathbb{R}^n$ define the *associated subgroup* $H^\dagger$ setting

$$H^\dagger := \{u \in \mathbb{R}^n : (\forall x \in H)(x|u) \in \mathbb{Z}\}.$$

Then obviously $(\mathbb{Z}^n)^\dagger = \mathbb{Z}^n$.

**Lemma 8.3.6.** *Let $H$ be a subgroup di $\mathbb{R}^n$. Then:*

1. *$H^\dagger$ is a closed subgroup of $\mathbb{R}^n$ and the correspondence $H \mapsto H^\dagger$ is decreasing;*
2. *$(\overline{H})^\dagger = H^\dagger$.*
3. *$H^o \subseteq H^\dagger$, equality holds if $H$ is a subspace.*
4. *for subgroup $H$ and $H_1$ of $\mathbb{R}^n$ one has $(H + H_1)^\dagger = H^\dagger \cap H_1^\dagger$.*

*Proof.* The map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $(x, y) \mapsto (x|u)$ is continuous.

(1) Let $q : \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the canonical homomorphism.Then for every fixed $a \in \mathbb{R}^n$ the assignment $x \mapsto (a|x) \mapsto f((a|x))$ is a continuous homomorphism $\chi_a : \mathbb{R}^n \to \mathbb{T}$. Hence the set $\chi_h^{-1}(0) = \{u \in \mathbb{R}^n : (\forall h \in H)(h|u) \in \mathbb{Z}\}$ is closed in $\mathbb{R}^n$. Therefore $H^\dagger = \bigcap_{h\in H} \chi_h^{-1}(0)$ is closed. The same equality proves that the correspondence $H \mapsto H^*$ is decreasing.

(2) From the second part of (a) one has $(\overline{H})^\dagger \subseteq H^\dagger$. Suppose that $u \in H^\dagger$ e $x \in \overline{H}$. By the continuity of the map $\chi_x(u) = \chi_u(x)$, as a function of $x$, one can deduce that $\chi_x(u) \in \mathbb{Z}$, being $\chi_u(h) \in \mathbb{Z}$ for every $h \in H$.

(3) The inclusion is obvious. Assume that $H$ is a subspace and $y \in H^\dagger$. To prove that $y \in H^o$ take any $x \in H$ and assume that $m = (x|y) \neq 0$. Then $z = \frac{1}{2m} \in H$ and $(z|y) = \frac{1}{2} \neq \mathbb{Z}$, a contradiction.

(4) The inclusion $(H + H_1)^\dagger \subseteq H^\dagger \cap H_1^\dagger$ follows from item (a). On the other hand, if $x \in H^\dagger \cap H_1^\dagger$, then obviously $x \in (H + H_1)^\dagger$. □

We study in the sequel the subgroup $H^\dagger$ associated to a closed subgroup $H$ of $\mathbb{R}^n$. According to Theorem 8.1.2 there exist a base $v_1, \dots, v_n$ of $\mathbb{R}^n$ and $k \leq n$, such that $H = V \oplus L$ where $V$ is the linear subspace generated by $v_1, \dots, v_s$ for some $0 \leq s \leq k$ and $L = \langle v_{s+1}, \dots, v_k\rangle$. Let $v_1', \dots, v_n'$ be the dual base of $v_1, \dots, v_n$.

**Lemma 8.3.7.** *In the above notation the subgroup $H^\dagger$ coincides with $\langle v_{s+1}', \dots, v_k'\rangle + W$, where $W$ is the linear subspace generated by $v_{k+1}', \dots, v_n'$.*

*Proof.* Let $V'$ be the linear subspace generated by $v'_1, \dots, v'_s$, $V''$ the linear subspace generated by $v'_{s+1}, \dots, v'_k$ and $L' = \langle v'_{s+1}, \dots, v'_k \rangle$. Then $L^\dagger = V' + L' + W$, while $V^\dagger = V'' + W$. Hence $H^\dagger = L^\dagger \cap V^\dagger$ by Lemma 8.3.6. Hence, $H^\dagger = L' + W$. □

**Corollario 8.3.8.** *$\overline{H} = (H^\dagger)^\dagger$ for every subgroup $H$ of $\mathbb{R}^n$.*

*Proof.* If $H$ is closed of the form $V + L$ in the notation of the previous lemma, then $H^\dagger = L' + W$ with $v'_1, \dots, v'_n$, $L'$ and $W$ defined as above. Now $H^\dagger = L' + W$ is a closed subgroup of $\mathbb{R}^n$ by Lemma 8.3.6 and $v_1, \dots, v_n$ is a dual base of $v'_1, \dots, v'_n$. Therefore, $H = V + L$ coincides with $(H^\dagger)^\dagger$. □

The next proposition is a particular case of the well-known Kronecker's theorem.

**Proposition 8.3.9.** *Let $v_1, \dots, v_n \in \mathbb{R}$. Then for $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ the subgroup $\langle v \rangle + \mathbb{Z}^n$ of $\mathbb{R}^n$ is dense iff $v_0 = 1, v_1, \dots, v_n \in \mathbb{R}$ are linearly independent as elements of the vector space $\mathbb{R}$ over $\mathbb{Q}$.*

*Proof.* Assume $v_0 = 1, v_1, \dots, v_n \in \mathbb{R}$ are linearly independent and let $H = \langle v \rangle + \mathbb{Z}^n$. Then $H^\dagger \subseteq \mathbb{Z}^n = (\mathbb{Z}^n)^\dagger$. It is easy to see now that some $z \in \mathbb{Z}^n$ belongs to $(\langle v \rangle)^\dagger$ iff $z = 0$. This proves that $H^\dagger = 0$. Consequently $H$ is dense in $\mathbb{R}^n$ by Corollary 8.3.8. If $\sum_{i=0}^n k_i v_i = 0$ is a non-trivial linear combination with $k_i \in \mathbb{Z}$, then the vector $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ is non-zero and obviously $k \in H^\dagger$. Thus $H^\dagger \neq 0$, hence $H$ is not dense. □

**Theorem 8.3.10.** *$\mathbb{T}^{\mathfrak{c}}$ is monothetic.*

*Proof.* Let $B$ be a Hamel base of $\mathbb{R}$ on $\mathbb{Q}$ that contains 1 and let $B_0 = B \setminus \{1\}$. Applying the previous proposition one can see that the element $x = (x_b)_{b \in B_0} \in \mathbb{T}^{B_0}$, defined by $x_b = b + \mathbb{Z} \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, is a generator of the group $\mathbb{T}^{B_0}$. To conclude note that $|B_0| = |\mathbb{R}| = \mathfrak{c}$. □

**Exercise 8.3.11.** *Determine for which of the following possible choices of the vector $v \in \mathbb{R}^4$*

$$(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}),\ (\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}),\ (\log 2, \log 3, \log 5, \log 6),$$

$$(\log 2, \log 3, \log 5, \log 7),\ (\log 3, \log 5, \log 7, \log 9)\ \textit{and}\ (\log 5, \log 7, \log 9, \log 11)$$

*the subgroup $\langle v \rangle + \mathbb{Z}^4$ of $\mathbb{R}^4$ is dense.*

**Exercise 8.3.12.** *Let $V$ be a hyperplain in $\mathbb{R}^n$ determined by the equation $\sum_{i=1}^n a_i x_i = 0$ such that there exists at least one coefficient $a_i = 1$. Then the subgroup $H = V + \mathbb{Z}^n$ of $\mathbb{R}^n$ is not dense iff all the coefficients $a_i$ are rational.*

(Hint. We can assume wlog that $i = n$. Suppose that $H$ is not dense in $\mathbb{R}^n$. Then $H^\dagger \neq 0$ by Corollary 8.3.8. Let $0 \neq z \in H^\dagger$. Since $\mathbb{Z}^n \leq H$, one has $H^\dagger \leq \mathbb{Z}^n = (\mathbb{Z}^n)^\dagger$, so $z \in \mathbb{Z}^n$. If $j < n$, then $a_j \in \mathbb{Q}$ as $v = (0, \dots, 0, 1, 0, \dots, 0, -a_j) \in V$.)

**Exercise 8.3.13.** *(a) Prove that a subgroup $H$ of $\mathbb{T}$ is dense iff $H$ is infinite.*

*(b) Determine the minimal (w.r.t. inclusion) dense subgroups $\mathbb{T}$.*

*(c)$^*$ Determine the minimal (w.r.t. inclusion) dense subgroups $\mathbb{T}^2$.*

# 9 Subgroups of the compact groups

For a subset $E$ of an abelian group $G$ we set $E_{(2)} = E - E$, $E_{(4)} = E - E + E - E$, $E_{(6)} = E - E + E - E + E - E$ and so on.

## 9.1 Big subsets of groups

A subset $X$ of an abelian group $(G, +)$ is *big*[8] if there exists a finite subset $F$ of $G$ such that $G = F + X$. Obviously, every non-empty set of a finite group is big; on the other hand, every big set in an infinite group is necessarily infinite.

**Example 9.1.1.** Let $B$ be an infinite subset of $\mathbb{Z}$. Show that $B$ is big iff the following two conditions hold:

(a) $B$ is unbounded from above and from below;

(b) if $B = \{b_n\}_{n=-\infty}^{\infty}$ is a one-to-one monotone enumeration of $B$ then the differences $b_{n+1} - b_n$ are bounded.

**Lemma 9.1.2.** *(a) Assume $B_\nu$ is a big set of the abelian group $G_\nu$ for $\nu = 1, 2, \dots, n$. Then $B_1 \times \dots \times B_n$ is a big set of $G_1 \times \dots \times G_n$.*

[8] Some authors use also the terminology *large*, *relatively dense*, or *syndetically dense*.

*(b) Let $f : G \to H$ be a surjective group homomorphism. Then:*

*($b_1$) if $B$ is a big subset of $H$, then $f^{-1}(B)$ is a big subset of $G$.*

*($b_2$) if $B'$ is a big subset of $G$, then $f(B')$ is a big subset of $H$.*

*Proof.* (a) and ($b_2$) follow directly from the definition

To prove ($b_1$) assume that exists a finite subset $F$ of $H$ such that $F + B = H$. Let $F'$ be a finite subset of $G$ such that $f(F') = F$. Then $G = F' + f^{-1}(B)$. Indeed, if $x \in G$, then there exists $a \in F$ such that $f(x) \in a + B$. Pick an $a' \in F'$ such that $f(a') = a$. Then $f(x) \in f(a') + B$, so that $f(x - a') \in B$. Hence, $x - a' \in f^{-1}(B)$. This proves that $x \in F' + f^{-1}(B)$. □

Note that if $f$ in item (b) of 9.1.2 is not surjective, then the property may fail. The next proposition gives an easy remedy to this.

**Proposition 9.1.3.** *Let $A$ be an abelian group and let $B$ be a big subset of $A$. Then $(B - B) \cap H$ is big with respect to $H$ for every subgroup $H$ of $A$.*

*If $a \in A$ then there exists a sufficiently large positive integer $n$ such that $na \in B - B$.*

*Proof.* There exists a finite subset $F$ of $A$ such that $F + B = A$. For every $f \in F$, if $(f + B \cap H$ is not empty, choose $a_f \in (f + B) \cap H$, and if $(f + B) \cap H$ is empty, choose an arbitrary $a_f \in H$. On the other hand, for every $x \in H$ there exists $f \in F$ such that $x \in f + B$; since $a_f \in f + B$, we have $x - a_f \in B - B$ and so $H \subseteq \{a_f : f \in F\} + (B - B) \cap H$, that is $(B - B) \cap H$ is big in $H$.

For the last assertion it suffices to take $H = \langle a \rangle$. If $H$ is finite, then there is nothing to prove as $0 \in B - B$. Otherwise $H \cong \mathbb{Z}$ so the first item of Example 9.1.1 applies. □

Combining Proposition 9.1.3 with item (b) of Lemma 9.1.2 we get:

**Corollary 9.1.4.** *For every group homomorphism $f : G \to H$ and every big subset $B$ of $H$, the subset $f^{-1}(B - B)$ of $G$ is a big.*

**Definition 9.1.5.** Call a subset $S$ of an infinite abelian group $G$ *small* if there exist (necessarily distinct) elements $g_1, g_2, \ldots, g_n, \ldots$ of $G$ such that $(g_n + S) \cap (g_m + S) = \emptyset$ whenever $m \neq n$.

**Lemma 9.1.6.** *Let $G$ be an infinite abelian group.*

*(a) Show that every finite subset of $G$ is small.*

*(b) Show that a subset $S$ of $G$ such that $S - S$ is not big is necessarily small.*

*(c) Show that the group $\mathbb{Z}$ is not a finite union of small sets.*

*Proof.* (a) is obvious.

(b) Build the sequence $(g_n)$ by induction, using the fact that at each stage $G \neq \bigcup g_n + S - S$ since $S - S$ is not big.

(c) So the next exercise. □

**Exercise 9.1.7.** * *Show that no infinite abelian group $G$ is a finite union of small sets.*

(Hint. Use a finitely additive invariant (Banach) measure[9] on $G$. For an elementary proof (due to U. Zannier), see [57, Exerc. 1.6.20].)

One can extend the notions of big and small for non-abelian groups as well (see the next definition), but then both versions, *left large* and *right large*, need not coincide. This creates some technical difficulties that we prefer to avoid since the second part of the next section is relevant only for abelian groups. The first half, including the characterization 9.2.5, remains valid in the non-abelian case as well (since, fortunately, the "left" and "right" versions of total boundedness coincide, see Exercise 9.2.2(b)).

**Definition 9.1.8.** Call a subset $B$ and a group $(G, \cdot)$:

(a) *left* (*right*) *big* if there exists a finite set $F \subseteq G$ such that $FB = G$ (resp., $BF = G$);

(b) *left* (*right*) *small* if there exist (necessarily distinct) elements $g_1, g_2, \ldots, g_n, \ldots$ of $G$ such that $g_nS \cap g_mS = \emptyset$ whenever $m \neq n$.

[9] This is finitely additive measure $m$ defined on the power-set of $G$, i.e., every subset is measurable and $m(G) = 1$. The existence of such a measure on the abelian groups can be proved using Hahn-Banach's theorem. Some non-abelian groups do not admit such measures (this is related to the Banach-Tarski paradox).

It is clear, that a subset $B$ is left big iff the subset $B^{-1}$ is right big. In the sequel we use to call simply big the sets that are simultaneously left and right big.

**Exercise 9.1.9.** *Prove that*

*(a) if $B_\nu$ is a left (right) big set of the group $G_\nu$ for $\nu = 1, 2, \ldots, n$, then $B_1 \times \ldots \times B_n$ is a left (right) big set of $G_1 \times \ldots \times G_n$.*

*(b) if $f : G \to H$ is a surjective group homomorphism, and*

*($b_1$) if $B$ is a left (right) big subset of $H$, then $f^{-1}(B)$ is a left (right) big subset of $G$;*

*($b_2$) if $B'$ is a left (right) big subset of $G$, then $f(B')$ is a left (right) big subset of $H$;*

*(c) if $B$ is a left (right) big subset of a group $G$, then $B^{-1}B \cap H$ (resp., $BB^{-1} \cap H$) is left (resp., right) big with respect to $H$ for every subgroup $H$ of $G$.*

*(d) Show that for an infinite group $G$ and a subgroup $H$ of $G$ the following are equivalent:*

*($d_1$) $H$ has infinite index;*

*($d_2$) $H$ is not left big;*

*($d_3$) $H$ is not right big;*

*($d_4$) $H$ is left small;*

*($d_5$) $H$ is right small.*

(Hint. (c) If $B$ is a left big subset of a group $G$, then there exists a finite subset $F$ of $G$ such that $FB = G$. For every $f \in F$, if $fB \cap H \neq \emptyset$, choose $a_f \in fB \cap H$, and if $fB \cap H = \emptyset$, choose an arbitrary $a_f \in H$. On the other hand, for every $h \in H$ there exists $f \in F$ such that $h \in fB$; since $a_f \in fB$, we have $a_f^{-1}h \in B^{-1}B$ and so $H \subseteq \{a_f : f \in F\}(B^{-1}B \cap H)$, that is $B^{-1}B \cap H$ is left big in $H$.)

**Exercise 9.1.10.** * *Every infinite abelian group has a small set of generators.*

This can be extended to arbitrary groups [58]. One can find in the literature also different (weaker) forms of smallness ([5, 14]).

## 9.2 Precompact groups

### 9.2.1 Totally bounded and precompact groups

**Definition 9.2.1.** A topological group $G$ is *totally bounded* if every open non-empty subset $U$ of $G$ is left big. A Hausdorff totally bounded group will be called *precompact*.

Clearly, compact groups are precompact. Let us underline the fact that the notions of total boundedness and precompactness defined by using *left* big sets is only apparently asymmetric. Indeed, a topological group $G$ is totally bounded iff every open non-empty subset $U$ of $G$ is right big. (Take any $V \in \mathcal{V}(e_G)$ such that $V^{-1} \subseteq U$. Since $V$ must be left big, $V^{-1}$ is right big, so $U$ is right big as well.)

**Exercise 9.2.2.** *Let $G$ be topological group. Prove that $G$ is totally bounded iff $G/\overline{\{e_G\}}$ is totally bounded (i.e., $G/\overline{\{e_G\}}$ is precompact).*

(Hint. Use Exercise 9.1.9, as well as the fact that $G/\overline{\{e_G\}}$ carries the initial topology w.r.t. the quotient map $G \to G/\overline{\{e_G\}}$.)

For the nice connection between totally boundedness and precompactness from this exercise we shall often prove a property for on of this property and this will easily imply that the properties holds (with very few exceptions) also for the other one.

**Lemma 9.2.3.** *If $G$ is a totally bounded group, then for every $U \in \mathcal{V}(e_G)$ there exists a $V \in \mathcal{V}(e_G)$ such that $g^{-1}Vg \subseteq U$ for all $g \in G$.*

*Proof.* Pick a symmetric $W \in \mathcal{V}(e_G)$ satisfying $W^3 \subseteq U$. Then $G = FW$ for some finite $F$ set $F$ in $G$. For every $a \in F$ pick a $V_a \in \mathcal{V}(e_G)$ such that $aV_aa^{-1} \subseteq W$ and let $V = \bigcap_{a\in F} V_a$. Then $g^{-1}Vg \subseteq U$ for every $g \in G$. Indeed, assume $g \in aW$ for some $a \in F$. Then $g^{-1} = w^{-1}a^{-1}$, so

$$g^{-1}Vg = w^{-1}a^{-1}Vaw \subseteq w^{-1}a^{-1}V_aaw \subseteq w^{-1}Ww \subseteq U.$$

□

Here comes the most important fact on precompact groups that we prove by means of the properties extablished in of Exercise 9.1.9.

**Corollary 9.2.4.** *Subgroups of precompact groups are precompact. In particular, all subgroups of compact groups are precompact.*

*Proof.* Let $H$ be a subgroup of the precompact group $G$. If $U$ is a neighborhood of $e_G$ in $H$, one can choose a neighborhood $W$ of 0 in $G$ such that $U = G \cap W$. Pick a neighborhood $V$ of $e_G$ in $G$ such that $V^{-1}V \subseteq W$. Then $V^{-1}V \cap G \subseteq W \cap G = U$ is big in $H$ by Exercise 9.1.9. Thus $U$ is big in $H$. □

One can show that the precompact groups are precisely the subgroups of the compact groups. This requires two steps as the next theorem shows:

**Theorem 9.2.5.** *(a) A group having a dense precompact subgroup is necessarily precompact.*

*(b) The compact groups are precisely the complete precompact groups.*

*Proof.* (a) Indeed, assume that $H$ is a dense precompact subgroup of a group $G$. Then for every $U \in \mathcal{V}_G(e_G)$ choose an open $V \in \mathcal{V}_G(0)$ with $VV \subseteq U$. By the precompactness of $H$ there exists a finite set $F \subseteq H$ such that $H = F(V \cap H)$. Then

$$G = HV \subseteq F(V \cap H)V \subseteq FVV \subseteq FU.$$

(b) Compact groups are complete and precompact. To prove the other implication take a complete precompact group $G$. To prove that $G$ is compact it sufficies to prove that every ultrafilter on $G$ converges. Assume $\mathcal{U}$ is such an ultrafilter. We show first that it is a Cauchy filter. Indeed, if $U \in \mathcal{V}_G(e_G)$, then $U$ is a big set of $G$ so there exists $g_1, g_2, \ldots, g_n \in G$ such that $G = \bigcup_{i=1}^n g_iU$. Since $\mathcal{U}$ is an ultrafilter, $g_iU \in \mathcal{U}$ for some $i$. Hence $\mathcal{U}$ is a Cauchy filter. According to Exercise 6.2.13 $\mathcal{U}$ converges. □

In this way we have described the precompact groups internally (as the Hausdorff topological groups having big non-empty open sets), or externally (as the subgroups of the compact groups).

**Lemma 9.2.6.** *For a topological group $G$ the following are equivalent:*

*(a) $G$ is not precompact;*

*(b) $H$ has a left small non-empty open set.*

*(c) $H$ has a right small non-empty open set.*

*Proof.* (a) → (b) If $U$ is a neighborhood of 0 that is not left big, choose a neighborhood $V$ of 0 such that $V^{-1}V \subseteq U$. Then $V$ is left small by (the obvious non-abelian version of) item (b) of Lemma 9.1.6. Similarly, (a) → (c). Since a left (right) small set is not left (resp., right) big, both (b) and (c) trivially imply (a). □

**Lemma 9.2.7.** *Let $G$ be a locally compact monothetic group. Then $G$ is either compact or is discrete.*

*Proof.* If $G$ is finite, then $G$ is both compact and discrete. So we can suppose without loss of generality that $\langle x\rangle \cong \mathbb{Z}$ is infinite and so also that $\mathbb{Z}$ is a subgroup of $G$.

If $G$ induces the discrete topology on $\mathbb{Z}$, then $\mathbb{Z}$ is closed and so $G = \mathbb{Z}$ is discrete.

Suppose now that $G$ induces on $\mathbb{Z}$ a non-discrete topology. Our aim is to show that it is totally bounded. Then the density of $\mathbb{Z}$ in $G$ yields that $G = \widetilde{\mathbb{Z}} = \overline{\mathbb{Z}}$ is compact, as $G$ is locally compact and so complete (see Lemma 7.2.6).

Every open subset of $G$ has no maximal element. Indeed, if $U$ is an open subset of $\mathbb{Z}$ that contains 0 and it has a maximal element, then $-U$ is an open subset of $\mathbb{Z}$ that contains 0 and it has a minimal element and $U \cap -U$ is an open finite neighborhood of 0 in $\mathbb{Z}$; thus $\mathbb{Z}$ is discrete against the assumption. Consequently every open subset of $\mathbb{Z}$ contains positive elements.

Let $U$ be a compact neighborhood of 0 in $G$ and $V$ a symmetric neighborhood of 0 in $G$ such that $V+V \subseteq U$. There exist $g_1, \dots, g_m \in G$ such that $U \subseteq \bigcup_{i=1}^m (g_i + V)$. Let $n_1, \dots, n_m \in \mathbb{Z}$ be positive integers such that $n_i \in g_i + V$ for every $i = 1, \dots, m$. Equivalently $g_i \in n_i - V = n_i + V$. Thus

$$U \subseteq \bigcup_{i=1}^m (g_i + V) \subseteq \bigcup_{i=1}^m (n_i + V + V) \subseteq \bigcup_{i=1}^m (n_i + U)$$

implies

$$U \cap \mathbb{Z} \subseteq \bigcup_{i=1}^m (n_i + U \cap \mathbb{Z}). \tag{1}$$

We show that $U \cap \mathbb{Z}$ is big with respect to $\mathbb{Z}$. Let $t \in \mathbb{Z}$; since $U \cap \mathbb{Z}$ has no maximal element, then there exists $s \in U \cap \mathbb{Z}$ such that $s \geq t$. Define $s_t = \min\{s \in U \cap \mathbb{Z} : s \geq t\}$. By (1) $s_t = n_i + u_t$ for some $i \leq m$ and $u_t \in U \cap \mathbb{Z}$. Since $n_i > 0$, then $u_t < s_t$ and so $u_t < t \leq s_t$. Now put $N = \max\{n_1, \dots, n_m\}$ and $F = \{1, \dots, N\}$. Hence $U \cap \mathbb{Z} + F = \mathbb{Z}$. This proves that the topology induced on $\mathbb{Z}$ by $G$ is totally bounded. □

**Corollary 9.2.8.** *Let $G$ be a locally compact abelian group and $x \in G$. Then $\overline{\langle x \rangle}$ is either compact or discrete.*

### 9.2.2 The Bohr compactification

**Proposition 9.2.9.** *(a) If $f : G \to H$ is a continuous surjective homomorphism of topological groups, then $H$ is totally bounded whenever $G$ is totally bounded. If $G$ carries the initial topology of $f$ and if $H$ is totally bounded, then also $G$ is totally bounded.*

*(b) If $\{G_i : i \in I\}$ is a family of topological groups, then $\prod_i G_i$ is totally bounded (precompact) iff each $G_i$ is totally bounded (precompact).*

*(c) Every group $G$ admits a finest totally bounded group topology $\mathcal{P}_G$.*

*Proof.* (a) Follows from item ($b_2$) of Lemma 9.1.2. The second assertion follows from the fact that the open sets in $G$ are preimages of the open sets on $H$, in case $G$ carries the initial topology of $f$. Now Exercise 9.1.9 applies.

(b) Follows from item (a) of Lemma 9.1.2 and the definition of the Tychonov topology.

(c) Let $\mathcal{T}(G) = \{\tau_i : i \in I\}$ be a the family of all totally bounded topologies on $G$. By Exercise 3.8.7 $(G, \sup\{\tau_i : i \in I\})$ is topologically isomorphic to the diagonal subgroup $\Delta = \{x = (x_i) \in G^I : x_i = x_j \text{ for all } i, j \in I\}$ of $\prod_{i \in I}(G_i, \tau_i)$. Hence $\sup\{\tau_i : i \in I\}$ is still totally bounded. Obviously, this is the finest totally bounded group topology on $G$. □

We shall see in Corollary 9.2.16 that if $G$ is abelian, then $\mathcal{P}_G$ is precompact (i.e., Hausdorff).

Using the same argument one can prove a version of (c) for topological groups. This will allow us to see that every topological abelian group $G$ admits a "universal" precompact continuous surjective homomorphic image $q : G \to G^+$:

**Proposition 9.2.10.** *(a) Every topological group $(G, \tau)$ admits a finest totally bounded group topology $\tau^+$ with $\tau^+ \leq \tau$.*

*(b) For every topological abelian group $(G, \tau)$ the quotient group $G^+ = G/\overline{\{e_G\}}^{\tau+}$ equipped with the quotient topology is precompact and for every continuous homomorphsm $f : G \to P$, where $P$ is a precompact group, factors through $q : G \to G^+$ ).*

*Proof.* (a) Use the argument from the proof of Proposition 9.2.9 (c).

(b) The precompactness of the quotient $G^+$ is obvious in view of Exercise 9.2.2. Let $\tau_1$ be the initial topology of $G$ w.r.t. $f : G \to P$. Then $\tau_1 \leq \tau$ and $\tau_1$ is totally bounded by Proposition 9.2.9 (a). Now item (a) implies that $\tau_1 \leq \tau^+$. Therefore $f : (G, \tau^+) \to P$ is continuous as well. Now we can factorize $f$ through the quotient map $q : G \to G^+$ according to Lemma 4.1.8. □

According to item (b) of the above proposition, the assignment $G \mapsto G^+$ is a functor from the category of all topological groups to the subcategory of all precompact groups.

**Theorem 9.2.11.** *Every topological group $G$ admits a compact group $bG$ and a continuous homomorphism $b_G : G \to bG$ of $G$, such that for every continuous homomorphism $f : G \to K$ into a compact group $K$ there exists a (unique) continuous homomorphism $f' : bG \to K$ with $f' \circ b_G = f$.*

*Proof.* Take the completion $bG$ of the group $G^+$ built in item (b) of the above proposition. Consider now a continuous homomorphism $f : G \to K$ into a compact group $K$. By the previous proposition, $f$ factorizes through $q : G \to G^+$, i.e., there exists a continuos homomorphism $h : G^+ \to K$ such that $f = h \circ q$. Since compact groups are complete, we can extend $h$ to the completion $bG$ of $G^+$. The continuous homomorphism $f' : bG \to K$ obtained in this way satisfies $f' \circ q = f$. The uniquencess of $f'$ follows from the fact that two homomorphisms $f', f'' : bG \to K$ with this property must coinside on the dense subgroup $G^+ = q(G)$ of $bG$, hence $f'' = f'$. □

The compact group $bG$ and the homomorphism $b_G : G \to bG$ from the above theorem are called *Bohr compactification* of the topological group $G$. Clearly, the assignment $G \mapsto bG$ is a functor from the category of all topological groups to the subcategory of all compact groups. In some sense the Bohr compactification of a topological group $G$ is the compact group $bG$ that *best approximates* $G$ in the sense of Theorem 9.2.11.

The terms Bohr topology and Bohr compactification have been chosen as a reward to Harald Bohr for his work [9] on almost periodic functions closely related to the Bohr compactification (see Theorems 11.4.7 and 11.4.9). Otherwise, Bohr compactification is due to A. Weil. More general results were obtained later by Holm [107] and Prodanov [124].

According to J. von Neumann, we adopt the following terminology concerning the injectivity of the map $b_G$:

**Definition 9.2.12.** A topological group $G$ is called

(a) *maximally almost periodic* (briefly, MAP), if $b_G$ is injective;

(b) *minimally almost periodic* , if $bG$ is a singleton.

According to Corollary 2.1.12, every discrete abelian group $G$ is MAP. We will prove that $G^+$ coincides with $G^\#$ and $bG$ coincides with the completion of $G^\#$.

The name MAP (maximally almost periodic) is justified by the notion of almost periodic *function*. For a (topological) group $G$ a complex-valued function $f \in B(G)$ is *almost periodic*[10] if the set $\{f_a : a \in G\}$ is relatively compact in the *uniform topology* of $B(G)$, where $f_a(x) = f(xa)$ for all $x \in G$ and $a \in G$ (i.e., if every sequence $(f_{a_n})$ of translations of $f$ has a subsequence that converges uniformly in $B(G)$, see also §11.4 for the case of abelian topological groups). The continuous almost periodic functions of a group $G$ are related to the Bohr compactification $bG$ of $G$ as follows. Every continuous almost periodic function $f : G \to \mathbb{C}$ admits an'extension' to $bG$ (see the proof of this fact in the abelian case in §11.4, Theorem 11.4.9). In other words, the continuous almost periodic function of $G$ are precisely the compositions of $b_G$ with continuous functions of the compact group $bG$. Therefore, the group $G$ is maximally almost periodic iff the continuous almost periodic functions of $G$ separate the points of $G$.

We give the following fact without a detailed proof:

**Fact 9.2.13.** *The set $A(G)$ of all almost periodic functions of a group $G$ is a closed $\mathbb{C}$-subalgebra of $B(G)$ closed under the complex conjugation.*

(Hint. To check that $A(G)$ is a $\mathbb{C}$-vector subspace take two almost periodic functions $f, g$ of $G$. We have to prove that $c_1 f + c_2 g$ is an almost periodic function of $G$ for every $c_1, c_2 \in \mathbb{C}$. It suffices to consider the case $c_1 = c_2 = 1$ since $c_1 f$ and $c_2 g$ are obviously almost periodic functions. Then the closures $K_f = \overline{\{f_a : a \in G\}}$ and $K_g = \overline{\{g_a : a \in G\}}$ taken in the uniform topology of $B(G)$ are compact. Hence $K_f + K_g$ is compact as well. Since $(f + g)_a = f_a + g_a \in K_f + K_g$ for every $a \in G$, we conclude that $f + g$ is almost periodic. The closedness of $A(G)$ under the complex conjugation is obvious.

To check that $A(G)$ is closed assume that $f$ can be uniformly approximated by almost periodic functions and pick a sequence $(g^{(m)})$ of almost periodic functions of $G$ such that

$$\|f - g^{(m)}\| \leq 1/m. \tag{$*$}$$

Then for every sequence $(f_{a_n})$ of translations of $f$ one can inductively define a sequence of subsequence of $(a_n)$ as follows. For the first one the subsequence $(g^{(1)}_{a_{n_k}})$ of the sequence $(g^{(1)}_{a_n})$ uniformly converges in $B(G)$. Then pick a subsequence $a_{n_{k_s}}$ of $a_{n_k}$ such that the subsequence $(g^{(2)}_{a_{n_{k_s}}})$ of the sequence $(g^{(2)}_{a_{n_k}})$ uniformly converges in $B(G)$, etc. Finally take a diagonal subsequence $a_\nu$, e.g., $a_1, a_{n_1}, a_{n_{k_1}}, \ldots$ such that for each subsequence $a_n$, $a_{n_k}$, $a_{n_{k_s}}$, etc. of $a_n$ has a tail contained in the subsequence $a_\nu$. Then for every $m$ the sequence $(g^{(m)}_{a_\nu})$ uniformly converges in $B(G)$. Therefore, by (*) also the sequence $(f_{a_\nu})$ uniformly converges in $B(G)$.)

In §11.4 we give a detailed alternative desciption of the almost periodic functions of an abelian group $G$.

**Exercise 9.2.14.** *Let $h : G \to H$ be a homomorphism and let $f : H \to \mathbb{C}$ be an almost periodic function. Then also $g = f \circ h : G \to \mathbb{C}$ is almost periodic.*

(Hint. Let $a_n$ be a sequence in $G$. Then for $b_n = h(a_n)$, the sequence $f_{b_n}$ has a uniformly convergent subsequence $f_{b_{n_k}}$ in $B(H)$. Then $g_{a_{n_k}}$ is a convergent subsequence of $g_{a_n}$ in $B(G)$. Thus $g \in A(G)$.)

[10] This definition, in the case of $G = \mathbb{R}$, is due to Bochner.

### 9.2.3 Precompactness of the topologies $\mathcal{T}_H$

Now we adopt a different approach to describe the precompact abelian groups, based on the use of characters. Our first aim will be to see that the topologies induced by characters are always totally bounded.

**Proposition 9.2.15.** *If $A$ is an abelian group, $\delta > 0$ and $\chi_1, \ldots, \chi_s \in A^*$ ($s \in \mathbb{N}_+$), then $U(\chi_1, \ldots, \chi_s; \delta)$ is big in $A$. Moreover for every $a \in A$ there exists a sufficiently large positive integer $n$ such that $na \in U(\chi_1, \ldots, \chi_s; \delta)$.*

*Proof.* Define $h : A \to \mathbb{T}^s$ such that $h(x) = (\chi_1(x), \ldots, \chi_s(x))$ and

$$B = \left\{(z_1, \ldots, z_n) \in \mathbb{S}^s : |\mathrm{Arg}\, z_i| < \frac{\delta}{2} \text{ for } i = 1, \ldots, s\right\} = \left\{z \in \mathbb{S} : |\mathrm{Arg}\, z| < \frac{\delta}{2}\right\}^s.$$

Then $B$ is big in $\mathbb{S}^s$ and

$$B - B \subseteq C = \{(z_1, \ldots, z_s) \in \mathbb{S}^s : \|\mathrm{Arg}\, z_i\| < \delta \text{ for } i = 1, \ldots, s\}.$$

Therefore $U(\chi_1, \ldots, \chi_s; \delta) = h^{-1}(C)$ is big in $A$ by Corollary 9.1.4.

The second statement follows from Proposition 9.1.3, since

$$U\left(\chi_1, \ldots, \chi_s; \frac{\delta}{2}\right) - U\left(\chi_1, \ldots, \chi_s; \frac{\delta}{2}\right) \subseteq U(\chi_1, \ldots, \chi_s; \delta).$$

$\square$

**Corollary 9.2.16.** *For an abelian group $G$ all topologies of the form $\mathcal{T}_H$, where $H \leq G^*$, are totally bounded. Moreover, $\mathcal{T}_H$ is precompact iff $H$ separates the points of $G$. Hence $\mathcal{P}_G$ is precompact.*

It requires a considerable effort to prove that, conversely, every totally bounded group topology has the form $\mathcal{T}_H$ for some $H$ (see Remark 11.1.3). At this stage we can prove only that every group $G$ admits a finest totally bounded group topology $\mathcal{P}_G$ (Exercise 9.2.10), moreover, it is precompact when $G$ is abelian. So the above corollary gives so far only the inequality $\mathcal{P}_G \geq \mathcal{T}_{G^*}$.

It follows easily from Corollary 9.2.16 and Proposition 9.2.15 that for every neighborhood $E$ of 0 in the Bohr topology (namely, a set $E$ containing a subset of the form $U(\chi_1, \ldots, \chi_n; \varepsilon)$ with characters $\chi_i : G \to \mathbb{S}$, $i = 1, 2, \ldots, n$, and $\varepsilon > 0$) there exists a big set $B$ of $G$ such that $B_{(8)} \subseteq E$ (just take $B = U(\chi_1, \ldots, \chi_n; \varepsilon/8)$). Surprisingly, the converse is also true. Namely, we shall obtain as a corollary of Følner's lemma that every set $E$ satisfying $B_{(8)} \subseteq E$ for some big set $B$ of $G$ must be a neighborhood of 0 in the Bohr topology of $G$ (see Corollary 10.2.5), i.e., $\mathcal{P}_G = \mathcal{T}_{G^*}$.

**Lemma 9.2.17.** *If $G$ is a countably infinite Hausdorff abelian group, then for every compact set $K$ in $G$ the set $K_{(2n)}$ is big for no $n \in \mathbb{N}$.*

*Proof.* By Lemma 7.2.1 every set $K_{(2n)}$ is compact. So if $K_{(2n)}$ were big for some $n$, then $G$ itself would be compact. Now Lemma 7.2.7 applies. $\square$

**Exercise 9.2.18.** *(a) If $S = (a_n)$ is a one-to-one $T$-sequence in an abelian group $G$, then for every $n \in \mathbb{N}$ the set $S_{(2n)}$ is small in $G$.*

*(b)* Show that the sequence $(p_n)$ of prime numbers in $\mathbb{Z}$ is not a $T$-sequence.*

(Hint. (a) Consider the (countable) subgroup generated by $S$ and note that if $a_n \to 0$ in some Hausdorff group topology $\tau$ on $G$, then the set $S \cup \{0\}$ would be compact in $\tau$, so item (a) and Lemma 9.2.17 apply. For (b) use (a) and the fact that there exists a constant $m$ such that every integer number is a sum of at most $m$. More precisely, according to the positive solution of the ternary Goldbach's conjecture there exists a constant $C > 0$ such that every odd integer $\geq C$ is a sum of three primes (see [143] for further details on Goldbach's conjecture).

## 9.3 Haar integral and unitary representations

According to a classical result of E. Følner, an abelian topological group $G$ is MAP iff for every $a \neq 0$ in $G$ there exists a big set $B$ such that $a$ does not belong to the closure of $B_{(4)} = B - B + B - B$. A weaker form of this theorem will be proved in §9.3, (with the bigger set $B_{(8)}$ in place of $B_{(4)}$).

The nice structure theory of locally compact groups (see §11.3) is due to the Haar measure and Haar integral in locally compact groups. Every locally compact group $G$ admits a *right Haar integral*, i.e., a positive linear functional $\int_G$ defined on the space $C_0(G)$ of all continuous complex-valued functions on $G$ with compact support that is right invariant (in the sense that $I(f_a) = I(f)$ for every $f \in C_0(G)$ and $a \in G$ [102, Theorem (15.5)], see also §§11.4.3, 11.4.2

for more detail in the abelian case). Moreover, if $\int_G^*$ is another right Haar integral of $G$ then there exists a positive $c \in \mathbb{R}$ such that $\int_G^* = c \int_G$. The measure $m$ induced by a right Haar integral on the family of all Borel sets of $G$ is called a *right Haar measure*. The group $G$ has finite measure iff $G$ is compact. In such a case the measure $m$ is determined uniquely by the additional condition $m(G) = 1$. Analogously, a locally compact group admits a unique, up to a positive multiplicative constant, *left Haar integral*. Every compact group $G$ admits a unique Haar integral that is right and left invariant, such that its Haar measure satisfies $m(G) = 1$.

Alternatively, the *Haar measure* of a compact group $G$ is a function $\mu : \mathcal{B}(G) \to [0,1]$ such that

(a) ($\sigma$-additivity) $\mu(\bigcup_{n=1}^\infty B_n) = \sum_{n=1}^\infty \mu(B_n)$ for every family $(B_n)$ of pairwise disjoint members of $\mathcal{B}(G)$;

(b) (left and right invariance) $m(aB) = m(Ba) = m(B)$ for every $B \in \mathcal{B}(G)$;

(c) $m(G) = 1$.

It easily follows from (b) and (c) that $m(U) > 0$ for every non-empty open set $U$ of $G$. The Haar measure is unique with the properties (a)–(c).

The representations of the locally compact groups are based on the Haar integral (one can see in §11.4 how these unitary representation arise in the case of compact abelian groups).

**Theorem 9.3.1.** (Gel′fand-Raĭkov ) *For every locally compact group $G$ and $a \in G$, $a \neq e$, there exists a continuous irreducible representation $V$ of $G$ by unitary operators of some Hilbert space $\mathcal{H}$, such that $V_a \neq e$.*

The proof of this theorem can be found in [102, 22.12]. If $G$ is compact, $\mathcal{H}$ can be chosen finite dimensional. Then the unitary group of $\mathcal{H}$ is compact. Note that the locally compact group groups with the last property (namely, those locally compact groups whose continuous irreducible unitary representations in finite-dimensional Hilbert space separate the points), are precisely the MAP locally compact groups.

It was proved by Freudenthal and Weil that the connected locally compact MAP groups have the form $\mathbb{R}^n \times G$, where $G$ is compact (and necessarily connected).

The case of Gel′fand-Raĭkov's theorem with compact group $G$ is known as *Peter-Weyl-van Kampen theorem*:

**Theorem 9.3.2.** *Let $G$ be a compact group. For every $a \in G$, $a \neq e$, there exists a continuous homomorphism $f : G \to U(n)$, such that $f(a) \neq e$ ($n$ may depend on $a$).*

In particular, a topological group $G$ is MAP iff the continuous homomorphisms $G \to U(n)$ (with $n$ varying in $\mathbb{N}$) separate the points of $G$.

**Corollary 9.3.3.** *If $G$ is a compact group, then $G$ is isomorphic to a (closed) subgroup of some power $\mathbb{U}^I$ of the group $\mathbb{U}$.*

*Proof.* Since the continuous homomorphims $f_i : G \to \mathbb{U}$ $(i \in I)$ separate the points of $G$, the diagonal map determined by all homomorphisms $f_i$ defines a continuous injective homomorphism $\Delta_I : G \hookrightarrow \mathbb{U}^I$. By the compactness of $G$ and the open mapping theorem, this is the required embedding. □

In the case of an abelian group $G$ the continuous irreducible unitary representations are simply the continuous characters $G \to \mathbb{T}$. Hence an abelian topological group $G$ is MAP iff the continuous homomorphisms $G \to \mathbb{T}$ separate the points of $G$, i.e., for every $x, y \in G$ with $x \neq y$, there exists $\chi \in \widehat{G}$ such that $\chi(x) \neq \chi(y)$.

Using Peter-Weyl's theorem in the abelian case one can prove that every locally compact abelian group is MAP. The proof of this fact (see Theorem 11.3.3) requires several ingredients that we develop in §11.

The standard exposition of Pontryagin-van Kampen duality exploits the Haar measure for the proof of Peter-Weyl's theorem. Our aim here is to obtain a proof of Peter-Weyl-van Kampen theorem in the abelian case without any recourse to Haar integration and tools of functional analysis. This elementary approach, based on Følner's theorem mentioned above and ideas of Iv. Prodanov, can be found in [57, Ch.1]). It makes no recourse to Haar measure at all – on the contrary, after giving a self-contained elementary proof of Peter-Weyl's theorem, one obtains as an easy consequence the existence of Haar measure on the locally compact abelian groups (see Theorem 11.4.17 for the compact case and Theorem 11.4.21 for the locally compact one).

# 10 Følner's theorem

This section is entirely dedicated to Følner's theorem.

## 10.1 Fourier theory for finite abelian groups

In the sequel $G$ will be a finite abelian group, so $G^* \cong G$, so in particular $|G^*| = |G|$.

Here we recall some well known properties of the scalar product in finite-dimensional complex spaces $V = \mathbb{C}^n$. Since our space will be "spanned" by a finite abelian group $G$ of size $n$ (i.e., $V = \mathbb{C}^G$), we have also an action of $G$ on $V$. We normalize the scalar product in a such way to let the vector $(1, 1, \dots, 1)$ (i. e., the constant function 1) to have norm 1. The reader familiar with Haar integration may easily recognize in this the Haar integral on $G$.

Define the scalar product by

$$(f, g) = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}.$$

Let us see first that the elements of the subset $G^*$ of $V$ are pairwise orthogonal and have norm 1:

**Proposition 10.1.1.** *Let $G$ be an abelian finite group and $\varphi, \chi \in G^*$, $x, y \in G$. Then:*

*(a)* $\frac{1}{|G|} \sum_{x \in G} \varphi(x)\overline{\chi(x)} = \begin{cases} 1 & \text{if } \varphi = \chi \\ 0 & \text{if } \varphi \neq \chi \end{cases}$;

*(b)* $\frac{1}{|G^*|} \sum_{\chi \in G^*} \chi(x)\overline{\chi(y)} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$.

*Proof.* (a) If $\varphi = \chi$ then $\chi(x)\overline{\chi(x)} = \chi(x)\chi(x)^{-1} = 1$.

If $\varphi \neq \chi$ there exists $z \in G$ such that $\varphi(z) \neq \chi(z)$. Therefore the following equalities

$$\sum_{x \in G} \varphi(x)\overline{\chi(x)} = \frac{\varphi(z)}{\chi(z)} \sum_{x \in G} \varphi(x-z)\overline{\chi(x-z)} = \frac{\varphi(z)}{\chi(z)} \sum_{x \in G} \varphi(x)\overline{\chi(x)}$$

imply that $\sum_{x \in G} \varphi(x)\overline{\chi(x)} = 0$.

(b) If $x = y$ then $\chi(x)\overline{\chi(x)} = \chi(x)\chi(x)^{-1} = 1$.

If $x \neq y$, by Corollary 2.1.12 there exists $\chi_0 \in G^*$ such that $\chi_0(x) \neq \chi_0(y)$. Now we can proceed as before, that is

$$\sum_{\chi \in G^*} \chi(x)\overline{\chi(y)} = \frac{\chi_0(x)}{\chi_0(y)} \sum_{\chi \in G^*} (\chi\chi_0)(x)\overline{(\chi\chi_0)(y)} = \frac{\chi_0(y)}{\chi_0(x)} \sum_{\chi \in G^*} \chi(x)\overline{\chi(y)}$$

yields $\sum_{\chi \in G^*} \chi(x)\overline{\chi(y)} = 0$. $\square$

If $G$ is a finite abelian group and $f$ is a complex valued function on $G$, then for every $\chi \in G^*$ we can define

$$c_\chi = (f, \chi) = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{\chi(x)},$$

that is the *Fourier coefficient* of $f$ corresponding to $\chi$.

For complex valued functions $f, g$ on a finite abelian group $G$ define the *convolution* $f * g$ by $(f * g)(x) = \frac{1}{|G|} \sum_{y \in G} \overline{f(y)} g(x+y)$.

**Proposition 10.1.2.** *Let $G$ be an abelian finite group and $f$ a complex valued function on $G$ with Fourier coefficients $c_\chi$ where $\chi \in G^*$. Then:*

*(a)* *$f(x) = \sum_{\chi \in G^*} c_\chi \chi(x)$ for every $x \in G$;*

*(b)* *if $\{a_\chi\}_{\chi \in G^*}$ is such that $f(x) = \sum_{\chi \in G^*} a_\chi \chi(x)$, then $a_\chi = c_\chi$ for every $\chi \in G^*$;*

*(c)* *$\frac{1}{|G|} \sum_{x \in G} |f(x)|^2 = \sum_{\chi \in G^*} |c_\chi|^2$;*

*(d)* *if $g$ is an other complex valued function on $G$ with Fourier coefficients $(d_\chi)_{\chi \in G^*}$, then $f * g$ has Fourier coefficients $(\overline{c_\chi} d_\chi)_{\chi \in G^*}$.*

*Proof.* (a) The definition of the coefficients $c_\chi$ yields

$$\sum_{\chi\in G^*} c_\chi \chi(x) = \sum_{\chi\in G^*} \frac{1}{|G|} \sum_{y\in G} f(y)\overline{\chi(y)}\chi(x).$$

Computing $\sum_{\chi\in G^*} \overline{\chi(y)}\chi(x)$ with Proposition 10.1.1(b) we get $\sum_{\chi\in G^*} c_\chi\chi(x) = \frac{|G^*|}{|G|} f(x)$ for every $x \in G$. Now $|G| = |G^*|$ gives $f(x) = \sum_{\chi\in G^*} c_\chi\chi(x)$ for every $x \in G$.

(b) By Proposition 10.1.1 the definition of the coefficients $c_\chi$ and the relation $f(x) = \sum_{\chi\in G^*} a_\chi\chi(x)$

$$c_\chi = \frac{1}{|G|} \sum_{\varphi\in G^*} a_\varphi \sum_{x\in G} \varphi(x)\overline{\chi(x)} = a_\chi.$$

(d) By item (a) $g(x) = \sum_{\varphi\in G^*} d_\varphi\varphi(x)$ for every $x \in G$. Therefore

$$\begin{aligned}\sum_{y\in G} \overline{f(y)}g(x+y) &= \sum_{y\in G} \Big(\sum_{\chi\in G^*} \overline{c_\chi}\overline{\chi(y)}\Big)\Big(\sum_{\varphi\in G^*} d_\varphi\varphi(x)\varphi(y)\Big) = \\ &= \sum_{\chi\in G^*}\sum_{\varphi\in G^*} \overline{c_\chi} d_\varphi\varphi(x) \sum_{y\in G} \overline{\chi(y)}\varphi(y) = |G| \sum_{\chi\in G^*} \overline{c_\chi} d_\chi\chi(x).\end{aligned}$$

(c) It is sufficient to apply (d) with $g = f$ and let $x = 0$. □

**Corollary 10.1.3.** *Let $G$ be a finite abelian group, $E$ be a non-empty subset of $G$ and let $f$ be the characteristic function of $E$. Then for the convolution $g = f * f$ one has*

*(a) $g(x) > 0$ iff $x \in E_{(2)}$;*

*(b) $g(x) = \sum_{\chi\in G^*} |c_\chi|^2\chi(x)$.*

*Proof.* (a) $g(x) > 0$ if and only if there exists $y \in E$ with $x + y \in E$, that is $x \in E - E = E_{(2)}$.

(b) follows obviously from Proposition 10.1.2(d). □

## 10.2 Bogoliouboff and Følner Lemmas

**Lemma 10.2.1** (Bogoliouboff lemma)**.** *If $F$ is a finite abelian group and $E$ is a non-empty subset of $F$, then there exist $\chi_1, \ldots, \chi_m \in F^*$, where $m = \big[\big(\frac{|F|}{|E|}\big)^2\big]$, such that $U(\chi_1, \ldots, \chi_m; \frac{\pi}{2}) \subseteq E_{(4)}$.*

*Proof.* Let $f$ be the characteristic function of $E$. By Proposition 10.1.2(a) we have

$$f(x) = \sum_{\chi\in F^*} c_\chi\chi(x), \text{ with } c_\chi = \frac{1}{|F|} \sum_{x\in F} f(x)\overline{\chi(x)}. \tag{1}$$

Define $g = f * f$ and $h = g * g$. The functions $f$ and $g = f * f$ have real values and by Corollary 10.1.3

$$g(x) = \sum_{\chi\in F^*} |c_\chi|^2\chi(x) \text{ and } h(x) = \sum_{\chi\in F^*} |c_\chi|^4\chi(x) \text{ for } x \in F. \tag{2}$$

Moreover, $g(x) > 0$ if and only if $x \in E - E = E_{(2)}$. Analogously $h(x) > 0$ if and only if $x \in E_{(4)}$.

By Proposition 10.1.2(c) $\sum_{\chi\in F^*} |c_\chi|^2 = \frac{|E|}{|F|}$. Set $a = \frac{|E|}{|F|}$ and order the Fourier coefficients of $f$ so that

$$|c_{\chi_0}| \geq |c_{\chi_1}| \geq \ldots \geq |c_{\chi_k}| \geq \ldots$$

(note that they are finitely many). Taking into account the fact that $f$ is the characteristic function of $E$, it easily follows from (1) that the maximum value of $|c_{\chi_i}|$ is attained for the trivial character $\chi_0 = 1$, namely $c_{\chi_0} = a$. Then $\sum_{i=0}^k |c_{\chi_i}|^2 \leq \sum_{\chi\in F^*} |c_\chi|^2 = a$ for every $k \geq 0$. Consequently $(k+1)|c_{\chi_k}|^2 \leq a$, so

$$|c_{\chi_k}|^4 \leq \frac{a^2}{(k+1)^2}. \tag{3}$$

Now let $m = [\frac{1}{a^2}]$. We are going to show now that with these $\chi_1, \ldots, \chi_m \in F^*$ one has

$$h(x) > 0 \text{ for every } x \in U(\chi_1 \ldots, \chi_m; \frac{\pi}{2}). \tag{4}$$

Clearly $Re\chi_k(x) \geq 0$ for $k = 1, 2, \ldots, m$ whenever $x \in U(\chi_1 \ldots, \chi_m; \frac{\pi}{2})$ thus

$$|a^4 + \sum_{k=1}^{m} |c_{\chi_k}|^4 \chi_k(x)| \geq Re(a^4 + \sum_{k=1}^{m} |c_{\chi_k}|^4 \chi_k(x)) \geq a^4. \tag{5}$$

On the other hand, (3) yields

$$\sum_{k \geq m+1} |c_{\chi_k}|^4 \leq \sum_{k \geq m+1} \frac{a^2}{(k+1)^2} < a^2 \sum_{k \geq m+1} \frac{1}{k(k+1)} \leq \frac{a^2}{m+1}. \tag{6}$$

Since $h$ has real values, (2), (5) and (6) give

$$h(x) = |h(x)| = |a^4 + |c_{\chi_1}|^4 \chi_1(x) + \ldots| \geq \left| a^4 + \sum_{k=1}^{m} |c_{\chi_k}|^4 \chi_k(x) \right| - \sum_{k \geq m+1} |c_{\chi_k}|^4 \geq a^4 - \frac{a^2}{m+1} \geq a^2(a^2 - \frac{1}{m+1}) > 0.$$

This proves (4). Therefore $U(\chi_1 \ldots, \chi_m; \frac{\pi}{2}) \subseteq E_{(4)}$. □

Let us note that the estimate for the number $m$ of characters is certainly non-optimal when $E$ is too small. For example, when $E$ is just the singleton $\{0\}$, the upper bound given by the lemma is just $|F|^2$, while one can certainly find at most $m = |F| - 1$ characters $\chi_1, \ldots, \chi_m$ (namely, all non-trivial $\chi_i \in F^*$) such that $U(\chi_1, \ldots, \chi_m; \frac{\pi}{2}) = \{0\}$. For certain groups (e.g., $F = \mathbb{Z}_2^k$) one can find even a much smaller number (say $m = \log_2 |F|$). Nevertheless, in the cases relevant for the proof of Følner's theorem, namely when the subset $E$ is relatively large with respect to $F$, this estimate seems more reasonable.

The next lemma will be needed in the following proofs.

**Lemma 10.2.2.** *Let $A$ be an abelian group and $\{A_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $A$ such that*

$$\lim_{n \to \infty} \frac{|(A_n - a) \cap A_n|}{|A_n|} = 1$$

*for every $a \in A$. If $k$ is a positive integer and $V$ is a subset of $A$ such that $k$ translates of $V$ cover $A$, then for every $\varepsilon > 0$ there exists $N > 0$ such that*

$$|V \cap A_n| > \left( \frac{1}{k} - \varepsilon \right) |A_n|$$

*for every $n \geq N$.*

*Proof.* Let $a_1, \ldots, a_k \in A$ be such that $\bigcup_{i=1}^{k} (a_i + V) = A$. If $\varepsilon > 0$, then there exists $N_1 > 0$ such that for every $n \geq N_1$

$$|(A_n - a_i) \cap A_n| > (1 - \varepsilon)|A_n|$$

and consequently,

$$|(A_n - a_i) \setminus A_n| < \varepsilon |A_n| \tag{7}$$

for every $i = 1, \ldots, k$. Since $A_n = \bigcup_{i=1}^{k} (a_i + V) \cap A_n$, for every $n$ there exists $i_n \in \{1, \ldots, k\}$ such that

$$\frac{1}{k}|A_n| \leq |(a_{i_n} + V) \cap A_n| = |V \cap (A_n - a_{i_n})|.$$

Since $V \cap (A_n - a_{i_n}) \subseteq (V \cap A_n) \cup ((A_n - a_{i_n}) \setminus A_n)$, (7) yields

$$\frac{1}{k}|A_n| \leq |V \cap (A_n - a_{i_n})| \leq |V \cap A_n| + |(A_n - a_{i_n}) \setminus A_n| < |V \cap A_n| + \varepsilon |A_n|.$$

□

**Lemma 10.2.3** (Bogoliouboff-Følner lemma)**.** *Let $A$ be a finitely generated abelian group and let $r = r_0(A)$. If $k$ is a positive integer and $V$ is a subset of $A$ such that $k$ translates of $V$ cover $A$, then there exist $\rho_1, \ldots, \rho_s \in A^*$, where $s = 3^{2r} k^2$, such that $U_A(\rho_1, \ldots, \rho_s; \frac{\pi}{2}) \subseteq V_{(4)}$.*

*Proof.* By Theorem 2.1.1 we have $A \cong \mathbb{Z}^r \times F$, where $F$ is a finite abelian group; so we can identify $A$ with the group $\mathbb{Z}^r \times F$. Define $A_n = (-n, n]^r \times F$, let $a = (a_1, \dots, a_r; f) \in \mathbb{Z}^r \times F$. Then $J_{ni} = (-n, n] \cap (-n - a_i, n - a_i]$ satisfies $|J_{ni}| \geq 2n - |a_i|$. In particular, $J_{ni} \neq \emptyset$ for every $n > n_0 = \max\{|a_i| : i = 1, 2, \dots, n\}$. As $(A_n - a) \cap A_n = \prod_{i=1}^r J_{ni} \times F$, we have

$$|(A_n - a) \cap A_n| \geq |F| \cdot \prod_{i=1}^r (2n - |a_i|)$$

or all $n > n_0$. Since $|A_n| = |F|(2n)^r$, we can apply Lemma 10.2.2. Thus for every $\varepsilon > 0$ we have

$$|V \cap A_n| > \left(\frac{1}{k} - \varepsilon\right)|A_n|. \tag{8}$$

for every sufficiently large $n$. For $n$ with (8) define $G = A/(6n\mathbb{Z}^r)$ and $E = q(V \cap A_n)$ where $q$ is the canonical projection of $A$ onto $G$. Observe that $q\restriction_{A_n}$ is injective, as $(A_n - A_n) \cap \ker q = \{0\}$. Then (8) gives

$$|E| = |V \cap A_n| > \left(\frac{1}{k} - \varepsilon\right)|A_n| = \left(\frac{1}{k} - \varepsilon\right)(2n)^r|F|$$

and so

$$\frac{|G|}{|E|} \leq \frac{(6n)^r|F|}{(\frac{1}{k} - \varepsilon)(2n)^r|F|} = \frac{3^r k}{1 - k\varepsilon}.$$

Fix $\varepsilon > 0$ sufficiently small to have $\left[\frac{3^{2r}k^2}{(1-k\varepsilon)^2}\right] = 3^{2r}k^2$ and pick sufficiently large $n$ to have (8). Now apply the Bogoliouboff Lemma 10.2.1 to find $s = 3^{2r}k^2$ characters $\xi_{1n}, \dots, \xi_{sn} \in G^*$ such that $U_G(\xi_{1n}, \dots, \xi_{sn}; \frac{\pi}{2}) \subseteq E_{(4)}$. For $j = 1, \dots, s$ define $\varrho_{jn} = \xi_{jn} \circ \pi \in A^*$. If $a \in A_n \cap U_A(\varrho_{1n}, \dots, \varrho_{sn}; \frac{\pi}{2})$ then $q(a) \in U_G(\xi_{1n}, \dots, \xi_{sn}; \frac{\pi}{2}) \subseteq E_{(4)}$ and so there exist $b_1, b_2, b_3, b_4 \in V \cap A_n$ and $c = (c_i) \in 6n\mathbb{Z}^r$ such that $a = b_1 - b_2 + b_3 - b_4 + c$. Now

$$c = a - b_1 + b_2 - b_3 + b_4 \in (A_n)_{(4)} + A_n$$

implies $|c_i| \leq 5n$ for each $i$. So $c = 0$ as $6n$ divides $c_i$ for each $i$. Thus $a \in V_{(4)}$ and so

$$A_n \cap U_A\left(\varrho_{1n}, \dots, \varrho_{sn}; \frac{\pi}{2}\right) \subseteq V_{(4)} \tag{9}$$

for all $n$ satisfying (8).

By Lemma 7.1.3 there exist $\varrho_1, \dots, \varrho_s \in A^*$ and a subsequence $\{n_l\}_l$ of $\{n\}_{n \in \mathbb{N}_+}$ such that $\varrho_i(a) = \lim_l \varrho_{in_l}(a)$ for every $i = 1, \dots, s$ and $a \in A$. We are going to prove now that

$$U_A\left(\varrho_1, \dots \varrho_s; \frac{\pi}{2}\right) \subseteq V_{(4)}. \tag{10}$$

Take $a \in U_A(\varrho_1, \dots, \varrho_s; \frac{\pi}{2})$. Since $A = \bigcup_{l=k}^\infty A_{n_l}$ for every $k \in \mathbb{N}_+$, there exists $n_0$ satisfying (8) and $a \in A_{n_0}$. As $\varrho_i(a) = \lim_l \varrho_{in_l}(a)$ for every $i = 1, \dots, s$, we can pick $l$ to have $n_l \geq n_0$ and $|Arg(\varrho_{in_l}(a))| < \pi/2$ for every $i = 1, \dots, s$, i.e., $a \in U_A(\varrho_{1n_l}, \dots, \varrho_{sn_l}; \frac{\pi}{2}) \cap A_{n_l}$. Now (9), applied to $n_l$, yields $a \in V_{(4)}$. This proves (10). □

Our next aim is to eliminate the dependence of the number $m$ of characters on the free rank of the group $A$ in Bogoliouboff - Følner's lemma. The price to pay for this is taking $V_{(8)}$ instead of $V_{(4)}$.

**Lemma 10.2.4** (Følner lemma)**.** *Let $A$ be an abelian group. If $k$ is a positive integer and $V$ be a subset of $A$ such that $k$ translates of $V$ cover $A$, then there exist $\chi_1, \dots, \chi_m \in A^*$, where $m = k^2$, such that $U_A(\chi_1, \dots, \chi_m; \frac{\pi}{2}) \subseteq V_{(8)}$.*

*Proof.* We consider first the case when $A$ is finitely generated. Let $r = r_0(A)$. By Lemma 10.2.3 there exist $\varrho_1, \dots, \varrho_s \in A^*$, where $s = 3^{2r}k^2$, such that

$$U_A\left(\varrho_1, \dots, \varrho_s; \frac{\pi}{2}\right) \subseteq V_{(4)}.$$

Since it is finitely generated, we can identify $A$ with $\mathbb{Z}^r \times F$, where $F$ is a finite abelian group. For $t \in \{1, \dots, r\}$ define a monomorphism $i_t : \mathbb{Z} \hookrightarrow A$ by letting

$$i_t(n) = (\underbrace{0, \dots, 0, n}_{t}, 0, \dots, 0; 0) \in A.$$

Then each $\kappa_{jt} = \varrho_j \circ i_t$, where $j \in \{1, \dots, s\}, t \in \{1, \dots, r\}$, is a character of $\mathbb{Z}$. By Proposition 9.2.15 the subset

$$L = U_{\mathbb{Z}}\left(\{\kappa_{jt} : j = 1, \dots, s, t = 1, \dots, r\}; \frac{\pi}{8r}\right)$$

of $\mathbb{Z}$ is infinite. Let $L^0 = \bigcup_{t=1}^r i_t(L)$, i.e., this is the set of all elements of $A$ of the form $i_t(n)$ with $n \in L$ and $t \in \{1, \dots, r\}$. Then obviously $L^0 \subseteq U_A\left(\varrho_1, \dots, \varrho_s; \frac{\pi}{8r}\right)$, therefore,

$$L^0_{(4r)} \subseteq U_A\left(\varrho_1, \dots, \varrho_s; \frac{\pi}{2}\right) \subseteq V_{(4)}. \qquad (\lambda)$$

Define $A_n = (-n, n]^r \times F$ and pick $\varepsilon > 0$ such that $\varepsilon < \frac{1}{6k^4}$. Then $\left[\left(\frac{k}{1-k\varepsilon}\right)^2\right] = k^2$. As in Lemma 10.2.3 $A_n$ satisfies the hypotheses of Lemma 10.2.2 and so $|V \cap A_n| > (\frac{1}{k} - \varepsilon)|A_n|$ for sufficiently large $n$. Moreover, we choose this sufficiently large $n$ from $L$. Let $G_n = A/(2n\mathbb{Z}^r) \cong \mathbb{Z}^r_{2n} \times F$ and $E = q(A_n \cap V)$ where $q$ is the canonical projection $A \to G_n$. Then $q\restriction_{A_n}$ is injective as $(A_n - A_n) \cap \ker q = 0$. So $q$ induces a bijection between $A_n$ and $G_n$ on one hand, and between $V \cap A_n$ and $E$. Thus $|A_n| = |G_n| = (2n)^r|F|$, $|E| > (\frac{1}{k} - \varepsilon)|A_n|$ and so

$$\left(\frac{|G_n|}{|E|}\right)^2 \leq \left(\frac{k}{\varepsilon k - 1}\right)^2, \text{ hence } \left[\left(\frac{|G_n|}{|E|}\right)^2\right] \leq \left[\left(\frac{k}{\varepsilon k - 1}\right)^2\right] = k^2.$$

To the finite group $G_n$ apply the Bogoliouboff Lemma 10.2.1 to get $\xi_{1n}, \dots, \xi_{mn} \in G_n^*$, where $m = k^2$, such that

$$U_{G_n}\left(\xi_{1n}, \dots, \xi_{mn}; \frac{\pi}{2}\right) \subseteq E_{(4)}.$$

Let $\chi_{jn} = \xi_{jn} \circ q \in A^*$. If $a \in A_n \cap U_A(\chi_{1n}, \dots, \chi_{mn}; \frac{\pi}{2})$, then $q(a) \in U_{G_n}(\xi_{1n}, \dots, \xi_{mn}; \frac{\pi}{2}) \subseteq E_{(4)}$. Therefore there exist $b_1, b_2, b_3, b_4 \in A_n \cap V$ and $c = (c_i) \in 2n\mathbb{Z}^r$ such that $a = b_1 - b_2 + b_3 - b_4 + c$. Since $2n$ divides $c_i$ for every $i$ and $|c_i| \leq 5n$, we conclude that $c_i \in \{0, \pm 2n \pm 4n\}$ for $i = 1, 2, \dots, r$. This means that $c$ can be written as a sum of at most $4r$ elements of $L^0$. This gives $c \in L^0_{(4r)} \subseteq V_{(4)}$ by $(\lambda)$, consequently $a \in V_{(8)}$. Therefore

$$A_n \cap U_A\left(\chi_{1n}, \dots, \chi_{mn}; \frac{\pi}{2}\right) \subseteq V_{(8)}$$

for $n \in L$ sufficiently large $n$. By Lemma 7.1.3 there exist $\chi_1, \dots, \chi_m \in A^*$ and a subsequence $\{n_l\}_l$ of $\{n\}_{n \in \mathbb{N}_+}$ such that $\chi_j(a) = \lim_l \chi_{jn_l}(a)$ for every $j = 1, \dots, m$ and for every $a \in A$. Being $A = \bigcup\{A_{n_l} : l > k, n_l \in L\}$ for every $k \in \mathbb{N}_+$ we can conclude as above that $U_A\left(\chi_1, \dots, \chi_m; \frac{\pi}{2}\right) \subseteq V_{(8)}$.

Consider now the general case. Let $g_1, \dots, g_k \in A$ be such that $A = \bigcup_{i=1}^k (g_i + V)$. Suppose that $G$ is a finitely generated subgroup of $A$ containing $g_1, \dots, g_k$. Then $G = \bigcup_{1=1}^k (g_i + V \cap G)$ and so $k$ translates of $V \cap G$ cover $G$. By the above argument there exist $\varphi_{1G}, \dots, \varphi_{mG} \in G^*$, where $m = k^2$, such that

$$U_G\left(\varphi_{1G}, \dots, \varphi_{mG}; \frac{\pi}{2}\right) \subseteq (V \cap G)_{(8)} \subseteq V_{(8)}.$$

By Corollary 2.1.11 we can extend each $\varphi_{iG}$ to a character of $A$, so that we assume from now on $\varphi_{1G}, \dots, \varphi_{mG} \in A^*$ and

$$G \cap U_A\left(\varphi_{1G}, \dots, \varphi_{mG}; \frac{\pi}{2}\right) = U_G\left(\varphi_{1G}, \dots, \varphi_{mG}; \frac{\pi}{2}\right) \subseteq V_{(8)}. \qquad (11)$$

Let $\mathcal{G}$ be the family of all finitely generated subgroups $G$ of $A$ containing $g_1, \dots, g_k$. It is a directed set under inclusion. So we get $m$ nets $\{\varphi_{jG}\}_{G \in \mathcal{G}}$ in $A^*$ for $j = 1, \dots, m$. By Lemma 7.1.3 there exist subnets $\{\varphi_{jG_\beta}\}_\beta$ and $\chi_1, \dots, \chi_m \in A^*$ such that

$$\chi_j(x) = \lim_\beta \varphi_{jG_\beta}(x) \text{ for every } x \in A \text{ and } j = 1, \dots, m. \qquad (12)$$

From (11) and (12) we conclude as before that $U_A(\chi_1, \dots, \chi_m; \frac{\pi}{2}) \subseteq V_{(8)}$. □

As a corollary of Følner's lemma we obtain the following *internal* description of the neighborhoods of 0 in the Bohr topology of $A$.

**Corollary 10.2.5.** *For a subset $E$ of an abelian group $A$ the following are equivalent:*

*(a) $E$ contains $V_{(8)}$ for some big subset $V$ of $A$;*

*(b) for every $n \in \mathbb{N}_+$ $E$ contains $V_{(2n)}$ for some big subset $V$ of $A$;*

*(c) $E$ is a neighborhood of 0 in the Bohr topology of $A$.*

*Proof.* The implication (a) ⇒ (c) follows from Følner's lemma. The implication (c) ⇒ (b) follows from Corollary 9.2.16 and Proposition 9.2.15. □

**Corollary 10.2.6.** *For an abelian group $G$ the Bohr topology $\mathcal{T}_{G^*}$ coincides with the finest precompact group topology $\mathcal{P}_G$.*

**Corollary 10.2.7.** *For a subgroup $H$ of an abelian group $G$ the Bohr topology of $G/H$ coincides with the quotient topology of $G^\#$.*

*Proof.* Let $q : G \to G/H$ be the quotient homomorphism. The quotient topology $\overline{\mathcal{T}}_{G^*}$ of the Bohr topology $\mathcal{T}_{G^*}$ is a precompact group topology on $G/H$ (as $H$ is closed in $G^\#$ by Theorem 3.4.7). Hence $\overline{\mathcal{T}}_{G^*} \leq \mathcal{P}_{G/H} = \mathcal{T}_{G/H}$. On the other hand, $q : G^\# \to (G/H)^\#$ is continuous, hence $\mathcal{T}_{G/H} \leq \overline{\mathcal{T}}_{G^*}$ by the properties of the quotient topology. Hence $\overline{\mathcal{T}}_{G^*} = \mathcal{T}_{G/H}$. □

**Exercise 10.2.8.** *Prove the above corollary using the explicit description of the neighborhoods of 0 in $G^\#$ given in Corollary 10.2.5.*

(Hint. Since $q : G^\# \to (G/H)^\#$ is continuous, it remains to show that it is also open. To this end take a neighborhood $U$ of 0 in $G^\#$. Then $U$ contains some $V_{(8)}$, where $V$ is a big set in $G$. Since $q(V)$ is big in $G/H$ and $q(V_{(8)}) = q(V)_{(8)} \subseteq q(U)$, we deduce from Corollary 10.2.5 that $q(U)$ is a neighborhood of 0 in $(G/H)^\#$.)

It follows from results of Følner [79] obtained by less elementary tools, that (a) can be replaced by the weaker assumption $V_{(4)} \subseteq E$ (see also Ellis and Keynes [76] or Cotlar and Ricabarra [30] for further improvements). Nevertheless the following old problems concerning the group $\mathbb{Z}$ is still open (see Cotlar and Ricabarra [30], Ellis and Keynes [76], Følner [79], Glasner [89], Pestov [120, Question 1025] or Veech [141]):

**Question 10.2.9.** Does there exist a big set $V \subseteq \mathbb{Z}$ such that $V - V$ is not a neighborhood of 0 in the Bohr topology of $G$?

It is known that every infinite abelian group $G$ admits a big set with empty interior with respect to the Bohr topology [5] (more precisely, these authors prove that every totally bounded group has a big subset with empty interior).

## 10.3 Prodanov's lemma and independence of characters

In the sequel various subspaces of the $\mathbb{C}$-algebra $B(G)$ of all bounded complex-valued functions on an abelian group $G$ will be used. We denote by $\mathfrak{X}(G)$ the $\mathbb{C}$-subspace of $B(G)$ consisting of all linear combinations of continuous characters of a topological abelian group $G$ with coefficients from $\mathbb{C}$ and by $\mathfrak{X}_0(G)$ its $\mathbb{C}$-subspace spanned by the continuous non-trivial characters of $G$. If $G$ carries no specific topology, we shall always assume that $G$ is discrete, so that $G^* = \widehat{G}$. Note that $\mathfrak{X}(G) = \mathbb{C} \cdot 1 \oplus \mathfrak{X}_0(G)$ and both $\mathcal{X}(G)$ and $\mathfrak{X}_0(G)$ are invariant under the action $f \mapsto f_a$ of the group $G$.

For an abelian group $G$ let $\mathfrak{A}(G)$ denote the set of all functions $f \in B(G)$ such that for every $\varepsilon > 0$ there exists a $g \in \mathfrak{X}(G)$ with $\|f - g\| \leq \varepsilon$, i.e., $\mathfrak{A}(G)$ is the closure of $\mathfrak{X}(G)$ in $B(G)$ with respect to the uniform convergence topology of $B(G)$. Hence $\mathfrak{A}(G)$ is a $\mathbb{C}$-subalgebra of $B(G)$ containing all constants and closed under complex conjugation. Furthermore, let $\mathfrak{A}_0(G)$ denote the set of all functions $f \in \mathfrak{A}(G)$ such that for every $\varepsilon > 0$ there exists a $g \in \mathfrak{X}_0(G)$ with $\|f - g\| \leq \varepsilon$, i.e., $\mathfrak{A}_0(G)$ is the closure of $\mathfrak{X}_0(G)$ in $B(G)$ with respect to the uniform convergence topology of $B(G)$. It is easy to see that $\mathfrak{A}_0(G)$ is $\mathbb{C}$-vector subspaces of $\mathfrak{A}(G)$ (hence of $B(G)$ as well). Moreover, $\mathfrak{A}(G) = \mathfrak{A}_0(G) + \mathbb{C} \cdot 1$, where $\mathbb{C} \cdot 1$ is the one-dimensional subalgebra consisting of the constant functions.

### 10.3.1 Prodanov's lemma

Let $C$ be a set in a real or complex vector space. Then $C$ is said to be *convex* if, for all $x, y \in C$ and all $t \in [0, 1]$, the point $(1 - t)x + ty \in C$.

The next lemma, due to Prodanov [127], allows us to eliminate the discontinuous characters in uniform approximations of continuous functions via linear combinations of characters. In [57, Lemma 1.4.1] it is proved for abelian groups $G$ that carry a topology $\tau$ such that for every $g \in G$ and $n \in \mathbb{Z}$ the functions $x \mapsto x + g$ and $x \mapsto nx$ are continuous in $(G, \tau)$. The fact that this topology is not assumed to be Hausdorff will be crucial in the applications of the lemma.

**Lemma 10.3.1** (Prodanov's lemma)**.** *Let $G$ be a topological abelian group, let $U$ be an open subset of $G$, $f$ a complex valued continuous function on $U$ and $M$ a convex closed subset of $\mathbb{C}$. Let $k \in \mathbb{N}_+$ and $\chi_1, \ldots, \chi_k \in G'$. Suppose that $c_1, \ldots, c_k \in \mathbb{C}$ are such that $\sum_{j=1}^k c_j\chi_j(x) - f(x) \in M$ for every $x \in U$. If $\chi_{m_1}, \ldots, \chi_{m_s}$, with $m_1 < \cdots < m_s, s \in \mathbb{N}, \{m_1, \ldots, m_s\} \subseteq \{1, \ldots, k\}$, are precisly all continuous characters among $\chi_1, \ldots, \chi_k$, then $\sum_{i=1}^s c_{m_i}\chi_{m_i}(x) - f(x) \in M$ for every $x \in U$.*

*Proof.* Let $\chi_k \in G^*$ be discontinuous. Then it is discontinuous at 0. Consequently there exists a net $\{x_\gamma\}_\gamma$ in $G$ such that $\lim_\gamma x_\gamma = 0$ and there exist $y_j = \lim_\gamma \chi_j(x_\gamma)$ for all $j = 1, \dots, k$, but $y_k \neq 1$. Notice that always $|y_j| = 1$. Moreover, $y_j = 1$ when $\chi_j$ is continuous because $x_\gamma \to 0$, so $y_j = \lim \chi_j(x_\gamma) = 1$.

Consider $\sum_{j=1}^k c_j\chi_j(x + tx_\gamma) - f(x + tx_\gamma)$, where $t \in \mathbb{Z}$. Since $\lim_\gamma x_\gamma = 0$, we have $x + tx_\gamma \in U$ for every $x \in U$ and for every sufficiently large $\gamma$. Thus $\sum_{j=1}^k c_j\chi_j(x)\chi_j(x_\gamma)^t - f(x + tx_\gamma) \in M$ and so passing to the limit $\sum_{j=1}^k c_j\chi_j(x)y_j^t - f(x) \in M$, because $f$ is continuous and $M$ is closed.

Take an arbitrary $n \in \mathbb{N}$. By the convexity of $M$ and the relation above for $t = 0, \dots, n$, we obtain

$$\frac{1}{n+1}\sum_{t=0}^{n}\left(\sum_{j=1}^{k} c_j\chi_j(x)y_j^t - f(x)\right) \in M.$$

Note that $\sum_{t=0}^n y_k^t = \frac{y_k^{n+1}-1}{y_k-1}$ because $y_k \neq 1$. Hence we get

$$\sum_{j=1}^{k-1} c_{jn}\chi_j(x) + \frac{c_k}{1+n}\frac{1-y_k^{n+1}}{1-y_k}\chi_k(x) - f(x) \in M$$

for every $x \in U$, where $c_{jn} = \frac{c_j}{n+1}\sum_{t=0}^n y_j^t$. Now for every $j = 1, 2, \dots, k-1$

- $|c_{jn}| \leq \frac{|c_j|}{n+1}\sum_{t=0}^n |y_j|^t = |c_j|$ (because $|y_j| = 1$), and
- if $y_j = 1$ then $c_{jn} = c_j$.

By the boundedness of the sequences $\{c_{jn}\}_{n=1}^\infty$ for $j = 1, \dots, k-1$, there exists a subsequence $\{n_m\}_{m=1}^\infty$ such that all limits $c'_j = \lim_m c_{jn_m}$ exist for $j = 1, \dots, k-1$. On the other hand, $|y_k| = 1$, so $|1 - y_k^{n+1}| \leq 1 + |y_k^{n+1}| \leq 2$ hence

$$\lim_n \frac{c_k}{n+1}\frac{1-y_k^{n+1}}{1-y_k} = 0.$$

Taking the limit for $m \to \infty$ in

$$\sum_{j=1}^{k-1} c_{jn_m}\chi_j(x) + \frac{c_k}{1+n_m}\frac{1-y_k^{n_m+1}}{1-y_k}\chi_k(x) - f(x) \in M$$

gives

$$\sum_{j=1}^{k-1} c'_j\chi_j(x) - f(x) \in M \quad \text{for } x \in U; \tag{13}$$

moreover $c'_j = c_j$ for every $j = 1, \dots, k-1$ such that $\chi_j$ is continuous.

The condition (13) is obtained by the hypothesis, removing the discontinuous character $\chi_k$ in such a way that the coefficients of the continuous characters remain the same. Iterating this procedure, we can remove all discontinuous characters among $\chi_1, \dots, \chi_k$. □

This lemma allows to "produce continuity out of nothing" in the process of approximation.

**Corollary 10.3.2.** *Let $G$ be a topological abelian group, $f \in C(G)$ and $\varepsilon > 0$. If $\| \sum_{j=1}^k c_j\chi_j - f\| \leq \varepsilon$ for some $k \in \mathbb{N}_+$, $\chi_1, \dots, \chi_k \in G^*$ and $c_1, \dots, c_k \in \mathbb{C}$, then also $\| \sum_{i=1}^s c_{m_i}\chi_{m_i} - f\| \leq \varepsilon$, where $\{\chi_{m_1}, \dots, \chi_{m_s}\} = \{\chi_1, \dots, \chi_n\} \cap \widehat{G}$, with $m_1 < \dots < m_s$.*

In particular, if $f = \sum_{j=1}^k c_j\chi_j$ for some $k \in \mathbb{N}_+$, $\chi_1, \dots, \chi_k \in G^*$ and $c_1, \dots, c_k \in \mathbb{C}$, then also $f = \sum_{i=1}^s c_{m_i}\chi_{m_i}$ with $\{\chi_{m_1}, \dots, \chi_{m_s}\}$ are the continuous characters in the linear combination. In other words, $C(G) \cap \mathfrak{X}(G_d)$ coincides with the $\mathbb{C}$-subalgebra $\mathfrak{X}(G)$ of $B(G)$ generated by $\widehat{G}$. For further use in the sequel we isolate also the following equality $C(G) \cap \overline{\mathfrak{X}(G_d)} = \overline{\mathfrak{X}(G)}$, i.e.,

**Corollary 10.3.3.** $C(G) \cap \mathfrak{A}(G_d) = \mathfrak{A}(G)$ *for every topological abelian group $G$.*

In other words, as far as continous functions are concerned, in the definition of $\mathfrak{A}(G)$ it is irrelevant whether one approximates via (linear combinations of) continuous or discontinuous characters.

Now we give an (apparently) topology-free form of the local version of the Stone-Weierstraß theorem 2.2.35.

**Proposition 10.3.4.** *Let $G$ be an abelian group and $H$ be a group of characters of $G$. If $X$ is a subset of $G$ and $f$ is a complex valued bounded function on $X$ then the following conditions are equivalent:*

*(a) $f$ can be uniformly approximated on $X$ by a linear combination of elements of $H$ with complex coefficients;*

*(b) for every $\varepsilon > 0$ there exist $\delta > 0$ and $\chi_1, \dots, \chi_m \in H$ such that $x - y \in U_G(\chi_1, \dots, \chi_m; \delta)$ yields $|f(x) - f(y)| < \varepsilon$ for every $x, y \in X$.*

*Proof.* (a)⇒(b) Let $\varepsilon > 0$. By (a) there exist $c_1, \dots, c_m \in \mathbb{C}$ and $\chi_1, \dots, \chi_m \in H$ such that $\| \sum_{i=1}^m c_1 \chi_i - f\|_\infty < \frac{\varepsilon}{4}$, that is $| \sum_{i=1}^m c_1 \chi_i(x) - f(x)| < \frac{\varepsilon}{4}$ for every $x \in X$.

On the other hand note that $| \sum_{i=1}^m c_i \chi_i(x) - \sum_{i=1}^m c_i \chi_i(y)| \leq \sum_{i=1}^m |c_i| \cdot |\chi_i(x) - \chi_i(y)|$ and that $|\chi_i(x - y) - 1| = |\chi_i(x)\chi_i(y)^{-1} - 1| = |\chi_i(x) - \chi_i(y)|$. If we take

$$\delta = \frac{\varepsilon}{2m \max_{i=1,\dots,m} |c_i|}$$

then $x - y \in U(\chi_1, \dots, \chi_m; \delta)$ implies $\sum_{i=1}^m |c_i| \cdot |\chi_i(x) - \chi_i(y)| < \frac{\varepsilon}{2}$ and so also $| \sum_{i=1}^m c_i \chi_i(x) - \sum_{i=1}^m c_i \chi_i(y)| < \frac{\varepsilon}{2}$. Consequently,

$$|f(x) - f(y)| \leq \left| f(x) - \sum_{i=1}^m c_i \chi_i(x) \right| + \left| \sum_{i=1}^m c_i \chi_i(x) - \sum_{i=1}^m c_i \chi_i(y) \right| + \left| \sum_{i=1}^m c_i \chi_i(y) - f(y) \right| < \varepsilon.$$

(b)⇒(a) Let $\beta X$ be the Čech-Stone compactification of $X$ endowed with the discrete topology. If $F : X \to \mathbb{C}$ is bounded, there exists a unique continuous extension $F^\beta$ of $F$ to $\beta X$. Let $\mathcal{S}$ be the collection of all continuous functions $g$ on $\beta X$ such that $g = \sum_{j=1}^n c_j \chi_j^\beta$ with $\chi_j \in H$, $c_j \in \mathbb{C}$ and $n \in \mathbb{N}_+$. Then $\mathcal{S}$ is a subalgebra of $\mathcal{C}(\beta X, \mathbb{C})$ closed under conjugation and contains all constants. In fact in $\mathcal{S}$ we have $\chi_k^\beta \chi_j^\beta = (\chi_k \chi_j)^\beta$ by definition and $\overline{\chi^\beta} = (\overline{\chi})^\beta$ because $\chi\overline{\chi} = 1$ and so $(\chi\overline{\chi})^\beta = \chi^\beta (\overline{\chi})^\beta = 1$, that is $(\overline{\chi})^\beta = (\chi^{-1})^\beta = \overline{\chi^\beta}$.

Now we will see that $\mathcal{S}$ separates the points of $\beta X$ separated by $f^\beta$, to apply the local Stone-Weierstraß Theorem 2.2.35. Let $x, y \in \beta X$ and $f^\beta(x) \neq f^\beta(y)$. Consider two nets $\{x_i\}_i$ and $\{y_i\}_i$ in $X$ such that $x_i \to x$ and $y_i \to y$. Since $f^\beta$ is continuous, we have $f^\beta(x) = \lim f(x_i)$ and $f^\beta(y) = \lim f(y_i)$. Along with $f^\beta(x) \neq f^\beta(y)$ this implies that there exists $\varepsilon > 0$ such that $|f(x_i) - f(y_i)| \geq \varepsilon$ for every sufficiently large $i$. By the hypothesis there exist $\delta > 0$ and $\chi_1, \dots, \chi_k \in H$ such that for every $u, v \in X$ if $u - v \in U_G(\chi_1, \dots, \chi_k; \delta)$ then $|f(u) - f(v)| < \varepsilon$. Assume $\chi_j^\beta(x) = \chi_j^\beta(y)$ holds true for every $j = 1, \dots, k$. Then $x_i - y_i \in U_G(\chi_1, \dots, \chi_k; \delta)$ for every sufficiently large $i$, this contradicts (a). So each pair of points of $\beta X$ separated by $f^\beta$ is also separated by $\mathcal{S}$. Since $\beta X$ is compact, one can apply the local version of the Stone-Weierstraß Theorem 2.2.35 to $\mathcal{S}$ and $f^\beta$ and so $f^\beta$ can be uniformly approximated by $\mathcal{S}$. To conclude note that if $g = \sum c_j \chi_j^\beta$ on $\beta X$ then $g \restriction_X = \sum c_j \chi_j$. □

The reader familiar with uniform spaces will note that item (b) is nothing else but uniform continuity of $f$ w.r.t. the uniformity on $X$ induced by the uniformity of the whole group $G$ determined by the topology $\mathcal{T}_H$.

The use of the Čech-Stone compactification in the above proof is inspired by Nobeling and Beyer [**?**] who proved that if $S$ is a subalgebra of $B(X)$, for some set $X$, containing the constants and stable under conjugation, then $g \in B(X)$ belongs to the closure of $S$ with respect to the norm topology if and only if for every net $(x_\alpha)$ in $X$ the net $g(x_\alpha)$ is convergent whenever the nets $f(x_\alpha)$ are convergent for all $f \in S$.

### 10.3.2 Proof of Følner's theorem

**Theorem 10.3.5** (Følner theorem)**.** *Let $G$ be a topological abelian group. If $k$ is a positive integer and $E$ is a subset of $G$ such that $k$ translates of $E$ cover $G$, then for every neighborhood $U$ of $0$ in $G$ there exist $\chi_1, \dots, \chi_m \in \widehat{G}$, where $m = k^2$, and $\delta > 0$ such that $U_G(\chi_1, \dots, \chi_m; \delta) \subseteq U - U + E_{(8)}$.*

*Proof.* We can assume, without loss of generality, that $U$ is open. By Følner's lemma 10.2.4, there exist $\varphi_1, \dots, \varphi_m \in G^*$ such that $U_G(\varphi_1, \dots, \varphi_m; \frac{\pi}{2}) \subseteq E_{(8)}$, where the characters $\varphi_j$ can be discontinuous. Our aim will be to replace these characters by continuous ones "enlarging" $E_{(8)}$ to $U - U + E_{(8)}$.

It follows from Lemma 3.4.2 that $C := \overline{E_{(8)} + U} \subseteq E_{(8)} + U - U$. Consider the open set $X = U \cup (G \setminus C)$ and the function $f : X \to \mathbb{C}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in G \setminus C \end{cases}$$

Then $f$ is continuous as $X = U \cup (G \setminus C)$ is a clopen partition of $X$.

Let $H$ be the group generated by $\varphi_1, \dots, \varphi_m$. Take $x, y \in X$ with $x - y \in U_G(\varphi_1, \dots, \varphi_m; \frac{\pi}{2}) \subseteq E_{(8)}$. So if $y \in U$ then $x \in E_{(8)} + U$ and consequently $x \notin G \setminus \overline{E_{(8)} + U}$, that is $x \in U$. In the same way it can be showed that $x \in U$

yields $y \in U$. This gives $f(x) = f(y)$ by the definition of $f$. So by Proposition 10.3.4 one can uniformly approximate $f$ on $X$ by characters of $H$. Hence one can find a finite number of $m$-uples $\tilde{j} = (j_1, \dots, j_m)$ of integers and $c_{\tilde{\jmath}} \in \mathbb{C}$ such that

$$\left|\sum_{\tilde{\jmath}} c_{\tilde{\jmath}} \varphi_1^{j_1}(x) \cdot \dots \cdot \varphi_m^{j_m}(x) - f(x)\right| \leq \frac{1}{3} \tag{13}$$

holds for every $x \in X$. Denote the product $\varphi_1^{j_1} \cdot \dots \cdot \varphi_m^{j_m}$ by $\xi_{\tilde{\jmath}}$. Since $X$ is open and $f$ is continuous, we can apply Lemma 10.3.1 to the convex closed set $M = \{z \in \mathbb{C} : |z| \leq \frac{1}{3}\}$ and this permits us to assume that all characters $\xi_{\tilde{\jmath}}$ are continuous. Letting $x = 0$ in (13) one gets $|\sum_{\tilde{\jmath}} c_{\tilde{\jmath}}| \leq \frac{1}{3}$, and consequently,

$$\frac{2}{3} \leq \left|\sum_{\tilde{\jmath}} c_{\tilde{\jmath}} - 1\right|. \tag{14}$$

Let now $\Phi$ be the subgroup of $H$ consisting of all *continuous* characters of $H$, i.e., $\Phi = H \cap \widehat{G}$. By Theorem 2.1.1 there exist $\chi_1, \dots, \chi_m \in \Phi$ that generate $\Phi$. Since the characters $\xi_{\tilde{\jmath}}$ are continuous, each $\xi_{\tilde{\jmath}}$ can be written as a product $\chi_1^{s_1(\tilde{\jmath})} \cdot \dots \cdot \chi_m^{s_m(\tilde{\jmath})}$ for appropriate $s_1(\tilde{\jmath}), \dots s_m(\tilde{\jmath}) \in \mathbb{Z}$. Choose $\varepsilon > 0$ with $\varepsilon \sum_{\tilde{\jmath}} |c_{\tilde{\jmath}}| < \frac{1}{3}$. Then there exists $\delta > 0$ such that $|\xi_{\tilde{\jmath}}(x) - 1| \leq \varepsilon$ for all summands $\xi_{\tilde{\jmath}}$ in (13) whenever $x \in U_G(\chi_1, \dots, \chi_m; \delta)$.

To prove

$$U_G(\chi_1, \dots, \chi_m; \delta) \subseteq U - U + E_{(8)}$$

assume for a contradiction that some $z \in U_G(\chi_1, \dots, \chi_m; \delta)$ and $z \notin U - U + E_{(8)}$. Since $C = \overline{E_{(8)} + U} \subseteq E_{(8)} + U - U$, then $z \in G \setminus C \subseteq X$. Thus, by the definition of $f$, (13), (14) and $|\xi_{\tilde{\jmath}}(z) - 1| \leq \varepsilon$, we have

$$\frac{2}{3} \leq \left|\sum_{\tilde{\jmath}} c_{\tilde{\jmath}} - 1\right| \leq \left|\sum_{\tilde{\jmath}} c_{\tilde{\jmath}}(1 - \xi_{\tilde{\jmath}}(z))\right| + \left|\sum_{\tilde{\jmath}} c_{\tilde{\jmath}} \xi_{\tilde{\jmath}}(z) - f(z)\right| \leq \varepsilon \sum_{\tilde{j}} |c_{\tilde{\jmath}}| + \frac{1}{3}.$$

These inequalities together give $\frac{2}{3} \leq \varepsilon \sum_{\tilde{\jmath}} |c_{\tilde{\jmath}}| + \frac{1}{3}$. This contradicts the choice of $\varepsilon$. □

### 10.3.3 Independence of characters

We apply now Prodanov's lemma for indiscrete $G$ and $U = G$. Note that this necessarily yields $f$ is a constant function.

**Corollary 10.3.6.** *Let $G$ be an abelian group, $g \in \mathfrak{X}_0(G)$ and $M$ be a closed convex subset of $\mathbb{C}$. If $g(x) + c \in M$ for some $c \in \mathbb{C}$ and for every $x \in G$, then $c \in M$.*

*Proof.* Assume that $g(x) = \sum_{j=1}^k c_j \chi_j(x)$ for some $c_1, \dots, c_k \in \mathbb{C}$ and non-constant $\chi_1, \dots, \chi_k \in G^*$. Apply Lemma 10.3.1 with $G$ indiscrete, $U = G$ and $f$ the constant function $c$. Since all characters $\chi_1, \dots, \chi_k$ are discontinuous, we conclude $c \in M$ with Lemma 10.3.1. □

**Corollary 10.3.7.** *Let $G$ be an abelian group, $g \in \mathfrak{X}_0$ and $\varepsilon \geq 0$. If $|g(x) - c| \leq \varepsilon$ for some $c \in \mathbb{C}$ and for every $x \in G$. Then $|c| \leq \varepsilon$.*

*Proof.* Follows from the above corollary with $M$ the closed disk with center 0 and radius $\varepsilon$. □

**Corollary 10.3.8.** *Let $G$ be an abelian group, and let $\chi_0, \chi_1, \dots, \chi_k \in G^*$ be distinct characters. Then $\chi_0, \chi_1, \dots, \chi_k$ are linearly independent.*

Using this corollary we shall see now that for an abelian group $G$ the characters $G^*$ not only span $\mathfrak{X}(G)$ as a base, but they have a much stronger independence property.

**Corollary 10.3.9.** *Let $G$ be an abelian group, and let $\chi_0, \chi_1, \dots, \chi_k \in G^*$ be distinct characters. Then $\|\chi_0 - \sum_{j=1}^k c_j \chi_j\| \geq 1$ for every $c_1, \dots, c_k \in \mathbb{C}$.*

*Proof.* Let $\varepsilon = \|\sum_{j=1}^k c_j \chi_j - \chi_0\|$. Then

$$\big|\sum_{j=1}^k c_j \chi_j(x) - \chi(x)\big| \leq \varepsilon \tag{1}$$

for every $x \in G$. According to the previous corollary, $\chi_0, \chi_1, \dots, \chi_k$ are linearly independent, hence $\varepsilon > 0$.

By our assumption $\xi_j = \chi_j \chi^{-1}$ is non-constant for every $j = 1, 2, \ldots, n$. So $g = \sum_{j=1}^{m} \xi_j \in \mathfrak{X}_0$ and (1) yields

$$|g(x) - 1| = \Big| \sum_{j=1}^{m} c_j \chi_j(x) \chi^{-1}(x) - 1 \Big| \leq \varepsilon$$

for every $x \in G$. According to the previous corollary $|1| \leq \varepsilon$. □

**Corollary 10.3.10.** *Let $G$ be an abelian group, $H \leq G^*$ and $\chi \in G^*$ such that there exist $k \in \mathbb{N}_+$, $\chi_1, \ldots, \chi_k \in H$ and $c_1, \ldots, c_k \in \mathbb{C}$ such that*

$$\left| \sum_{j=1}^{k} c_j \chi_j(x) - \chi(x) \right| \leq \frac{1}{2} \tag{2}$$

*for every $x \in G$. Then $\chi = \chi_i$ for some $i$ (hence $\chi \in H$).*

*Proof.* We can assume without loss of generality that $\chi_1, \ldots, \chi_k$ are pairwise distinct. Assume for a contradiction that $\chi \neq \chi_j$ for all $j = 1, 2, \ldots, k$. Then the previous corollary applied to $\chi, \chi_1, \ldots, \chi_k$ yields $\| \sum_{j=1}^{k} c_j \chi_j - \chi \| \geq 1$. This contradicts (2). Therefore, $\chi = \chi_j$ for some $j = 1, 2, \ldots, k$, so $\chi \in H$. □

We obtain as an immediate consequence of Corollary 10.3.10 the following fact of independent interest: the continuous characters of $(G, \mathcal{T}_H)$ are precisely the characters of $H$.

**Corollary 10.3.11.** *Let $G$ be an abelian group. Then $H = \widehat{(G, \mathcal{T}_H)}$ for every $H \leq G^*$.*

*Proof.* Obviously, $H \subseteq \widehat{(G, \mathcal{T}_H)}$. Now let $\chi \in \widehat{(G, \tau_H)}$. For every fixed $\varepsilon > 0$ the set $O = \{a \in \mathbb{S} : |a - 1| < \varepsilon\}$ is an open neighborhood of 1 in $\mathbb{S}$. Hence $W = \chi^{-1}(O)$ is $\mathcal{T}_H$-open in $G$. So there exist $\chi_1, \ldots, \chi_m \in H$ and $\delta > 0$ such that $U_G(\chi_1, \ldots, \chi_m; \delta) \subseteq W$. Now, if $x - y \in U_G(\chi_1, \ldots, \chi_m; \delta)$ then $\chi(x - y) \in O$, so $|\chi(x)\chi^{-1}(y) - 1| < \varepsilon$. So $|\chi(x) - \chi(y)| < \varepsilon$. In other words, $\chi$ satisfies condition (b) of Proposition 10.3.4. Hence there exist $\chi_1, \ldots, \chi_m \in H$ and $c_1, \ldots, c_m \in \mathbb{C}$ such that $\big| \sum_{j=1}^{m} c_j \chi_j(x) - \chi(x) \big| \leq \frac{1}{2}$ for every $x \in G$. By Corollary 10.3.10 this yields $\chi \in H$. □

# 11 Applications of Følner's theorem

In this section we prove Peter-Weyl's theorem using Følner's theorem. Moreover, we use Prodanov's lemma to describe the precompact topologies of the abelian groups and to easily build the Haar integral of a compact abelian group.

## 11.1 Precompact group topologies on abelian groups

Let us recall here that for an abelian group $G$ and a subgroup $H$ of $G^*$, the group topology $\mathcal{T}_H$ generated by $H$ is the coarsest group topology on $G$ that makes every character from $H$ continuous. We recall its description and properties in the next proposition:

**Proposition 11.1.1.** *Let $G$ be an abelian group and let $H$ be a group of characters of $G$. A base of the neighborhoods of 0 in $(G, \mathcal{T}_H)$ is given by the sets $U(\chi_1, \ldots, \chi_m; \delta)$, where $\chi_1, \ldots, \chi_m \in H$ and $\delta > 0$. Moreover $(G, \mathcal{T}_H)$ is a Hausdorff if and only if $H$ separates the points of $G$.*

Now we can characterize the precompact topologies on abelian groups.

**Theorem 11.1.2.** *Let $(G, \tau)$ be an abelian group. The following conditions are equivalent:*

*(a) $\tau$ is precompact;*

*(b) $\tau$ is Hausdorff on $G$ and the neighborhoods of 0 in $G$ are big subsets;*

*(c) there exists a group $H$ of continuous characters of $G$ that separates the points of $G$ and such that $\tau = \mathcal{T}_H$.*

*Proof.* (a)⇒(b) is the definition of precompact topology.

(b)⇒(c) If $H = \widehat{(G, \tau)}$ then $\mathcal{T}_H \subseteq \tau$. Let $U$ and $V$ be open neighborhoods of 0 in $(G, \tau)$ such that $V_{(10)} \subseteq U$. Then $V$ is big and by Følner's Theorem 10.3.5 there exist continuous characters $\chi_1, \ldots, \chi_m$ of $G$ such that $U_G(\chi_1, \ldots, \chi_m; \delta) \subseteq V_{(10)} \subseteq U$ for some $\delta > 0$. Thus $U \in \mathcal{T}_H$ and $\tau \subseteq \mathcal{T}_H$.

(c)⇒(a) Even if this implication is contained in Corollary 9.2.16, we give a direct proof here. Let $i : G \to \mathbb{S}^H$ be defined by $i(g) = i_g : H \to \mathbb{S}$ (if $g \in G$) with $i_g(\chi) = \chi(g)$ for every $\chi \in H$. Since $H$ separates the points of $G$, the function $i$ is injective. The product $\mathbb{S}^H$ endowed with the product topology is compact and so $i$ is a topological immersion by Proposition 11.1.1. The closure of $i(G)$ in $\mathbb{S}^H$ is compact and $\widetilde{G}$ is isomorphic to it, hence $\widetilde{G}$ is compact. □

**Remark 11.1.3.** The above theorem essentially belongs to Comfort and Ross [29]. It can be given in the following simpler "Hausdorff-free" version: $\tau$ is totally bounded iff $\tau = \mathcal{T}_H$ for some group $H$ of continuous characters of $G$. Indeed, let $N$ denote the closure of 0 in $(G, \tau)$ and let $\bar{\tau}$ denote the quotient topology of $G/N$. Then $(G, \bar{\tau})$ is precompact iff $(G, \tau)$ totally bounded. On the other hand, if $(G, \tau)$ totally bounded, then the neighborhoods of 0 in $(G, \tau)$ are big. Finally, if the neighborhoods of 0 in $(G, \tau)$ are big, then an application of Følner's Theorem 10.3.5 gives, as above, $\tau \subseteq \mathcal{T}_{\widehat{(G,\tau)}}$.

Theorem 11.1.4 will allow us to sharpen this property (see Corollary 11.2.3).

**Theorem 11.1.4.** *Let $G$ be an abelian group. Let $\mathcal{D}(G)$ be the set of all groups of characters of $G$ separating the points of $G$ and $\mathcal{P}$ be the set of all precompact group topologies on $G$. Then the map $T : \mathcal{D}(G) \to \mathcal{P}$, $\mathcal{D}(G) \ni H \overset{T}{\mapsto} \mathcal{T}_H \in \mathcal{P}$, is an order preserving bijection (if $H_1, H_2 \in \mathcal{D}(G)$ then $\mathcal{T}_{H_1} \subseteq \mathcal{T}_{H_2}$ if and only if $H_1 \subseteq H_2$).*

*Proof.* The equivalence (a)$\Leftrightarrow$(c) of Theorem 11.1.2 yields that $\mathcal{T}_H \in \mathcal{P}$ for every $H \in \mathcal{D}(G)$ and that $T$ is surjective.

By Corollary 10.3.11, $\mathcal{T}_{H_1} = \mathcal{T}_{H_2}$ for $H_1, H_2 \in \mathcal{H}$ yields $H_1 = H_2$. Therefore, $T$ is a bijection.

The last statement of the theorem is obvious. □

We proved in Corollary 10.3.11 that for a subgroup $H$ of $G^*$ the continuous characters of $(G, \mathcal{T}_H)$ are precisely the characters of $H$. This allows us to prove that $w(G) = \chi(G) = |\widehat{G}|$ for precompact abelian groups:

**Corollary 11.1.5.** *If $G$ an abelian group and $H \leq G^*$, then $w(G, \mathcal{T}_H) = \chi(G, \mathcal{T}_H) = |H|$.*

*Proof.* According to Exercise 6.1.13, $w(G, \mathcal{T}_H) \leq |H|$. Let $\kappa = \chi(G, \mathcal{T}_H)$. We aim to prove that $\kappa \geq |H|$. Then we obtain $\chi(G, \mathcal{T}_H) \geq \kappa \geq |H| \geq w(G, \mathcal{T}_H) \geq \chi(G, \mathcal{T}_H)$, thus $\chi(G, \mathcal{T}_H) = w(G, \mathcal{T}_H) = |H|$. Pick a base $\mathcal{B}$ of the neighborhoods at 0 of $\mathcal{T}_H$ of size $\leq \kappa$. By the definition of $\mathcal{T}_H$, every element $B \in \mathcal{B}$ can be written as $B = U_G(\chi_{1,B}, \ldots, \chi_{n_B,B}; 1/m_B)$, where $n_B, m_B \in \mathbb{N}$ and $\chi_{i,B} \in H$ for $i = 1, \ldots, n_B$. Then the subset $H' = \{\chi_{i,B} : B \in \mathcal{B}, i = 1, \ldots, n_B\}$ of $H$ has $|H'| \leq \kappa$ and produces the topology $\mathcal{T}_{H'}$ that is finer than $\mathcal{T}_H$, by the choice of $\mathcal{B}$. On the other hand, $\mathcal{T}_{H'} \leq \mathcal{T}_{\langle H' \rangle} \leq \mathcal{T}_H$. Therefore, $\mathcal{T}_{\langle H' \rangle} = \mathcal{T}_H$. By Theorem 11.1.4, $\langle H' \rangle = H$. This gives $|H| = |H'| \leq \kappa$, as desired. □

**Corollary 11.1.6.** *Let $G$ an abelian group and $H \leq G^*$ such that $\mathcal{T}_H$ is metrizable. Then $H$ is countable.*

## 11.2 Peter-Weyl's theorem for compact abelian groups

Let us start with the following important consequence of Theorem 11.1.2.

**Corollary 11.2.1** (Peter-Weyl's theorem)**.** *If $G$ is a compact abelian group, then $\widehat{G}$ separates the points of $G$.*

*Proof.* Let $\tau$ be the topology of $G$. By Theorem 11.1.2 there exists a group $H$ of continuous characters of $G$ (i.e., $H \subseteq \widehat{G}$) such that $\tau = \mathcal{T}_H$. Since $\tau \supseteq \mathcal{T}_{\widehat{G}}$ and $H \subseteq \widehat{G}$ we conclude that $H = \widehat{G}$ separates the points of $G$. □

**Corollary 11.2.2.** *If $G$ is a compact abelian group, then $G$ is isomorphic to a (closed) subgroup of the power $\mathbb{T}^{\widehat{G}}$.*

*Proof.* Since the characters $\chi \in \widehat{G}$ separate the points of $G$, the diagonal map determined by all characters defines a continuous injective homomorphism $\Delta_{\widehat{G}} : G \hookrightarrow \mathbb{T}^{\widehat{G}}$. By the compactness of $G$ and the open mapping theorem, this is the required embedding. □

Let us note here that the power $\mathbb{T}^{\widehat{G}}$ is the smallest possible one with this property. Indeed, if $G$ embedds into some power $\mathbb{T}^\kappa$, then $\kappa = w(\mathbb{T}^\kappa) \geq w(G) = |\widehat{G}|$.

As a corollary of Theorem 11.1.4 we obtain the following useful fact that completes Corollary 11.2.1. It will be essentially used in the proof of the duality theorem.

**Corollary 11.2.3.** *If $(G, \tau)$ is a compact abelian group and $H \leq \widehat{G}$ separates the points of $G$, then $H = \widehat{G}$.*

*Proof.* By Theorem 11.1.2 it holds $\tau = \mathcal{T}_{\widehat{G}}$. Since $\mathcal{T}_H \subseteq \mathcal{T}_{\widehat{G}}$ by Theorem 11.1.4 and $\mathcal{T}_H$ is Hausdorff, then $\mathcal{T}_H = \mathcal{T}_{\widehat{G}}$. Now again Theorem 11.1.4 yields $H = \widehat{G}$. □

We show now that every compact abelian group is an inverse limit of elementary compact abelian groups (see Definition 8.3.1).

**Proposition 11.2.4.** *Let $G$ be a compact abelian group and let $U$ be an open neighborhood of 0 in $G$. Then there exists a closed subgroup $C$ of $G$ such that $C \subseteq U$ and $G/C$ is an elementary compact abelian group. In particular, $G$ is an inverse limit of elementary compact abelian groups.*

*Proof.* By the Peter-Weyl Theorem 11.2.1 $\bigcap_{\chi\in\widehat{G}}\ker\chi = \{0\}$ and each $\ker\chi$ is a closed subgroup of $G$. By the compactness of $G$ there exists a finite subset $F$ of $\widehat{G}$ such that $C = \bigcap_{\chi\in F}\ker\chi \subseteq U$. Define now $g = \prod_{\chi\in F}\chi : G \to \mathbb{T}^F$. Thus $\ker g = C$ and $G/C$ is topologically isomorphic to the closed subgroup $g(G)$ of $\mathbb{T}^F$ by the compactness of $G$. So $G/C$ is elementary compact abelian by Lemma 8.3.3.

To prove the last statement, fix for every open neighborhood $U_i$ of 0 in $G$ a closed subgroup $C_i$ of $G$ with $C_i \subseteq U$ and such that $G/C_i$ is elementary compact abelian. Note that for $C_i$ and $C_j$ obtained in this way the subgroup $C_i\cap C_j$ has the same property as $G/C_i\cap C_j$ is isomorphic to a closed subgroup of the product $G/C_i \times G/C_j$ which is again an elementary compact abelian group. Enlarging the family $(C_i)$ with all finite intersections we obtain an inverse system of elementary compact abelian groups $G/C_i$ where the connecting homomorphisms $G/C_i \to G/C_j$, when $C_i \leq C_j$, are simply the induced homomorphisms. Then the inverse limit $G'$ of this inverse system is a compact abelian group together with a continuous homomorphism $f : G \to G'$ induced by the projections $p_i : G \to G/C_i$. Assume $x \in G$ is non-zero. Pick on open neighborhood $U$ of 0. By the first part of the proof, there exists $C_i \subseteq U$, hence $x \notin C_i$. Therefore, $p_i(x) \neq 0$, so $f(x) \neq 0$ as well. This proves that $f$ is injective. To check surjectivity of $f$ take an element $x' = (x_i + C_i)$ of the inverse limit $G'$. Then the family of closed cosets $x_i + C_i$ in $G$ has the finite intersection property, so has a non-empty intersection. For every element $x$ of that intersection one has $f(x) = x'$. Finally, the continuous isomorphism $f : G \to G'$ must be open by the compactness of $G$. $\square$

For a topological abelian group $G$ we say that $G$ has *no small subgroups*, or shortly, $G$ is *NSS*, if there exists a neighborhood $U$ of 0 such that $U$ contain no non-trivial subgroups of $G$. The next corollary follows immediately from the above proposition:

**Corollary 11.2.5.** *A compact abelian group $G$ has no small subgroups precisely when $G$ is an elementary compact abelian group.*

## 11.3 On the structure of compactly generated locally compact abelian groups

From now on all groups are Hausdorff; quotients are taken for closed subgroups and so they are still Hausdorff.

**Proposition 11.3.1.** *Let $G$ be a compactly generated locally compact abelian group. Then there exists a discrete subgroup $H$ of $G$ such that $H \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$ and $G/H$ is compact.*

*Proof.* Suppose first that there exist $g_1, \dots, g_m \in G$ such that $G = \overline{\langle g_1, \dots, g_m\rangle}$. We proceed by induction. For $m = 1$ apply Lemma 9.2.7: if $G$ is infinite and discrete take $H = G$ and if $G$ is compact $H = \{0\}$. Suppose now that the property holds for $m \geq 1$ and $G = \overline{\langle g_1, \dots, g_{m+1}\rangle}$. If every $\overline{\langle g_i\rangle}$ is compact, then so is $G$ and $H = \{0\}$. If $\langle g_{m+1}\rangle$ is discrete, consider the canonical projection $\pi : G \to G_1 = G/\langle g_{m+1}\rangle$. Since $G_1$ has a dense subgroup generated by $m$ elements, by the inductive hypothesis there exists a discrete subgroup $H_1$ of $G_1$ such that $H_1 \cong \mathbb{Z}^n$ and $G_1/H_1$ is compact. Therefore $H = \pi^{-1}(H_1)$ is a closed countable subgroup of $G$. Thus $H$ is locally compact and countable, hence discrete by Lemma 7.2.7.

Since $H$ is finitely generated, it is isomorphic to $H_2 \times F$, where $H_2 \cong \mathbb{Z}^s$ for some $s \in \mathbb{N}$ and $F$ is a finite abelian group (see Theorem 2.1.1). Now $G/H$ is isomorphic to $G_1/H_1$ and $H/H_2$ is finite, so $G/H_2$ is compact thanks to Lemma 7.2.4.

Now consider the general case. There exists a compact subset $K$ of $G$ that generates $G$. By Lemma 7.2.11 we can assume wlog that $K = \overline{U}$, where $U$ is a symmetric neighborhood of 0 in $G$ with compact closure. We show now that there exists a finite subset $F$ of $G$ such that

$$K + K \subseteq K + \langle F\rangle. \tag{2}$$

In fact, pick a symmetric neighborhood $V$ of 0 in $G$ such that $V + V \subseteq U$. For the compact set $K$ satisfying $K \subseteq \bigcup_{x\in K}(x + V)$ there exists a finite subset $F$ of $K$ such that $K \subseteq \bigcup_{x\in F}(x + V) = F + V$. Then

$$K + K \subseteq F + F + V + V \subseteq \langle F\rangle + U \subseteq \langle F\rangle + K.$$

gives (2). An easy inductive argument shows that $\langle K\rangle = G$ and (2) imply $G = \langle K\rangle \subseteq K + \langle F\rangle$.

Let $G_1 = \langle F\rangle$. By $G = \langle F\rangle + K$ the quotient $\pi(K) = G/G_1$ is compact. By the first part of the proof there exists a discrete subgroup $H$ of the locally compact subgroup $G_1$ of $G$, such that $H \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$ and $G_1/H$ is compact. Since $G_1/H$ is a compact subgroup of $G/H$ such that $(G/H)/(G_1/H) \cong G/G_1$ is compact, we conclude that also $G/H$ is compact. $\square$

**Proposition 11.3.2.** *Let $G$ be a compactly generated locally compact abelian group. Then there exists a compact subgroup $K$ of $G$ such that $G/K$ is elementary locally compact abelian.*

*Proof.* By Proposition 11.3.1 there exists a discrete subgroup $H$ of $G$ such that the quotient $G/H$ is compact. Consider the canonical projection $\pi$ of $G$ onto $G/H$. Let $U$ be a compact symmetric neighborhood of 0 in $G$ such that $(U+U+U)\cap H=\{0\}$. So $\pi(U)$ is a neighborhood of 0 in $G/H$ and applying Lemma 11.2.4 we find a closed subgroup $L\supseteq H$ of $G$ such that the closed subgroup $C=L/H$ of $G/H$ satisfies

$$C=L/H\subseteq\pi(U) \text{ and } (G/H)/(L/H)=G/L\cong\mathbb{T}^t\times F, \tag{4}$$

where $F$ is a finite abelian group and $t\in\mathbb{N}$, i.e., $G/L$ is elementary compact abelian.

The set $K=L\cap U$ is compact being closed in the compact neighborhood $U$. Let us see now that $K$ is a subgroup of $G$. To this end take $x,y\in K$. Then $x-y\in L$ and $\pi(x-y)\in C\subseteq\pi(U)$. Thus $\pi(x-y)=\pi(u)$ for some $u\in U$. As $\pi(x-y-u)=0$ in $G/H$, one has $x-y-u\in(U+U+U)\cap H=\{0\}$. Hence $x-y=u\in L\cap U=K$.

Now take $x\in L$; consequently $\pi(x)\in C\subseteq\pi(U)$ so $\pi(x)=\pi(u)$ for some $u\in U$. Clearly, $u\in L\cap U=K$, hence $\pi(L)=\pi(K)$. Thus $L=K+H$ and $K\cap H=\{0\}$ yields that the canonical projection $l:G\to G/K$ restricted to $H$ is a continuous isomorphism of $H$ onto $l(H)=l(L)$. Let us see now that $l(H)$ is discrete. To this end we apply Lemma 7.2.3 to deduce first that $l(H)$ is closed. Since $H$ is discrete, $\{0_G\}$ is open in $H$, so $A:=H\setminus\{0_G\}$ is a closed subset of $H$, hence of $G$ as well. Again by Lemma 7.2.3, $l(A)=l(H)\setminus\{0_{G/K}\}$ is closed in $G/K$. Hence, $\{0_{G/K}\}$ is open in $l(H)$. Thus

$$l(L)=l(H)\cong H\cong\mathbb{Z}^s$$

is discrete in $G/K$.

Observe that (4) yields the following isomorphisms:

$$(G/K)/l(L)=(G/K)/(L/K)\cong G/L\cong\mathbb{T}^t\times F.$$

Denote by $\varrho$ the composition $G/K\to G/L\to\mathbb{T}^t\times F$ and note that $l(L)=\ker\varrho$ is a discrete subgroup $G/K$. Hence to $\varrho:G/K\to G/L$ and the composition $q:\mathbb{R}^t\to G/L$ of the canonical projection $\mathbb{R}^t\to\mathbb{T}^t$ and the obvious inclusion of $\mathbb{T}^t$ in $G/L$ one can apply Lemma 8.0.7[11] to obtain an open continuous homomorphism $f:\mathbb{R}^t\to G/K$ such that $\varrho\circ f=q$. In particular, $N=f(\mathbb{R}^t)$ is an open subgroup of $G/K$ as has a non-empty interior (as $q$ and $\varrho$ are local homeomorphisms). As $f:\mathbb{R}^t\to N$ is open by Theorem 7.3.1, $N$ is isomorphic to a quotient of $\mathbb{R}^t$, so $N$ is an elementary locally compact abelian. Since $N$ is divisible (as a quotient of $\mathbb{R}^t$), by Lemma 2.1.13 $G/K=N\times B$ where $B$ is a discrete subgroup of $G/K$ because $N\cap B=\{0\}$ and $N$ is open. Moreover $B$ is compactly generated as it is a quotient of $G$. Since it is also discrete, $B$ is finitely generated. Therefore, $G/K=N\times B$ is an elementary locally compact abelian as well. □

To prove the Pontryagin-van Kampen duality theorem in the general case (for $G\in\mathcal{L}$), we need Theorem 11.3.3, which generalizes the Peter-Weyl Theorem 11.2.1.

**Theorem 11.3.3.** *If $G$ is a locally compact abelian group, then $\widehat{G}$ separates the points of $G$.*

*Proof.* Let $V$ be a compact neighborhood of 0 in $G$. Take $x\in G\setminus\{0\}$. Then $G_1=\langle V\cup\{x\}\rangle$ is an open (it has non-void interior) compactly generated subgroup of $G$. In particular $G_1$ is locally compact. By Proposition 11.3.1 there exists a discrete subgroup $H$ of $G_1$ such that $H\cong\mathbb{Z}^m$ for some $m\in\mathbb{N}$ and $G_1/H$ is compact. Thus $\bigcap_{n\in\mathbb{N}_+}nH=\{0\}$ and so there exists $n\in\mathbb{N}_+$ such that $x\notin nH$. Since $H/nH$ is finite, the quotient $G_2=G_1/nH$ is compact by Lemma 7.2.4. Consider the canonical projection $\pi:G_1\to G_2$ and note that $\pi(x)=y\neq 0$ in $G_2$. By the Peter-Weyl Theorem 11.2.1 there exists $\xi\in\widehat{G}$ such that $\xi(y)\neq 0$. Consequently $\chi=\xi\circ\pi\in\widehat{G_1}$ and $\chi(x)\neq 0$. By Theorem 2.1.10 there exists $\overline{\chi}\in\widehat{G}$ such that $\overline{\chi}\restriction_{G_1}=\chi$. Since $G_1$ is an open subgroup of $G$, this extension will be continuous (as its restriction to $G_1$ is continuous). □

**Corollary 11.3.4.** *Let $G$ be a locally compact abelian group and $K$ a compact subgroup of $G$. Then for every $\chi\in\widehat{K}$ there exists $\xi\in\widehat{G}$ such that $\xi\restriction_K=\chi$.*

*Proof.* Define $H=\{\chi\in\widehat{K}$ : there exists $\xi\in\widehat{G}$ with $\xi\restriction_K=\chi\}$. By Theorem 11.3.3 the continuous characters of $G$ separate the points of $G$. Therefore $H$ separate the points of $K$. Now apply Corollary 11.2.3 to conclude that $H=\widehat{K}$. □

Here is another corollary of Theorem 11.3.3:

**Corollary 11.3.5.** *A $\sigma$-compact and locally compact abelian group is totally disconnected iff for every continuous character $\chi$ of $G$ the image $\chi(G)$ is a proper subgroup of $\mathbb{T}$.*

[11]The reader who is familiar with covering maps may deduce the existence of such a lifting from the facts that $\varrho$ is a covering homomorphism and $\mathbb{R}^t$ is simply connected.

*Proof.* Assume that $G$ is a locally compact abelian group such that $\chi(G)$ is a proper subgroup of $\mathbb{T}$ for every continuous character $\chi$ of $G$. According to Theorem 11.3.3 the diagonal homomorphism $f : G \to \prod\{\chi(G) : \chi \in \widehat{G}\}$ of all $\chi \in \widehat{G}$ is injective. Since the proper subgroups of $\mathbb{T}$ are totally disconnected, the whole product will be totally disconnected, so also $G$ will be totally disconnected. Now assume that $G$ is $\sigma$-compact, locally compact and totally disconnected. Consider $\chi \in \widehat{G}$ and assume for a contradiction that $\chi(G) = \mathbb{T}$. Then $\chi : G \to \mathbb{T}$ will be an open map by the open mapping theorem, so $\mathbb{T}$ will be a quotient of $G$. As total disconnectedness is inherited by quotiens of locally compact groups (see Corollary 7.4.3), we conclude that $\mathbb{T}$ must be totally disconnected, a contradiction. □

One cannot remove "$\sigma$-compact" in the above corollary. Indeed, let $G$ denote $\mathbb{T}$ equipped with the discrete topology. Then $G$ is totally disconnected, although the identity map $\chi : G \to \mathbb{T}$ provides a character with $\chi(G) = \mathbb{T}$.

Algebraic properties of the dual group $\widehat{G}$ of a compact abelian group $G$ can be described in terms of topological properties of the group $G$. We prove in Corollary 11.3.6 that $\widehat{G}$ is torsion precisely when $G$ is totally disconnected:

**Corollary 11.3.6.** *A compact abelian group is totally disconnected iff every continuous character of $G$ is torsion.*

*Proof.* For a compact abelian group $G$ the image $\chi(G)$ under a continuous character $\chi$ of $G$ is a compact, hence closed subgroup of $\mathbb{T}$. Hence $\chi(G)$ is a proper subgroup of $\mathbb{T}$ precisely when it is finite. This means that the character $\chi$ is torsion. □

Compactness plays an essential role here. We shall see examples of totally disconnected $\sigma$-compact and locally compact abelian groups $G$ such that no continuous character of $G$ is torsion (e.g., $G = \mathbb{Q}_p$).

Here is the counterpart of this property in the connected case:

**Proposition 11.3.7.** *Let $G$ be a topological abelian group.*

(a) *If $G$ is connected, then the dual group $\widehat{G}$ is torsion-free.*

(b) *If $G$ is compact, then the dual group $\widehat{G}$ is torsion-free iff $G$ is connected.*

*Proof.* (a) Since for every non-zero continuous character $\chi : G \to \mathbb{T}$ the image $\chi(G)$ is a non-trivial connected subgroup of $\mathbb{T}$, we deduce that $\chi(G) = \mathbb{T}$ for every non-zero $\chi \in \widehat{G}$. Hence $\widehat{G}$ is torsion-free.

(b) If the group $G$ is compact and disconnected, then by Theorem 7.4.2 there exists a proper open subgroup $N$ of $G$. Take any non-zero character $\xi$ of the finite group $G/N$. Then $m\xi = 0$ for some positive integer $m$ (e.g., $m = [G : N]$). Now the composition $\chi$ of $\xi$ and the canonical homomorphism $G \to G/N$ satisfies $m\chi = 0$ as well. So $\widehat{G}$ has a non-zero torsion character. This proves the implication left open by item (a). □

## 11.4 Almost periodic functions and Haar integral in compact abelian groups

### 11.4.1 Almost periodic functions of the abelian groups

**Example 11.4.1.** Let $f : \mathbb{R} \to \mathbb{C}$ be a function. One says that $a \in \mathbb{R}$ is a period of $f$, if $f(x + a) = f(x)$ for every $x \in \mathbb{R}$ (i.e., $f_a = f$). Clearly, if $a \in \mathbb{R}$ is a period of $f$, then also $ka$ is a period of $f$ for every $k \in \mathbb{Z}$. More precisely, the periods of $f$ form a subgroup $\Pi(f)$ of $\mathbb{R}$. Call $f$ *periodic* if $\Pi(f) \neq \{0\}$.

It is easy to see that $f$ has period $a$ iff $f$ factorizes through the quotient homomorphism $\mathbb{R} \to \mathbb{R}/\langle a\rangle$. Since $\mathbb{R}/\langle a\rangle \cong \mathbb{T}$ is compact, this explains the great importance of the periodic functions, i.e., these are the functions that can be factorized through the compact circle group $\mathbb{T}$.

**Exercise 11.4.2.** *Let $G$ be an abelian group. Call $a \in G$ a period of a function $f : G \to \mathbb{C}$ if $f(x+a) = f(x)$ for every $x \in G$. Prove that:*

(a) *the subset $\Pi(f)$ of all periods of $f$ is a subgroup of $G$ and $f$ factorizes through the quotient map $G \to G/\Pi(f)$;*

(b) *$\Pi(f)$ is the largest subgroup such that $f$ is constant on each coset of $\Pi(f)$;*

(c) *if $G$ is a topological group and $f$ is continuous, then $\Pi(f)$ is a closed subgroup of $G$.*

(d) *if $f : \mathbb{R} \to \mathbb{C}$ is a continuous non-constant function, then there exists a smallest positive period $a$ of $f$.*

**Definition 11.4.3.** For an abelian group $G$, a function $f : G \to \mathbb{C}$ and $\varepsilon > 0$ an element $a \in G$ is called an *$\varepsilon$-almost period* of $f$ if $\|f - f_a\| \leq \varepsilon$.

Let

$$T(f, \varepsilon) = \{a \in G : a \text{ is an } \varepsilon\text{-almost period of } f\}.$$

**Exercise 11.4.4.** *Let $G$ be an abelian group and let $f : G \to \mathbb{C}$ be a function. Prove that $\{T(f,\varepsilon) : \varepsilon > 0\}$ is a filter-base of neighborhoods of 0 in a group topology $T_f$ on $G$.*

(Hint. Note that $-T(f,\varepsilon) = T(f,\varepsilon)$ and $T(f,\varepsilon/2) + T(f,\varepsilon/2) \subseteq T(f,\varepsilon)$ for every $\varepsilon$.)

Now we use the group topology $T_f$ to find an equivalent description of almost periodicity of $f$.

**Proposition 11.4.5.** *Let $G$ be an abelian group. Then for every function $f : G \to \mathbb{C}$ the following are equivalent:*

*(a) $f$ is almost periodic*

*(b) $T_f$ is totally bounded.*

*Proof.* Clearly, $T_f$ is totally bounded iff for every $\varepsilon > 0$ the set $T(f,\varepsilon)$ is big, i.e., for every $\varepsilon > 0$ there exist $a_1, \dots, a_n \in G$ such that $G = \bigcup_{k=1}^n a_k + T(f,\varepsilon)$.

(a) $\to$ (b) Arguing for a contradiction assume that $T_f$ is not totally bounded. Then by Lemma 9.2.6 there exists some $\varepsilon > 0$ such that $T(f,\varepsilon)$ is small, so there exists a sequence $(b_n)$ in $G$ such that the sets $b_n + T(f,\varepsilon)$ are pairwise disjoint. By the almost periodicity of $f$ the sequence of translates $(f_{b_m})$ admits a subsequence that is Cauchy w.r.t. the uniform topology of $B(G)$. In particular, one can find two *distinct* indexes $n < m$ such that $\|f_{b_m} - f_{b_n}\| = \|f_{b_m - b_n} - f\| \leq \varepsilon$, i.e., $b_m - b_n \in T(f,\varepsilon) \subseteq T(f,\varepsilon) - T(f,\varepsilon)$. Hence $(b_m + T(f,\varepsilon)) \cap (b_n + T(f,\varepsilon)) \neq \emptyset$, a contradiction.

(b) $\to$ (a) It suffices to check that every infinite sequence of translates $(f_{b_m})$ of $f$ admits a subsequence that is Cauchy w.r.t. the uniform topology of $B(G)$. From the completeness of $B(G)$, we deduce then that this subsequence converges in the uniform topology of $B(G)$, so $f$ is almost periodic.

Assume for a contradiction that some sequence of translates $(f_{b_m})$ admits no Cauchy subsequence. That is, for every subsequence $(f_{b_{m_k}})$ there exists an $\varepsilon > 0$ such that for some subsequence $m_{k_s}$ of $m_k$ one has

$$\|f_{b_{m_{k_s}}} - f_{b_{m_{k_t}}}\| = \|f_{b_{m_{k_s}} - b_{m_{k_t}}} - f\| \geq 2\varepsilon \text{ for all } s \neq t. \tag{3}$$

Since our hypothesis implies that $T(f,\varepsilon/2)$ is big, there exist $a_1, \dots, a_n \in G$ such that $G = \bigcup_{k=1}^n a_k + T(f,\varepsilon/2)$. Since infinitely many $b_{m_{k_s}}$ will fall in the same $a_k + T(f,\varepsilon/2)$ for some $k$, we deduce that for distinct $s$ and $t$ with $b_{m_{k_s}}, b_{m_{k_t}} \in a_k + T(f,\varepsilon/2)$ one has

$$b_{m_{k_s}} - b_{m_{k_t}} \in T(f,\varepsilon/2) - T(f,\varepsilon/2) \subseteq T(f,\varepsilon), \text{ i.e., } \|f_{b_{m_{k_s}} - b_{m_{k_t}}} - f\| \leq \varepsilon.$$

This contradicts (3). □

**Example 11.4.6.** Let $\chi \in G^*$. Then $T(\chi,\varepsilon) = \{a \in G : |\chi(a) - 1| < \varepsilon\}$ contains $U_G(\chi;\delta)$ for an appropriate $\delta$, hence $T(\chi,\varepsilon)$ is big. Consequently, $\chi$ is almost periodic.

**Theorem 11.4.7** (Bohr-von Neumann Theorem). *$A(G) = \mathfrak{A}(G)$ for every abelian group $G$, i.e., $f \in B(G)$ is almost periodic if and only if $f$ can be uniformly approximated by linear combinations, with complex coefficients, of characters of $G$ (i.e., functions from $\mathfrak{X}(G)$).*

*Proof.* We give a brief sketch of the proof, for more details see [57, Theorem 2.2.2].

According to Example 11.4.6 every character is almost periodic. It follows from Fact 9.2.13 that every linear combination of characters is an almost periodic function. This implies that that every function from $\mathfrak{X}(G)$ is almost periodic. Moreover, by this and by the proof of Fact 9.2.13 it follows that every function from $\mathfrak{A}(G)$ is almost periodic. This proves the inclusion $\mathfrak{A}(G) \subseteq A(G)$.

To establish the inclusion $\mathfrak{A}(G) \supseteq A(G)$ we assume that the function $f$ is almost periodic. Fix an $\varepsilon > 0$. By Proposition 11.4.5 the set $T(f,\varepsilon/8)$ is big. Hence we can apply Følner's theorem to the set $T(f,\varepsilon)$ containing $T(f,\varepsilon/8)_{(8)}$ and find $\chi_1, \dots, \chi_n \in G^*$, $\delta > 0$ such that $U_G(\chi_1, \dots, \chi_n; \delta) \subseteq T(f,\varepsilon)$. Now if $x, y \in G$ satisfy $x - y \in U_G(\chi_1, \dots, \chi_n; \delta)$, then $x - y \in T(f,\varepsilon)$, so $\|f_{x-y} - f\| \leq \varepsilon$. In particular, $|f(x) - f(y)| \leq \varepsilon$. Then $f$ satisfies condition (b) of Proposition 10.3.4 with $H = G^*$. Hence $f \in \mathfrak{A}(G)$ according to the conclusion of that proposition. □

**Corollary 11.4.8.** *Every continuous function on a compact abelian group is almost periodic.*

*Proof.* It follows immediately from Stone-Weierstraß Theorem and Peter-Weyl's Theorem that every $f \in C(G)$ can be uniformly approximated by linear combinations, with complex coefficients, of characters of $G$ when $G$ is compact. Hence the above theorem applies. □

Now we are in position to prove that the continuous almost periodic functions of a topological abelian group $G$ are precisely those that factorize through the Bohr compactification $b_G : G \to bG$.

**Theorem 11.4.9.** *Let $G$ be a topological abelian group. Then a continuous function $f: G \to \mathbb{C}$ is almost periodic iff there exists a continuous function $\tilde{f}: G \to \mathbb{C}$ such that $f = \tilde{f} \circ b_G$.*

*Proof.* Assume there exists a continuous function $\tilde{f}: G \to \mathbb{C}$ such that $f = \tilde{f} \circ b_G$. Then $\tilde{f}$ is almost periodic by Corollary 11.4.8. Now Exercise 9.2.14 implies that $f \in A(G)$.

Now assume that $f \in A(G)$. Then by Theorem 11.4.7 $f$ can be uniformly approximated by functions from $\mathfrak{X}(G)$. By Theorem 2.2.35 $\mathcal{X}(G)$ separates the points of $G$ separated by $f$. Since $g(x) = g(y)$ for $x, y \in G$ and all $g \in \mathfrak{X}(G)$ is equivalent to $b_G(x) = b_G(y)$, we conclude that $f(x) = f(y)$ whenever $b_G(x) = b_G(y)$. This means that $f$ can be factorized as $f = \tilde{f} \circ b_G$ for some function $\tilde{f}: G \to \mathbb{C}$. Note that the continuity of $f$ yields that $\tilde{f}$ is continuous. □

We recall that for an abelian group $G$, $\mathfrak{A}(G) = \mathfrak{A}_0(G) + \mathbb{C}\cdot 1$, where $\mathbb{C}\cdot 1$ is the one-dimensional subalgebra consisting of the constant functions. We shall see below that $\mathfrak{A}_0(G) \cap \mathbb{C}\cdot 1 = 0$, so $\mathfrak{A}_0(G)$ has co-dimension one in $\mathfrak{A}(G)$.

The next lemma is a corollary of Corollary 10.3.6:

**Lemma 11.4.10.** *Let $G$ be an abelian group, $g \in \mathfrak{A}_0(G)$ and let $M$ be a closed convex subset of $\mathbb{C}$. If $g(x) - c \in M$ for some $c \in \mathbb{C}$ and for every $x \in G$, then $c \in M$.*

*Proof.* Assume for contradiction that $c \notin M$. Since $M$ is closed there exists $\varepsilon > 0$ such that $c \notin M + D$, where $D$ is the closed (so compact) ball with center 0 and radius $\varepsilon$. Let $h \in \mathfrak{X}_0(G)$ with $\|g - h\| \leq \varepsilon/2$. Since $M + D$ is still a closed convex set of $\mathbb{C}$ and $h(x) - c \in M + D$, we conclude with Corollary 10.3.6 that $c \in M + D$, a contradiction. □

**Lemma 11.4.11.** *For every abelian group $G$*

$$\mathfrak{A}(G) = \mathfrak{A}_0(G) \oplus \mathbb{C}\cdot 1 \qquad (4).$$

*Moreover, if $f \in C(G)$ is written as $f(x) = g_f(x) + c_f$, with $g_f \in C_z(G)$ and $c_f \in \mathbb{C}$ a constant function, then $|c_f| \leq \|f\|$ and $c_f \geq 0$, whenever $f$ satisfies $f(x) \geq 0$ for all $x \in G$.*

*Proof.* Assume $c \cdot 1 = g \in \mathfrak{A}_0(G)$ for some $c \in \mathbb{C}$. For $M = \{0\}$ apply Lemma 11.4.11 to $c - g = 0 \in M$. The conclusion of the lemma gives $c = 0$. Hence $\mathfrak{A}_0(G) \cap \mathbb{C}\cdot 1 = 0$. This proves (4).

For $f \in \mathfrak{A}(G)$ the projections $f \mapsto g_f \in \mathfrak{A}_0(G)$ and $f \mapsto c_f \in \mathbb{C}\cdot 1$ related to this factorization (4) can be described as follows. By the definition of $\mathfrak{A}(G)$, for every $n \in \mathbb{N}_+$ there exist $h_n \in \mathfrak{X}(G)$, $h_n = c_n + g_n$, with $g_n \in \mathfrak{X}_0(G)$, $c_n \in \mathbb{C}$ such that

$$|f(x) - c_n - g_n(x)| \leq \frac{1}{n} \qquad (*_n)$$

for every $x \in G$. Applying the triangle inequality to $(*_n)$ and $(*_k)$ one gets

$$|c_n - c_k - g_n(x) + g_k(x)| \leq \frac{1}{n} + \frac{1}{k}$$

for every $x \in G$. By Lemma 11.4.11 applied to the closed disk $M$ with center 0 and radius $\frac{1}{n} + \frac{1}{k}$ we conclude $|c_n - c_k| \leq \frac{1}{n} + \frac{1}{k}$. Hence $(c_n)$ is a Cauchy sequence in $\mathbb{C}$. Let $c_f := \lim_n c_n$. Then $g_f := f - c_f \in \mathfrak{A}_0(G)$. Indeed, according to $(*_n)$ and the definition of $c_f$, $\|f - c_f - g_n\| \leq \|f - c_n - g_n + (c_n - c_f)\| \leq \frac{1}{n} + |c_n - c_f|$ becomes arbitrarly small when $n \to \infty$.

If $f = 0$, then $c_f = 0$ and there is nothing to prove. Assume $f \neq 0$ and let $\varepsilon = \|f\|$. Then $\|f\| = \|g_f + c_f\| \leq \varepsilon$ yields $|c_f| \leq \varepsilon$ by Lemma 11.4.10.

To prove the last assertion, apply Lemma 11.4.10 to the closed convex subset of $\mathbb{C}$ consisting of all non-negatrive real numbers. □

According to this lemma the projection $\mathfrak{A}(G) \to \mathbb{C}$ defined by $f \mapsto c_f$ is a continuous positive linear functional. We show in the sequel that this is the Haar integral on $\mathfrak{A}(G)$ (Theorem 11.4.12).

### 11.4.2 Haar integral of the compact abelian groups

Let $G$ be an abelian group and let $J(G)$ be a translation-invariant $\mathbb{C}$-subspace of $B(G)$ containing all constant functions and closed under complex conjugation. The Haar integral on $J(G)$ is a linear functional $\int$ defined on the space $J(G)$ which is

(a) positive (i.e., if $f \in J(G)$ is real-valued and $f \geq 0$, then also $\int f \geq 0$);

(b) invariant (i.e., $\int f_a = \int f$ for every $f \in J(G)$ and $a \in G$, where $f_a(x) = f(x + a)$);

(c) $\int 1 = 1$.

The last item can be announced also as $m(G) = 1$ in terms of the measure $m$ associated to $\int$. In the presence of the Haar intergral one can define also a scalar product in $J(G)$ by $(f, g) = \int f(x)\overline{g(x)}$. This makes $J(G)$ a Hilbert space. Moreover, the scalar product is invariant, i.e., $(f_a, g_a) = (f, g)$ for every $a \in G$. Hence the action $f \mapsto f_a$ of $G$ in the Hilbert space $C(G)$ is given by unitary operators of the Hilbert space $J(G)$.

The Haar intregral in a compact abelian group $G$ is obtained with $J(G) = C(G)$.

Now we check that the assignment $f \mapsto c_f$ defines a Haar integral on the algebra $\mathfrak{A}(G)$ of almost periodic functions of an abelian group $G$.

**Theorem 11.4.12.** *For every abelian group $G$ the assignment $f \mapsto c_f$ $(f \in \mathfrak{A}(G))$ defines a Haar integral $\int$ on $\mathfrak{A}(G)$.*

*Proof.* Fix a function $f \in \mathfrak{A}(G)$ and consider $c_f \in \mathbb{C}$ as defined above. The fact that $f \mapsto c_f$ is linear is obvious from the definition. Positivity was established in Lemma 11.4.11. To check invariance note that if $f = g_f + c_f$ with $g_f \in \mathfrak{A}_0(G)$, then $g_f(a + x) = (g_f)_a(x) \in \mathfrak{A}_0(G)$ and $f_a(x) = f(a + x) = g_f(a + x) + c_f$. Hence $\int f_a = \int f$. Finally, for $f = 1$ one obviously has $c_1 = 1$. □

Next we see that the Haar integral on $\mathfrak{A}(G)$ is unique.

**Proposition 11.4.13.** *Let $G$ be an abelian group, let $\int$ be a Haar integral on $\mathfrak{A}(G)$ and $\varphi, \chi \in G^*$. Then:*

- $\int \varphi(x)\overline{\chi(x)} = \begin{cases} 1 & \text{if } \varphi = \chi \\ 0 & \text{if } \varphi \neq \chi \end{cases}.$

*In particular, $\int \varphi(x) = 0$ when $\varphi$ is non-trivial.*

From the above proposition we get:

**Corollary 11.4.14.** *If $G$ is an abelian group and $\int$ is a Haar integral on $\mathfrak{A}(G)$, then one has $\int f = 0$ for every $f \in \mathfrak{X}_0(G)$.*

*Proof.* The first assertion follows from Proposition 11.4.13. The second one from property (c) and Lemma 11.4.11 that guarantees that the functionals $\int$ and $f \mapsto c_f$ coincide once they coinide on $\mathbb{C} \cdot 1$ and have as kernel $\mathfrak{A}_0(G)$. □

**Exercise 11.4.15.** *Let $G$ be an abelian group and let $\int$ be a Haar integral on $J(G)$. If $f, g \in J(G)$ and $\|f - g\| \leq \varepsilon$, then also $|\int f - \int g| \leq \varepsilon$.*

**Corollary 11.4.16.** *Let $G$ be an abelian group and let $\int$ be a Haar integral on $\mathfrak{A}(G)$. Then $\int f = 0$ for every $f \in \mathfrak{A}_0(G)$. Consequenly, $\int f = c_f$ for every $f \in \mathfrak{A}(G)$.*

*Proof.* Let $f \in \mathfrak{A}_0(G)$. For every $\varepsilon > 0$ there exists $g \in \mathfrak{X}_0(G)$ such that $\|f - g\| \leq \varepsilon$. Then by Corollary 11.4.14 and Exercise 11.4.15 we get $|\int f| \leq \varepsilon$. Therefore, $\int f = 0$. □

According to Corollary 11.4.8 every continuous function on a compact abelian group is almost periodic. This fact gives an easy and natural way to define the Haar intregral in a compact abelian groups by using the construction of the functional $f \mapsto c_f$ from (4).

**Theorem 11.4.17.** ([57, Lemma 2.4.2]) *For every compact abelian group $G$ the assignment $f \mapsto c_f$ $(f \in C(G))$ defines a (unique) Haar integral on $G$.*

### 11.4.3 Haar integral of the locally compact abelian groups

Analogously, we can define a Haar integral on a locally compact (abelian) group $G$ as follows. A *Haar integral* on $G$ is a linear functional $I = \int_G : C_0(G) \longrightarrow \mathbb{C}$ such that:

(i) $\int f \geq 0$ for any real-valued $f \in C_0(G)$ with $f \geq 0$;

(ii) $\int f_a = \int f$ for any $f \in C_0(G)$ and any $a \in G$;

(iii) there exists $f \in C_0(G)$ with $\int f \neq 0$.

In the remaining part of this section we will show that every LCA group $G$ admits a Haar integral $\int_G$.

We begin with a simple property of Haar integrals that will be useful later on.

**Lemma 11.4.18.** *Let $I = \int_G$ be a Haar integral on a LCA group $G$. Then for any real-valued $h \in C_0(G)$ with $h \geq 0$ on $G$ and $h(x) > 0$ for at least one $x \in G$ we have $I(h) > 0$.*

*Proof.* Let $h \in C_0(G)$ be a real-valued function such that $h \geq 0$ on $G$ and $h(x_0) > 0$ for some $x_0 \in G$. Then there exists a neighbourhood $V$ of 0 in $G$ such that $h(x) \geq a = h(x_0)/2$ for all $x \in x_0 + V$.

By property (iii) of Haar integrals, there exists $f \in C_0(G)$ with $I(f) \neq 0$. Then $f = u + \imath v$ for some real-valued $u, v \in C_0(G)$, so we must have either $I(u) \neq 0$ or $I(v) \neq 0$. So, without loss of generality we may assume that $f$ is real-valued. Setting $f_+(x) = \max\{f(x), 0\}$, $x \in G$ and $f_-(x) = \max\{-f(x), 0\}$, we get functions $f_+, f_- \in C_0(G)$ such that $f_+ \geq 0$ and $f_- \geq 0$ on $G$ and $f = f_+ - f_-$. Thus, either $I(f_+) \neq 0$ or $I(f_-) \neq 0$.

So, we may assume that $f \geq 0$ and $I(f) \neq 0$; then by (i) we must have $I(f) > 0$. Since $f \in C_0(G)$, there exists a compact $K \subset G$ with $f(x) = 0$ on $G \setminus K$. So one can find a finite $F \subset G$ such that $K \subset F + V$. If $A = \max_{x \in G} f(x)$, then $A > 0$ and for every $g \in F$ we have $h_{x_0 - g}(x) \geq a$ for all $x \in g + V$. Thus, $f(x) \leq \frac{A}{a} \sum_{g \in F} h_{x_0 - g}(x)$ for all $x \in G$, and therefore $0 < I(f) \leq \frac{A}{a} |F| \, I(h)$, which shows that $I(h) > 0$. □

The following three lemmas are the main steps in the proof of existence of Haar integrals.

**Lemma 11.4.19.** *If $G$ is a discrete abelian group, then $G$ admits a Haar integral.*

*Proof.* Setting

$$\int_G f = \sum_{x \in G} f(x) \quad , \quad f \in C_0(G) \, ,$$

one checks easily that $\int_G$ is a Haar integral on $G$. □

**Lemma 11.4.20.** *If $G \in \mathcal{L}$ and $H$ is a closed subgroup of $G$ such that both $H$ and $G/H$ admit a Haar integral, then also $G$ admits a Haar integral.*

*Proof.* Let $f \in C_0(G)$. Then $f_y \restriction_H \in C_0(H)$ for every $y \in G$. Let $F(y) = \int_H f_y \restriction_H$. Then $F : G \to \mathbb{C}$ is a continuous function. Indeed, let $y_0 \in G$ and $\varepsilon > 0$. There exists a compact $K \subset G$ such that $f = 0$ on $G \setminus K$. Let $U$ be an arbitrary compact symmetric neighbourhood of 0 in $G$. There exists $h \in C_0(G)$ such that $0 \leq h(x) \leq 1$ for all $x \in G$ and $h(x) = 1$ for all $x \in y_0 + U + K$.

Since $f$ is continuous and $U + K$ is compact, there exists a symmetric neighbourhood $V$ of 0 in $G$ such that $V \subset U$ and

$$|f(x) - f(y)| \leq \varepsilon \quad , \quad x, y \in U + K \; , \; x - y \in V \; . \tag{4}$$

We will now show that $|F(y) - F(y_0)| \leq \varepsilon$ for all $y \in y_0 + V$. Given $y \in y_0 + V$ let us first check that

$$|f(x - y) - f(x - y_0)| \leq \varepsilon \, h(x) \quad , \quad x \in G \; . \tag{5}$$

Indeed, if $x \in G$ is such that $f(x - y) = f(x - y_0) = 0$, then (2) is obviously true. Assume that either $f(x - y) \neq 0$ or $f(x - y_0) \neq 0$. Then either $x - y \in K$ or $x - y_0 \in K$, so either $x \in y + K \subset y_0 + V + K$ or $x \in y_0 + K$. In both cases $x \in y_0 + U + K$ and $x - y, x - y_0 \in U + K$. Moreover,

$$(x - y) - (x - y_0) = y_0 - y \in y_0 - (y_0 + V) = V \; ,$$

so (1) and $h(x) = 1$ imply

$$|f(x - y) - f(x - y_0)| \leq \varepsilon \leq \varepsilon \, h(x) \; .$$

This proves (2).

From (2) it follows that $|f_y \restriction_H - f_{y_0} \restriction_H| \leq \varepsilon \, h \restriction_H$, so

$$|F(y) - F(y_0)| \leq \int_H |f_y \restriction_H - f_{y_0} \restriction_H| \leq \varepsilon \int_H h \restriction_H \quad .$$

This proves the continuity of $F$ at $y_0$.

Next, for any $x, y \in G$ with $x - y \in H$ we have

$$F(x) = \int_H f_x \restriction_H = \int_H (f_y)_{x-y} \restriction_H = \int_H f_y \restriction_H = F(y) \; ,$$

using the invariance of $\int_H$ in $H$. Then there exists a continuous function $\widetilde{F} : G/H \to \mathbb{C}$ such that $F = \widetilde{F} \circ p$, where $p : G \to G/H$ is the natural projection. Moreover, $\widetilde{F} \in C_0(G/H)$.

Set $\int_G f := \int_{G/H} \widetilde{F}$ for any $f \in C_0(G)$. It is now easy to check that $\int_G$ is a Haar integral on $G$. Indeed, the linearity of $\int_G$ follows from that of $\int_{G/H}$ and the fact that $(\alpha \, f_1 + \beta \, f_2)^\sim = \alpha \, \widetilde{F}_1 + \beta \, \widetilde{F}_2$ for any $\alpha, \beta \in \mathbb{C}$ and any $f_1, f_2 \in C_0(G)$.

If $f \geq 0$, then $\widetilde{F} \geq 0$, too, so $\int_G = \int_{G/H} \widetilde{F} \geq 0$. To check invariance, notice that for any $x \in G$ we have $(f_x)^{\sim} = (\widetilde{F})_{p(x)}$, so

$$\int_G f_x = \int_{G/H} (f_x)^{\sim} = \int_{G/H} (\widetilde{F})_{p(x)} = \int_{G/H} \widetilde{F} = \int_G f \,.$$

We will now show that $\int_G$ is non-trivial, i.e. it satisfies (iii). Take an arbitrary compact neighbourhood $U$ of 0 in $G$. There exists a real-valued $f \in C_0(G)$ with $f \geq 0$ on $G$ such that $f(x) \geq 1$ for all $x \in U$. Then $f \geq 1$ on $U \cap H$, so by Lemma 11.4.18, $F(0) = \int_H f \restriction H > 0$, which gives $\widetilde{F}(0) > 0$. Moreover $f \geq 0$ implies $\widetilde{F} \geq 0$ on $G/H$, so using Lemma 11.4.18 again, $\int_G f = \int_{G/H} \widetilde{F} > 0$.

Thus, $\int_G$ is a Haar integral on $G$. □

We are now ready to prove existence of Haar integrals on general LCA groups.

**Theorem 11.4.21.** *Every locally compact abelian group admits a Haar integral.*

*Proof.* Let $G$ be a LCA group. If $G$ is compact or discrete, then Theorem 11.4.17 or Lemma 11.4.19 apply. In case $G$ is compactly generated, $G$ has a discrete subgroup $H$ such that $G/H$ is compact by Proposition 11.3.1. So both $H$ and $G/H$ admit a Haar integral. It follows from Lemma 11.4.20 that $G$ admits a Haar integral, too.

In the general case $G$ has an open subgroup $H$ which is compactly generated – just take the subgroup generated by an arbitrary compact neighbourhood of 0 in $G$. Such a subgroup $H$ is locally compact (and compactly generated), hence admits a Haar integral by the above argument, while $G/H$ is discrete, so it also admits a Haar integral by Lemma 11.4.19. Finally, Lemma 11.4.20 implies that $G$ admits a Haar integral, too. □

## 11.5 Precompact group topologies determined by sequences

Large and lacunary sets (mainly in $\mathbb{Z}$ or elsewhere) are largely studied in number theory, harmonic analysis and dynamical systems ([76], [30], [120], [86], [87], [89], [90], [94]).

Let us consider a specific problem. For a strictly increasing sequence $\underline{u} = (u_n)_{n\geq 1}$ of integers, the interest in the distribution of the multiples $\{u_n\alpha : n \in \mathbb{N}\}$ of a non-torsion element $\alpha$ of the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has roots in number theory (Weyl's theorem of uniform distribution modulo 1) and in ergodic theory (Sturmian sequences and Hartman sets [146]). According to Weyl's theorem, the set $\{u_n\alpha : n \in \mathbb{N}\}$ will be uniformly dense in $\mathbb{T}$ for almost all $\alpha \in \mathbb{T}$. One can consider the subset $t_{\underline{u}}(\mathbb{T})$ of all elements $\alpha \in \mathbb{T}$ such that $\lim_n u_n\alpha = 0$ in $\mathbb{T}$. Clearly it will have measure zero. Moreover, it is a subgroup of $\mathbb{T}$ as well as a Borel set, so it is either countable or has size $\mathfrak{c}$. It was observed by Armacost [4] that when $u_n = p^n$ for all $n$ and some prime $p$, then $t_{\underline{u}}(\mathbb{T}) = \mathbb{Z}(p^\infty)$. He posed the question of describing the subgroup $t_{\underline{u}}(\mathbb{T})$ for the sequence $u_n = n!$, this was done by Borel [22] (see also [57] and [39] for the more general problem concerning sequences $\underline{u}$ with $u_{n-1}|u_n$ for every $n$).

Another motivation for the study of the subgroups of the form $t_{\underline{u}}(\mathbb{T})$ come from the fact that they lead to the description of precompact group topologies on $\mathbb{Z}$ that make the sequence $u_n$ converge to 0 in $\mathbb{Z}$ (see the comment after proposition 11.5.1). Let us start by an easy to prove general fact:

**Proposition 11.5.1.** [10] *A sequence $A = \{a_n\}_n$ in a precompact abelian group $G$ converges to 0 in $G$ iff $\chi(a_n) \to 0$ in $\mathbb{T}$ for every continuous character of $G$.*

In the case of $G = \mathbb{Z}$ the characters of $G$ are simply elements of $\mathbb{T}$, i.e., a precompact group topology on $\mathbb{Z}$ has the form $\mathcal{T}_H$ for some subgroup $H$ of $\mathbb{T}$. Thus the above proprosition for $G = \mathbb{Z}$ can be reformulated as: a sequence $A = \{a_n\}_n$ in $(\mathbb{Z}, \mathcal{T}_H)$ converges to 0 iff $a_n x \to 0$ for every $x \in H$, i.e., simply $H \subseteq t_{\underline{a}}(\mathbb{T})$.

Now we can discuss a counterpart of the notion of $T$-sequences (introduced in §3.5), defined with respect to topologies induced by characters, i.e., precompact topologies.

**Definition 11.5.2.** [10, 12] A sequence $A = \{a_n\}_n$ in an abelian group $G$ is called a *TB-sequence* is there exists a precompct group topology on $G$ such that $a_n \to 0$.

Clearly, every TB-sequence is a T-sequence (see Example 11.5.4 for a T-sequence in $\mathbb{Z}$ that is not a TB-sequence). The advantage of TB-sequences over the T-sequences is in the easier way of determining sufficient condition for a sequence to be a TB-sequence [10, 12]. For example, a sequence $(a_n)$ in $\mathbb{Z}$ is a TB-sequence iff the subgroup $t_{\underline{a}}(\mathbb{T})$ of $\mathbb{T}$ is infinite.

Egglestone [73] proved that the asymptotic behavior of the sequence of ratios $q_n = \frac{u_{n+1}}{u_n}$ may have an impact on the size of the subgroup $t_{\underline{u}}(\mathbb{T})$ in the following remarkable "dichotomy":

**Theorem 11.5.3.** *Let $(a_n)$ be a sequence in $\mathbb{Z}$.*

- *If $\lim_n \frac{a_{n+1}}{a_n} = +\infty$, then $(a_n)$ is a TB-sequence and $|t_{\underline{a}}(\mathbb{T})| = \mathfrak{c}$.*

- *If $\frac{a_{n+1}}{a_n}$ is bounded, then $t_{\underline{a}}(\mathbb{T})$ is countable.*

**Example 11.5.4.** *[12] There exists a $TB$-sequence $(a_n)$ in $\mathbb{Z}$ with $\lim_n \frac{a_{n+1}}{a_n} = 1$ .*

Here is an example of a T-sequence in $\mathbb{Z}$ that is not a TB-sequence.

**Example 11.5.5.** For every $TB$-sequence $A = \{a_n\}$ in $\mathbb{Z}$ such that $t_{\underline{a}}(\mathbb{T})$ is countable, there exists a sequence $\{c_n\}$ in $\mathbb{Z}$ such that the sequence $q_n$ defined by $q_{2n} = c_n$ and $q_{2n-1} = a_n$, is a $T$-sequence, but not a $TB$-sequence.

*Proof.* Let $\{z_1, \ldots, z_n, \ldots\}$ be an enumeration of $t_{\underline{a}}(\mathbb{T})$.

According to Lemma 4.3.5 there exists a sequence $b_n$ in $\mathbb{Z}$ such that for every choice of the sequence $(e_n)$, where $e_n \in \{0,1\}$, the sequence $q_n$ defined by $q_{2n} = b_n + e_n$ and $q_{2n-1} = a_n$, is a $T$-sequence. Now we define the sequence $q_n$ with $q_{2m-1} = a_m$ and $q_{2m} = b_m$ when $m$ is not a prime power. Let $p_1, \ldots, p_n, \ldots$ be all prime numbers enumerated one-to-one. Now fix $k$ and define $e_k \in \{0,1\}$ depending on $\lim_n b_{p_k^n} z_k$ as follows:

- if $\lim_n b_{p_k^n} z_k = 0$, let $e_k = 1$,
- if $\lim_n b_{p_k^n} z_k \neq 0$ (in particular, if the limit does not exists) let $e_k = 0$.

Now let $q_{2p_k^n} = b_{p_k^n} + e_k$ for $n \in \mathbb{N}$. Hence for every $k \in \mathbb{N}$

$$\lim_n q_{2p_k^n} z_k = 0 \implies e_k = 1. \tag{$*$}$$

To see that $(q_n)$ is not a $TB$-sequence assume that $\chi : \mathbb{Z} \to \mathbb{T}$ a character such that $\chi(q_n) \to 0$ in $\mathbb{T}$. Then $x = \chi(1) \in \mathbb{T}$ satisfies $q_n x \to 0$, so $x \in t_{\underline{q}}(\mathbb{T}) \subseteq t_{\underline{a}}(\mathbb{T})$. So there exists $k \in \mathbb{N}$ with $x = z_k$. By (*) $e_k = 1$. Hence $q_{2p_k^n} = b_{p_k^n} + 1$ and $\lim_n b_{p_k^n} z_k = 0$, so $x \in t_{\underline{q}}(\mathbb{T})$ yields $0 = \lim_n q_{2p_k^n} x = 0 + x$, i.e., $x = 0$. This proves that every character $\chi : \mathbb{Z} \to \mathbb{T}$ such that $\chi(q_n) \to 0$ in $\mathbb{T}$ is trivial. In particular, $(q_n)$ not a $TB$-sequence. $\square$

Let us note that the above proof gives much more. Since $q_n \to 0$ in $\tau_{(q_n)}$, it shows that every $\tau_{(q_n)}$-continuous character of $\mathbb{Z}$ is trivial, i.e., $\widehat{(\mathbb{Z}, \tau_{(q_n)})} = 0$.

The information accumulated on the properties of the subgroups $t_{\underline{u}}(\mathbb{T})$ of $\mathbb{T}$ motivated the problem of describing those subgroups $H$ of $\mathbb{T}$ that can be characterized as $H = t_{\underline{u}}(\mathbb{T})$ for some sequence $\underline{u}$. As already mentioned, such an $H$ can be only countable or can have size $\mathfrak{c}$ being of measure zero. A measure zero subgroup $H$ of $\mathbb{T}$ of size $\mathfrak{c}$ that is not even contained in any proper subgroup of $\mathbb{T}$ of the form $t_{\underline{u}}(\mathbb{T})$ was built in [10] (under the assumption of Martin Axiom) and in later in [96, 97] (in ZFC). Much earlier Borel [22] had already resolved in the positive the remaining part of the problem showing that every countable subgroup of $\mathbb{T}$ can be characterized (in the above sense). Unaware of his result, Larcher [111], and later Kraaikamp and Liardet [108], proved that some cyclic subgroups of $\mathbb{T}$ are characterizable (see also [19, 18, 15, 17, 16] for related results). The paper [12] describes the algebraic structure of the subgroup $t_{\underline{u}}(\mathbb{T})$ when the sequence $\underline{u} := (u_n)$ verifies a linear recurrence relation of order $\leq k$,

$$u_n = a_n^{(1)} u_{n-1} + a_n^{(2)} u_{n-2} + \ldots + a_n^{(k)} u_{n-k}$$

for every $n > k$ with $a_n^{(i)} \in \mathbb{Z}$ for $i = 1, \ldots, k$.

Three proofs of Borel's theorem of characterizability of the countable subgroups of $\mathbb{T}$ were given in [16]. These author mentioned that the theorem can be extended to compact abelian groups in place of $\mathbb{T}$, without giving any precise formulation. There is a natural way to extend the definition of $t_{\underline{u}}(\mathbb{T})$ to an arbitrary topological abelian group $G$ by letting $t_{\underline{u}}(G) = \{x \in G : \lim_n u_n x = 0 \text{ in } G\}$. Actually, for the sequence $u_n = p^n$ (resp., $u_n = n!$) an element $x$ satisfying $\lim_n u_n x = 0$ has been called *topologically p-torsion* (resp., *topologically torsion*) by Braconnier and Vilenkin in the forties of the last century and these notions played a prominent role in the development of the theory of locally compact abelian groups. One can easily reduce the computation of $t_{\underline{u}}(G)$ for an arbitrary locally compact abelian group to that of $t_{\underline{u}}(\mathbb{T})$ [36]. Independently on their relevance in other questions, the subgroups $t_{\underline{u}}(G)$ turned out to be of no help in the characterization of countable subgroups of the compact abelian groups. Indeed, a much weaker condition, turned out the characterize the circle group $\mathbb{T}$ in the class of all locally compact abelian groups:

**Theorem 11.5.6.** [39] *In a locally compact abelian group $G$ every cyclic subgroup of the group $G$ is an intersection of subgroups of the form $t_{\underline{u}}(G)$ iff $G \cong \mathbb{T}$.*

Actually, one can remove the "abelian" restraint in the theorem remembering that in the non-abelian case $t_{\underline{u}}(G)$ is just a subset of $G$, not a subgroup in general [39].

The above theorem suggested to use in [53] a different approach to the problem, replacing the sequence of integers $u_n$ (characters of $\mathbb{T}$!) by a sequence $u_n$ in the Pontryagin-van Kampen dual $\widehat{G}$. Then the subgroup $s_{\underline{u}}(G) = \{x \in G : \lim_n u_n(x) = 0 \text{ in } \mathbb{T}\}$ of $G$ really can be used for such a characterization of all countable subgroups of the compact metrizable groups (see [53, 49, 20] for major detail).

# 12 Pontryagin-van Kampen duality

## 12.1 The dual group

In the sequel we shall write the circle additively as $(\mathbb{T},+)$ and we denote by $q_0:\mathbb{R}\to\mathbb{T}=\mathbb{R}/\mathbb{Z}$ the canonical projection. For every $k\in\mathbb{N}_+$ let $\Lambda_k=q_0((-\frac{1}{3k},\frac{1}{3k}))$. Then $\{\Lambda_k : k\in\mathbb{N}_+\}$ is a base of the neighborhoods of 0 in $\mathbb{T}$, because $\{(-\frac{1}{3k},\frac{1}{3k}) : k\in\mathbb{N}_+\}$ is a base of the neighborhoods of 0 in $\mathbb{R}$.

For abelian group $G$ we let as usual $G^*=\mathrm{Hom}\,(G,\mathbb{T})$. For a subset $K$ of $G$ and a subset $U$ of $\mathbb{T}$ let

$$W_{G^*}(K,U)=\{\chi\in G^*:\chi(K)\subseteq U\}.$$

For any subgroup $H$ of $G^*$ we abbreviate $H\cap W(K,U)$ to $W_H(K,U)$. When there is no danger of confusion we shall write only $W(K,U)$ in place of $W_{G^*}(K,U)$. The group $G^*$ will be considered only with one topology, namely the induced from $\mathbb{T}^G$ compact topology (see Remark 7.1.2).

If $G$ is a topological abelian group, $\widehat{G}$ will denote the subgroup of $G^*$ consisting of continuous characters.

The group $\widehat{G}$ will carry the *compact open topology* that has as basic neighborhoods of 0 the sets $W_{\widehat{G}}(K,U)$, where $K$ is a compact subset of $G$ and $U$ is neighborhood of 0 in $\mathbb{T}$. We shall see below that when $U\subseteq\Lambda_1$, then $W_{\widehat{G}}(K,U)$ coincides with $W_{G^*}(K,U)$ in case $K$ is a neighborhood of 0 in $G$. Therefore we shall use mainly the notation $W(K,U)$ when the group $G$ is clear from the context.

Let us start with an easy example.

**Example 12.1.1.** Let $G$ be an abelian topological group.

(1) If $G$ is compact, then $\widehat{G}$ is discrete.

(2) If $G$ is discrete, then $\widehat{G}$ is compact.

Indeed, to prove (1) it is sufficient to note that $W_{\widehat{G}}(G,\Lambda_1)=\{0\}$ as $\Lambda_1$ contains no subgroup of $\mathbb{T}$ beyond 0.

(2) Suppose that $G$ is discrete. Then $\widehat{G}=\mathrm{Hom}\,(G,\mathbb{T})$ is a subgroup of the compact group $\mathbb{T}^G$. The compact-open topology of $\widehat{G}$ coincides with the topology inherited from $\mathbb{T}^G$: let $F$ be a finite subset of $G$ and $U$ an open neighborhood of 0 in $\mathbb{T}$, then

$$\begin{aligned}\bigcap_{x\in F}\pi_x^{-1}(U)\cap\mathrm{Hom}\,(G,\mathbb{T})&=\{\chi\in\mathrm{Hom}\,(G,\mathbb{T}):\pi_x\in U\text{ for every }x\in F\}\\&=\{\chi\in\mathrm{Hom}\,(G,\mathbb{T}):\chi(x)\in U\text{ for every }x\in F\}=W(F,U).\end{aligned}$$

Moreover $\mathrm{Hom}\,(G,\mathbb{T})$ is closed in the compact product $\mathbb{T}^G$ by Remark 7.1.2 and we can conclude that $\widehat{G}$ is compact.

Now we prove that the dual group is always a topological group. Moreover, if the group $G$ is locally compact, then its dual is locally compact too (Corollary 12.1.4). This is the starting point of the Pontryagin-van Kampen duality theorem.

**Theorem 12.1.2.** *For an abelian topological group $G$ the following assertions hold true:*

*(a) if $x\in\mathbb{T}$ and $k\in\mathbb{N}_+$, then $x\in\Lambda_k$ if and only if $x,2x,\dots,kx\in\Lambda_1$;*

*(b) $\chi\in\mathrm{Hom}\,(G,\mathbb{T})$ is continuous if and only if $\chi^{-1}(\Lambda_1)$ is a neighborhood of 0 in $G$;*

*(c) $\{W_{\widehat{G}}(K,\Lambda_1):K$ compact $\subseteq G\}$ is a base of the neighborhoods of 0 in $\widehat{G}$, in particular $\widehat{G}$ is a topological group.*

*(d) $W_{\widehat{G}}(A,\Lambda_s)+W_{\widehat{G}}(A,\Lambda_s)\subseteq W_{\widehat{G}}(A,\Lambda_{[s/2]})$ and $W_{\widehat{G}}(\overline{A},\Lambda_s)+W_{\widehat{G}}(\overline{A},\Lambda_s)\subseteq W_{\widehat{G}}(\overline{A},\Lambda_{[s/2]})$ for every $A\subseteq G$ and $s>1$.*

*(e) if $F$ is a closed subset of $\mathbb{T}$, then for every $K\subseteq G$ the subset $W_{G^*}(K,F)$ of $G^*$ is closed (hence, compact);*

*(f) if $U$ is neighborhoodof 0 in $G$, then*

*($f_1$) $W_{\widehat{G}}(\overline{U},V)=W_{G^*}(\overline{U},V)$ for every neighborhood of 0 $V\subseteq\Lambda_1$ in $\mathbb{T}$;*

*($f_2$) $W(\overline{U},\Lambda_4)$ has compact closure;*

*($f_3$) if $U$ has compact closure, then $W(\overline{U},\Lambda_4)$ is a neighborhood of 0 in $\widehat{G}$ with compact closure, so $\widehat{G}$ is locally compact.*

*Proof.* (a) Note that for $s \in \mathbb{N}$, $sx \in \Lambda_1$ if and only if $x \in A_{s,t} = \Lambda_s + q_0(\frac{t}{s})$ for some integer $t$ with $0 \leq t \leq s$. On the other hand, $A_{s,0} = \Lambda_s$ and $\Lambda_s \cap A_{s+1,t}$ is non-empty if and only if $t = 0$. Hence, if $x \in \Lambda_s$ and $(s+1)x \in \Lambda_1$, then $x \in \Lambda_{s+1}$ and this holds in particular for $1 \leq s < k$. This proves that $sx \in \Lambda_1$ for $s = 1, \ldots, k$ if and only if $x \in \Lambda_k$.

(b) Suppose that $\chi^{-1}(\Lambda_1)$ is a neighborhood of 0 in $G$. So there exists an open neighborhood $U$ of 0 in $G$ such that $U \subseteq \chi^{-1}(\Lambda_1)$. Moreover, there exists another neighborhood $V$ of 0 in $G$ with $\underbrace{V + \cdots + V}_{k} \subseteq U$ where $k \in \mathbb{N}_+$. Now $s\chi(y) \in \Lambda_1$ for every $y \in V$ and $s = 1, \ldots, k$. By item (a) $\chi(y) \in \Lambda_k$ and so $\chi(V) \subseteq \Lambda_k$.

(c) Let $k \in \mathbb{N}_+$ and $K$ be a compact subset of $G$. Define $L = \underbrace{K + \cdots + K}_{k}$, which is a compact subset of $G$ because it is a continuous image of the compact subset $K^k$ of $G^k$. Take $\chi \in W(L, \Lambda_1)$. For every $x \in K$ we have $s\chi(x) \in \Lambda_1$ for $s = 1, \ldots, k$ and so $\chi(x) \in \Lambda_k$ by item (a). Hence $W(L, \Lambda_1) \subseteq W(K, \Lambda_k)$.

(d) obvious.

(e) If $\pi_x : \mathbb{T}^G \to \mathbb{T}$ is the projection defined by the evaluation at $x$, for $x \in G$, then obviously

$$W_{G^*}(K, F) = \bigcap_{x \in K} \{\chi \in G^* : \chi(x) \in F\} = \bigcap_{x \in K} (\pi_x^{-1}(F) \cap G^*)$$

is closed as each $(\pi_x^{-1}(F) \cap G^*)$ is closed in $G^*$.

($f_1$) follows immediately from item (b).

($f_2$) To prove that the closure of $W_0 = W(\overline{U}, \Lambda_4)$ is compact it is sufficient to note that $W_0 \subseteq W_1 := W(\overline{U}, \overline{\Lambda_4})$ and prove that $W_1$ is compact. Let $\tau_s$ denote the subspace topology of $W_1$ in $\widehat{G}$. We prove in the sequel that $(W_1, \tau_s)$ is compact.

Consider on the set $W_1$ also the weaker topology $\tau$ induced from $G^*$ and consequently from $\mathbb{T}^G$. By (e) $(W_1, \tau)$ is compact.

It remains to show that both topologies $\tau_s$ and $\tau$ of $W_1$ coincide. Since $\tau_s$ is finer than $\tau$, it suffices to show that if $\alpha \in W_1$ and $K$ is a compact subset of $G$, then $(\alpha + W(K, \Lambda_1)) \cap W_1$ is also a neighborhood of $\alpha$ in $(W_1, \tau)$.

Since $\bigcup\{a + U : a \in K\} \supseteq K$ and $K$ is compact, $K \subseteq F + U$, where $F$ is a finite subset of $K$. We prove now that

$$(\alpha + W(F, \Lambda_2)) \cap W_1 \subseteq (\alpha + W(K, \Lambda_1)) \cap W_1. \qquad (*)$$

Let $\xi' \in W(F, \Lambda_2)$, so that $\alpha + \xi' \in W_1 = W(\overline{U}, \overline{\Lambda_4})$. As $\alpha \in W_1$ as well, we deduce from items (c) and (d) that $\xi = (\alpha + \xi') - \alpha \in W_1 - W_1$. Hence $\xi(\overline{U}) \subseteq \overline{\Lambda_2}$ and consequently

$$\xi(K) \subseteq \xi(F + U) \subseteq \Lambda_2 + \overline{\Lambda_2} \subseteq \Lambda_1.$$

This proves that $\xi \in W(K, \Lambda_1)$ and (*).

($f_3$) Follows obviously from ($f_2$) and the definition of the compact open topology. □

**Corollary 12.1.3.** *Let $G$ be a locally compact abelian group. Then:*

*(a) $\widehat{G}$ is locally compact;*

*(b) if $G$ is metrizable, then $\widehat{G}$ is $\sigma$-compact;*

*(c) if $G$ is $\sigma$-compact, then $\widehat{G}$ is metrizable;*

*Proof.* (a) Follows immediately from the above theorem.

(b) Let $(U_n)$ be a countable base of the filter of neighborhoods of 0 in $G$. By item ($f_2$) of the above theorem $W(\overline{U}_n, \Lambda_4)$ has compact closure $K_n$. Let $\chi \in \widehat{G}$. Then by the continuity of $\chi$, there exists $n$ such that $\chi(\overline{U}_n) \subseteq \Lambda_4$, i.e., $\chi \in K_n$. Therefore $\widehat{G} = \bigcup_{n=1} K_n$ is $\sigma$-compact.

(c) If $G$ is $\sigma$-compact, then $G$ is also hemicompact by Exercise 2.2.25, so $G = \bigcup_{n=1} K_n$ where each $K$ is compact and every compact subset $K$ of $G$ is contained in some $K_n$. Then $W(K, \Lambda_1) \supseteq W(K_n, \Lambda_1)$. Hence the neighborhoods $W(K_n, \Lambda_1)$ form a countable base of the filter of neighborhoods of 0 in $\widehat{G}$. By Birkhoff-Kakutani theorem $\widehat{G}$ is metrizable. □

The proof of Theorem 12.1.2 shows another relevant fact. The neighborhood $W(\overline{U}, \Lambda_4)$ of 0 in the dual group $\widehat{G}$ carries the same topology in $\widehat{G}$ and $G^*$, nevertheless the inclusion map $j : \widehat{G} \hookrightarrow G^*$ need not be an embedding:

**Corollary 12.1.4.** *For a locally compact abelian group $G$ the following are equivalent:*

*(a) the inclusion map $j : \widehat{G} \hookrightarrow G^*$ is an embedding;*

*(b) $G$ is discrete;*

*(c) $\widehat{G} = G^*$ is compact.*

*Proof.* Since $G^*$ is compact, $j$ can be an embedding iff $\widehat{G}$ itself is compact. According to Example 12.1.1 this occurs precisely when $G$ is discrete. In that case $\widehat{G} = G^*$ is compact. □

Actually, it can be proved, once the duality theorem is available, that $j : \widehat{G} \hookrightarrow G^*$ need not be even a local homeomorphism. (If $j$ is a local homeomorphism, then the topological subgroup $j(\widehat{G})$ of $G^*$ will be locally compact, hence closed in $G^*$. This would yield that $j(\widehat{G})$ is compact. On the other hand, the topology of $j(\widehat{G})$ is precisely the initial topology of all projections $p_x$ restricted to $\widehat{G}$. By the Pontryagin duality theorem, these projections form the group of all continuous characters of $\widehat{G}$. So this topology coincides with $\mathcal{T}_{\widehat{\widehat{G}}}$. By a general theorem of Glicksberg, a locally compact abelian groups $H$ and $(H, \mathcal{T}_{\widehat{H}})$ have the same compact sets. In particular, compactness of $(H, \mathcal{T}_{\widehat{H}})$ yields compactness of $H$. This proves that if $j : \widehat{G} \hookrightarrow G^*$ is a local homeomorphism, then $\widehat{G}$ is compact and consequently $G$ is discrete.)

## 12.2 Computation of some dual groups

In the sequel we denote by $k \cdot id_G$ the endomorphism of an abelian group $G$ obtained by the map $x \mapsto kx$, for a fixed $k \in \mathbb{Z}$. The next lemma will be used for the computation of the dual groups in Example 12.2.4.

**Lemma 12.2.1.** *Every continuous homomorphism $\chi : \mathbb{T} \to \mathbb{T}$ has the form $k \cdot id_{\mathbb{T}}$, for some $k \in \mathbb{Z}$. In particular, the only topological isomorphisms $\chi : \mathbb{T} \to \mathbb{T}$ are $\pm id_{\mathbb{T}}$.*

*Proof.* We give two proofs of this fact.

*First proof.* Let prove first that the only topological isomorphisms $\chi : \mathbb{T} \to \mathbb{T}$ are $\pm id_{\mathbb{T}}$. This completely self-contained proof will exploit the fact that the arcs are the only connected sets of $\mathbb{T}$. Hence $\chi$ sends any arc of $\mathbb{T}$ to an arc, sending end points to end points. Denote by $\varphi$ the canonical homomorphism $\mathbb{R} \to \mathbb{T}$ and for $n \in \mathbb{N}$ let $c_n = \varphi(1/2^n)$ be the generators of the Prüfer subgroup $\mathbb{Z}(2^\infty)$ of $\mathbb{T}$. Then, $c_1$ is the only element of $\mathbb{T}$ of order 2, hence $g(c_1) = c_1$. Therefore, the arc $A_1 = \varphi([0, 1/2])$ either goes onto itself, or goes onto its symmetric image $-A_1$. Let us consider the first case. Clearly, either $g(c_2) = c_2$ or $g(c_2) = -c_2$ as $o(g(c_2)) = 4$ and being $\pm c_2$ the only elements of order 4 of $\mathbb{T}$. By our assumption $g(A_1) = A_1$ we have $g(c_2) = c_2$ since $c_2$ is the only element of order 4 on the arc $A_1$. Now the arc $A_2 = [0, c_2]$ goes onto itself, hence for $c_3$ we must have $g(c_3) = c_3$ as the only element of order 8 on the arc $A_2$, etc. We see in the same way that $g(c_n) = c_n$. Hence $g$ is identical on the whole subgroup $\mathbb{Z}(2^\infty)$. As this subgroup is dense in $\mathbb{T}$, we conclude that $g$ coincides with $id_{\mathbb{T}}$. In the case $g(A_1) = -A_1$ we replace $g$ by $-g$ and the previous proof gives $-g = id_{\mathbb{T}}$, i.e., $g = -id_{\mathbb{T}}$.

For $k \in \mathbb{N}_+$ let $\pi_k = k \cdot id_{\mathbb{T}}$. Then $\ker \pi_k = \mathbb{Z}_k$ and $\pi_k$ is surjective. Let now $\chi : \mathbb{T} \to \mathbb{T}$ be a non-trivial continuous homomorphism. Then $\ker \chi$ is a closed proper subgroup of $\mathbb{T}$, hence $\ker \chi = \mathbb{Z}_k$ for some $k \in \mathbb{N}_+$. Let $q : \mathbb{T} \to \mathbb{T}/\mathbb{Z}_k$ be the quotient homomorphism. Since $\chi(\mathbb{T})$ is a connected non-trivial subgroup of $\mathbb{T}$, one has $\chi(\mathbb{T}) = \mathbb{T}$. Now we apply Proposition 3.6.4 with $G = H_1 = H_2 = \mathbb{T}$, $\chi_2 = \chi$ and $\chi_1 = \pi_k$. Since $\ker \chi_1 = \ker \chi_2 = \mathbb{Z}_k$, $q_1 = q_2 = q$ and the homomorphism $t$ in Proposition 3.6.4 becomes the identity of $\mathbb{T}/\mathbb{Z}_k$ and we obtain the following commutative diagram:

$$
\begin{tikzcd}
 & G \arrow[dl, "\chi_1"'] \arrow[d, "q"] \arrow[dr, "\chi_2"] & \\
\mathbb{T} \arrow[rr, bend right, "\iota"'] & \mathbb{T}/\mathbb{Z}_k \arrow[l, "j_1"] \arrow[r, "j_2"'] & \mathbb{T}
\end{tikzcd}
\tag{6}
$$

According to the first part of the argument the isomorphism $\iota = j_2 \circ j_1^{-1} : \mathbb{T} \to \mathbb{T}$ coincides with $\pm id_{\mathbb{T}}$. Therefore, $\chi = \pm \pi_k$.

*Second proof.* Applying Lemma 8.0.7 to the composition $q = \varphi \circ \chi : \mathbb{R} \to \mathbb{T}$ and $\varphi : \mathbb{R} \to \mathbb{T}$ we can find a continuous homomorphism $\eta : \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ \eta = q = \varphi \circ \chi$, i.e., one has the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "\eta"'] \arrow[d, "\varphi"'] \arrow[dr, "q"] & \mathbb{R} \arrow[d, "\varphi"] \\
\mathbb{T} \arrow[r, "\chi"'] & \mathbb{T}
\end{tikzcd}
\tag{7}
$$

As $\chi(\varphi(\mathbb{Z})) = 0$, we deduce that $\varphi(\eta(\mathbb{Z})) = 0$ as well. Therefore, $\eta(\mathbb{Z}) \leq \ker \varphi = \mathbb{Z}$. It is easy to prove that there exists a real number $\rho$ such that $\eta(x) = \rho x$ for every $x \in \mathbb{R}$. Therefore, $\eta(\mathbb{Z}) \leq \mathbb{Z}$ yields $\rho \in \mathbb{Z}$. Hence, $\chi(y) = \rho x$ for every $y \in \mathbb{T}$. □

Obviously, $\chi = \pm\xi$ for characters $\chi, \xi : G \to \mathbb{T}$ implies $\ker \chi = \ker \xi$ and $\chi(G) = \xi(G)$. More generally, if $\chi = k \cdot \xi$ for some $k \in \mathbb{Z}$, then $\ker \chi \geq \ker \xi$ and $\chi(G) \leq \xi(G)$. Now we see that this implication can be (partially) inverted under appropriate hypotheses.

**Corollary 12.2.2.** *Let $G$ be a locally compact and $\sigma$–compact abelian group and let $\chi_i : G \to \mathbb{T}$, $i = 1, 2$, be continuous surjective characters. Then there exists an integer $m \in \mathbb{Z}$ such that $\chi_2 = m\chi_1$ iff $\ker \chi_1 \leq \ker \chi_2$. If $\ker \chi_1 = \ker \chi_2$ then $\chi_2 = \pm\chi_1$.*

*Proof.* Argue as in the final part of the above proof, applying Proposition 3.6.4 with $G = H_1 = H_2 = \mathbb{T}$ and use the diagram (8) to conclude as above. □

**Corollary 12.2.3.** *Let $G$ be a $\sigma$-compact locally compact abelian group and let $\chi, \xi : G \to \mathbb{T}$ be continuous characters such that $\ker \chi \geq \ker \xi$ and $\chi(G) \leq \xi(G)$.*

*(a) If $G$ is compact and $|\xi(G)| = m$ for some $m \in \mathbb{N}_+$, then $\chi = k\xi$ for some $k \in \mathbb{Z}$; moreover, $\ker \chi = \ker \xi$ iff $\chi(G) = \xi(G)$, in such a case $k$ must be coprime to $m$.*

*(b) If $\ker \chi = \ker \xi$ is open and $H = \chi(G) = \xi(G)$, then $\chi = \iota \circ \xi$, where $\iota : H \to H$ is an appropriate automorphism of the subgroup $H$ of $\mathbb{T}$ equipped with the discrete topology.*

*Proof.* (a) If $G$ is compact and $|\xi(G)| = m$ for some $m \in \mathbb{N}_+$, then $\xi(G)$ is a cyclic subgroup of $\mathbb{T}$ of order $m$. Note that $\mathbb{T}$ has a unique such cyclic subgroup. By Proposition 3.6.4 there exists a homomorphism $\iota : \xi(G) \to \chi(G)$ such that $\chi = \iota \circ \xi$. The hypothesis $\chi(G) \leq \xi(G)$ implies that there such a $\iota$ must be the multiplication by some $k \in \mathbb{Z}$. In case $\chi(G) = \xi(G)$ this $k$ is coprime to $m$.

(b) Since $G$ be a $\sigma$-compact and $\ker \chi = \ker \xi$ is open, the group $H = \chi(G) = \xi(G)$ is countable. Proposition 3.6.4 applies again. □

**Example 12.2.4.** Let $p$ be a prime. Then $\widehat{\mathbb{Z}(p^\infty)} \cong \mathbb{J}_p$, $\widehat{\mathbb{J}_p} \cong \mathbb{Z}(p^\infty)$, $\widehat{\mathbb{T}} \cong \mathbb{Z}$, $\widehat{\mathbb{Z}} \cong \mathbb{T}$ and $\widehat{\mathbb{R}} \cong \mathbb{R}$.

*Proof.* The first isomorphism $\widehat{\mathbb{Z}(p^\infty)} = \mathbb{J}_p$ follows from our definition $\mathbb{J}_p = \mathrm{End}(\mathbb{Z}(p^\infty)) = \mathrm{Hom}(\mathbb{Z}(p^\infty), \mathbb{T}) = \widehat{\mathbb{Z}(p^\infty)}$.

To verify the isomorphism $\widehat{\mathbb{J}_p} \cong \mathbb{Z}(p^\infty)$ consider first the quotient homomorphism $\eta_n : \mathbb{J}_p \to \mathbb{J}_p/p^n\mathbb{J}_p \cong \mathbb{Z}_{p^n} \leq \mathbb{T}$. With this identifications we consider $\eta_n \in \mathbb{J}_p^*$. As $\ker \eta_n = p^n\mathbb{J}_p$ is open, one has actually $\eta_n \in \widehat{\mathbb{J}_p}$. It is easy to see that under this identification $p\eta_n = \eta_{n-1}$. Therefore, the subgroup $H$ of $\widehat{\mathbb{J}_p}$ generated by the characters $\eta_n$ is isomorphic to $\mathbb{Z}(p^\infty)$. Let us see that $H = \widehat{\mathbb{J}_p}$. Indeed, take any non-trivial character $\chi : \mathbb{J}_p \to \mathbb{T}$. Then $N = \ker \chi$ is a closed proper subgroup of $\mathbb{J}_p$. Moreover, $N \neq 0$ as $\mathbb{J}_p$ is not isomorphic to a subgroup of $\mathbb{T}$ by Exercise 8.3.13. Thus $N = p^n\mathbb{J}_p$ for some $n \in \mathbb{N}_+$. Since $N = \ker \eta_n$, we conclude with (b) of Corollary 12.2.3 that $\chi = k\eta_n$ for some $k \in \mathbb{Z}$. This proves that $\chi \in H$ and consequently $\widehat{\mathbb{J}_p} \cong \mathbb{Z}(p^\infty)$.

The isomorphism $g : \widehat{\mathbb{Z}} \to \mathbb{T}$ is obtained by setting $g(\chi) := \chi(1)$ for every $\chi : \mathbb{Z} \to \mathbb{T}$. It is easy to check that this isomorphism is topological.

According to 12.2.1 every $\chi \in \widehat{\mathbb{T}}$ has the form $\chi = k \cdot id_{\mathbb{T}}$ for some $k \in \mathbb{Z}$. This gives a homomorphism $\widehat{\mathbb{T}} \to \mathbb{Z}$ assigning $\chi \mapsto k$. It is obviously injective and surjective. This proves $\widehat{\mathbb{T}} \cong \mathbb{Z}$ since both groups are discrete.

To prove $\widehat{\mathbb{R}} \cong \mathbb{R}$ consider the character $\chi_1 : \mathbb{R} \to \mathbb{T}$ obtained simply by the canonical map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. For every $r \in \mathbb{R}$ consider the map $\rho_r : \mathbb{R} \to \mathbb{R}$ defined by $\rho_r(x) = rx$. Then its composition $\chi_r = \chi_1 \circ \rho_r$ with $\chi_1$ gives a continuous character of $\mathbb{R}$. If $r \neq 0$, then $\chi_r \neq 0$, so the homomorphism $g : \mathbb{R} \to \widehat{\mathbb{R}}$ defined by $g(r) = \chi_r$ has $\ker g = 0$. To see that $g$ is surjective consider any continuous non-trivial character $\chi \in \widehat{\mathbb{R}}$. Applying Lemma 8.0.7 to the homorphisms $\chi : \mathbb{R} \to \mathbb{T}$ and $\varphi : \mathbb{R} \to \mathbb{T}$ we can find a continuous homomorphism $\eta : \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ \eta = \chi$, i.e., one has the following commutative diagram:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\eta"'] \arrow[dr, "\chi"'] & \mathbb{R} \arrow[d, "\varphi"] \\ & \mathbb{T} \end{tikzcd} \tag{8}$$

Let $r := \eta(1)$. It is easy to check that $\eta(x) = rx$ for every $x \in \mathbb{R}$. This means that $\chi = \chi_r$. This proves that the assignment $g : r \mapsto \chi_r$ is an isomorphism $\mathbb{R} \to \widehat{\mathbb{R}}$. Its continuity immediately follows from the definition of the compact-open topology of $\widehat{\mathbb{R}}$. As $\mathbb{R}$ is $\sigma$-compact, this isomorphism is also open by the open mapping theorem. □

**Proposition 12.2.5.** *Let $G$ be a totally disconnected locally compact abelian group. Then $\ker\chi$ is an open subgroup of $G$ for every $\chi \in \widehat{G}$.*

*Proof.* According to Theorem 7.4.2, by the continuity of $\chi$ and the total disconnectedness of $G$ there exists an open subgroup $O$ of $G$ such that $\chi(O) \subseteq \Lambda_1$. Since $\Lambda_1$ contains no non-trivial subgroup, $\chi(O) = \{1\}$, so $O \subseteq \ker\chi$. Therefore, $\ker\chi$ is open. □

**Exercise 12.2.6.** *Let $G$ be an abelian group and $p$ be a prime. Prove that*

*(a) $\chi \in p\widehat{G}$ iff $\chi(G[p]) = 0$.*

*(b) $p\chi = 0$ in $\widehat{G}$ iff $\chi(pG) = 0$.*

*Conclude that*

*(i) a discrete abelian group $G$ is divisible (resp., torsion-free) iff $\widehat{G}$ is torsion-free (resp., divisible).*

*(ii) the groups $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{Q}_p}$ are torsion-free and divisible.*

**Example 12.2.7.** Let $p$ be a prime. Then $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$, where $\mathbb{Q}_p$ denotes the field of all $p$-adic numbers.

To prove $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$ consider the character $\chi_1 : \mathbb{Q}_p \to \mathbb{T}$ obtained simply by the canonical map $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{J}_p \cong \mathbb{Z}(p^\infty) \leq \mathbb{T}$. As $\mathbb{J}_p$ is open in $\mathbb{Q}_p$, $\chi_1 \in \widehat{\mathbb{Q}_p}$. For every $\xi \in \mathbb{Q}_p$ consider the map $\rho_\xi : \mathbb{Q}_p \to \mathbb{Q}_p$ defined by $\rho_\xi(x) = \xi x$. Then its composition $\chi_\xi = \chi_1 \circ \rho_\xi$ with $\chi_1$ gives a continuous character of $\mathbb{Q}_p$. If $\xi \neq 0$, then $\chi_\xi \neq 0$, so the homomorphism $g : \mathbb{Q}_p \to \widehat{\mathbb{Q}_p}$ defined by $g(\xi) = \chi_\xi$ has $\ker g = 0$. To see that $g$ is surjective consider any continuous non-trivial character $\chi \in \widehat{\mathbb{Q}_p}$. By Proposition 12.2.5, $N = \ker\chi$ is an open subgroup of $\mathbb{Q}_p$. Hence, $N = p^m\mathbb{J}_p$ for some $m \in \mathbb{Z}$. Let $\chi'$ be defined by $\chi'(x) = \chi(p^{-m}x)$ for all $x \in \mathbb{Q}_p$. Then $\ker\chi' = \mathbb{J}_p$. In this way $\ker\chi' = \ker\chi_1 = \mathbb{J}_p$. On the other hand, $\chi_1(\mathbb{Q}_p) = \chi'(\mathbb{Q}_p) = \mathbb{Z}(p^\infty)$. By Corollary 12.2.3 (b), there exists an automorphism $\iota$ of $\mathbb{Z}(p^\infty)$, such that $\chi' = \iota \circ \chi_1$. There exists $\xi \in \mathbb{J}_p$, such that $\iota(x) = \xi x$ for every $x \in \mathbb{Z}(p^\infty)$. Since all three homomorphisms $\chi_1 : \mathbb{Q}_p \to \mathbb{Z}(p^\infty)$, $\chi' : \mathbb{Q}_p \to \mathbb{Z}(p^\infty)$ and $\iota : \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)$, are $\mathbb{J}_p$-module homomorphims, we deduce that $\chi'(x) = \chi_1(\xi x)$ for all $x \in \mathbb{Q}_p$. Consequently, $\chi(x) = \chi'(p^m\xi x)$ for all $x \in \mathbb{Q}_p$. In other words, $\chi = \chi_{p^m\xi} = g(p^m\xi)$. Therefore, $g : \mathbb{Q}_p \to \widehat{\mathbb{Q}_p}$ is an isomorphism. To check its continuity note first that every compact set of $\mathbb{Q}_p$ is contained in some of the compact open subgroups $p^m\mathbb{J}_p$. Then the basic neighborhood $U_m := W(p^m\mathbb{J}_p, \Lambda_1)$ coincides with all $\chi$ that vanish on $p^m\mathbb{J}_p$ (as $\Lambda_1$ contains no non-trivial subgroups). Hence, $g^{-1}(U_m)$ is open, as it contains the open subgroup $p^{-m}\mathbb{J}_m$ of $\mathbb{Q}_p$. This proves the continuity of $g$. As $\mathbb{Q}_p$ is $\sigma$-compact, this isomorphism is also open by the open mapping theorem.

**Exercise 12.2.8.** *Let $H$ be a subgroup of $\mathbb{R}^n$. Prove that every $\chi \in \widehat{H}$ extends to a continuous character of $\mathbb{R}^n$.*

## 12.3 Some general properties of the dual

### 12.3.1 The dual of direct products and direct sums

We prove next that the dual group of a finite product of abelian topological groups is the product of the dual groups of each group.

**Lemma 12.3.1.** *If $G$ and $H$ are topological abelian groups, then $\widehat{G \times H}$ is isomorphic to $\widehat{G} \times \widehat{H}$.*

*Proof.* Define $\Phi : \widehat{G} \times \widehat{H} \to \widehat{G \times H}$ by $\Phi(\chi_1, \chi_2)(x_1, x_2) = \chi_1(x_1) + \chi_2(x_2)$ for every $(\chi_1, \chi_2) \in \widehat{G} \times \widehat{H}$ and $(x_1, x_2) \in G \times H$. Then $\Phi$ is a homomorphism, in fact $\Phi(\chi_1 + \psi_1, \chi_2 + \psi_2)(x_1, x_2) = (\chi_1 + \psi_1)(x_1) + (\chi_2 + \psi_2)(x_2) = \chi_1(x_1) + \psi_1(x_1) + \chi_2(x_2) + \psi_2(x_2) = \Phi(\chi_1, \chi_2)(x_1, x_2) + \Phi(\psi_1, \psi_2)(x_1, x_2)$.

Moreover $\Phi$ is injective, because

$$\begin{aligned}
\ker\Phi &= \{(\chi, \psi) \in \widehat{G} \times \widehat{H} : \Phi(\chi, \psi) = 0\} \\
&= \{(\chi, \psi) \in \widehat{G} \times \widehat{H} : \Phi(\chi, \psi)(x, y) = 0 \text{ for every } (x, y) \in G \times H\} \\
&= \{(\chi, \psi) \in \widehat{G} \times \widehat{H} : \chi(x) + \psi(y) = 0 \text{ for every } (x, y) \in G \times H\} \\
&= \{(\chi, \psi) \in \widehat{G} \times \widehat{H} : \chi(x) = 0 \text{ and } \psi(y) = 0 \text{ for every } (x, y) \in G \times H\} \\
&= \{(0, 0)\}.
\end{aligned}$$

To prove that $\Phi$ is surjective, take $\psi \in \widehat{G \times H}$ and note that $\psi(x_1, x_2) = \psi(x_1, 0) + \psi(0, x_2)$. Now define $\psi_1(x_1) = \psi(x_1, 0)$ for every $x_1 \in G$ and $\psi_2(x_2) = \psi(0, x_2)$ for every $x_2 \in H$. Hence $\psi_1 \in \widehat{G}$, $\psi_2 \in \widehat{H}$ and $\psi = \Phi(\psi_1, \psi_2)$.

Now we show that $\Phi$ is continuous. Let $W(K,U)$ be an open neighborhood of 0 in $\widehat{G\times H}$ ($K$ is a compact subset of $G\times H$ and $U$ is an open neighborhood of 0 in $\mathbb{T}$). Since the projections $\pi_G$ and $\pi_H$ of $G\times H$ onto $G$ and $H$ are continuous, $K_G=\pi_G(K)$ and $K_H=\pi_H(K)$ are compact in $G$ and in $H$ respectively. Taking an open neighborhood $V$ of 0 in $\mathbb{T}$ with $V+V\subseteq U$, it follows $\Phi(W(K_G,V)\times W(K_H,V))\subseteq W(K,U)$.

It remains to prove that $\Phi$ is open. Consider two open neighborhoods $W(K_G,U_G)$ of 0 in $\widehat{G}$ and $W(K_H,U_H)$ of 0 in $\widehat{H}$, where $K_G\subseteq G$ and $K_H\subseteq H$ are compact and $U_G,U_H$ are open neighborhoods of 0 in $\mathbb{T}$. Then $K=(K_G\cup\{0\})\times(K_H\cup\{0\})$ is a compact subset of $G\times H$ and $U=U_G\cap U_H$ is an open neighborhood of 0 in $\mathbb{T}$. Thus $W(K,U)\subseteq\Phi(W(K_G,U_G)\times W(K_H,U_H))$, because if $\chi\in W(K,U)$ then $\chi=\Phi(\chi_1,\chi_2)$, where $\chi_1(x_1)=\chi(x_1,0)\in U\subseteq U_G$ for every $x_1\in K_G$ and $\chi_2(x_2)=\chi(0,x_2)\in U\subseteq U_H$ for every $x_2\in K_H$. $\square$

It follows from Example 12.2.4 that the groups $\mathbb{T}$, $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$, $\mathbb{J}_p$ e $\mathbb{R}$ satisfy $\widehat{\widehat{G}}\cong G$, namely the Pontryagin-van Kampen duality theorem. Using the Lemma 12.3.1 this propertiy extends to all finite direct products of these groups.

Call a topological abelian group $G$ *autodual*, if $G$ satisfies $\widehat{G}\cong G$. We have seen already that $\mathbb{R}$ and $\mathbb{Q}_p$ are autodual. By Lemma 12.3.1 finite direct products of autodual groups are autodual. Now using this observation and Lemma 12.3.1 we provide a large supply of groups for which the Pontryagin-van Kampen duality holds true.

**Proposition 12.3.2.** *Let $P_1,P_2$ and $P_3$ be finite sets of primes, $m,n,k,k_p\in\mathbb{N}$ ($p\in P_3$) and $n_p,m_p\in\mathbb{N}_+$ ($p\in P_1\cup P_2$). Then every group of the form*

$$G=\mathbb{T}^n\times\mathbb{Z}^m\times\mathbb{R}^k\times F\times\prod_{p\in P_1}\mathbb{Z}(p^\infty)^{n_p}\times\prod_{p\in P_2}\mathbb{J}_p^{m_p}\times\prod_{j\in P_3}\mathbb{Q}_p^{k_p},$$

*where $F$ is a finite abelian group, satisfies $\widehat{\widehat{G}}\cong G$.*

*Moreover, such a group is autodual iff $n=m$, $P_1=P_2$ and $n_p=m_p$ for all $p\in P_1=P_2$. In particular, $\widehat{\widehat{G}}\cong G$ holds true for all elementary locally compact abelian groups.*

*Proof.* Let us start by proving $\widehat{F}=F^*\cong F$. Recall that $F$ has the form $F\cong\mathbb{Z}_{n_1}\times\ldots\times\mathbb{Z}_{n_m}$. So applying Lemma 12.3.1 we are left with the proof of the isomorphism $\mathbb{Z}_n^*\cong\mathbb{Z}_n$ for every $n\in\mathbb{N}_+$. The elements $x$ of $\mathbb{T}$ satisfying $nx=0$ are precisely those of the unique cyclic subgroup of order $n$ of $\mathbb{T}$, we shall denote that subgroup by $\mathbb{Z}_n$. Therefore, the group $Hom(\mathbb{Z}_n,\mathbb{Z}_n)$ of all homomorphisms $\mathbb{Z}_n\to\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_n$.

It follows easily from Lemma 12.3.1 that if $\widehat{\widehat{G_i}}\cong G_i$ (resp., $\widehat{G_i}\cong G_i$) for a finite family $\{G_i\}_{i=1}^n$ of topological abelian groups, then also $G=\prod_{i=1}^n G_i$ satisfies $\widehat{\widehat{G}}\cong G$ (resp., $\widehat{G}\cong G$). Therefore, it suffices to verify that the groups $\mathbb{T}$, $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$, and $\mathbb{J}_p$ e satisfy $\widehat{\widehat{G}}\cong G$, while $\widehat{\mathbb{R}}\cong\mathbb{R}$, $\widehat{\mathbb{Q}_p}\cong\mathbb{Q}_p$ were already checked.

It follows from Proposition 12.2.4 that $\widehat{\mathbb{Z}}\cong\mathbb{T}$ and $\widehat{\mathbb{T}}\cong\mathbb{Z}$, hence $\mathbb{Z}\cong\widehat{\widehat{\mathbb{Z}}}$ and $\mathbb{T}\cong\widehat{\widehat{\mathbb{T}}}$. Analogously, $\widehat{\mathbb{Z}(p^\infty)}\cong\mathbb{J}_p$ and $\widehat{\mathbb{J}_p}\cong\mathbb{Z}(p^\infty)$ yield $\mathbb{Z}(p^\infty)\cong\widehat{\widehat{\mathbb{Z}(p^\infty)}}$ and $\mathbb{J}_p\cong\widehat{\widehat{\mathbb{J}}}_p$. $\square$

The problem of characterizing all autodual locally compact abelian groups is still open [81, 82].

**Theorem 12.3.3.** *Let $\{D_i\}_{i\in I}$ be a family of discrete abelian groups and let $\{G_i\}_{i\in I}$ be a family of compact abelian groups. Then*

$$\widehat{\bigoplus_{i\in I}D_i}\cong\prod_{i\in I}\widehat{D_i}\quad\text{and}\quad\widehat{\prod_{i\in I}G_i}\cong\bigoplus_{i\in I}\widehat{G_i}.\tag{5}$$

*Proof.* Let $\chi:\bigoplus_{i\in I}D_i\to\mathbb{T}$ be a character and let $\chi_i:D_i\to\mathbb{T}$ be its restriction to $D_i$. Then $\chi\mapsto(\chi_i)\in\prod_{i\in I}\widehat{D_i}$ is the first isomorphism in (5).

Let $\chi:\prod_{i\in I}G_i\to\mathbb{T}$ be a continuous character. Pick a neighborhood $U$ of 0 containing no non-trivial subgroups of $\mathbb{T}$. Then there exists a neighborhood $V$ of 0 in $G=\prod_{i\in I}G_i$ with $\chi(V)\subseteq U$. By the definition of the Tychonov topology there exists a finite subset $F\subseteq I$ such that $V$ contains the subproduct $B=\prod_{i\in I\setminus F}G_i$. Being $\chi(B)$ a subgroup of $\mathbb{T}$, we conclude that $\chi(B)=0$ by the choice of $U$. Hence $\chi$ factorizes through the projection $p:G\to\prod_{i\in F}G_i=G/B$; so there exists a character $\chi':\prod_{i\in F}G_i\to\mathbb{T}$ such that $\chi=\chi'\circ p$. Obviously, $\chi'\in\bigoplus_{i\in I}\widehat{G_i}$. Then $\chi\mapsto\chi'$ is the second isomorphism in (5). $\square$

In order to extend the isomorphism (5) to the general case of locally compact abelian groups one has to consider a specific topology on the direct sum. This will not be done here.

**Example 12.3.4.** Using the isomorphism $\mathbb{Q}/\mathbb{Z}\cong\bigoplus_p\mathbb{Z}(p^\infty)$, Example 12.2.4 and Theorem 12.3.3, we obtain $\widehat{\mathbb{Q}/\mathbb{Z}}\cong\prod_p\mathbb{J}_p$.

### 12.3.2 Extending the duality functor $\widehat{\ }$ to homomorphisms

Let $G$ and $H$ be abelian topological groups. If $f : G \to H$ is a continuous homomorphism, define $\widehat{f} : \widehat{H} \to \widehat{G}$ putting $\widehat{f}(\chi) = \chi \circ f$ for every $\chi \in \widehat{H}$.

**Lemma 12.3.5.** *If $f : G \to H$ is a continuous homomorphism of topological abelian group, then $\widehat{f}(\chi) = \chi \circ f$ is a continuous homomorphism as well.*

*(a) If $f(G)$ is dense in $H$, then $\widehat{f}$ is injective.*

*(b) If $f$ is an embedding and $f(G)$ is either open or dense in $H$, then $\widehat{f}$ is surjective.*

*(c) if $f$ is a surjective homomorphism, such that every compact subset of $H$ is covered by some compact subset of $G$, then $\widehat{f}$ is an embedding.*

*(d) if $f$ is a quotient homomorphism and $G$ is locally compact, then $\widehat{f}$ is an embedding.*

*Proof.* Assume $K$ is a compact subset of $G$ and $U$ a neighborhood of 0 in $\mathbb{T}$. Then $f(K)$ is a compact set in $H$, so $W = W_{\widehat{H}}(f(K), U)$ is a neighborhood of 0 in $\widehat{H}$ and $\widehat{f}(W) \subseteq W_{\widehat{G}}(K, U)$. This proves the continuity of $\widehat{f}$.

(a) If $\widehat{f}(\chi) = 0$, then $\chi \circ f = 0$. By the density of $f(G)$ in $H$ this yields $\chi = 0$.

(b) Let $\chi \in \widehat{G}$. If $f(G)$ is open in $H$, then any extension $\xi : H \to \mathbb{T}$ of $\chi$ will be continuous on $f(G)$. There exists at least one such extension $\xi$ by Corollary 2.1.11. Hence $\xi \in \widehat{H}$ and $\chi = \widehat{f}(\xi)$. Now consider the case when $f(G)$ is dense in $H$. Then $\widetilde{H} = \widetilde{G}$ and the characters of $H$ can be extended to characters of $G$ (see Theorem 6.2.3).

(c) Assume $L$ is a compact subset of $H$ and $U$ a neighborhood of 0 in $\mathbb{T}$. Let $K$ be a compact set in $G$ such that $f(K) = L$. Then $\widehat{f}(W_{\widehat{H}}(L, U)) = \mathrm{Im}\widehat{f} \cap W_{\widehat{G}}(K, U)$, so $\widehat{f}$ is an embedding.

(d) Follows from (c) and Lemma 7.2.5.

□

If $\mathcal{H}$ denote the category of all Hausdorff abelian topological groups, the *Pontryagin-van Kampen duality functor*, defined by

$$G \mapsto \widehat{G} \text{ and } f \mapsto \widehat{f}$$

for objects $G$ and morphisms $f$ of $\mathcal{H}$, is a contravariant functor $\widehat{\ }: \mathcal{H} \to \mathcal{H}$ (see Lemma 12.3.5). In particular, if $f$ is a topological isomorphism, then $\widehat{f}$ is a topological isomorphism too.

**Corollary 12.3.6.** *If $G$ is an abelian group and $H$ is a subgroup of $G$, then $|\widehat{H}| \leq |\widehat{G}|$.*

Now we use this corollary in order to compute the size of the dual $\widehat{G}$ of a discrete abelian group.

**Theorem 12.3.7** (Kakutani)**.** *For every infinite discrete abelian group $|\widehat{G}| = 2^{|G|}$.*

*Proof.* The inequality $|\widehat{G}| \leq 2^{|G|}$ is obvious since $\widehat{G}$ is contained in the Cartesian power $\mathbb{T}^G$ which has cardinality $2^{|G|}$. It remains to prove the inequality $|\widehat{G}| \geq 2^{|G|}$. We consider several cases using each time the inequality $\widehat{G} \geq \widehat{H}$ from Corollary 12.3.6 for an appropriate subgroup $H$ of $G$.

**Case 1.** $G$ is countable, so we have to check that $|\widehat{G}| \geq \mathfrak{c}$.

Assume first that $G$ is a $p$-group. If $r_p(G) = n$ is finite, then by Example 2.1.22 $G$ contains a subgroup $H \cong \mathbb{Z}(p^\infty)$. Since $|\widehat{\mathbb{Z}(p^\infty)}| = \mathfrak{c}$, from the above corollary we conclude that $|G| \geq \mathfrak{c}$. If $r_p(G)$ is infinite, then $G$ contains a subgroup $H \cong \bigoplus_{\mathbb{N}} \mathbb{Z}_p$ (namely, $H = G[p]$). Since $\widehat{H} \cong \mathbb{Z}_p^{\mathbb{N}}$ by Theorem 12.3.3, we conclude again that $|\widehat{G}| \geq |\widehat{H}| = \mathfrak{c}$.

Now assume that $G$ is torsion. If $r_p(G)$ is positive for infinitely many primes $p_1, p_2, \ldots, p_n, \ldots$, then $G$ contains a subgroup $H \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}_{p_n}$. Since $\widehat{H} \cong \prod_{n=1}^{\infty} \mathbb{Z}_{p_n}$ by Theorem 12.3.3, we conclude again that $|\widehat{G}| \geq |\widehat{H}| = \mathfrak{c}$.

Finally, assume that $G$ is not torsion. Then $G$ contains a subgroup $H \cong \mathbb{Z}$. Since $\widehat{H} \cong \mathbb{T}$, we conclude that $|\widehat{G}| \geq |\widehat{H}| = \mathfrak{c}$.

**Case 2.** $G$ is uncountable. Now, with $|G| = \kappa$, the group $G$ contains a subgroup $H$ of the form $H \cong \bigoplus_{i \in I} C_i$, where $|I| = \kappa$ and each $C_i$ is a cyclic group. Indeed, let $M$ be a maximal independent subset of $G$, so that

$$\langle M \rangle = \bigoplus_{x \in M} \langle x \rangle \cong \bigoplus_{|M|} \mathbb{Z}$$

is a free abelian group. Then, with

$$Soc(G) = \bigoplus_p G[p] \cong \bigoplus_p \left( \bigoplus_{r_p(G)} \mathbb{Z}_p \right)$$

let $H = \langle M \rangle \oplus Soc(G)$. It is easy to see now that for every non-zero $x \in G$ there exists $k \in \mathbb{Z}$ such that $kx \in H$ and $kx \neq 0$. Let

$$D = \left( \bigoplus_M \mathbb{Q} \right) \oplus \bigoplus_p \left( \bigoplus_{r_p(G)} \mathbb{Z}(p^\infty) \right).$$

Then $D$ is divisible and there is an obvious injective homomorphism $j : H \to D$. Let us see that $|H| = |D|$. Indeed, $|\langle M \rangle| = |\left(\bigoplus_M \mathbb{Q}\right)|$. This ends up the argument when $|G| = r_0(G) = |\langle M \rangle|$. Assume now that $|G| > r_0(G)$, so $|G| = \sup r_p(G)$, hence at least one of the $p$-ranks is infinite. It remains to note now that $|G[p]| = r_p(G) = \left|\bigoplus_{r_p(G)} \mathbb{Z}(p^\infty)\right|$ whenever $r_p(G)$ is infinite, so $|G| = \sup r_p(G) = |H|$ again.

By the divisibility of $D$ $j$ can be extended to a homomorphism $j_1 : G \to D$. Assume $j_1(x) = 0$ for some non-zero $x \in G$. Then $kx \in H$ and $kx \neq 0$ for some $k \in \mathbb{Z}$. This gives $j(kx) = j_1(kx) = kj_1(x) = 0$, a contradiction. Hence, $G \cong j_1(G) \leq D$. This gives $|H| = |D| = |G|$. Therefore, $H$ is a direct sum of $|G|$-many cyclic groups, i.e., has the desired form. By the above theorem, $\widehat{H} \cong \prod_{i\in I} \widehat{C_i}$. Since each $\widehat{C_i}$ is either a finite cyclic group, or a copy of $\mathbb{T}$, we conclude that $|\widehat{G}| \geq |\widehat{H}| = 2^{|I|} = 2^{|G|}$. □

**Remark 12.3.8.** As we shall see in the sequel, every compact abelian group $K$ has the form $K = \widehat{G}$ for some discrete abelian group. Moreover, $G$ can be taken to be dual $\widehat{K}$. Hence, one can re-write Kakutani's theorem also as $|K| = 2^{w(K)}$, where $K$ is a compact abelian group. This property can be established for *arbitrary compact groups*. Since the inequality $|K| \leq 2^{w(K)}$ holds true for every Hausdorff topological group, it remains to use the deeply non-trivial fact that a compact group $K$ contains a copy of the Cantor cube $\{0,1\}^{w(K)}$ having size $2^{w(K)}$. The compactness plays a relevant role in this embedding theorem. Indeed, there are precompact groups that contain no copy of $\{0,1\}^{\aleph_0}$ (e.g., all groups of the form $G^\#$, as they contain no non-trivial convergent sequences by Glicksberg's theorem, whereas $\{0,1\}^{\aleph_0}$ contains non-trivial convergent sequences).

Now we shall see that the group $\mathbb{Q}$ satisfies the duality theorem (see item (b) below).

**Example 12.3.9.** Let $K$ denote the compact group $\widehat{\mathbb{Q}}$. Then:

(a) $K$ contains a closed subgroup $H$ isomorphic to $\widehat{\mathbb{Q}/\mathbb{Z}}$ such that $K/H \cong \mathbb{T}$;

(ii) $\widehat{K} \cong \mathbb{Q}$.

(a) Denote by $H$ the subgroup of all $\chi \in K$ such that $\chi(\mathbb{Z}) = 0$. We prove that $H$ is a closed subgroup of $K$ such that $K/H$ is isomorphic to $\mathbb{T}$. To this end consider the continuous map $\rho : K \to \widehat{\mathbb{Z}}$ obtained by the restriction to $\mathbb{Z}$ of every $\chi \in K$ (i.e., $\rho = \widehat{j}$, where $j : \mathbb{Z} \hookrightarrow \mathbb{Q}$). Then $\rho$ is surjective by Lemma 12.3.5. Obviously, $\ker \rho = H$, so $\mathbb{T} \cong \widehat{\mathbb{Z}} \cong K/H$. To see that $H \cong \widehat{\mathbb{Q}/\mathbb{Z}}$ note that the characters of $\mathbb{Q}/\mathbb{Z}$ correspond precisely to those characters of $\mathbb{Q}$ that vanish on $\mathbb{Z}$, i.e., precisely $H$.

(b) By Exercise 12.2.6 $K$ is a divisible torsion-free group, every non-zero $r \in \mathbb{Q}$ defines a continuous automorphism $\lambda_r$ of $K$ by setting $\lambda_r(x) = rx$ for every $x \in K$ (see Exercise 7.3.3). Then the composition $\rho \circ \lambda_r : K \to \mathbb{T}$ defines a character $\chi_r \in \widehat{K}$ with $\ker \chi_r = r^{-1}H$. For the sake of completeness let $\chi_0 = 0$. By Exercise 12.3.4 $\widehat{\mathbb{Q}/\mathbb{Z}} \cong \prod_p \mathbb{J}_p$ is totally disconnected, so by Corollary 11.3.5 $H$ has no surjective characters $\chi : H \to \mathbb{T}$. Now let $\chi \in \widehat{K}$ be non-zero. Then $\chi(K)$ will be a non-zero closed divisible subgroup of $\mathbb{T}$, hence $\chi(K) = \mathbb{T}$. On the other hand, $N = \ker \chi$ is a proper closed subgroup of $K$ such that $N + H \neq K$, as $\chi(H)$ is a proper closed subgroup of $\mathbb{T}$ by the previous argument. Hence, $\chi(H)$ is finite, say of order $m$. Then $N + H$ contains $N$ is a finite-index subgroup, more precisely $[(N+H) : N] = [H : (N \cap H)] = m$. Then $mH \leq N$. Consider the character $\chi_{m^{-1}}$ of $K$ having $\ker \chi_{m^{-1}} = mH \leq N$. By Corollary there exists $k \in \mathbb{Z}$ such that $\chi = k\chi_{m^{-1}} = \chi_r$, where $r = km^{-1} \in \mathbb{Q}$. This shows that $\widehat{K} = \{\chi_r : r \in \mathbb{Q}\} \cong \mathbb{Q}$.

The compact group $\widehat{\mathbb{Q}}$ is closely related to the adele ring[12] $\mathbf{A}_{\mathbb{Q}}$ of the field $\mathbb{Q}$, more detail can be found in [52, 68, 114, 144].

**Exercise 12.3.10.** *Prove that a discrete abelian group $G$ satisfies $\widehat{\widehat{G}} \cong G$ whenever*

[12] $\mathbf{A}_{\mathbb{Q}}$ is the subring of $\mathbb{R} \times \prod_p \mathbb{Q}_p$ consisting of those elements $x = (r, (\xi_p)) \in \mathbb{R} \times \prod_p \mathbb{Q}_p$ ($r \in \mathbb{R}$, $\xi_p \in \mathbb{Q}_p$ for each $p$) such that all but finitely many $\xi_p \in \mathbb{J}_p$. Then the subgroup $Q = \{x = (r, (\xi_p)) \in \mathbb{R} \times \prod_p \mathbb{Q}_p : \xi_p = r \in \mathbb{Q}$ for all $p\}$ of $\mathbf{A}_{\mathbb{Q}}$ is discrete and $\mathbf{A}_{\mathbb{Q}}/Q \cong \widehat{\mathbb{Q}}$, according to A. Weil's theorem.

*(a) $G$ is divisible;*

*(b) $G$ is free;*

*(c) $G$ is of finite exponent;*

*(d) $G$ is torsion and every primary component of $G$ is of finite exponent.*

(Hint. (a) Use Examples 12.2.4 and 12.3.9 (b) and the fact that every divisible group is a direct sum of copies of $\mathbb{Q}$ and the groups $\mathbb{Z}(p^\infty)$.

(c) and (d) Use that fact that every abelian group of finite exponent is a direct sum of cyclic subgroups (i.e., Prüfer's theorem, see (c) of Example 2.1.3).

## 12.4 The natural transformation $\omega$

Let $G$ be a topological abelian group. Define $\omega_G : G \to \widehat{\widehat{G}}$ such that $\omega_G(x)(\chi) = \chi(x)$, for every $x \in G$ and for every $\chi \in \widehat{G}$. We show now that $\omega_G(x) \in \widehat{\widehat{G}}$.

**Proposition 12.4.1.** *If $G$ is a topological abelian group. Then $\omega_G(x) \in \widehat{\widehat{G}}$ and $\omega_G : G \to \widehat{\widehat{G}}$ is a homomorphism.*

*If $G$ is locally compact, then the homomorphism $\omega_G$ is a continuous.*

*Proof.* In fact,
$$\omega_G(x)(\chi + \psi) = (\chi + \psi)(x) = \chi(x) + \psi(x) = \omega_G(x)(\chi) + \omega_G(x)(\psi),$$
for every $\chi, \psi \in \widehat{G}$. Moreover, if $U$ is an open neighborhood of 0 in $\mathbb{T}$, then $\omega_G(x)(W(\{x\}, U)) \subseteq U$. This proves that $\omega_G(x)$ is a character of $\widehat{G}$, i.e., $\omega_G(x) \in \widehat{\widehat{G}}$. For every $x, y \in G$ and for every $\chi \in \widehat{G}$ we have $\omega_G(x+y)(\chi) = (\chi)(x+y) = \chi(x) + \chi(y) = \omega_G(\chi)(x) + \omega_G(\chi)(y)$ and so $\omega_G$ is a homomorphism.

Now assume $G$ is locally compact. To prove that $\omega_G$ is continuous, pick an open neighborhood $A$ of 0 in $\mathbb{T}$ and a compact subset $K$ of $\widehat{G}$. Then $W(K, A)$ is an open neighborhood of 0 in $\widehat{\widehat{G}}$. Let $U$ be an open neighborhood of 0 in $G$ with compact closure. Take an open symmetric neighborhood $B$ of 0 in $\mathbb{T}$ with $B + B \subseteq A$. Thus $W(\overline{U}, B)$ is an open neighborhood of 0 in $\widehat{G}$. Since $K$ is compact, there exist finitely many characters $\chi_1, \ldots, \chi_m$ of $G$ such that $K \subseteq (\chi_1 + W(\overline{U}, B)) \cup \cdots \cup (\chi_m + W(\overline{U}, B))$. For every $i = 1, \ldots, m$ there is an open neighborhood $V_i$ of 0 in $G$ such that $\chi_i(V_i) \subseteq B$ and $V_i \subseteq U$. Define $V = U \cap V_1 \cap \cdots \cap V_m \subseteq U$ and note that $\chi_i(V) \subseteq B$ for every $i = 1, \ldots, m$. Thus $\omega_G(V) \subseteq W(K, A)$. Indeed, if $x \in V$ and $\chi \in K$, then $\chi_i(x) \in B$ for every $i = 1, \ldots, m$ and there exists $i_0 \in \{1, \ldots, m\}$ such that $\chi \in \chi_{i_0} + W(\overline{U}, B)$; so $\chi(x) = \chi_{i_0}(x) + \psi(x)$ with $\psi \in W(\overline{U}, B)$ and then $\omega_G(x)(\chi) = \chi(x) \in B + B \subseteq A$. □

Let us see that local compactness is essential in the above proposition

**Example 12.4.2.** For a countably infinite abelian group $G$ consider the topological group $G^\#$. Then $\widehat{G^\#} = G^*$ is compact since the only compact subsets of $\widehat{G^\#}$ are the finite ones.[13] Therefore, $\widehat{\widehat{G^\#}} = G$ discrete. Hence $\omega : G^\# \to \widehat{\widehat{G^\#}}$ is not continuous when $G$ is infinite (as $G^\#$ is precompact, while the discrete group $G$ is not precompact).

In this chapter we shall adopt a more precise approach to Pontryagin-van Kampen duality theorem, by asking $\omega_G$ to be a topological isomorphism. To this end we give the following

**Definition 12.4.3.** A topological abelian groups $G$ is said to satisfy Pontryagin-van Kampen duality theorem, or shortly, to be *reflexive*, if $\omega_G$ is a topological isomorphism.

**Lemma 12.4.4.** *If the topological abelian groups $G_i$ are reflexive for $i = 1, 2, \ldots, n$, then also $G = \prod_{i=1}^n G_i$ is reflexive.*

*Proof.* Apply Lemma 12.3.1 twice to obtain an isomorphism $j : \prod_{i=1}^n \widehat{\widehat{G}}_i \to \widehat{\widehat{G}}$. It remains to verify that the product $\pi : G \to \prod_{i=1}^n \widehat{\widehat{G}}_i$ of the isomorphisms $\omega_{G_i} : G_i \to \widehat{\widehat{G}}_i$ given by our hypothesis composed with the isomorphism $j$ gives precisely $\omega_G$. □

Let $\mathcal{L}$ be the full subcategory of $\mathcal{H}$ having as objects all locally compact abelian groups. According to Proposition 12.1.2, the functor $\widehat{\ }: \mathcal{H} \to \mathcal{H}$ sends $\mathcal{L}$ to itself, i.e., defines a functor $\widehat{\ }: \mathcal{L} \to \mathcal{L}$. The Pontryagin-van Kampen duality theorem states that $\omega$ is a natural equivalence from $id_{\mathcal{L}}$ to $\widehat{\widehat{\ }}: \mathcal{L} \to \mathcal{L}$. We start by proving that $\omega$ is a natural transformation.

[13] This non-trivial fact is a particular case of Glicksberg's theorem: a locally compact abelian group $G$ and its Bohr modification $G^+$ have the same compact sets. For a proof in the specific countable case see Theorem 12.4.9.

**Proposition 12.4.5.** *$\omega$ is a natural transformation from $id_{\mathcal{L}}$ to $\widehat{\widehat{\ }}: \mathcal{L} \to \mathcal{L}$.*

*Proof.* By Proposition 12.4.1 $\omega_G$ is continuous for every $G \in \mathcal{L}$. Moreover for every continuous homomorphism $f : G \to H$ the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\omega_G"'] & H \arrow[d, "\omega_H"] \\ \widehat{\widehat{G}} \arrow[r, "\widehat{\widehat{f}}"'] & \widehat{\widehat{H}} \end{tikzcd}$$

In fact, if $x \in G$ and $\xi \in \widehat{H}$, then $\omega_H(f(x))(\xi) = \xi(f(x))$. On the other hand,

$$(\widehat{\widehat{f}}(\omega_G(x)))(\xi) = (\omega_G(x) \circ \widehat{f})(\xi) = \omega_G(x)(\widehat{f}(\xi)) = \omega_G(x)(\xi \circ f) = \xi(f(x)).$$

Hence $\omega_H(f(x)) = \widehat{\widehat{f}}(\omega_G(x))$ for every $x \in G$. $\square$

**Remark 12.4.6.** Note that $\omega_G$ is a monomorphism if and only if $\widehat{G}$ separates the points of $G$. Hence, by Theorem 11.3.3, $\omega_G$ is a monomorphism for every locally compact abelian group. Moreover, $\omega_G(G)$ is a subgroup of $\widehat{\widehat{G}}$ that separates the points of $\widehat{G}$.

### 12.4.1 Proof of the compact-discrete case of Pontryagin-van Kampen duality theorem

Now we can prove the Pontryagin-van Kampen duality theorem in the case when $G$ is either compact or discrete.

**Theorem 12.4.7.** *If the abelian topological group $G$ is either compact or discrete, then $\omega_G$ is a topological isomorphism.*

*Proof.* If $G$ is discrete, then $\widehat{G}$ separates the points of $G$ by Corollary 2.1.12 and if $G$ is compact, then $\widehat{G}$ separates the points of $G$ by the Peter-Weyl Theorem 11.2.1. Therefore $\omega_G$ is injective by Remark 12.4.6. If $G$ is discrete, then $\widehat{G}$ is compact and the characters from $\omega_G(G)$ separate the points of $\widehat{G}$. Hence, $\omega_G(G) = \widehat{\widehat{G}}$ by Corollary 11.2.3. Since $\widehat{\widehat{G}}$ is discrete, $\omega_G$ is a topological isomorphism.

Let now $G$ be compact. Then $\omega_G$ is a continuous monomorphism by Proposition 12.4.1 and Remark 12.4.6. Moreover, $\omega_G$ is open, by Theorem 7.3.1. Suppose that $\omega_G(G)$ is a proper subgroup of $\widehat{\widehat{G}}$. By the compactness of $G$, $\omega_G(G)$ is compact, hence closed in $\widehat{\widehat{G}}$. By the Peter-Weyl Theorem 11.2.1 applied to $\widehat{\widehat{G}}/\omega_G(G)$, there exists $\xi \in \widehat{\widehat{\widehat{G}}} \setminus \{0\}$ such that $\xi(\omega_G(G)) = \{0\}$. Since $\widehat{G}$ is discrete, $\omega_{\widehat{G}}$ is a topological isomorphism and so there exists $\chi \in \widehat{G}$ such that $\omega_{\widehat{G}}(\chi) = \xi$. Thus for every $x \in G$ we have $0 = \xi(\omega_G(x)) = \omega_{\widehat{G}}(\chi)(\omega_G(x)) = \omega_G(x)(\chi) = \chi(x)$. It follows that $\chi \equiv 0$ and so that also $\xi \equiv 0$, a contradiction. $\square$

Our next step is to prove the Pontryagin-van Kampen duality theorem when $G$ is elementary locally compact abelian:

**Theorem 12.4.8.** *If $G$ is an elementary locally compact abelian group, then $\omega_G$ is a topological isomorphism of $G$ onto $\widehat{\widehat{G}}$.*

*Proof.* According to Lemma 12.4.4 and Theorem 12.4.7 it suffices to prove that $\omega_{\mathbb{R}}$ is a topologically isomorphism. Of course, by the fact that $\widehat{\mathbb{R}}$ is topologically isomorphic to $\mathbb{R}$, one concludes immediately that also $\mathbb{R}$ and $\widehat{\widehat{\mathbb{R}}}$ are topologically isomorphic. A more careful analysis of the dual $\widehat{\mathbb{R}}$ shows the crucial role of the ($\mathbb{Z}$-)bilnear map $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{T}$ defined by $\lambda(x, y) = \chi_1(xy)$, where $\chi_1 : \mathbb{R} \to \mathbb{T}$ is the character determined by the canonical quotient map $\mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Indeed, for every $y \in \mathbb{R}$ the map $\chi_y : \mathbb{R} \to \mathbb{T}$ defined by $x \mapsto \lambda(x, y)$ is an element of $\widehat{\mathbb{R}}$. Hence the second copy $\{0\} \times \mathbb{R}$ of $\mathbb{R}$ in $\mathbb{R} \times \mathbb{R}$ can be identified with $\widehat{\mathbb{R}}$. On the other hand, every element $x \in \mathbb{R}$ gives a continuous character$\mathbb{R} \to \mathbb{T}$ defined by $y \mapsto \lambda(x, y)$, so can be considered as the element $\omega_{\mathbb{R}}(x)$ of $\widehat{\widehat{\mathbb{R}}}$. We have seen that every $\xi \in \widehat{\widehat{\mathbb{R}}}$ has this form. This means that $\omega_{\mathbb{R}}$ is surjective. Since continuity of $\omega_{\mathbb{R}}$, as well as local compactness of $\widehat{\widehat{\mathbb{R}}}$ are already established, $\omega_{\mathbb{R}}$ is a topological isomorphism by the open mapping theorem. $\square$

As a first application of the duality theorem we can prove:

**Theorem 12.4.9.** *Let $G$ be a countably infinite abelian group. Then the topological group $G^{\#}$ has no infinite compact sets.*

*Proof.* According to Theorem 2.2.17, it suffices to see that $G^{\#}$ has no non-trivial convergent sequences. Assume that $x_n$ is a null sequence in $G^{\#}$. Let $K$ be the compact dual of $G$, and consider the characters $\chi_n = \omega_G(x_n)$ of $K$. Then

$$\chi_n(x) \to 0 \ \text{ in } \ \mathbb{T} \ \text{ for every } \ x \in K. \tag{$*$}$$

Hence, letting

$$F_n = \{x \in K : (\forall m \geq n)\chi_m(x) \in \overline{\Lambda}_4\},$$

we get an increasing chain $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \subseteq \ldots$ of closed sets in $K$ with $K = \bigcup_n F_n$. Since $K$ is compact, from the Baire category theorem we deduce that some $F_n$ must have non-empty interior, i.e., there exists $y \in K$ and $U \in \mathcal{V}_K(0)$ with $x+U \subseteq F_n$. Hence, $\chi_m(x+U) \subseteq \overline{\Lambda}_4$ for all $m \geq n$. From (*) we deduce that there exists $n_1$ such that $\chi_m(x) \in \overline{\Lambda}_4$ for all $m \geq n_1$. Therefore, for all $\chi_m(U) \subseteq \overline{\Lambda}_2$ for all $m \geq n_2 = \max\{n, n_1\}$.

From (*) we deduce that $\chi_n(x) \in \overline{\Lambda}_2$ for all $n \geq k_x$. By the compactness of $K = \bigcup_x x + U$, there exist a finite number of points $x_1, \ldots, x_s \in K$ such that $K = \bigcup_{i=1}^s x_i + U$. Let $k_0 = \max\{k_1, \ldots, k_s\}$ and $n_0 = \max\{n_2, k_0\}$. Then $\chi_m(K) \subseteq \Lambda_1$ for all $m \geq n_0$. As $\Lambda_1$ contains no non-trivial subgroups, we deduce that $\chi_m = 0$ for all $m \geq n_0$. This entails $x_m = 0$ for all $m \geq n_0$. □

### 12.4.2 Exactness of the functor $\widehat{\ }$

For a subset $X$ of $G$ the *annihilator* of $X$ in $\widehat{G}$ is $A_{\widehat{G}}(X) = \{\chi \in \widehat{G} : \chi(X) = \{0\}\}$ and for a subset $Y$ of $\widehat{G}$ the *annihilator* of $Y$ in $G$ is $A_G(Y) = \{x \in G : \chi(x) = 0 \text{ for every } \chi \in Y\}$. When no confusion is possible we shall omit the subscripts $_{\widehat{G}}$ and $_G$.

The next lemma will help us in computing the dual of a subgroup and a quotient group.

**Lemma 12.4.10.** *Let $G$ be a locally compact abelian group. If $M$ is a subset of $G$, then $A_{\widehat{G}}(M)$ is a closed subgroup of $\widehat{G}$.*

*Proof.* It suffices to note that

$$A_{\widehat{G}}(M) = \bigcap_{x \in M} \{\chi \in \widehat{G} : \chi(x) = 0\} = \bigcap \{\ker \omega_G(x) : x \in M\},$$

where each $\ker \omega(x)$ is a closed subgroup of $\widehat{G}$. □

Call a continuous homomorphism $f : G \to H$ of topological groups *proper* if $f : G \to f(G)$ is open, whenever $f(G)$ carries the topology inherited from $H$. In particular, a surjective continuous homomorphism is proper iff it is open.

A short sequence $0 \to G_1 \xrightarrow{f} G \xrightarrow{h} G_2 \to 0$ in $\mathcal{L}$, where $f$ and $h$ are continuous homomorphisms, is *exact* if $f$ is injective, $h$ is surjective and $\operatorname{im} f = \ker h$. It is *proper* if $f$ and $h$ are proper.

**Lemma 12.4.11.** *Let $G$ be a locally compact abelian group, $H$ a subgroup of $G$ and $i : H \to G$ the canonical inclusion of $H$ in $G$. Then*

*(a) $\widehat{i} : \widehat{G} \to \widehat{H}$ is surjective if $H$ is dense or open or compact;*

*(b) $\widehat{i}$ is injective if and only if $H$ is dense in $G$;*

*(c) if $H$ is closed and $\pi : G \to G/H$ is the canonical projection, then the sequence*

$$0 \to \widehat{G/H} \xrightarrow{\widehat{\pi}} \widehat{G} \xrightarrow{\widehat{i}} \widehat{H}$$

*is exact, $\widehat{\pi}$ is proper and $\operatorname{im} \widehat{\pi} = A_{\widehat{G}}(H)$. If $H$ is open or compact, then $\widehat{i}$ is open and surjective.*

*Proof.* (a) Note that $\widehat{i}$ is surjective if and only if for every $\chi \in \widehat{H}$ there exists $\xi \in \widehat{G}$ such that $\xi \restriction_H = \chi$. If $H$ is compact apply Corollary 11.3.4. Otherwise Lemma 12.3.5 applies.

(b) If $H$ is dense, then $\widehat{i}$ is injective by Lemma 12.3.5. Conversely, assume that $\overline{H}$ is a proper subgroup of $G$ and let $q : G \to G/\overline{H}$ be the canonical projection. By Theorem 11.3.3 there exists $\chi \in \widehat{G/\overline{H}}$ not identically zero. Then $\xi = \chi \circ q \in \widehat{G}$ is non-zero and satisfies $\xi(H) = \{0\}$, i.e., $\widehat{i}(\xi) = 0$. This implies that $\widehat{i}$ is not injective.

(c) According to Lemma 12.3.5 $\widehat{\pi}$ is a monomorphism, since $\pi$ is surjective. We have that $\widehat{i} \circ \widehat{\pi} = \widehat{\pi \circ i} = 0$. If $\xi \in \ker \widehat{i} = \{\chi \in \widehat{G} : \chi(H) = \{0\}\}$, then $\xi(H) = \{0\}$. So there exists $\xi_1 \in \widehat{G/H}$ such that $\xi = \xi_1 \circ \pi$ (i.e. $\xi = \widehat{\pi}(\xi_1)$) and we can conclude that $\ker \widehat{i} = \operatorname{im} \widehat{\pi}$. So the sequence is exact and $\operatorname{im} \widehat{\pi} = \ker \widehat{i} = A_{\widehat{G}}(H)$.

To show that $\widehat{\pi}$ is proper it suffices to apply Lemma 12.3.5.

If $H$ is open or compact, (a) implies that $\widehat{i}$ is surjective. It remains to show that $\widehat{i}$ is open. If $H$ is compact then $\widehat{H}$ is discrete by Example 12.1.1(2), so $\widehat{i}$ is obviously open. If $H$ is open, let $K$ be a compact neighborhood of 0 in $G$ such that $K \subseteq H$. Then $W = W_{\widehat{G}}(K, \overline{\Lambda_4})$ is a compact neighborhood of 0 in $\widehat{G}$. Since $\widehat{i}$ is surjective, $V = \widehat{i}(W) = W_{\widehat{H}}(K, \overline{\Lambda_4})$ is a neighborhood of 0 in $\widehat{H}$. Now $M = \langle W \rangle$ and $M_1 = \langle V \rangle$ are open compactly generated subgroups respectively of $\widehat{G}$ and $\widehat{H}$, and $\widehat{i}(M) = M_1$. Since $M$ is $\sigma$-compact by Lemma 7.2.9, we can apply Theorem 7.3.1 to the continuous surjective homomorphism $\widehat{i}\restriction_M \colon M \to M_1$ and so also $\widehat{i}$ is open. □

The lemma gives these immediate corollaries:

**Corollary 12.4.12.** *Let $G$ be a locally compact abelian group and let $H$ be a closed subgroup of $G$. Then $\widehat{G/H} \cong A_{\widehat{G}}(H)$. Moreover, if $H$ is open or compact, then $\widehat{H} \cong \widehat{G}/A_{\widehat{G}}(H)$.*

**Corollary 12.4.13.** *Let $G$ be a locally compact abelian group and $H$ a closed subgroup of $G$. If $a \in G \setminus H$ then there exists $\chi \in A(H)$ such that $\chi(x) \neq 0$.*

*Proof.* Let $\rho : \widehat{G/H} \to A(H)$ be the topological isomorphism of Corollary 12.4.12. By Theorem 11.3.3 there exists $\psi \in \widehat{G/H}$ such that $\psi(a + H) \neq 0$. Therefore $\chi = \rho(\psi) \in A(H)$ and $\chi(a) = \rho(\psi)(a) = \psi(a + H) \neq 0$. □

This gives the following immediate

**Corollary 12.4.14.** *Let $f : G \to H$ be a continous homomorphism of locally compact abelian groups. Then $f(G)$ is dense iff $\widehat{f}$ is injective.*

The next corollary says that the duality functor preserves proper exactness for some sequences.

**Corollary 12.4.15.** *If the sequence $0 \to G_1 \xrightarrow{f} G \xrightarrow{h} G_2 \to 0$ in $\mathcal{L}$ is proper exact, with $G_1$ compact or $G_2$ discrete, then $0 \to \widehat{G_2} \xrightarrow{\widehat{h}} \widehat{G} \xrightarrow{\widehat{f}} \widehat{G_1} \to 0$ is proper exact with the same property.*

### 12.4.3 Proof of Pontryagin-van Kampen duality theorem: the general case

Now we can prove prove the Pontryagin-van Kampen duality theorem, namely $\omega$ is a natural equivalence from $id_{\mathcal{L}}$ to $\widehat{\widehat{\ }} \colon \mathcal{L} \to \mathcal{L}$.

**Theorem 12.4.16.** *If $G$ is a locally compact abelian group, then $\omega_G$ is a topological isomorphism of $G$ onto $\widehat{\widehat{G}}$.*

*Proof.* We know by Proposition 12.4.5 that $\omega$ is a natural transformation from $id_{\mathcal{L}}$ to $\widehat{\widehat{\ }} \colon \mathcal{L} \to \mathcal{L}$. Our plan is to chase the given locally compact abelian group $G$ into an appropriately chosen proper exact sequence

$$0 \to G_1 \xrightarrow{f} G \xrightarrow{h} G_2 \to 0$$

in $\mathcal{L}$, with $G_1$ compact or $G_2$ discrete, such that both $G_1$ and $G_2$ satisfy the duality theorem. By Corollary 12.4.15 the sequences

$$0 \to \widehat{G}_2 \xrightarrow{\widehat{h}} \widehat{G} \xrightarrow{\widehat{f}} \widehat{G}_2 \to 0 \quad \text{and} \quad 0 \to \widehat{\widehat{G}}_1 \xrightarrow{\widehat{\widehat{f}}} \widehat{\widehat{G}} \xrightarrow{\widehat{\widehat{h}}} \widehat{\widehat{G}}_2 \to 0$$

are proper exact. According to Proposition 12.4.5 the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & G_1 \arrow[r, "f"] \arrow[d, "\omega_{G_1}"'] & G \arrow[r, "h"] \arrow[d, "\omega_G"] & G_2 \arrow[r] \arrow[d, "\omega_{G_2}"] & 0 \\
0 \arrow[r] & \widehat{\widehat{G}}_1 \arrow[r, "\widehat{\widehat{f}}"'] & \widehat{\widehat{G}} \arrow[r, "\widehat{\widehat{h}}"'] & \widehat{\widehat{G}}_2 \arrow[r] & 0
\end{tikzcd}$$

According to Remark 12.4.6, $\omega_{G_1}$, $\omega_G$, $\omega_{G_2}$ are injective. Moreover, $\omega_{G_1}$ and $\omega_{G_2}$ are surjective by our choice of $G_1$ and $G_2$. Then $\omega_G$ must be surjective too, by Lemma 2.1.26.

If $G$ is locally compact abelian and compactly generated, by Proposition 11.3.2 we can choose $G_1$ compact and $G_2$ elementary locally compact abelian. Then $G_1$ and $G_2$ satisfy the duality theorem by Theorems 12.4.7 and 12.4.8, hence $\omega_G$ is surjective, by Lemma 2.1.26. Since $\omega_G$ is a continuous isomorphism and $G$ is $\sigma$-compact, we conclude with Theorem 7.3.1 that $\omega_G$ is a topological isomorphism.

In the general case of locally compact abelian group $G$, we can take an open compactly generated subgroup $G_1$ of $G$. This will produce a proper exact sequence $0 \to G_1 \xrightarrow{f} G \xrightarrow{h} G_2 \to 0$ with $G_1$ compactly generated and $G_2 \cong G/G_1$ discrete. By the previous case $\omega_{G_1}$ is a topological isomorphism and $\omega_{G_2}$ is an isomorphism thanks to Theorem 12.4.7. Therefore $\omega_G$ is a continuous isomorphism by Lemma 2.1.26.

Moreover $\omega_G \restriction_{f(G_1)}: f(G_1) \to \widehat{\widehat{f}}(\widehat{\widehat{G}}_1)$ is a topological isomorphism and $f(G_1)$ and $\widehat{\widehat{f}}(\widehat{\widehat{G}}_1)$ are open subgroups respectively of $G$ and $\widehat{\widehat{G}}$. Thus $\omega_G$ is a topological isomorphism. □

### 12.4.4 First applications of the duality theorem

As a first applications of the duality theorem we describe now the structure of the some classes of compact abelian groups, such as monothetic ones or bounded torsion ones.

**Theorem 12.4.17.** *Let $K$ be a compact group. Then $K$ is monothetic if and only if the dual group $\widehat{K}$ admits an injective homomorphism into $\mathbb{T}$.*

*Proof.* The group $G = \widehat{K}$ is discrete.

Assume there exists an injective homomorphism $j : G \to \mathbb{T}$. Taking the duals we obtain a homomorphism $\widehat{j} : \mathbb{Z} = \widehat{\mathbb{T}} \to \widehat{G} = \widehat{\widehat{K}} \cong K$ with dense image (Corollary 12.4.14). Hence $K$ is monothetic. Viceversa, if $K$ is monothetic, then there exists a homomorphism $f : \mathbb{Z} = \widehat{\mathbb{T}} \to K$ with dense image. By Corollary 12.4.14 the homomorphism $\widehat{f} : G \to \widehat{Z} = \mathbb{T}$ is injective. □

The above theorem gives the following corollary:

**Corollary 12.4.18.** *Let $K$ be a compact group.*

*(a) If $K$ is connected, then $K$ is monothetic if and only if $w(K) \leq \mathfrak{c}$.*

*(b) If $K$ is totally disconnected, then $K$ is monothetic if and only if $K$ is isomorphic to a quotient group of $\prod_{p\in\mathbb{P}} \mathbb{J}_p$.*

*Proof.* (a) By Proposition 11.3.7, $G = \widehat{K}$ is torsion-free. Since a torsion-free group $G$ admits an injective homomorphism into $\mathbb{T}$ precisely when $|G| \leq$, it remains to recall that $w(K) = |G|$.

(b) By Corollary 11.3.6, $G = \widehat{K}$ is torsion. Since $t(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$, the torsion group $G$ admits an injective homomorphism into $\mathbb{T}$ if and only if $G$ admits an injective homomorphism into $\mathbb{Q}/\mathbb{Z}$. This is equivalent to have $K$ is isomorphic to a quotient group of $\prod_{p\in\mathbb{P}} \mathbb{J}_p \cong \widehat{\mathbb{Q}/\mathbb{Z}}$. □

Now we describe the torsion compact abelian groups.

**Theorem 12.4.19.** *Every torsion compact abelian group $G$ is bounded. More precisely, there exists natural numbers $m_1, \ldots, m_n$ and cardinals $\alpha_1, \ldots, \alpha_n$ such that $G \cong \prod_{i=1}^n \mathbb{Z}_{m_i}^{\alpha_i}$.*

*Proof.* Let us note first that $G = \bigcup_{n=1}^\infty G[n!]$ is a union of closed subgroups. Using the Baire category theorem we conclude that $G[n!]$ is open for some $n$, so must have finite index by the compactness of $G$. This yields $mG = 0$ for some $m$. Show that this yields also $m\widehat{G} = 0$. Now apply Prüfer's theorem to $\widehat{G}$ and the fact that $G \cong \widehat{\widehat{G}}$. □

Next we compute the density character of a compact group $K$ as a function of its weight $w(K) = |\widehat{K}|$. More precisely, given the already known inequality $w(K) \leq 2^{d(K)}$, valid for all topological groups, we see now that for compact $K$ the density character $d(K)$ has the smallest possible value (w.r.t. $w(K)$).

**Proposition 12.4.20.** *For a compact abelian group $K$*

$$d(K) = \log|\widehat{K}| = \min\{\kappa : 2^\kappa \geq |\widehat{K}|\}.$$

*Proof.* Let $\kappa = \min\{\kappa : 2^\kappa \geq |\widehat{K}|\}$ and $\lambda = d(G)$. Since $r_p(\mathbb{T}^\kappa) = r_0(\mathbb{T}^\kappa) = 2^\kappa$, the inequality $|\widehat{K}| \leq 2^\kappa$ and the divisibility of $\mathbb{T}^\kappa$ ensure that there exists an injective homomorphism $j : \widehat{K} \to \mathbb{T}^\kappa$. Therefore, the continuous homomorphism $\widehat{j} : \bigoplus_\kappa \mathbb{Z} \cong \widehat{\mathbb{T}^\kappa} \to \widehat{\widehat{K}} \cong K$ has a dense image. This proves $\lambda \leq \kappa$.

Now assume that $D$ is a dense subgroup of $K$ of size $\lambda$. Then there exists a surjective homomorphism $q : \bigoplus_\lambda \mathbb{Z} \to D$, hence we get a homomorphism $j : \bigoplus_\lambda \mathbb{Z} \to K$ with dense image. Therefore $\widehat{j} : \widehat{K} \to \mathbb{T}^\lambda$ is injective, by Corollary 12.4.14. Since $|\mathbb{T}^\lambda| = 2^\lambda$, this yields $2^\lambda \geq |\widehat{K}|$, i.e., $\lambda \geq \kappa$. □

## 12.5 The annihilator relations and further applications of the duality theorem

Our last aim is to prove that the annihilators define an inclusion-inverting bijection between the family of all closed subgroups of a locally compact group $G$ and the family of all closed subgroups of $\widehat{G}$. We use the fact that one can identify $G$ and $\widehat{\widehat{G}}$ by the topological isomorphism $\omega_G$. In more precise terms:

**Exercise 12.5.1.** *Let $G$ be a locally compact abelian group and $Y$ be a subset of $\widehat{G}$. Then $A_{\widehat{\widehat{G}}}(Y) = \omega_G(A_G(Y))$.*

**Lemma 12.5.2.** *If $G$ is a locally compact abelian group and $H$ a closed subgroup of $G$, then*

$$H = A_G(A_{\widehat{G}}(H))) = \omega_G^{-1}(A_{\widehat{\widehat{G}}}(A_{\widehat{G}}(H))).$$

*Proof.* The first equality follows immediately from Corollary 12.4.13.

The last equality follows from the equality $H = A_G(A_{\widehat{G}}(H)))$ and Exercise 12.5.1. □

By Lemma 12.4.12 the equality $H = A_G(A_{\widehat{G}}(H)))$ holds if and only if $H$ is a closed subgroup of $G$.

**Proposition 12.5.3.** *Let $G$ be a locally compact abelian group and $H$ a closed subgroup of $G$. Then $\widehat{H} \cong \widehat{G}/A(H)$.*

*Proof.* Since $H = \omega_G^{-1}(A_{\widehat{\widehat{G}}}(A_{\widehat{G}}(H)))$ by Lemma 12.5.2 we have a topological isomorphism $\phi$ from $H$ to $\widehat{\widehat{G}/A(H)}$ given by $\phi(h)(\alpha + A(H)) = \alpha(h)$ for every $h \in H$ and $\alpha \in \widehat{G}$. This gives rise to another topological isomorphism $\widehat{\phi} : \widehat{\widehat{\widehat{G}/A(H)}} \to \widehat{H}$. By Pontryagin's duality theorem 12.4.16 $\omega_{\widehat{G}/A(H)}$ is a topological isomorphism from $\widehat{G}/A(H)$ to $\widehat{\widehat{\widehat{G}/A(H)}}$. The composition gives the desired isomorphism. □

Finally, let us resume for reader's benefit some of the most relevant points of Pontryagin-van Kampen duality theorem established so far:

**Theorem 12.5.4.** *Let $G$ be a locally compact abelian group. Then $\widehat{G}$ is a locally compact abelian group and:*

*(a) the correspondence $H \mapsto A_{\widehat{G}}(H)$, $N \mapsto A_G(N)$, where $H$ is a closed subgroup of $G$ and $N$ is a closed subgroup of $\widehat{G}$, defines an order-inverting bijection between the family of all closed subgroups of $G$ and the family of all closed subgroups of $\widehat{G}$;*

*(b) for every closed subgroup $H$ of $G$ the dual group $\widehat{H}$ is isomorphic to $\widehat{G}/A(H)$, while $A(H)$ is isomorphic to the dual $\widehat{G/H}$;*

*(c) $\omega_G : G \to \widehat{\widehat{G}}$ is a topological isomorphism;*

*(d) $G$ is compact (resp., discrete) if and only if $\widehat{G}$ is discrete (resp., compact);*

*Proof.* The first sentence is proved in Theorem 12.1.2. (a) is Lemma 12.5.2 while (b) is Proposition 12.5.3. (c) is Theorem 12.4.16. To prove (d) apply Theorem 12.4.16 and Lemma 12.1.1. □

Using the full power of the duality theorem one can prove the following structure theorem on compactly generated locally compact abelian groups.

**Theorem 12.5.5.** *Let $G$ be a locally compact compactly generated abelian group. Prove that $G \cong \mathbb{R}^n \times \mathbb{Z}^m \times K$, where $n, m \in \mathbb{N}$ and $K$ is a compact abelian group.*

*Proof.* According to Theorem 11.3.2 there exists a compact subgroup $K$ of $G$ such that $G/K$ is an elementary locally compact abelian group. Taking a bigger compact subgroup one can get the quotient $G/K$ to be of the form $\mathbb{R}^n \times \mathbb{Z}^m$ for some $n, m \in \mathbb{N}$. Now the dual group $\widehat{G}$ has an open subgroup $A(K) \cong \widehat{G/K} \cong \mathbb{R}^n \times \mathbb{T}^m$. Since this subgroup is divisible, one has $\widehat{G} \cong \mathbb{R}^n \times \mathbb{T}^m \times D$, where $D \cong \widehat{G}/A(K)$ is discrete and $D \cong \widehat{K}$. Taking duals gives $G \cong \widehat{\widehat{G}} \cong \mathbb{R}^n \times \mathbb{Z}^m \times K$. □

Making sharper use of the annihilators one can prove the structure theorem on locally compact abelian groups (see [102, 57] for a proof).

**Theorem 12.5.6.** *Let $G$ be a locally compact abelian group. Then $G \cong \mathbb{R}^n \times G_0$, where $G_0$ is a closed subgroup of $G$ containing an open compact subgroup $K$.*

This is the strongest structure theorem concerning the locally compact abelian groups.

**Exercise 12.5.7.** *Deduce Theorem 12.5.5 from Theorem 12.5.6.*

(Hint. Let $G$ be a locally compact compactly generated abelian group and let $C$ be a compact subset of $G$ generating $G$. By Theorem 12.5.6 we can write $G = \mathbb{R}^n \times G_0$, where $G_0$ is a closed subgroup of $G$ containing an open compact subgroup $K$. Since the quotient group $G_0/K \cong G/\mathbb{R}^n \times K$ is discrete, the image of $C$ in $G/\mathbb{R}^n \times K$ is finite. Since $G$ is generated by $C$, this yields that the quotient group $G/\mathbb{R}^n \times K$ is finitely generated, so isomorphic to $\mathbb{Z}^d \times F$ for some finite group $F$ and $d \in \mathbb{N}$. Hence, by taking a compact subgroup $K_1$ of $G$ containing $K$, we can assume that $G/\mathbb{R}^n \times K \cong \mathbb{Z}^d$. Since the group $\mathbb{Z}^d$ is free, the group $G$ splis as $G = \mathbb{R}^n \times K_1 \times \mathbb{Z}^d$.)

As another corollary of Theorem 12.5.6 one obtains:

**Corollary 12.5.8.** *Every locally compact abelian group is isomorphic to a closed subgroup of a group of the form $\mathbb{R}^n \times D \times C$, where $n \in \mathbb{N}$, $D$ is a discrete abelian group and $C$ is a compact abelian group.*

*Proof.* Let $G \cong \mathbb{R}^n \times G_0$ with $n$, $G_0$ and $K$ as in Theorem 12.5.6. Then there exists a cardinal $\kappa$ and an embedding $j : K \to \mathbb{T}^\kappa$. Since $\mathbb{T}^\kappa$ is divisible, one can extend $j$ to a homomorphism $j_1 : G_0 \to \mathbb{T}^\kappa$. It will be continuous by the continuity of $j$ and by the openness of $K$ in $G_0$. Let $j_2 : G_0/K \to D$ be an injective homomorphism with $D$ is discrete divisible group. Then the diagonal map $f : (j_1, j_2) : G_0 \to \mathbb{T}^\kappa \times D$ is injective and continuous. Since $K$ is compact, its restriction to $K$ is an embedding. Since $K$ is open in $G_0$ this yields that $f : G_0 \to \mathbb{T}^\kappa \times D$ is an embedding. This provides an embedding $\nu$ of $G \cong \mathbb{R}^n \times G_0$ into the group $\mathbb{R}^n \times \mathbb{T}^\kappa \times D$. The image $\nu(G) \cong G$ will be a closed subgroup of $\mathbb{R}^n \times \mathbb{T}^\kappa \times D$ since locally compact groups are complete. □

The next lemma follows directly from the definitions.

**Lemma 12.5.9.** *Let $G$ be a topological abelian group. Then for $\chi_1, \dots, \chi_n \in \widehat{G}$ and $\delta > 0$ one has*

$$U_G(\chi_1, \dots, \chi_n; \delta) = \omega_G^{-1}(W_{\widehat{\widehat{G}}}(\{\chi_1, \dots, \chi_n\}, U),$$

*where $U$ is the neighborhood of 0 in $\mathbb{T} \cong \mathbb{S}$ determined by $|\mathrm{Arg}\ z| < \delta$.*

Using this lemma we can prove now that another duality can be obtained for precompact abelian groups, if the dual $\widehat{G}$ of the group is equipped with the topology of the pointwise convergence instead of the finer compact-open topology. In the sequel we shall denote by $\widehat{G}_{pw}$ the the dual $\widehat{G}$ equipped with the pointwise convergence topology.

**Theorem 12.5.10.** *The assigment $G \mapsto \widehat{G}_{pw}$ defines a duality in the category of precompact abelian groups, more precisely $\omega_G \to \widehat{(\widehat{G}_{pw})}_{pw}$ is a topological isomorphism for every precompact abelian group $G$.*

*Proof.* Note that by the definition of the group $\widehat{G}_{pw}$, its topology coincides with $\mathcal{T}_{\omega_G(G)}$. This proves that $\omega_G$ is surjective in view of Corollary 10.3.11. The injectivity of $\omega_G$ follows from the fact that $G$ is precompact, so its continuous characters separate to points of $G$. The fact that $\omega_G$ is a homeomorphism follows from the preceding lemma and the fact that a typical neighborhood of 0 in $\widehat{(\widehat{G}_{pw})}_{pw}$ has the form $W_{\widehat{G}}(\{\chi_1, \dots, \chi_n\}, U)$ for some $\chi_1, \dots, \chi_n \in \widehat{G}$ and a neighborhood $U$ of 0 in $\mathbb{T} \cong \mathbb{S}$. □

**Proposition 12.5.11.** *For a compact connected abelian group $G$ the subgroup $t(G)$ is dense in $G$ iff $\widehat{G}$ is reduced. Consequently, every compact connected abelian group $G$ has the form $G \cong G_1 \times \widehat{\mathbb{Q}}^\alpha$ for some cardinal $\alpha$, where the compact subgroup $G_1$ coincides with the closure of the subgroup $t(G)$ of $G$.*

*Proof.* Note first that $\widehat{G}$ is torsion-free by Proposition 11.3.7. Hence $\widehat{G}$ is reduced iff $\bigcap_{n=1}^\infty n\widehat{G} = 0$. It is easy to see that this equality is equivalent to density of $t(G) = \bigcup_{n=1}^\infty G[n]$ in $G$. To prove the second assertion consider the torsion-free dual $\widehat{G}$ and its decomposition $\widehat{G} = D(\widehat{G}) \times R$, where $R$ is a reduced subgroup of $\widehat{G}$. Since $\widehat{G}$ is torsion-free, there exists a cardinal $\alpha$ such that $D(\widehat{G}) \cong \bigoplus_\alpha \mathbb{Q}$. Therefore, $\widehat{D(\widehat{G})} \cong \widehat{\mathbb{Q}}^\alpha$. On the other hand, by the first part of the proof, the compact group $G_1 = \widehat{R}$ has a dense $t(G_1)$. Since $G \cong \widehat{\widehat{G}} \cong \widehat{\mathbb{Q}}^\alpha \times G_1$ and $\widehat{\mathbb{Q}}^\alpha$ is torsion-free, the torsion subgroup of $\widehat{\widehat{G}}$ coincides with $t(G_1)$, so its closure gives $G_1$. □

**Exercise 12.5.12.** *Give example of a reduced abelian group $G$ such that $\bigcap_{n=1}^\infty nG \neq 0$.*

(Hint. Fix a prime number $p$ and take an appropriate quotient of the group $\bigoplus_{n=1}^\infty \mathbb{Z}(p^n)$.

# 13 Appendix

## 13.1 Topological rings and fields

Let us start with the definition of a topological ring:

**Definition 13.1.1.** Let $A$ be a ring.

- A topology $\tau$ on $G$ is said to be a *ring topology* if the maps $f : G \times G \to G$ and $m : G \times G \to G$ defined by $f(x, y) = x - y$ and $m(x, y) = xy$, are continuous when $A \times A$ carries the product topology.
- A *topological ring* is a pair $(A, \tau)$ of a ring $A$ and a ring topology $\tau$ on $A$.

Obviously, a topology $\tau$ on a ring $A$ is a ring topology iff $(A, +, \tau)$ is a topological group and the map $m : A \times A \to A$ is continuous.

Here are some examples, starting with two trivial ones: for every ring $A$ the discrete topology and the indiscrete topology on $A$ are ring topologies. Non-trivial examples of a topological ring are provided by the fields $\mathbb{R}$ and $\mathbb{C}$ of reals and complex numbers, respectively.

**Example 13.1.2.** For every prime $p$ the group $\mathbb{J}_p$ of $p$-adic integers carries also a ring structure and its compact group topology is also a ring topology.

Other examples of ring topologies will be given in §13.1.5.

We shall exploit the fact that for a topological ring $A$ the pair $(A, +, \tau)$ is a topological group. In particular, for $a \in A$ we shall make use of the fact that the filter $\mathcal{V}_{G,\tau}(a)$ of all neighborhoods of the element $a$ of $A$ coincides with the filter $a + \mathcal{V}_{G,\tau}(0)$, obtained by translation of the filter $\mathcal{V}_{G,\tau}(0)$.

The following theorem is a counterpart of Theorem 3.1.5:

**Theorem 13.1.3.** *Let $A$ be a ring and let $\mathcal{V}(0_A)$ be the filter of all neighborhoods of $0_G$ in some ring topology $\tau$ on $G$. Then:*

**(a)** *for every $U \in \mathcal{V}(0_G)$ there exists $V \in \mathcal{V}(0_G)$ with $V + V \subseteq U$;*

**(b)** *for every $U \in \mathcal{V}(0_G)$ there exists $V \in \mathcal{V}(0_G)$ with $-V \subseteq U$;*

**(c)** *for every $U \in \mathcal{V}(0_G)$ and for every $a \in G$ there exists $V \in \mathcal{V}(0_G)$ with $Va \subseteq U$ and $aV \subseteq U$;*

**(d)** *for every $U \in \mathcal{V}(0_G)$ there exists $V \in \mathcal{V}(0_G)$ with $VV \subseteq U$.*

*Conversely, if $\mathcal{V}$ is a filter on $A$ satisfying (a), (b), (c) and (d), then there exists a unique ring topology $\tau$ on $G$ such that $\mathcal{V}$ coincides with the filter of all $\tau$-neighborhoods of $0_G$ in $A$.*

*Proof.* Since $(A, +, \tau)$ is a topological group, (a) and (b) hold true by Theorem 3.1.5. To prove (d) it suffices to apply the definition of the continuity of the multiplication $m : A \times A \to A$ at $(0_A, 0_A) \in A \times A$. Analogously, for (c) use the continuity of the multiplication $m : A \times A \to A$ at $(0_A, a) \in A \times A$ and $(a, 0_A) \in A \times A$.

Let $\mathcal{V}$ be a filter on $G$ satisfying all conditions (a), (b), (c) and (d). It can be proved as in the proof of Theorem 3.1.5 that every $U \in \mathcal{V}$ contains $0_A$. Define a topology $\tau$ on $A$ as the group topology on $(A, +)$ having as a filter of neighborhoods at $0_A$ the filter $\mathcal{V}$ (i.e., the $\tau$-open sets $O$ are the subsets $O \subseteq G$, such that for all $a \in O$ there exists some $U \in \mathcal{V}$ with $a + U \subseteq O$). Since this is a group topology on $(A, +)$ having as a filter of neighborhoods at $0_A$ the filter $\mathcal{V}$, it only remains to check that this is a ring topology, i.e., the multiplication map $m : A \times A \to A$ is continuous at every pair $(a, b) \in A \times A$. Pick a neighborhood of $ab \in A$, it is not restrictive to take it of the form $ab + U$, with $U \in \mathcal{V}$. Next, choose $V \in \mathcal{V}$ such that $V + V + V \subseteq U$ and pick a $W \in \mathcal{V}$ with $WW \subseteq V$, $aW \subseteq V$ and $Wb \subseteq V$. Then

$$m((a + W) \times (b + W)) = (a + W)(b + W) = ab + aW + Wb + WW \subseteq ab + V + V + V \subseteq ab + U.$$

This proves the continuity of the multiplication $m : A \times A \to A$ at $(a, b)$. □

### 13.1.1 Examples and general properties of topological rings

Let $\mathcal{V} = \{J_i : i \in I\}$ be a filter base consisting of two-sided ideals of a ring $A$. Then $\mathcal{V}$ satisfies (a)–(d) from the above theorem, hence generates a ring topology on $A$ having as basic neighborhoods of a point $a \in A$ the family of cosets $\{a + J_i : i \in I\}$. Ring topologies of this type will be called *linear ring topologies*.

Let $(A, \tau)$ be a topological ring and let $I$ be a two-sided ideal of $A$. The quotient ring $A/I$, equipped with the quotient topology of the underlying abelian group $(A/I, +)$ is a topological ring, that we call quotient ring.

If $(A, \tau)$ is a topological ring, then the closure of a two-sided (left, right) ideal $I$ of $A$ is again a two-sided (resp., left, right) ideal of $A$. In particular, the closure $J$ of the ideal $\{0\}$ is a closed two-sided ideal. As we already know, the quotient ring $A/I$ is Hausdorff and shares many of the properties of the initial topological ring $(A, \tau)$. This is why we consider exclusively Hausdorff topological rings.

A Hausdorff topological ring $(A, \tau)$ is called complete, if it is complete as a topological group. In general, if $(A, \tau)$ is a Hausdorff topological ring, the completion $\widetilde{A}$ of the topological group $(A, +, \tau)$ carries a natural ring structure, obtained by the extension of the ring operation of $A$ to $\widetilde{A}$ by continuity[14]. In this way, the completion $\widetilde{A}$ becomes a topological ring.

As fas as connectedness if concerned, one has the following easy to prove fact:

**Theorem 13.1.4.** *The connected component of a topological ring is a two-sided ideal. Hence, every topological ring that is a division ring is either connected, or totally disconnected.*

Let us see now some basic examples of a linear ring topologies.

**Example 13.1.5.** Let $A$ be a ring and $\mathfrak{A}$ be a two-sided ideal of $A$. Then the powers $\{\mathfrak{A}^n : n \in \mathbb{N}\}$ form a filter base of a ring topology named $\mathfrak{A}$-*adic topology.*

(a) The $p$-adic topology of the ring $\mathbb{J}_p$ coincides also with the $p\mathbb{J}_p$-adic topology of the ring $\mathbb{J}_p$, generated by the ideal $p\mathbb{J}_p$.

(b) Let $k$ be a field and let $A = k[x]$ be the polynomial ring over $k$. Take $\mathfrak{A} = (x)$, then the $\mathfrak{A}$-adic topology has as basic neighborhoods of 0 the ideals $(x^n)$.

(c) The completion $\widetilde{A}$ of the ring $A = k[x]$, equipped with the $(x)$-adic topology is the ring $k[[x]]$ of formal power series over $k$ (elements of $k[[x]]$ are the formal power series of the form $\sum_{n=0}^{\infty} a_n x^n$, with $a_n \in k$ for all $n$). The topology of the completion $\widetilde{A}$ coincides with the $(x)_{\widetilde{A}}$-adic topology of $\widetilde{A}$ (here the principal ideal is taken in $\widetilde{A}$).

(d) Let $k$ be a field, $n \in \mathbb{N}_+$ and let $A = k[x_1, \ldots, x_n]$ be the ring of polynomials of $n$-variables over $k$. Take $\mathfrak{A} = (x_1, \ldots, x_n)$, then the $\mathfrak{A}$-adic topology has as basic neighborhoods of 0 the ideals $(\mathfrak{A}^n)$, where the power $\mathfrak{A}^n$ consists of all polynomials having no terms of degree less than $n$.

(e) Similarly, for every $n \in \mathbb{N}_+$ the completion $\widetilde{A}$ of the ring $A = k[x_1, \ldots, x_n]$, equipped with the $(x_1, \ldots, x_n)$-adic topology is the ring $k[[x_1, \ldots, x_n]]$ of formal power series of $n$-variables over $k$. The topology of the completion $\widetilde{A}$ coincides with the $(x_1\widetilde{A} + \ldots + x_n\widetilde{A})$-adic topology of $\widetilde{A}$ (the principal ideals are obviously taken in $\widetilde{A}$). The topological ring $\widetilde{A}$ is compact precisely when $k$ is finite.

A subset $B$ of a topological ring $A$ is *bounded* if for every $U \in \mathcal{V}(0)$ there exists a $V \in \mathcal{V}(0)$ such that $VB \subseteq U$ and $BV \subseteq U$.

**Exercise 13.1.6.** *Let $A$ be a topological ring. Prove that*

*(a) the family of bounded subsets of $A$ is stable under taking subsets and finite unions;*

*(b) every compact subset of $A$ is bounded;*

*(c) if $A$ has a linear topology, then $A$ is bounded.*

*(d*) if $A$ is a compact unitary ring, then $A$ has a linear topology.*

### 13.1.2 Topological fields

Now comes the definition of a topological field:

**Definition 13.1.7.** Let $K$ be a field.

- A topology $\tau$ on $K$ is said to be a *field topology* if the maps $d : G\times G \to G$, $m : G\times G \to G$ and $\iota : K\setminus\{0\} \to K\setminus\{0\}$, defined by $d(x, y) = x - y$, $m(x, y) = xy$ and $\iota(x) = x^{-1}$ are continuous when $A \times A$ carries the product topology.

[14] This can be done in two steps. First one defined an $A$-module structure on $\widetilde{A}$ as follows. For a fixed $a \in A$ the map $x \mapsto ax$ in $A$ is uniformly continuous, so extends to a continuous map $z \mapsto az$ of $\widetilde{A}$. This makes $\widetilde{A}$ a topological left $A$-module. Analogously $\widetilde{A}$ becmes a topological left $A$-module. Now consider a fixed element $y \in \widetilde{A}$. Then $y = \lim a_i$ for some net $(a_i)$ in $A$. For every $z \in \widetilde{A}$ the nets $a_i z$ and $za_i$ are Cauchy nets in $\widetilde{A}$. Put $yz := \lim_i a_i z$ and $zy := \lim_i za_i$. This multiplications makes $\widetilde{A}$ a ring, containing $A$ as a subring. Moreover, $\widetilde{A}$ is a topological ring, when equipped with its comlpetion topology, containing $A$ as a dense subring.

- A *topological field* is a pair $(A, \tau)$ of a field $A$ and a field topology $\tau$ on $A$.

**Exercise 13.1.8.** *Every compact topological field is finite.*

(Hint. Apply item (d) of Exercise 13.1.6.)

The next example provides instances of infinite locally compact topological fields.

**Example 13.1.9.** (a) Obviously, $\mathbb{R}$ and $\mathbb{C}$ are (connected) locally compact topological fields.

(b) For every prime $p$ the field $\mathbb{Q}_p$ equipped with the $p$-adic topology is a locally compact topological field.

(c) Let $K$ be a finite extension of $\mathbb{Q}_p$, equipped with the Tichonov topology, induced by the isomorphism $K \cong \mathbb{Q}_p^d$ of $\mathbb{Q}_p$-vectors spaces, where $d = [K : \mathbb{Q}_p]$. Then $K$ is a locally compact topological field.

It turns out that the example of item (a) gives all connected locally compact topological fields:

**Theorem 13.1.10** (Pontryagin)**.** $\mathbb{R}$ *and* $\mathbb{C}$ *are the only locally compact connected topological fields.*

The locally compact topological fields from Example 13.1.9 (a) and (b) have characteristic 0. It was proved by Kowalski that the totally disconnected locally compact topological fields of characteristic 0 are necessarily the form given in item (b) of the example.

It is possible to build locally compact topological fields, by taking the compact ring $k[[x]]$, where $k$ is a finite field, and its field of fractions $k((x))$, consisting of formal power series of the form $\sum_{n=n_0}^{\infty} a_n x^n$, $n_0 \in \mathbb{N}$. By declaring the subring $k[[x]]$ of $k((x))$ open, with its compact topology, one obtains a locally compact field topology on $k((x))$ having the same characteristic as $k$. Obviously, finite extensions of $k((x))$ will still be locally compact fields of finite characteristic. One can prove that these are all locally compact fields of finite characteristic have this form.

## 13.2 Uniqueness of Pontryagin-van Kampen duality

For topological abelian groups $G, H$ denote by $Chom(G, H)$ the group of all continuous homomorphisms $G \to H$ equipped with the compact-open topology. It was pointed out already by Pontryagin that the group $\mathbb{T}$ is the unique locally compact group $L$ with the property $Chom(Chom(\mathbb{T}, L), L) \cong \mathbb{T}$ (note that this is much weaker than asking $Chom(-, L)$ to define a duality of $\mathcal{L}$). Much later Roeder [136] proved that Pontryagin-van Kampen dualityis the unique functorial duality of $\mathcal{L}$, i.e., the unique involutive contravariant endofunctor $\mathcal{L} \to \mathcal{L}$. Several years later Prodanov [128] rediscovered this result in the following much more general setting. Let $R$ be a locally compact commutative ring and $\mathcal{L}_R$ be the category of locally compact topological $R$-modules. A *functorial duality* $^\# : \mathcal{L}_R \to \mathcal{L}_R$ is a contravariant functor such that $^\# \cdot {}^\#$ is naturally equivalent to the identity of $\mathcal{L}_R$ and for each morphism $f : M \to N$ in $\mathcal{L}_R$ and $r \in R$ $(rf)^\# = rf^\#$ (where, as usual, $rf$ is the morphism $M \to N$ defined by $(rf)(x) = rf(x)$). It is easy to see that the restriction of the Pontryagin-van Kampen duality functor on $\mathcal{L}_R$ is a functorial duality, since the Pontryagin-van Kampen dual $\widehat{M}$ of an $M \in \mathcal{L}_R$ has a natural structure of an $R$-module. So *there is always a functorial duality in* $\mathcal{L}_R$. This stimulated Prodanov to raise the question *how many* functorial dualities can carry $\mathcal{L}_R$ and extend this question to other well known dualities and adjunctions, such as Stone duality[15], the spectrum of a commutative rings [129], etc. at his Seminar on dualities (Sofia University, 1979/83). Uniqueness of the functorial duality was obtained by L. Stoyanov [139] in the case of a compact commutative ring $R$. In 1988 Gregorio [91] extended this result to the general case of compact (not necessarily commutative) ring $R$ (here left and right $R$-modules should be distinguished, so that the dualities are no more *endo*functors). Later Gregorio jointly with Orsatti [93] offered another approach to this phenomenon.

Surprisingly the case of a discrete ring $R$ turned out to be more complicated. For each functorial duality $^\# : \mathcal{L}_R \to \mathcal{L}_R$ the module $T = R^\#$ (the *torus* of the duality $^\#$) is compact and for every $X \in \mathcal{L}_R$ the module $\Delta_T(X) := Chom_R(X, T)$ of all continuous $R$-module homomorphisms $X \to T$, equipped with the compact-open topology, is algebraically isomorphic to $X^\#$. The duality $^\#$ is called *continuous* if for each $X$ this isomorphism is also topological, otherwise $^\#$ is *discontinuous*. Clearly, continuous dualities are classified by their tori, which in turn can be classified by means of the Picard group $Pic(R)$ of $R$. In particular, the unique continuous functorial duality on $\mathcal{L}_R$ is the Pontryagin-van Kampen duality if and only if $Pic(R) = 0$ ([54, Theorem 5.17]). Prodanov [128] (see also [57, §3.4]) proved that every functorial duality on $\mathcal{L} = \mathcal{L}_{\mathbb{Z}}$ is continuous, which in view of $Pic(\mathbb{Z}) = 0$ gives another proof of Roeder's theorem of uniqueness. Continuous dualities were studied in the non-commutative context by Gregorio [92]. While the Picard group provides a good tool to measure the failure of uniqueness for continuous dualities, there is still no efficient way to capture it for discontinuous ones. The first example of a discontinuous duality was given in [54, Theorem 11.1]. Discontinuous dualities of $\mathcal{L}_{\mathbb{Q}}$ and its subcategories are discussed in [52]. It was conjectured by Prodanov that in case

[15]his conjecture that the Stone duality is the unique functorial duality between compact totally disconnected Hausdorff spaces and Boolean algebras was proved to be true by Dimov [69].

$R$ is an algebraic number ring uniqueness of dualities is available if and only if $R$ is a principal ideal domain. This conjecture was proved to be true for real algebraic number rings, but Prodanov's conjecture was shown to fail in case $R$ is an order in an imaginary quadratic number field [33].

We will not touch other well-known dualities for module categories such as Morita duality (see [115]) or more general setting of dualities of (representable dualities, adjunctions rather than involutions, etc. [70], [71] and [123]).

## 13.3 Non-abelian or non-locally compact groups

The Pontryagin-van Kampen duality theorem was extended to some non-locally compact abelian topological groups (e.g., infinite powers of the reals, the underlying additive groups of certain linear topological spaces, etc.). The underlying topological group of a Banach space is reflexive. Characterizations of the reflexive groups were proposed by Venkatamaran [142] and Kye [110], but they contained flaws. These gaps were removed in the recent paper of Hernández [98]. An important class of abelian groups (nuclear groups) were introduced and studied in the monograph [9] (see also [6]) in relation to the duality theorem. Further reference can be found also in [24, 85, 100]

We do not discuss here non-commutative versions of duality for locally compact groups. The difficulties arise already in the compact case – there is no appropriate (or at least, comfortable) structure on the set of irreducible unitary representations of a compact non-abelian group. The reader is referred to [102] for a historical panorama of this trend (Tanaka-Keĭn duality, etc.). In the locally compact case one should see the pioneering paper of H. Chu [25], as well as the monograph of Heyer [103] (see also [104]). The reader can find the last achievements in this field in the survey of Galindo, Hernández, and Wu [87] (see also [99] and [48]).

## 13.4 Relations to the topological theory of topological groups

The Pontryagin-van Kampen dual of a compact abelian group $K$ carries a lot of useful information about the topology of $H$. For example,

- $w(K) = |\widehat{K}|$ (this is true for every precompact group $K$, Corollary 11.1.6),
- $d(K) = \log|\widehat{K}| = \min\{\kappa : 2^\kappa \geq |\widehat{K}|\}$ (Proposition 12.4.20),
- $K$ is connected iff $\widehat{K}$ is torsion-free (Proposition 11.3.7),
- $K$ is totally disconnected iff $\widehat{K}$ is torsion (Corollary 11.3.6),
- $c(K) = A(t(\widehat{K}))$, where $t(\widehat{K})$ is the torsion subgroup of $\widehat{K}$,
- $\dim K = r_0(\widehat{K})$,
- $H^1(K,\mathbb{Z}) \cong \widehat{K}$ if $K$ is connected (here $H^1(K,\mathbb{Z})$ denotes the first cohomology group),
- for two compact connected abelian groups $K_1$ and $K_2$ the following are equivalent: (i) $K_1$ and $K_2$ are homotopically equivalent as topological spaces; (ii) $K_1$ and $K_2$ are homeomorphic as topological spaces; (iii) $\widehat{K_1} \cong \widehat{K_2}$; (iv) $K_1 \cong K_2$ as topological groups.

The first equality can be generalized to $w(K) = w(\widehat{K})$ for all locally compact abelian groups $K$.

The Pontryagin-van Kampen duality can be used to easily build the *Bohr compactification* $bG$ of a locally compact abelian group $G$. In the case when $G$ is discrete, $bG$ is simply the completion of $G^{\#}$, the group $G$ equipped with its Bohr topology. One can prove that $bG \cong \widehat{\widehat{G}_d}$, where $\widehat{G}_d$ denotes the group $\widehat{G}$ equipped with the discrete topology. For a comment on the non-abelian case see [38, 87].

Many nice properties of $\mathbb{Z}^{\#}$ can be found in Kunnen and Rudin [109]. For a fast growing sequence $(a_n)$ in $\mathbb{Z}^{\#}$ the range is a closed discrete set of $\mathbb{Z}^{\#}$ (see [87] for further properties of the lacunary sets in $\mathbb{Z}^{\#}$), whereas for a polynomial function $n \mapsto a_n = P(n)$ the range has no isolated points [109, 44, Theorem 5.4]. Moreover, the range $P(\mathbb{Z})$ is closed when $P(x) = x^k$ is a monomial. For quadratic polynomials $P(x) = ax^2 + bx + c$, $(a, b, c, \in \mathbb{Z}, a \neq 0)$ the situation is already more complicated: the range $P(\mathbb{Z})$ is closed iff there is at most one prime that divides $a$, but does not divide $b$ [109, 44, Theorem 5.6]. This leaves open the general question [36, Problem 954].

**Problem 13.4.1.** *Characterize the polynomials $P(x) \in \mathbb{Z}[x]$ such that $P(\mathbb{Z})$ is closed in $Z^{\#}$.*

## 13.5 Countably compact and pseudocompact groups

Let us see first that countably compact groups and pseudocompact groups are precompact.

**Theorem 13.5.1.** *Every countably compact group is precompact.*

*Proof.* Apply the above lemma to conclude that a non-precompact group has a symmetric neighborhood $V$ of 1 and a sequence $(g_n)$ of elements of $G$ such that such that $g_nV \cap g_mV = \emptyset$ whenever $m \neq n$. Let us see that the sequence $(g_n)$ has no accumulation points. Indeed, for every $x \in G$ the neighborhood $xV$ of $x$ may contain at most one of the elements of the sequence $(g_n)$. This proves that a non-precompact group cannot be countably compact. $\square$

We can prove even the stronger property:

**Theorem 13.5.2.** *Every pseudocompact group is precompact.*

*Proof.* Using again Lemma 9.2.6 as in the above proof, we can produce a symmetric neighborhood $V$ of 1 and a sequence $(g_n)$ of elements of $G$ such that such that $g_nV \cap g_mV = \emptyset$ whenever $m \neq n$. Pick a symmetric neighborhood $W$ of 1 with $W^2 \subseteq V$. Since $G$ is a Tychonov space, for each $n \in \mathbb{N}$ we can find a continuous function $f_n : G \to [0,1]$ such that $f_n(g_n) = 1$ and $f_n(X \setminus g_nW) = 0$. Since the family $(g_nW)$ is locally finite, the sum $f(x) = \sum_n nf_n(x)$ is continuous and obviously unbounded. □

Deduce from this an alternative proof of the fact that countably compact groups are precompact.
of $Y$. The necessity of the next theorem follows from the previous theorem and Remark 2.2.19.

**Theorem 13.5.3** (Comfort and Ross)**.** *A topological group $G$ is pseudocompact if and only if $G$ is precompact and $G_\delta$-dense in its (compact) completion.*

For the sufficiency one can make use of the following steps:

(a) every $G_\delta$-subset $O$ of a compact group $G$ containing $e_G$ contains also a closed $G_\delta$-subgroup $N$ of $G$;

(b) if $f : G \to \mathbb{R}$ is a continuous function, then there exists a second countable group $M$, a continuous homomorphism $h : G \to M$ and a continuous function $f' : M \to \mathbb{R}$, such that $f = f' \circ h$;

(c) for $h : G \to M$ from (b) take the extension $\widetilde{h} : \widetilde{G} \to \widetilde{M}$ to the respective extensions. Then by the density of $h(G)$ in $\widetilde{M}$ and the compactness of $\widetilde{G}$, we deduce that $\widetilde{h}$ is surjective and $M$ is compact. Hence, the range of $f'$ (and consequently, taht of $f$, as well) is bounded.

The factorization property used in item (b) was introduced by Tkachenko. Topological groups with this property he called $\mathbb{R}$-*factorizable*. In particular, precompact groups are $\mathbb{R}$-factorizable.

The structure of groups admitting pseudocompact group topologies as well as many other features of pseudocompact groups are discussed in [60].

## 13.6 Relations to dynamical systems

Among the known facts relating the dynamical systems with the topic of these notes let us mention just two.

- A compact group $G$ admits ergodic translations $t_a(x) = xa$ iff $G$ is monothetic. The ergodic rotations $t_a$ of G are precisely those defined by a topological generator a of G.
- A continuous surjective endomorphism $T : K \to K$ of a compact abelian group is ergodic iff the dual $\widehat{T} : \widehat{K} \to \widehat{K}$ has no periodic points except $x = 0$.

The Pontryagin-van Kampen duality has an important impact also on the computation of the entropy of endomorphisms of (topological) abelian groups. Adler, Konheim, and McAndrew introduced the notion of topological entropy of continuous self-maps of compact topological spaces in the pioneering paper [1]. In 1975 Weiss [145] developed the definition of entropy for endomorphisms of abelian groups briefly sketched in [1]. He called it "algebraic entropy", and gave detailed proofs of its basic properties. His main result was that the topological entropy of a continuous endomorphism $\phi$ of a profinite abelian group coincides with the algebraic entropy of the adjoint map $\widehat{\phi}$ of $\phi$ (note that pro-finite abelian groups are precisely the Pontryagin duals of the torsion abelian groups).

In 1979 Peters [122] extended Weiss's definition of entropy for automorphisms of a discrete abelian group $G$. He generalized Weiss's main result to metrizable compact abelian groups, relating again the topological entropy of a continuous automorphism of such a group $G$ to the entropy of the adjoint automorphism of the dual group $\widehat{G}$. The definition of entropy of automorphisms given by Peters is easily adaptable to endomorphisms of Abelian groups, but it remains unclear whether his theorem can be extended to the computation of the topological entropy of a continuous endomorphism of compact abelian groups. The algebraic entropy is extensively studied in [47]. In particular, the above mentioned results of Weiss and Peters were extended in [47] to arbitrary continuous endomorphisms of compact abelian groups. Recently, this result was further extended to some locally compact abelian groups.

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