Mode Interactions with Symmetry

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To my Parents,
for all they taught me
and all they let me discover.

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Summary

This thesis deals with steady-state mode interaction problems with symmetry. We prove several results concerning problems invariant under the action of an arbitrary compact Lie group Γ. These include the existence of mixed-mode solutions and secondary Hopf bifurcations. We also consider the unfolding of the equations characterizing such problems. Where appropriate, we distinguish the case when Γ acts trivially on one of the modes. We then apply the results to the problems of the $(1,3)$-, $(1,5)$- and $(1,3,5)$-mode interactions with spherical symmetry. We also consider the $(3,5)$- and the $(1,3,5)$- mode interaction problems with $SO(3)$ symmetry.

Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

Chapter 1

Introduction

The interest in bifurcation problems in a spherically symmetric setting goes at least as far back as Poincaré. In 1885, Poincaré [25] studies the configurations of a fluid in a spherical boundary, when subject to rotation around an axis, and their stability. In 1932, Lamb [20] refers to spherical harmonics as 'the most important' method to solve the Laplace equation in three dimensions. He applies the method to solve problems concerning irrotational motion of a liquid within a spherical boundary. Later, in 1961, Chandrasekhar relates steady-state bifurcation with spherical symmetry to convection in the earth's mantle (Chandrasekhar [8]). In the seventies, steady-state bifurcation with spherical symmetry is extensively applied to model physical problems. It has been used to model problems in plate tectonics (Le Mouel [22] and Le Pichon and Huchon [24]), in astrophysics (Nice proceedings [26]) and in the buckling of a spherical shell (Knightly and Sather [19]). However, the problem that has interested mathematicians the most is that of convection in spherical shells. It is the spherical version of the Bénard convection problem between two horizontal layers. In the spherical case, we assume the fluid is contained between two concentrical spherical boundaries with radii R_1 and R_2 ($R_1 \neq R_2$) and is heated from within. There exists a static equilibrium for all values of the temperature difference ΔT between the boundaries. If the temperature increases in the direction of gravity, the static equilibrium becomes unstable when ΔT exceeds a finite value and other solutions may appear (see Busse [4] for a full description of the problem). Chossat [10] considers a variation of this problem, namely the rotation of the spherical boundaries with constant angular velocity.

The study made by Busse in 1975 is complemented by that in Busse and Riahi [5] where the previous analysis is extended to odd degrees l of spherical harmonics. Mixed-mode patterns are first considered in Busse and Riahi [6] where the study is made for two consecutive modes, i.e., neighbouring degrees l and l^* of spherical harmonics.

In the meantime, group theory related to $O(3)$ was developed and Cicogna and Vanderbauwhede proved the *Equivariant Branching Lemma* (see Cicogna [11] and Vanderbauwhede [29]). These allow a more systematic approach to the problem. In 1982, Golubitsky and Schaeffer [14] gave a full description of the bifurcation with spherical symmetry on the 5-dimensional space of spherical harmonics of order 2, already using group theory to simplify the problem. In 1984, Ihrig and Golubitsky [18] gave an extensive account of the group theory concerning $O(3)$, namely a list of closed subgroups of $O(3)$, isotropy subgroups and lattice, fixed-point subspaces and maximal isotropy. They also proved that solutions obtained using the Equivariant Branching Lemma are unstable. More recently, a family of stable equilibria was found by Barany and Melbourne [3]. More complicated behaviour, such as the existence of heteroclinic orbits, was found by Friedrich and Haken [13] and Armbruster and Chossat [1] when two modes interact. A different perspective was introduced by Field, Golubitsky and Stewart [12] who consider problems on the hemisphere, hence with $O(2)$ symmetry, and their extension to $O(3)$ on the full sphere. They consider reaction-diffusion equations on the hemisphere with Neumann boundary conditions along the equator.

A global account of results concerning steady-state bifurcation with spherical symmetry can be found in Chossat, Lauterbach and Melbourne [23] and in Golubitsky *et al* [16],chapters XIII and XV.

In this work, we consider mode interaction problems in spherically invariant systems. Our interest is not based on any specific application although one could eventually be thought of. In order to study problems with spherical symmetry, we first consider the group $SO(3)$. It is known that $SO(3)$ has precisely one irreducible representation, up to isomorphism, in each odd dimension $2l+1$ and that these representations can be realized in terms of *spherical harmonics.* The space V_l of spherical harmonics has dimension $2l+1$. To obtain from this representation the irreducible representations of $O(3)$, we recall that

$$O(3) = SO(3) \oplus \mathbb{Z}_2^c,$$

where $\mathbb{Z}_2^c = \pm I$. We then have two representations of $O(3)$ on V_l, according to whether the element of order 2 in $\mathbb{Z}_2^c$ acts as plus or minus the identity. They are called the *plus* and the *minus* representation, respectively. Usually, the action of $O(3)$ is induced from the natural action on $\mathbb{R}^3$ which leads to the representation on V_l whose sign is $(-1)^l$. (See Golubitsky *et al* [16] for more detail and further results.)

We study the interactions involving the modes of order $l = 0$, $l = 1$ and $l = 2$. As in previous works, we assume that a centre manifold reduction has been performed and so, we study differential equations on spaces which contain 1-, 3- or 5-dimensional modes. If the interacting modes have dimensions m and n, we shall refer to the problem as an (m, n)-mode interaction. In the last two chapters, we consider the $(1, 3, 5)$-mode interaction.

Our approach consists in the use of group theory to establish the most appropriate representation for the $O(3)$-action. In the case of the $(3, 5)$-mode interaction, this leads to a problem with $SO(3)$ symmetry rather than $O(3)$ (note that the latter problem has already been studied by Armbruster and Chossat [1]). Given a representation, we use invariant theory to obtain the corresponding invariants and equivariants which allow us to write the bifurcation equations. Singularity theory then provides an unfolding for the equations and finally, we do the bifurcation analysis.

While trying to solve the spherically symmetric mode interaction steady-state bifurcation problems, we have come across some general results concerning mode interactions (including an 'intuitive' definition of genericity) and unfoldings for such problems. We restrict our study to generic problems and shall not consider any mode interactions involving Hopf bifurcation.

We chose not to include a chapter on background definitions or results. Instead, we provide any background knowledge as it becomes relevant. For a complete and concise overview of the background see Golubitsky *et al* [16]. We shall use the concept of codimension as in this reference, that is, a bifurcation problem described by the equation

$$\dot{x} + f(x, \lambda) = 0,$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter, is codimension 0 if it does not need any unfolding parameters (some authors do not distinguish λ and consider this problem to be codimension 1).

This thesis is organized as follows:

- in chapter 2, we prove several results concerning mode interaction problems. We include a definition of genericity and a result on how to unfold such a problem. We consider the particular case of mode interactions involving one trivial mode and prove results concerning Birkhoff normal form equations, determinacy and the isotropy lattice associated to such problems. We then return to ordinary mode interaction problems to prove that a mixed-mode branch always exists and that, along this branch, there are parameter values for which a secondary Hopf bifurcation occurs.
- chapter 3 deals with the $(1, 3)$-mode interaction. We simplify the problem by reducing the action of $O(3)$ to that of $\mathbb{Z}_2$, preserving stability results. This problem has already been solved by Golubitsky *et al* [16] and hence, there are hardly any calculations to be done.

- in chapter 4, we study the $(1,5)$-mode interaction. Again, we simplify the problem by reducing the group action to that of S_3. The bifurcation analysis shows that there exist secondary Hopf bifurcations of equilibria to stable limit cycles, which then disappear in a heteroclinic connection.
- chapter 5 approaches the $(3,5)$- and the $(1,3,5)$-mode interaction problems. Trying to simplify the group action, we find that an appropriate setting reduces the action of $O(3)$ to that of $SO(3)$. Then, we study the problems under the $SO(3)$-action. This is done by comparison with the results obtained by Armbruster and Chossat [1] concerning the $(3,5)$-mode interaction with $O(3)$ symmetry.
- finally, in chapter 6, we give a brief description of the possible bifurcation features in the $(1,3,5)$-mode interaction with $O(3)$ symmetry. Again we use Armbruster and Chossat [1].

In the Appendices, we give a brief explanation of the program KAOS and present the print-outs both from KAOS and from MAPLE used in our study.

Chapter 2

General results on mode interactions

2.1. Introduction

This chapter is concerned with providing the generic setting for a mode interaction problem. The term 'mode interaction' is widely used in the applied literature but has not been fully formalized. Therefore, we start by giving an 'intuitive' definition of a generic mode interaction bifurcation problem. The proof that this is actually the 'right' definition would take us into singularity theory grounds, out of the scope of this work. It would however be interesting and useful to produce a rigorous definition. We shall, from now on, be concerned only with generic problems. The unfolding of the generic mode interaction bifurcation problem completes section 2.2. The following section addresses a particular kind of mode interaction which is obtained by considering the action of the group Γ on $\mathbb{R} \times V$, trivially on $\mathbb{R}$, where V is an n-dimensional space which we often identify with $\mathbb{R}^n$. In section 2.4, we prove the existence of mixed-mode solutions. The final section is devoted to proving that it is always possible to find coefficient values for which a secondary Hopf bifurcation exists along a branch of mixed-mode solutions.

2.2. Mode interactions

We start this section with a definition of mode interaction. Consider the system described by

$$\dot{x} + g(x, \lambda) = 0,$$

where $x \in U$, an n-dimensional space, and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Assume that g is equivariant under the action of the Lie group Γ. Without loss of generality, let us from now on assume that any bifurcation occurs at the origin, for $\lambda = 0$. Then, we have a *steady-state mode* when the 0-eigenspace of $(dg)_{0,0}$ is Γ-irreducible. Because we have assumed that a centre manifold reduction has taken place, U is that 0-eigenspace.

Definition 2.1 (Golubitsky *et al* [16], chapter XII, section 2). *A space V is said to be Γ-**irreducible** if the only Γ -invariant subspaces of V are 0 and V itself.*

In order to have the interaction of two modes, we need to be able to decompose the aforementioned 0-eigenspace into two Γ-irreducible components. Let us then consider the system of equations

$$\dot{x} + g_x(x, y, \lambda) = 0$$
$$\dot{y} + g_y(x, y, \lambda) = 0,$$

where $x \in U$, $y \in V$ (respectively, n- and m-dimensional spaces) and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Define

$$g : U \times V \times \mathbb{R} \to U \times V$$

to be $g \equiv (g_x, g_y)$ and assume it is Γ-equivariant.

Remark Here, and in all that follows, we use the index notation g_* to indicate the component of g in the subspace to which $*$ belongs and not a derivative. It shall be explicitly stated if the meaning is a different one.

Definition 2.2 (Golubitsky *et al* [16], chapter XX, section 0). *We say that the bifurcation problem is a* ***mode interaction*** *if the* 0*-eigenspace of* $(dg)_{0,0,0}$ *decomposes as the direct sum of two* Γ*-irreducible subspaces.*

As before, we assume that the two Γ-irreducible components are U and V. We refer to such a problem as an (n,m)-mode interaction. The natural definition of the interaction of k modes is to consider a problem for which the 0-eigenspace of the linear part decomposes as the direct sum of k Γ-irreducible components.

Note that the decomposition into two subspaces corresponds to a linear degeneracy at the origin. This is characteristic of mode interaction problems. This degeneracy can be avoided by introducing an extra parameter, say $\alpha \in \mathbb{R}$, on the linear level. Then, one mode bifurcates at the origin and the other away from it, at a value of λ depending on α. This splitting of the bifurcation gives rise to some interesting behaviour which cannot be predicted by analyzing only the single-mode problems. Later, we shall prove that 'generically' one such extra parameter α is also sufficient to unfold the mode interaction bifurcation equations.

Next, we give an *intuitive* definition of a generic mode interaction bifurcation problem. First, however, we need to introduce a few concepts, for which we use Golubitsky *et al* [16], chapters XIV and XV as a reference.

Notation:

(1) $\mathcal{E}_{x,\lambda}(\Gamma)$ denotes the ring of Γ-invariant germs $V \times \mathbb{R} \to \mathbb{R}$. It is generated by a set of functions of Γ-invariant polynomials.

(2) $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$ denotes the space of Γ-equivariant germs of mappings of $V\times\mathbb{R}$ into V. It is a module over $\mathcal{E}_{x,\lambda}(\Gamma)$ and generated by Γ-equivariant polynomials.

(3) $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$ is a finitely generated module over $\mathcal{E}_{x,\lambda}(\Gamma)$ and consists of Γ-equivariant matrix germs, i.e., $n \times n$ matrices S such that

$$S(\gamma x, \lambda)\gamma = \gamma S(x, \lambda) \ \ \forall \gamma \in \Gamma \ \forall (x, \lambda) \in V \times \mathbb{R}.$$

(4) $\vec{\mathcal{M}}_{x,\lambda}(\Gamma)$ is the set of germs in $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$ which vanish at the origin.

Definition 2.3. *Let $g(x,\lambda) \in \vec{\mathcal{E}}_{x,\lambda}(\Gamma)$ be a Γ-equivariant bifurcation problem. We define a k-parameter Γ-**unfolding** of g to be a Γ-equivariant map germ $G(x,\lambda,\alpha) \in \vec{\mathcal{E}}_{x,\lambda,\alpha}(\Gamma)$, where $\alpha \in \mathbb{R}^k$ and*

$$G(x,\lambda,0) = g(x,\lambda).$$

Definition 2.4. *The Γ-**equivariant tangent space** $T(g,\Gamma)$ is the subspace of $\vec{\mathcal{E}}_{x,\lambda}(\Gamma)$ generated by*

$$\{S_i g; (dg)X_j\} \text{ over } \mathcal{E}_{x,\lambda}(\Gamma),$$
$$\{(dg)Y_k\} \text{ over } \mathbb{R}$$

and

$$\{\partial g/\partial\lambda\} \text{ over } \mathcal{E}_\lambda,$$

where the S_i generate $\overleftrightarrow{\mathcal{E}}_{x,\lambda}(\Gamma)$, the X_j generate $\vec{\mathcal{M}}_{x,\lambda}(\Gamma)$ and the Y_k are such that

$$\vec{\mathcal{E}}_{x,\lambda}(\Gamma) = \vec{\mathcal{M}}_{x,\lambda}(\Gamma) \oplus \mathbb{R}\{Y_k\}_k.$$

Definition 2.5. *G is said to be a **versal unfolding** of g if it is an unfolding and*

$$\vec{\mathcal{E}}_{x,\lambda}(\Gamma) = T(g,\Gamma) + \mathbb{R}\{G_{\alpha_1}(x,\lambda,0), ..., G_{\alpha_k}(x,\lambda,0)\},$$

*where G_{α_i} denotes the derivative with respect to α_i. If k is the minimum number of parameters for which the equality holds, G is called **universal** and k the **codimension** of g.*

Now we can proceed to establish our definition. Consider the mode interaction bifurcation problem defined by

$$\dot{u} + g_u(u,v,\lambda) = 0$$
$$\dot{v} + g_v(u,v,\lambda) = 0$$

such that

$$g \equiv (g_u, g_v) : U \times V \times \mathbb{R} \to U \times V$$

is Γ-equivariant and verifies the equalities

$$g(0) = 0 \text{ and } Dg(0) = 0.$$

Without loss of generality, we suppose that the equations are in Birkhoff normal form, that is, g is a polynomial. We do not place any restrictions on the degree of these polynomials. It is then obvious that terms of the form $(\lambda u, \lambda v)$ must be part of the equations. We shall return to this issue.

By definition, this problem has a linear degeneracy at the origin which we unfold by an extra parameter $\alpha \in \mathbb{R}$. Suppose we unfold the linear part of g_v so that λv becomes $(\lambda - \alpha)v$. Let $\lambda' \equiv \lambda - \alpha$. Then we can rewrite the mode interaction problem as follows

$$\dot{u} + g_u(u, v, \lambda, \lambda') = 0 \qquad \textbf{2.2.1.}$$

$$\dot{v} + g_v(u, v, \lambda, \lambda') = 0 \qquad \textbf{2.2.2.}$$

such that

$$g \equiv (g_u, g_v) : U \times V \times \mathbb{R} \times \mathbb{R} \to U \times V$$

is Γ-equivariant and satisfies the same equalities as above.

Note that g_u does not in fact depend on λ' but that does not affect what follows.

From the above equations, we can define two single-mode bifurcation problems. This corresponds to looking at each mode independently, i.e., when they are not interacting. Consider the following equations

$$\dot{u} + g_u(u, 0, \lambda, 0) = 0 \qquad \textbf{2.2.3.}$$

and

$$\dot{v} + g_v(0, v, 0, \lambda') = 0. \qquad \textbf{2.2.4.}$$

The restrictions on g imply that both 2.2.3 and 2.2.4 define steady-state bifurcation problems, provided that

$$g_v(u, 0, \lambda, 0) = 0 \ \ \forall\, u \in U \ \ \forall\, \lambda \in \mathbb{R}$$

and

$$g_u(0, v, 0, \lambda') = 0 \ \ \forall\, v \in V \ \ \forall\, \lambda' \in \mathbb{R}$$

We note that some mode interactions do not satisfy the above equalities, for example when Γ acts trivially on a 1-dimensional mode or in the $(3,5)$-mode interaction studied in Chapter 5. This restriction however still applies to a considerable number of mode interaction problems and we assume it holds for the definition below.

It is clear that

$$g_u(.,0,.,0) : U \times \mathbb{R} \to U$$

and

$$g_v(0,.,0,.) : V \times \mathbb{R} \to V$$

are Γ-equivariant.

Definition 2.6. *A mode interaction problem such as (2.2.1, 2.2.2) is called* ***generic*** *if and only if the single-mode bifurcation problems within it, such as 2.2.3 and 2.2.4, are codimension* 0.

Intuitively, we see that if

$$\vec{\mathcal{E}}_{u,\lambda}\,(\Gamma) = T(g_u(u, 0, \lambda, 0), \Gamma)$$

and

$$\vec{\mathcal{E}}_{v,\lambda'}\,(\Gamma) = T(g_v(0, v, 0, \lambda'), \Gamma)$$

then, putting these two problems together in a mode interaction should lead to the least degenerate mode interaction possible. The proof of this statement would make the definition of generic mode interaction rigorous. However, we did not find a proof. We shall return to this point.

We remark that if $\alpha = 0$, that is, before the problem is unfolded, λ' is just λ. It is important that the bifurcation parameters can be varied independently for each problem however, so that $g_u(u, 0, \lambda, 0)$ and $g_v(0, v, 0, \lambda')$ are well defined. Before the problem has been unfolded, i.e., when the equations are

$$\dot{u} + g_u(u, v, \lambda) = 0$$
$$\dot{v} + g_v(u, v, \lambda) = 0,$$

the definition is equivalent to saying that $g_u(u, 0, \lambda)$ and $g_v(0, v, \lambda)$ are codimension 0. We consider the first setting to be clearer but use either one, depending on which happens to be more adequate.

Next we discuss an example with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-symmetry as a motivation for what follows. This example can be found in Golubitsky and Schaeffer [15], chapter X. Consider the normal form equations

$$\dot{x} + \epsilon_1 x^3 + mxy^2 + \epsilon_2 \lambda x = 0$$
$$\dot{y} + nx^2 y + \epsilon_3 y^3 + \epsilon_4 \lambda y = 0,$$

where $\epsilon_i = \pm 1; i = 1, \cdots, 4$ and $m \neq \epsilon_2\epsilon_3\epsilon_4$, $n \neq \epsilon_1\epsilon_2\epsilon_4$, $mn \neq \epsilon_1\epsilon_3$. These define a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-symmetric mode interaction problem $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. Singularity theory provides an unfolding for this problem, which is

$$\dot{x} + \epsilon_1 x^3 + \tilde{m}xy^2 + \epsilon_2 \lambda x = 0$$
$$\dot{y} + \tilde{n}x^2 y + \epsilon_3 y^3 + \epsilon_4 (\lambda - \alpha) y = 0,$$

where $(\tilde{m}, \tilde{n}, \alpha)$ varies on a neighbourhood of $(m, n, 0)$. In what follows, we shall refer to $\tilde{m}$ and $\tilde{n}$ simply as m and n. We note that these two parameters are special since they are coefficients of terms already present in the original equations and therefore vary in the neighbourhood of a point which is not necessarily the origin. The fact that these parameters are present both in the unfolded and the original equations, makes it reasonable to believe that the changes they introduce are not as noticeable as those introduced by the other unfolding parameters. We shall show that, for most of the parameter space, they do not change the topological type of the problem. It is only in a distinguished set that important changes occur; and we shall avoid this set.

Returning to the example, we now draw the bifurcation diagrams when the coefficients take the following values

$$\epsilon_1 = \epsilon_3 = 1, \epsilon_2 = \epsilon_4 = -1,$$

to which correspond the nondegeneracy conditions

$$mn \neq 1, m \neq 1 \text{ and } n \neq 1. \qquad \mathbf{2.2.5.}$$

We note that for the unperturbed case, all branches bifurcate from the origin at $\lambda = 0$. In the unfolded case, the y-mode branch bifurcates at $\lambda = \alpha$ and therefore, the sign of α is relevant to the bifurcation diagrams.

We draw schematic bifurcation diagrams, where the full lines indicate stable branches and the dotted lines unstable ones. We avoid the distinguished set of values for the parameters m and n defined by the nondegeneracy conditions 2.2.5 together with $m = 0$ and $n = 0$. For parameter values as above, this set is indicated by the solid lines in Figure 2.1.

This distinguished set of values divides the plane into 12 regions as indicated in the figure. In each region, the bifurcation diagrams are topologically equivalent. We can further simplify the task of drawing all bifurcation diagrams by noting that interchanging x and y is the same as interchanging m and n and reversing the sign of α. Thus the diagrams in regions $(2')$, $(3')$, $(3a')$ and $(4b')$ can be easily obtained from those in regions (2), (3), $(3a)$ and $(4b)$.

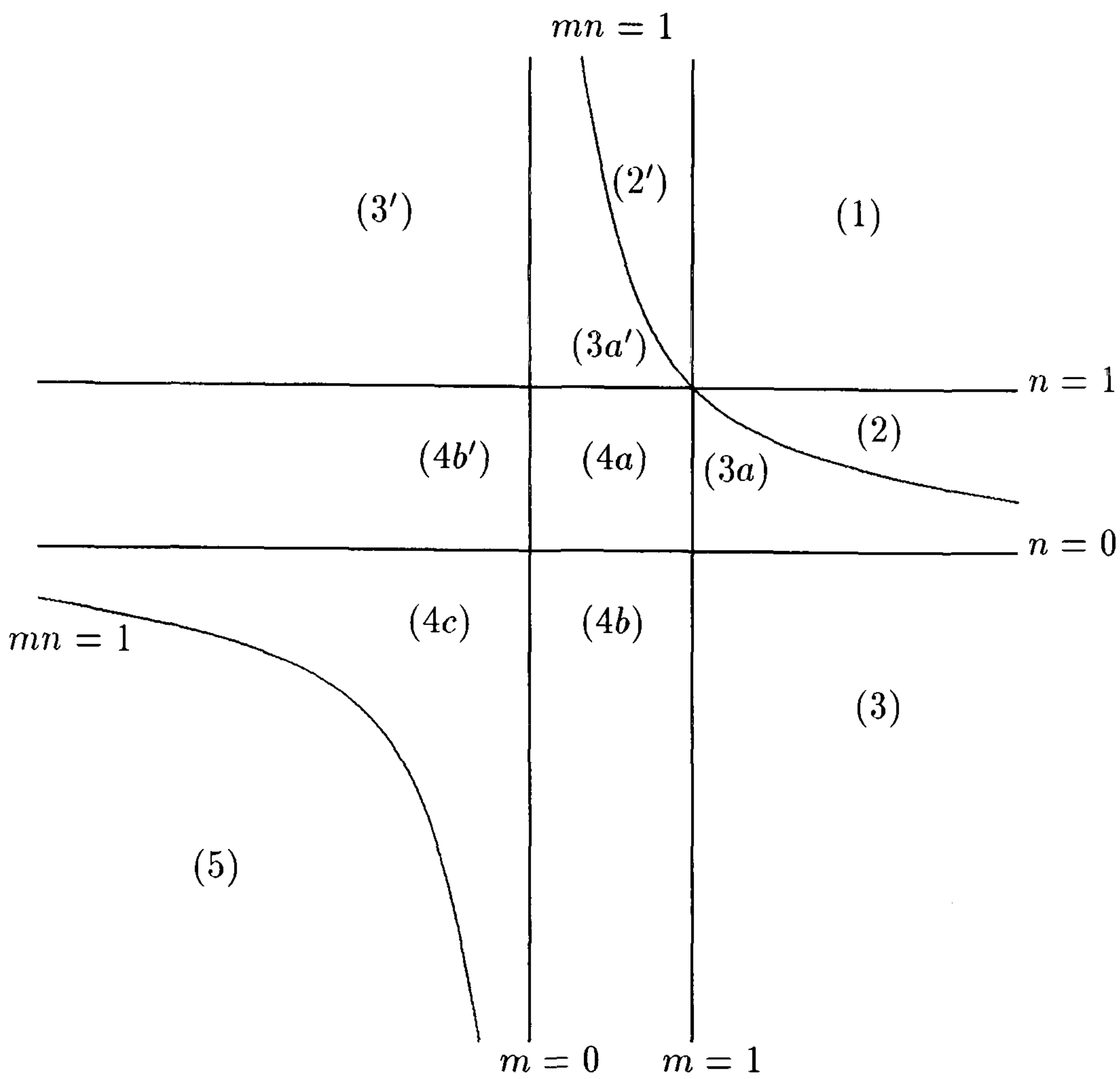


Figure 2.1.

The branching equations are as follows:

(a)	trivial solutions	$x = y = 0$
(b)	x – mode solutions	$\lambda = x^2;\ y = 0$
(c)	y – mode solutions	$x = 0;\ \lambda = y^2 + \alpha$
(d)	mixed-mode solutions	$\begin{cases} \lambda = x^2 + my^2 \\ (1-n)x^2 + (m-1)y^2 = \alpha \end{cases}$

Remark Figure 2.1 differs from Figure X, 4.1 in Golubitsky and Schaeffer [15] in that it includes two extra regions, namely, $(3a)$ and $(3a')$. These are in fact distinct from region (3). We also correct the diagram for region (3) when $\alpha > 0$. The difference between (3) and $(3a)$ is whether the mixed-mode branch bifurcates from the x-mode branch before or after the y-mode bifurcates from the origin, respectively.

Note also that in the absence of the nondegeneracy conditions which tell us the exact location of the distinguished set of parameters, it is still possible to determine the same 12 regions by varying the parameters and looking for topological changes in the bifurcation diagrams. Such parameters as m and n, which do not change the topological type of the problem for most values, we call modal parameters and discuss later.

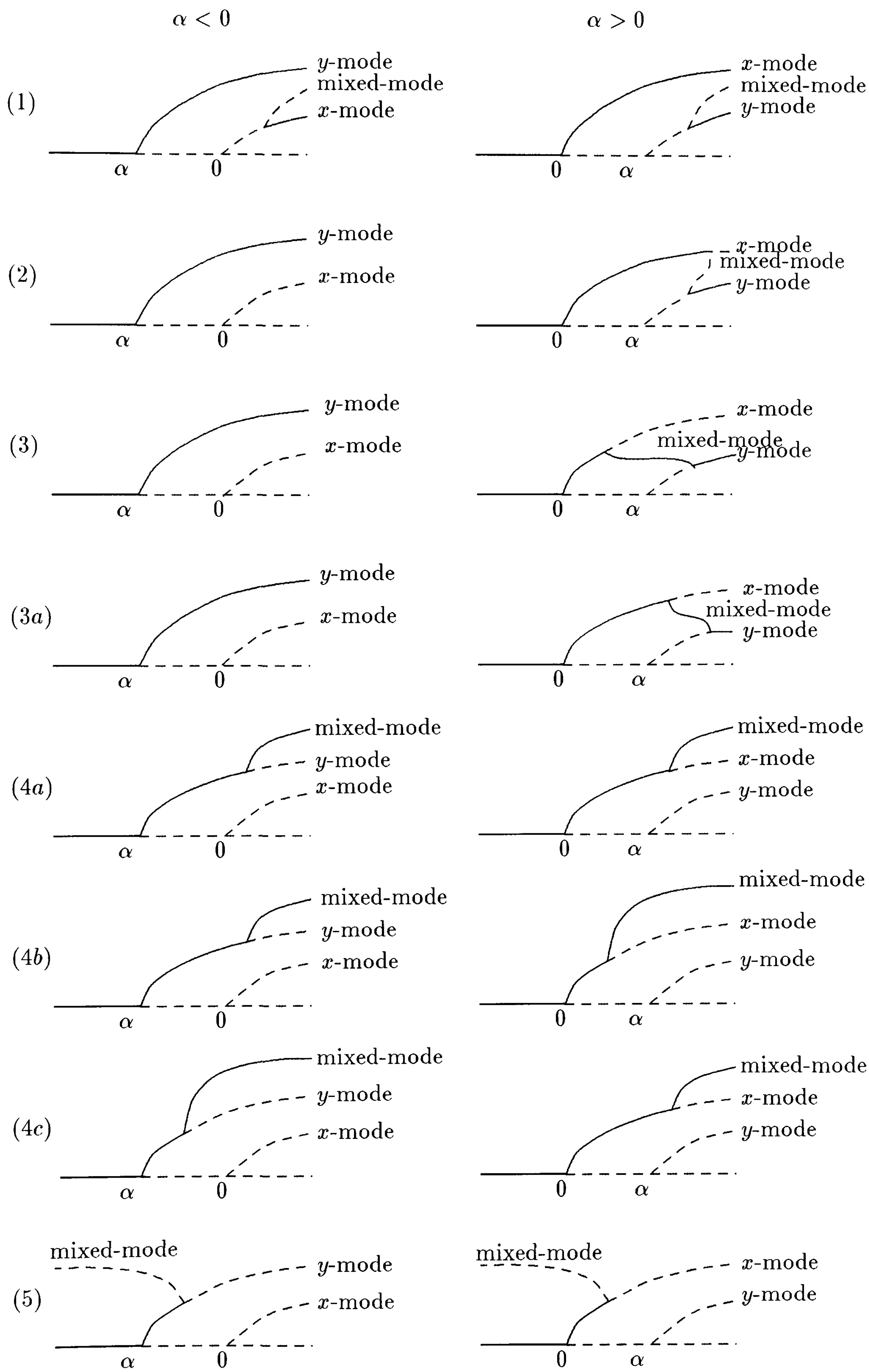
As can be seen in Figure 2.2, it is often the case that we have a mixed-mode branch of solutions together with the single-mode branches which existed in each single-mode problem. We point out that the single mode solutions come in pairs and the mixed-mode come in four at a time.

If we define $\lambda' \equiv \lambda - \alpha$, then this mode interaction problem takes the form used to define generic mode interactions. The separate single-mode bifurcation problems which can be obtained from the mode interaction by making $x = \lambda = 0$ or $y = \lambda' = 0$ are of the form

$$\dot{z} + \delta_1 z^3 + \delta_2 \sigma z = 0,$$

where $\sigma = \lambda$ or λ', $z = x$ or y and $\delta_i = \pm 1$; $i = 1, 2$, which defines a codimension 0, $\mathbb{Z}_2$-symmetric bifurcation problem.

This is an example which suggests that our definition of generic mode interaction is correct. Making it a rigorous definition, requires the proof of two results. The first states that given two single-mode bifurcation problems, both of codimension 0, we can obtain a generic mode interaction problem by 'putting them together'. 'Putting them together' would have to be defined but a reasonable idea seems to be to include in the equations all mixed equivariants that would arise from the group acting on the two spaces, without however including any single-mode equivariants absent from the single-mode problems. The second result, in a sense the converse of this one, saying that given a set of equations satisfying the linear degeneracy condition, i.e., defining a mode interaction, it corresponds to a generic mode interaction if in addition, it verifies an extra set of nondegeneracy conditions. We prove a version of this last result, and provide a universal unfolding, in Proposition 2.1; but we leave the rigorous definition as an open problem.


$\alpha < 0$
$\alpha > 0$
(1)
y-mode
mixed-mode
x-mode
α
0
x-mode
mixed-mode
y-mode
0
α
(2)
y-mode
x-mode
α
0
x-mode
mixed-mode
y-mode
0
α
(3)
y-mode
x-mode
α
0
x-mode
mixed-mode
y-mode
0
α
(3a)
y-mode
x-mode
α
0
x-mode
mixed-mode
y-mode
0
α
(4a)
mixed-mode
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mixed-mode
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α
(4b)
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y-mode
x-mode
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mixed-mode
x-mode
y-mode
0
α
(4c)
mixed-mode
y-mode
x-mode
α
0
mixed-mode
x-mode
y-mode
0
α
(5)
mixed-mode
y-mode
x-mode
α
0
mixed-mode
x-mode
y-mode
0
α

Figure 2.2.

Before stating and proving the proposition, we make a few comments on topological codimension and modal parameters. We have stated before that codimension is the minimum number of parameters in a versal unfolding. The notion of unfolding requires another one, that of equivalence: we say that g and h are two **Γ-equivalent** problems if there exist a change of coordinates

$$(x, \lambda) \to (X(x, \lambda), \Lambda(\lambda))$$

and a matrix-valued germ $S(x, \lambda)$, satisfying certain Γ -equivariant conditions and such that

$$g(x, \lambda) = S(x, \lambda) h(X(x, \lambda), \Lambda(\lambda)).$$

If the change of coordinates is C^∞ then the equivalence is said to be smooth. However, sometimes this notion is too strong and we allow the change of coordinates to be only continuous. The equivalence is then said to be **topological.** This allows a distinction between two kinds of parameters: those that do not change the topological type of the singularity on an open subset of the parameter space, which we call **modal**, and those that do, to which we refer simply as unfolding parameters. We shall avoid and not be concerned with the set of modal parameters for which there is a change in topological type. We define the topological codimension of g to be the C^∞-codimension minus the number of moduli, as long as the universal unfolding G of g is *topologically trivial.*

To define what we mean by topologically trivial, let g be C^∞-codimension k and let $G(x, \lambda, \alpha)$ be a universal unfolding of g. We define the **codimension constant variety** of G to be

$$\begin{aligned} \mathcal{C} = \{\alpha \in \mathbb{R}^k : \exists (x_0, \lambda_0) \text{ near } (0,0) \text{ such that} \\ \text{codim } G(x + x_0, \lambda + \lambda_0, \alpha) = \text{codim } g\}. \end{aligned}$$

We note that the parameters satisfying this condition are the modal parameters only. Let dim $\mathcal{C} = k - l$ and choose G so that

$$\mathcal{C} = \{\alpha \in \mathbb{R}^k : \alpha_i = 0; i = 1, \cdots, l\}$$

and denote $\alpha = (\beta, m)$; $\beta \in \mathbb{R}^l$, $m \in \mathbb{R}^{k-l}$. Let Σ be the transition variety, i.e., the set of bifurcation, hysteresis and double limit points, source of nonpersistence in bifurcation diagrams, and define

$$\Sigma_0 = \{\beta \in \mathbb{R}^l : (\beta, 0) \in \Sigma\},$$

which is the intersection of Σ with the space of non-modal parameters.

Definition 2.7 (Golubitsky and Schaeffer [15], V, 6.1). *The universal unfolding G is **topologically trivial** if Σ is (locally) homeomorphic to $C \times \Sigma_0$.*

Topological triviality implies that for (every) small m, the persistent perturbations of $G(x, \lambda, 0, m)$ are identical to those of $g(x, \lambda) = G(x, \lambda, 0, 0)$ meaning that the changes induced by the modal parameters are irrelevant. An important observation concerning modal parameters is that the moduli space divides into finitely many regions on which the universal unfolding is topologically trivial. These regions constitute the open subset of the parameter space referred to earlier (cf. Golubitsky and Schaeffer [15], chapter V, 7(d)), which, from now on, we call the **triviality set**.

Proposition 2.1. *Consider the Γ-symmetric mode interaction bifurcation problem on $U \times V$ defined by*

$$\dot{u} + g_u(u, v, \lambda) = 0$$

$$\dot{v} + g_v(u, v, \lambda) = 0,$$

where

$$g \equiv (g_u, g_v) : U \times V \times \mathbb{R} \to U \times V$$

is such that

$$g(0) = 0 \text{ and } Dg(0) = 0.$$

Let $\overrightarrow{\mathcal{E}}_{u,v,\lambda}(\Gamma) = < E_i(u,v) >_{i\in I}$ *and define*

$$k = max\,\{degE_i(u,v); i \in I\}.$$

Assume g is in Birkhoff normal form up to order k, that the coefficients of equivariants of degree between 2 and k are non-zero and in the triviality set. Then, the problem is generic, has topological codimension 1 and a universal unfolding is

$$\dot{u} + g_u(u,v,\lambda) = 0$$

$$\dot{v} + \alpha v + g_v(u,v,\lambda) = 0,$$

where g_u and g_v may include any number of modal parameters.

Remark The triviality set mentioned in the proposition above is any of the open regions away from the set of parameter values determined by the nondegeneracy conditions on modal parameters.

Proof Recall the definition of equivariant tangent space. The restrictions $g(0) = 0$ and $Dg(0) = 0$ imply that the lowest order terms in the equations are 2^{nd} order. It is obvious that both u and v are equivariant, in the respective domains, and therefore are in the equivariant space, hence in the tangent space. Given the generators of the tangent space, we see that u and v can only be obtained by differentiation with respect to λ, that is, we have

$$\frac{\partial g_u}{\partial \lambda} = u + \cdots$$

and

$$\frac{\partial g_v}{\partial \lambda} = v + \cdots.$$

When we calculate the tangent space for the mode interaction problem, we obtain

$$\frac{\partial g}{\partial \lambda} = (\frac{\partial g_u}{\partial \lambda}, \frac{\partial g_v}{\partial \lambda}) = (u + \cdots, v + \cdots)$$

and this is the only generator including first order terms. That it in fact exists is our hypothesis since $(\lambda u, \lambda v)$ is of order 2. It is then clear that the only way to obtain the two terms

$$(u, 0) \text{ and } (0, v)$$

belonging to $T(g, \Gamma)$ is by introducing an unfolding term, either

$$(u, 0) \text{ or } (0, v).$$

Remarks:

(1) This unfolding term will also eliminate the linear degeneracy characteristic of all mode interaction problems.

(2) We chose $(0, v)$ as the unfolding term simply by a question of tradition.

(3) We often use $(-\alpha v)$ instead of αv for convenience.

(4) The hypotheses on the coefficients are generically true.

Finally, note that there are no restrictions on any term of degree 2 or higher in the equations, so because we assume that, the equations include all equivariants of degree between 2 and k, any terms required to complement the tangent space will originate modal, but not unfolding, parameters.

When looking at each single-mode problem separately, the same argument and the fact that we obtain u and v from the derivative with respect to λ, guarantee that no unfolding parameters are needed. Hence, each single-mode problem is codimension 0.

This concludes the proof since 1 is the least number of parameters that can be necessary to unfold the mode interaction. □

Note that this result does not solve completely the problem of unfolding mode interaction bifurcation problems because it does not provide information about which, or how many, the modal parameters will be, or what the triviality set looks like. On the other hand, modal parameters typically change the problem into one which is topologically equivalent and this difference is usually not relevant. Also, this proposition avoids the computation of the tangent space which is often time-consuming. Furthermore, it gives ground to a procedure often used by many authors, consisting in unfolding the linear part of the equations and then using singularity theory methods to prove that the equations thus obtained are codimension zero. In fact, the proposition proves that it is always so for topological codimension. Finally, as was stated before, in examples any topological changes that do occur can be found using the bifurcation diagrams,so it is not necessary to calculate the triviality set in advance.

2.3. Mode interactions on $\mathbb{R} \times V$

In this section, we consider a particular type of mode interaction: that where the group Γ acts trivially on $\mathbb{R}$ and on an n-dimensional space V as follows

$$\gamma.(x, y) = (x, \gamma y) \;\; \forall\, y \in V \;\; \forall\, \gamma \in \Gamma. \qquad \textbf{2.3.1.}$$

We sometimes identify V with $\mathbb{R}^n$ for convenience. Note that, with such a group action, functions of x are both invariant and equivariant. Furthermore, a function $g \equiv (g_x, g_y)$ defining a bifurcation problem on $\mathbb{R} \times V$ is such that

$$g(\gamma.(x, y)) = g(x, \gamma y) = (g_x(x, \gamma y), g_y(x, \gamma y)) =$$
$$\gamma g(x, y) = (g_x(x, y), \gamma g_y(x, y)) \;\; \forall\, \gamma \in \Gamma \;\; \forall\, (x, y) \in \mathbb{R} \times V.$$

In other words, the first component of the function is always Γ- invariant both in x and in y, whereas the second is Γ- equivariant in y.

The main disadvantage of this type of problem is that we can no longer use the Equivariant Branching Lemma to prove existence of solutions for it is always

$$\mathrm{Fix}(\Gamma) = \mathbb{R} \neq \{0\}.$$

However, there are several results characteristic of this type of mode interaction problem which simplify the bifurcation analysis considerably. We describe them in the subsections below.

2.3.1. The equations in Birkhoff normal form

Suppose we are given a problem

$$\dot{x} + g_x(x, y, \lambda) = 0$$
$$\dot{y} + g_y(x, y, \lambda) = 0,$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ is the bifurcation parameter. We assume that

$$g \equiv (g_x, g_y) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n$$

is Γ-equivariant for the Γ-action described above and that $g(0) = 0$ and $Dg(0) = 0$.

We know then that there exists a Γ-equivariant change of coordinates which transforms g into a polynomial with some extra symmetry. See Golubitsky *et al* [16], chapter XVI, section 5 for a full description. In the particular case we are considering now, we have the following

Lemma 2.1. *Let g be a generic mode interaction problem as described above. Then the Γ-equivariant Birkhoff normal form equations for g are of the form*

$$\begin{aligned} \dot{x} + a\lambda + p(x,y) &= 0 \\ \dot{y} + q(x,y) &= 0, \end{aligned}$$

where p and q are Γ-equivariant polynomials of degree ≥ 2 in x and y.

Proof Extend the bifurcation equations to $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ as follows

$$\begin{aligned} \dot{x} + g_x(x,y,\lambda) &= 0 \\ \dot{y} + g_y(x,y,\lambda) &= 0 \\ \dot{\lambda} &= 0, \end{aligned}$$

defining $g^* \equiv (g_x, g_y, 0)$ and perform a normal form reduction. Using theorems XVI, 5.8 and 5.9 in [16], we know that the non-linear terms can be chosen to commute with the action of $\Gamma \times S$, where S is defined as follows: let

$$L \equiv dg^*(0,0,0) = \begin{pmatrix} & \frac{\partial g_x}{\partial \lambda}(0) \\ & \frac{\partial g_y^1}{\partial \lambda}(0) \\ Dg(0) & \vdots \\ & \frac{\partial g_y^n}{\partial \lambda}(0) \\ 0 & 0 \end{pmatrix}.$$

Because of the Γ-equivariance, we can write

$$g_y^i(x,y,\lambda) = \sum_k \bar{g}_y^{ik}(x,y,\lambda).P_e^k(y),$$

where $\bar{g}^{ik}(x, y, \lambda)$ is invariant in x and y and $P_e^k(y)$ is an equivariant polynomial in y. So,

$$\frac{\partial g_y^i}{\partial \lambda}(0) = \sum_k \frac{\partial \bar{g}_y^{ik}}{\partial \lambda}(0).P_e^k(0) = 0,$$

since $P_e^k(0) = 0$ as the group action is absolutely irreducible on $\mathbb{R}^n$. Hence,

$$L = \begin{pmatrix} 0_{(n+1)\times(n+1)} & \frac{\partial g_x}{\partial \lambda}(0) \\ 0_{1\times(n+1)} & 0_{(n+1)\times 1} \end{pmatrix} = \begin{pmatrix} 0_{(n+1)\times(n+1)} & a \\ 0_{1\times(n+1)} & 0_{(n+1)\times 1} \end{pmatrix},$$

where the indices correspond to the dimension of the submatrices. We define

$$S = Cl\{\exp(sL^t); s \in \mathbb{R}\} = Cl\left\{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ sa & 0 & \cdots & 1 \end{pmatrix}; s \in \mathbb{R}\right\},$$

where Cl stands for the closure of the set. Let $(p_0(x, y, \lambda), \cdots, p_{n+1}(x, y, \lambda))$ be a k^{th}-order $\Gamma \times S$-equivariant polynomial. The S-equivariance condition implies that

$$\begin{aligned} p_0(x, y, \lambda) &= p_0(x, y, sax + \lambda) \\ &\vdots \\ p_n(x, y, \lambda) &= p_n(x, y, sax + \lambda) \\ p_{n+1}(x, y, \lambda) &= p_{n+1}(x, y, sax + \lambda) - sap_0(x, y, \lambda). \end{aligned}$$

Since $s \in \mathbb{R}$ is arbitrary, the first $n + 1$ equalities hold if and only if they are independent of λ. According to theorem XVI, 5.8 in [16], the Birkhoff normal form equations are then given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + L \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p_x(x, y) \\ p_y(x, y) \end{pmatrix} = 0,$$

since we are not interested in the equation for λ. We write $p \equiv p_x$ and $q \equiv p_y = (p_1, \cdots, p_n)$ to finish the proof. □

Next, we discuss an example to show the scope of this result. Consider the bifurcation problem with $\mathbb{Z}_2$-symmetry as described in Golubitsky *et al* [16], chapter XIX, sections 2 and 3. As we shall show later, the equations are those of a mode interaction problem with $\mathbb{Z}_2$ acting on $\mathbb{R} \times \mathbb{R}$, trivially on the first component. After calculating the invariants and equivariants, we can use Lemma 2.1 to write the equations in Birkhoff normal form to order k. If $k = 2$, these are

$$\dot{x} + a\lambda + b_1 x^2 + b_2 y^2 = 0$$
$$\dot{y} + c_1 xy = 0,$$

which are exactly the ones in [16], Table XIX,2.1.

One way of studying the equations is to start by restricting them to $\text{Fix}(\mathbb{Z}_2)$. We obtain

$$\dot{x} + a\lambda + b_1 x^2 = 0.$$

So, depending on the sign of a and b_1, we do not have any equilibria for λ either positive or negative and then we have a branch of fully symmetric equilibria described by

$$\lambda = -\frac{b_1 x^2}{a}.$$

There is no symmetry-breaking up to this point, although there is a turning point at the origin.

To further study the equations, we need to unfold them. We have proved that it suffices to unfold the linear part, provided the equations are of high enough order. We also said that we chose to introduce the unfolding parameter in the second equation. It so happens that, in this case, the linear part of the second equation is x, which then becomes $(x - \alpha)$, $\alpha \in \mathbb{R}$, as in [16], Table XIX, 3.1. This means that the y-mode bifurcates from the fully symmetric branch. In other words, x becomes the bifurcation parameter for the y-mode, λ being implicit.

This is what happens generically in a problem in $\mathbb{R} \times V$. As it can be seen from Lemma 2.1 , restricting the equations to Fix(Γ), we obtain

$$\dot{x} + a\lambda + p(x, 0) = 0$$

and we can write $\lambda \equiv \lambda(x)$ along a branch which has the full symmetry of the problem. Moving down in the isotropy lattice and restricting the equations to the fixed-point space of a maximal isotropy subgroup, we see that any symmetry-breaking branch bifurcates at a value of x depending on $\alpha \in \mathbb{R}$, from which we can obtain the value of the bifurcation parameter λ.

2.3.2. Determinacy

In the previous subsection, when looking at the example with $\mathbb{Z}_2$-symmetry, we assumed that 2 was a high enough order for the equations to correspond to a generic problem. In this subsection, we prove that, under certain circumstances, it is always so.

We shall need a few auxiliary results, the first of which is stated in Golubitsky and Schaeffer [15], Facts VI, 2.4(*iii*) and is the equivariant version of Nakayama's lemma.

Lemma 2.2. *Let $\vec{\mathcal{I}}$ and $\vec{\mathcal{J}}$ be submodules with $\vec{\mathcal{I}}$ finitely generated. Then $\vec{\mathcal{I}} \subset \vec{\mathcal{J}}$ if and only if $\vec{\mathcal{I}} \subset \vec{\mathcal{J}} + \mathcal{M}.\, \vec{\mathcal{I}}$.*

The next result is the equivariant version of Corollary II, 5.4 in [15].

Lemma 2.3. *(a) Let $\vec{\mathcal{I}} = < p_1, \cdots, p_l >$ be a module in $\vec{\mathcal{E}}\,(\Gamma)$ and suppose that $q_1, \cdots, q_l$ are in $\mathcal{M}.\, \vec{\mathcal{I}}$. Then $\vec{\mathcal{I}}$ is also generated by $p_1 + q_1, \cdots, p_l + q_l$.*
***(b)** If g is a germ such that $\vec{\mathcal{M}}_k \subset RT(g, \Gamma)$ then g is k-determined, i.e., strongly equivalent to its Taylor polynomial of degree k, $j^k g$.*

Remark $\vec{\mathcal{M}}_k$ is the set of Γ-equivariant germs which have zero derivative at the origin up to order $(k-1)$.

Proof

(a) We have $q_j \in \mathcal{M}.\vec{\mathcal{I}} \subset \vec{\mathcal{I}}$ so, $p_j + q_j \in \vec{\mathcal{I}}$ and hence $< p_1 + q_1, \cdots, p_l + q_l > \subset \vec{\mathcal{I}}$.

For each p_j we can write

$$p_j = (p_j + q_j) - q_j \; \in \; \mathcal{M}.\vec{\mathcal{I}}$$

so,

$$\vec{\mathcal{I}} \subset < p_1 + q_1, \cdots, p_l + q_l > + \mathcal{M}.\vec{\mathcal{I}}$$

which, by Lemma 2.2, is equivalent to

$$\vec{\mathcal{I}} \subset < p_1 + q_1, \cdots, p_l + q_l >,$$

proving part (a).

(b) Write $g = j^k g - r$, where $r \in \vec{\mathcal{M}}_{k+1}$. We want to prove that

$$RT(g, \Gamma) = RT(g + tr, \Gamma) \text{ for } 0 \leq t \leq 1,$$

which for $t = 1$ gives a proof of part (b).

An element in $RT(g + tr, \Gamma)$ is of the following form, where $S \in \overleftrightarrow{\mathcal{E}}(\Gamma)$ and $X \in \vec{\mathcal{M}}(\Gamma)$,

$$S(g + tr) + (d(g + tr))X = [Sg + tSr] + [(dg)X + t(dr)X] \; =$$

$$[Sg + (dg)X] + t[Sr + (dr)X] \; \in \; RT(g, \Gamma) + \vec{\mathcal{M}}_{k+1} \, .$$

By hypothesis, $\vec{\mathcal{M}}_{k+1} \subset \mathcal{M}.RT(g, \Gamma)$ which implies that

$$RT(g, \Gamma) = RT(g + tr, \Gamma),$$

by part (a). □

Finally, we need to know more about the form of the equivariant matrices for the mode interaction problem. To that effect, we have the following

Lemma 2.4. *Let Γ act on $\mathbb{R} \times \mathbb{R}^n$ as in 2.3.1. Then any $S \in \overleftrightarrow{\mathcal{E}}_{x,y,\lambda}(\Gamma)$ is one of the following*

$$S_1 = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{pmatrix} \quad S_{2i} = \begin{pmatrix} 0 & E_i^t \\ 0_{n\times 1} & 0_{n\times n} \end{pmatrix}$$

$$S_{3i} = \begin{pmatrix} 0 & 0_{1\times n} \\ E_i & 0_{n\times n} \end{pmatrix} \quad S_4 = \begin{pmatrix} 0 & 0_{1\times n} \\ 0_{n\times 1} & S(y) \end{pmatrix}$$

where the indices correspond to the dimensions of the submatrices and $S(y)$ is the set of matrices which generates $\overleftrightarrow{\mathcal{E}}_{y,\lambda}(\Gamma)$ with coefficients in $\mathcal{E}_{x,y,\lambda}(\Gamma)$. Note that $S(y)$ includes the $n \times n$ identity matrix.

Proof $S \in \overleftrightarrow{\mathcal{E}}_{x,y,\lambda}(\Gamma)$ satisfies the equivariance condition

$$\forall\, \gamma \in \Gamma \;\; S(\gamma.(x,y),\lambda)\gamma = \gamma S(x,y,\lambda).$$

Decompose S in four blocks which distinguish between the two modes. The equivariance condition becomes

$$\begin{pmatrix} s_1(x,\gamma y,\lambda)_{1\times 1} & s_2(x,\gamma y,\lambda)_{1\times n} \\ s_3(x,\gamma y,\lambda)_{n\times 1} & s_4(x,\gamma y,\lambda)_{n\times n} \end{pmatrix} \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & \gamma_{n\times n} \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & \gamma_{n\times n} \end{pmatrix} \begin{pmatrix} s_1(x,y,\lambda)_{1\times 1} & s_2(x,y,\lambda)_{1\times n} \\ s_3(x,y,\lambda)_{n\times 1} & s_4(x,y,\lambda)_{n\times n} \end{pmatrix},$$

which is equivalent to the following set of equations

$$\begin{aligned} s_1(x,\gamma y,\lambda) &= s_1(x,y,\lambda) \\ s_2(x,\gamma y,\lambda)\gamma &= s_2(x,y,\lambda) \\ s_3(x,\gamma y,\lambda) &= \gamma s_3(x,y,\lambda) \\ s_4(x,\gamma y,\lambda)\gamma &= \gamma s_4(x,y,\lambda). \end{aligned}$$

The first equation means that s_1 is Γ-invariant. To lowest- order, it is just the identity. Because $\overleftrightarrow{\mathcal{E}}_{x,y,\lambda}(\Gamma)$ is a module over $\mathcal{E}_{x,y,\lambda}(\Gamma)$ and $(1,0)$ is equivariant, it is always just the identity. The last equation states that s_4 is a Γ-equivariant matrix for the group action on $\mathbb{R}^n$ and Γ-invariant in x. Because $\overleftrightarrow{\mathcal{E}}_{x,y,\lambda}(\Gamma)$ is a module over $\mathcal{E}_{x,y,\lambda}(\Gamma)$, it need not include x. Again, to lowest order it is the identity. The other two equations state that s_3 and s_2 are Γ-equivariant polynomials or their transpose, respectively. □

Lemma 2.5. *Let $f : U \times V \to \mathbb{R}$ be an invariant function under the action of an orthogonal group Γ, i.e.,*

$$f(\gamma x, \gamma y) = f(x, y) \;\; \forall \, \gamma \in \Gamma \; \forall \, (x, y) \in U \times V.$$

Then the gradient of f is an equivariant function.

Proof Define f_x to be the derivative of f with respect to the variables in U. Then, for $\gamma \in \Gamma$,

$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} f(\gamma x, \gamma y) = \frac{\partial}{\partial(\gamma x)} f(\gamma x, \gamma y).\gamma = f_x(\gamma x, \gamma y).\gamma.$$

Transpose this equality to obtain

$$f_x(x, y)^{\mathrm{t}} = \gamma^{-1} f_x(\gamma x, \gamma y)^{\mathrm{t}} \Leftrightarrow \gamma f_x(x, y)^{\mathrm{t}} = f_x(\gamma x, \gamma y)^{\mathrm{t}}.$$

Analogously for $f_y \equiv \frac{\partial}{\partial y} f$. □

This result is stated in Buzano and Russo [7] but not proved. It implies that for such a symmetric problem, there is only one invariant of degree 2, for we know that x and y are the only linear equivariants.

We shall use this result to prove the following

Proposition 2.2. *Let $g \equiv (g_x, g_y)$ define a mode interaction bifurcation problem on $\mathbb{R} \times V$ with Γ acting as in 2.3.1. Assume the Γ-action is such that there exists one and only one equivariant of degree 2 in y. Then G_α, the universal unfolding of g, is 2-determined.*

Proof By Lemma 2.1 , we know that the bifurcation equations are of the form

$$\dot{x} + g_x(x,y) + a\lambda = 0$$
$$\dot{y} + g_y(x,y) = 0.$$

We may write g_x and g_y as polynomials such as the following

$$g_x(x,y) = bx^2 + I(y^2) + \mathcal{O}(3)$$
$$g_y(x,y) = cxy + E(y^2) + \mathcal{O}(3),$$

where $I(y^2)$ and $E(y^2)$ represent, respectively, the only invariant and equivariant of order 2 in y. $\mathcal{O}(3)$ stands for terms of degree 3 or higher. Then by Proposition 2.1, the universal unfolding G_α is of the form

$$g_x^*(x,y,\alpha) = g_x(x,y)$$
$$g_y^*(x,y,\alpha) = c(x-\alpha)y + E(y^2) + \mathcal{O}(3).$$

To prove the proposition, it suffices to prove that

$$\vec{\mathcal{M}}_2 \subset RT(G_\alpha, \Gamma) + \vec{\mathcal{M}}_3 \qquad \textbf{2.3.2.}$$

and then use Lemma 2.3, part (b) and Lemma 2.2. We start by computing the generators of the restricted tangent space. We use Lemma 2.4, stressing the fact that S_4 includes the sub-identity matrix by writing

$$S_{41} = \begin{pmatrix} 0 & 0_{1\times n} \\ 0_{n\times 1} & I_{n\times n} \end{pmatrix}.$$

Note that $X \in \vec{\mathcal{M}}$ is of the form $(x,0)^t$ $(\lambda,0)^t$ or $(0,E)^t$, where E is Γ-equivariant. Since what we have to prove is 2.3.2, we do not have to worry about terms of degree 3 or higher, which we can cancel out with elements in $\vec{\mathcal{M}}_3$. We then have the following list of generators

(1) $S_1 G_\alpha = (a\lambda + bx^2 + I(y^2) + \mathcal{O}(3), 0)$

(2) $S_2 G_\alpha = (-c\alpha I(y^2) + \mathcal{O}(3), 0)$ or $(\mathcal{O}(3), 0)$

(3) $S_3 G_\alpha = (0, a\lambda y + \mathcal{O}(3))$ or $(0, \mathcal{O}(3))$

(4) $S_{41} G_\alpha = (0, c(x-\alpha)y + E(y^2) + \mathcal{O}(3))$

(5) $S_4 G_\alpha = (0, \mathcal{O}(3))$

(6) $(dG_\alpha)(\lambda, 0)^{\mathrm{t}} = (ax\lambda + \mathcal{O}(3), cy\lambda + \mathcal{O}(3))$

(7) $(dG_\alpha)(x, 0)^{\mathrm{t}} = (2bx^2 + \mathcal{O}(3), cxy + \mathcal{O}(3))$

(8) $(dG_\alpha)(0, E)^{\mathrm{t}} = (2I(y^2) + \mathcal{O}(3), c(x-\alpha)y + 2E(y^2) + \mathcal{O}(3))$ or $(\mathcal{O}(3), \mathcal{O}(3))$.

As it was pointed out before, we can ignore all $\mathcal{O}(3)$ terms and, since $RT(G_\alpha, \Gamma)$ is a module over $\mathcal{E}(\Gamma)$, we can use elements thereof as coefficients for the generators above.

We introduce the symbol '$\sim$' meaning 'equal to modulo a constant coefficient'. We simplify the list of generators to obtain

(6) $\sim (x\lambda, y\lambda)$

(2) $\sim (I(y^2), 0)$

(3) $\sim (0, \lambda y)$

(8') = **(8)** - 2**(2)** -**(4)** $\sim (0, E(y^2))$

(4') = **(4)** - **(8')** $\sim (0, (x-\alpha)y)$

(4") = x**(4')** $\sim (0, xy)$

(7') = **(7)** -c**(4")** $\sim (x^2, 0)$

(1') = **(1)** - **(2)** -b**(7')** $\sim (\lambda, 0)$

(6') = **(6)** - **(3)** $\sim (x\lambda, 0)$

Note that all the calculations above involve the implicit use of elements of $\vec{\mathcal{M}}_3$. This concludes the proof since $\vec{\mathcal{M}}_2$ is generated by { **(2)**, **(3)**, **(6')**, **(8')**, **(4")**, **(7')**, **(1')** }. □

This proposition ends the subsection. This result will be used in the two subsequent chapters for the study of the $(1,3)$- and $(1,5)$-mode interactions.

2.3.3. Adding a trivial mode to the equations

In this subsection we study how, given the equations for a codimension 0, single-mode, Γ-equivariant bifurcation problem in any n-dimensional space, we can write the equations for the $(1, n)$- mode interaction involving this problem. We start by proving a result concerning single-mode bifurcation problems.

Lemma 2.6. *Let $\dot{y} + g(y, \lambda) = 0$ be a Γ-equivariant codimension 0 bifurcation problem such that Fix$(\Gamma) = \{0\}$. Then its Birkhoff normal form does not include terms of the form*

$$\lambda^k E(y), \textit{for } k > 1 \textit{ or } \lambda E(y), \textit{when } E(y) \neq y.$$

Proof We have to show that $\lambda E(y)$ and $\lambda^k E(y)$ are higher order terms when $E(y) \neq y$ and $k > 1$, respectively, and we do so in three steps. First recall from Golubitsky *et al* [16], chapter XIV, 7.5(b) that

$$\text{Fix}(\Gamma) = \{0\} \Rightarrow \text{Itr}\mathcal{K}(g, \Gamma) = \mathcal{P}(g, \Gamma),$$

where $\mathcal{P}(g, \Gamma)$is the set of higher order terms and $\mathcal{K}(g, \Gamma)$ is generated by

$$\mathcal{M}RT(g, \Gamma),\ (dg)X_i\ (\text{deg}X_i \geq 2),\ S_j g\ (\text{deg}S_j \geq 1)$$

over $\mathcal{E}(\Gamma)$ and

$$\lambda^2 g_\lambda$$

over $\mathcal{E}_\lambda$.

By hypothesis, g is codimension 0, that is

$$\vec{\mathcal{E}}_{y,\lambda}(\Gamma) = T(g, \Gamma) = \qquad \textbf{2.3.3.}$$

$$RT(g, \Gamma) + \mathcal{E}_\lambda\{g_\lambda\} = < S_k g, (dg)X_l > + \mathcal{E}_\lambda\{g_\lambda\}.$$

We have proved that 2.3.3 implies that λy appears in the equation for g (see the proof of Proposition 2.1).

(1) Suppose that $X_l = E(y)$ and $E(y) \neq y$. We have

$$g(y,\lambda) = \lambda y + p(y,\lambda) \Rightarrow dg = \lambda + dp(y,\lambda)$$

and then

$$\lambda E(y) + dp(y,\lambda)E(y) \in RT(g,\Gamma).$$

Because $\lambda E(y) \in \vec{\mathcal{E}}_{y,\lambda}\ (\Gamma)$ and g is codimension 0, $\lambda E(y) \in RT(g,\Gamma)$. Given that $dp(y,\lambda)E(y)$ has degree higher than $\lambda E(y)$, it belongs to the submodule spanned by $(dg)X_i$ and $S_j g$ in the definition of $\mathcal{K}(g,\Gamma)$.

Hence, $\lambda E(y) \in \mathrm{Itr}\mathcal{K}(g,\Gamma)$ (cf. [16], XIV, 6.2) if $E(y) \neq y$, that is, it is a higher-order term and does not appear in the recognition problem for $g(y,\lambda)$.

(2) Consider

$$g(y,\lambda) = \lambda y + p(y,\lambda) \Rightarrow g_\lambda = y + p_\lambda(y,\lambda)$$

where $p_\lambda(y,\lambda)$ has degree ≥ 2 and so can be cancelled using terms in $RT(g,\Gamma)$. It is obvious that $y \in \vec{\mathcal{E}}_{y,\lambda}\ (\Gamma)$. By definition,

$$\lambda^2 y + \lambda^2 p_\lambda(y,\lambda) \in \mathcal{K}(g,\Gamma),$$

which implies that $\lambda^2 p_\lambda(y,\lambda) \in \mathcal{M}RT(g,\Gamma)$. So, $\lambda^k y$, for $k > 1$, can be obtained from

$$\mathcal{E}_\lambda\{\lambda^2 g_\lambda\}$$

and hence, is a higher-order term.

(3) Finally, we want to show that $\lambda^k E(y) \in \mathcal{P}(g,\Gamma)$ for $k > 1$. We know that $E(y) \in \vec{\mathcal{E}}_{y,\lambda}\ (\Gamma)$ so, it must appear in $T(g,\Gamma)$. It does not come from $\mathcal{E}_\lambda\{g_\lambda\}$ because, in (1), we have shown that $\lambda E(y) \in \mathcal{P}(g,\Gamma)$. So, it must come from $RT(g,\Gamma)$ and then $\lambda E(y) \in \mathcal{M}RT(g,\Gamma)$, that is, $\lambda^k E(y)$ is a higher-order term. □

Remark Another way of stating this result is to say that $g(y, \lambda)$ is equivalent to

$$c\lambda y + p(y),$$

where $c \in \mathbb{R}$ and $p(y)$ is Γ-equivariant.

Now we are able to consider how a trivial mode can be added to already existing bifurcation equations.

Lemma 2.7. *Let $\dot{y} + g_y(y, \lambda) = 0$ be a bifurcation problem as in the previous lemma. The mode interaction problem which involves g_y and a trivial 1-dimensional mode is defined by the following equations*

$$\dot{x} + a\lambda + g_x(x, y) = 0$$
$$\dot{y} + g_y(y, x) = 0,$$

where g_x is Γ- invariant in y and x replaces λ in g_y.

Proof The result follows from Lemma 2.1 and the definition of generic mode interactions. □

In fact,we can write the equations more explicitly, using the previous lemma, as follows

$$\dot{x} + a\lambda + g_x(x, y) = 0 \qquad \textbf{2.3.4.}$$

$$\dot{y} + cxy + p(y) = 0. \qquad \textbf{2.3.5.}$$

This result emphasizes the fact that, in such a mode interaction, the variable x acts as a bifurcation parameter for the non-trivial mode.

We end this subsection and the section with a result about what happens to the isotropy subgroups when a trivial 1-dimensional mode is added as above.

Lemma 2.8. *Let the group* Γ *act on* $\mathbb{R} \times V$*, trivially on* $\mathbb{R}$*. Let* $\{v_\alpha : \alpha \in A\}$ *be a list of orbit representatives for the action of* Γ *on* V *and let* $\Sigma_\alpha \subset \Gamma$ *be the isotropy subgroup of* v_α*. Then*

$$\{(x, v_\alpha) : \alpha \in A\}$$

is a list of orbit representatives for Γ *acting on* $\mathbb{R} \times V$ *and* Σ_α *is the isotropy subgroup of* (x, v_α)*.*

Proof Take $\sigma \in \Sigma_\alpha$; then,

$$\sigma.(x, v_\alpha) = (x, v_\alpha)$$

because Γ acts trivially on $\mathbb{R}$ and Σ_α is the isotropy subgroup of v_α.

Take $\gamma \in \Gamma$ and suppose it fixes (x, v_α), i.e.,

$$\gamma.(x, v_\alpha) = (x, \gamma v_\alpha) = (x, v_\alpha).$$

Then $\gamma \in \Sigma_\alpha$. □

With this result, the computation of isotropy subgroups when the mode interaction problem involves a trivial mode is simplified. It suffices to find the isotropy subgroups for the non-trivial component. Note that if this involves more than one mode, we can use Proposition XX, 2.3 in Golubitsky *et al* [16].

2.4. Existence of mixed-mode solutions

In this section, we prove that, under a few assumptions, a mode interaction bifurcation problem always has solutions involving both modes. These are called *mixed-mode solutions*. The assumptions we make are similar to those in the Equivariant Branching Lemma (see also Theorem XIII, 3.5 in Golubitsky *et al* [16]). Although the existence of mixed-mode solutions is valid both for problems on $\mathbb{R} \times V$, as those treated in the previous section, and for those in $U \times V$, we prove two different but analogous results. We believe that the proofs are clearer this way.

Proposition 2.3. *Let the group Γ act on $\mathbb{R} \times V$, trivially on $\mathbb{R}$ and absolutely irreducibly on V, where V is an n-dimensional space. Assume that the mode interaction problem defined by $g \equiv (g_x, g_v)$ is generic and that*

(i) $Fix(\Gamma)_{|V} = \{0\}$

(ii) $\Sigma \subset \Gamma$ *is an isotropy subgroup for* Γ *acting on* V *such that*

$$dim\ Fix(\Sigma)_{|V} = 1$$

(iii) $g_v : \mathbb{R} \times V \to V$ *is a* Γ*- equivariant problem satisfying*

$$(dg_v)_x(\alpha, 0)(v_0) \neq 0 \ \ for\ v_0 \in Fix(\Sigma)_{|V}.$$

Then there exists a branch of solutions in the unfolded equations of the form

$$\{(x, v(x), \lambda(x)) : x \in \mathbb{R}, v \in Fix(\Sigma)_{|V}, \lambda \in \mathbb{R}\}$$

with isotropy Σ.

Proof Using (2.3.4, 2.3.5), we write the unfolded equations as follows

$$\dot{x} + a\lambda + p(x, v) = 0$$

$$\dot{v} + c(x - \alpha)v + q(x, v) = 0,$$

where $g(x, v, \lambda, \alpha) = (a\lambda + p(x, v), c(x - \alpha)v + q(x, v))$. By Lemma 2.8, we know that the isotropy subgroups for the action of Γ on V are the same as those for the action on $\mathbb{R} \times V$. Then,

$$\text{Fix}(\Sigma) = \{(x, v) : x \in \mathbb{R}, v \in \text{Fix}(\Sigma)_{|V}\}$$

and

$$\dim \text{Fix}(\Sigma) = 2.$$

Restrict the above equations to Fix(Σ). For $v = 0$, we can write $\lambda \equiv \lambda(x)$ and, for $x = \alpha$, define

$$L = (dg)(\alpha, 0, \lambda(\alpha), \alpha) = \begin{pmatrix} \frac{\partial p}{\partial x}(\alpha, 0) & \frac{\partial p}{\partial v}(\alpha, 0) \\ \frac{\partial q}{\partial x}(\alpha, 0) & (x - \alpha) + \frac{\partial q}{\partial v}(\alpha, 0) \end{pmatrix}$$

which has rank $1 = \dim \text{Fix}(\Sigma) - 1$.

Remark $\frac{\partial p}{\partial v}(\alpha, 0) = \frac{\partial q}{\partial v}(\alpha, 0) = 0$ because p and q are at least of degree 2 in v and therefore have a zero derivative at the origin. Also, $\frac{\partial q}{\partial x}(\alpha, 0) = 0$ since $q(x, v)$ is of the form $I(x).E(v)$.

A Liapunov-Schmidt reduction at this point introduces the equivalent 1-dimensional equation defined by

$$\phi \ : \mathrm{ker}L \times \mathbb{R} \ \rightarrow \ \mathrm{ker}E$$

where

$$E : \mathrm{Fix}(\Sigma) \ \rightarrow \ \mathrm{range}L$$

is the projection. So,

$$\mathrm{ker}L \ = \ \mathrm{ker}E \ = \ \mathrm{Fix}(\Sigma)_{|V}.$$

Hence, by assumptions $(i)-(iii)$ above and Theorem XIII, 3.5 in [16], at $x = \alpha$ there exists a smooth branch of solutions in $\mathrm{Fix}(\Sigma)$ bifurcating from $\lambda \equiv \lambda(x)$.
□

Remark In the proof above, because the second equation does not depend on λ, we could have applied the Equivariant Branching Lemma to this equation. Then, it would suffice to substitute the solution thus obtained into the first equation to solve for λ. The proof, as it stands, is merely a preview of the proof of Proposition 2.4 .

We note that, in mode interaction problems involving one trivial mode, the only single-mode branch which exists is the trivial one. All other branches are mixed-mode because all other fixed-point subspaces include both modes.

To establish the next result we need the following

Definition 2.8. *Let the group Γ act on two n-dimensional spaces V and W. We say that the actions of Γ on V and W, or that the spaces, are Γ-* ***isomorphic*** *if there exists a (linear) isomorphism $A : V \rightarrow W$ such that*

$$A(\gamma.v) \ = \ \gamma.(Av) \ \forall v \in V \ \ \forall \gamma \in \Gamma.$$

We also say that V and W define isomorphic representations of Γ.

Proposition 2.4. *Let* Γ *act absolutely irreducibly on* U *and* V, *so that* U *and* V *are non-isomorphic representations of* Γ. *Consider the generic mode interaction bifurcation problem on* $U \times V$ *defined by* $g \equiv (g_u, g_v)$ *and assume that*

(i) $Fix(\Sigma')_{|V} = \{0\}$

(ii) $\Sigma \subset \Sigma'$ *is an isotropy subgroup such that*

$$dim\ Fix(\Sigma)_{|V} \ = \ 1,$$

$$dim\ Fix(\Sigma)_{|U} \ = \ Fix\ (\Sigma')$$

and such that $Fix(\Sigma') \subset U$ *and* $dim\ Fix(\Sigma') = 1$, *so that there exists a branch* $\lambda \equiv h(u)$ *with isotropy* Σ'

(iii) g *is such that*

$$\frac{\partial}{\partial u}(dg)_v(u, 0, h(u))_{|u=u^*}(v_0) \ \neq \ 0 \ \ for\ v_0 \in Fix(\Sigma)_{|V}.$$

Then

$$Fix(\Sigma) = \{(u, v) \mid u \in Fix(\Sigma'), y \in Fix(\Sigma)_{|V}\}.$$

Let $l(u, \lambda)$ *be the coefficient of the linear term in the bifurcation equations. If, along* $\lambda \equiv h(u)$, *there exists a point* (u^*, λ^*) *such that*

$$l(u^*, \lambda^*) = 0,$$

then $\{(u, v)\}$ *defines a mixed-mode branch with isotropy* Σ, *branching off the* Σ'*-symmetric branch at* $(u^*, 0, \lambda^*)$.

Proof Let us write the unfolded equations as follows

$$\dot{u} + g_u(u, v, \lambda) = 0$$

$$\dot{v} + [f(u, \lambda) - \alpha]v + \bar{g}_v(u, v, \lambda) = 0,$$

where $\bar{g}_v$ contains no linear terms in v and $l(u, \lambda) = f(u, \lambda) - \alpha$.

As in the previous proof, restrict the equations to $\mathrm{Fix}(\Sigma)$ which is such that

$$\dim \mathrm{Fix}(\Sigma) \;=\; \dim \mathrm{Fix}(\Sigma') + 1 \;=\; 2$$

and compute the derivative of g at $(u^*, 0, \lambda^*, \alpha)$ where $f(u^*, \lambda^*) = \alpha$ to obtain

$$L = (dg)(u^*, 0, \lambda^*, \alpha) \;=\; \begin{pmatrix} \frac{\partial g_u}{\partial u} & \frac{\partial g_u}{\partial v} \\ \frac{\partial g_v}{\partial u} & (f(u,\lambda) - \alpha) + \frac{\partial \bar{g}_v}{\partial v} \end{pmatrix}_{(u^*,0,\lambda^*,\alpha)} .$$

Now, note the following

(a) $f(u^*, \lambda^*) - \alpha = 0$, by a previous assumption,

(b) $\frac{\partial g_u}{\partial u}(u^*, 0, \lambda^*, \alpha) \;\neq\; 0$ because of the existence of a branch with symmetry Σ'

and

(c) $\frac{\partial \bar{g}_v}{\partial v}(u^*, 0, \lambda^*, \alpha) \;=\; 0$ because there are no linear terms in $\bar{g}_v$.

If we can prove that

$$\frac{\partial g_v}{\partial u}(u^*, 0, \lambda^*, \alpha) \;= 0 \quad \text{or} \quad \frac{\partial g_u}{\partial v}(u^*, 0, \lambda^*, \alpha) \;=\; 0, \qquad \textbf{2.4.1.}$$

i.e., rank $L = \dim \mathrm{Fix}\, \Sigma - 1$, we can perform a Liapunov-Schmidt reduction in $\mathrm{Fix}(\Sigma)$ at $(x^*, 0, \lambda^*, 0)$, as in the previous proof, and hence prove the existence of the mixed-mode branch bifurcating from the Σ'-symmetric branch.

We prove 2.4.1. If there are no equivariant maps from U to V, or V to U, then $\frac{\partial g_v}{\partial u}$ includes v in every term and hence is zero at $v = 0$. Also, $\frac{\partial g_u}{\partial v}$ includes v because invariants in v are at least of degree 2 and there are no equivariants from V to U.

Suppose $E : U \to V$ is equivariant. Then $\frac{\partial g_v}{\partial u}$ is not necessarily zero at $(u^*, 0, \lambda^*, \alpha)$ but, because U and V are non-isomorphic representations, E is non-linear. Hence, $\frac{\partial E}{\partial v} = 0$ at $v = 0$ and so the rank of the matrix is still equal to $\dim \mathrm{Fix}(\Sigma) - 1$. □

Remark In the above proposition, we can assume dim Fix(Σ') $= k$, when $k \in \mathbb{N}$, $k > 1$, as long as we can prove the existence of a branch with symmetry Σ'. The hypotheses of Proposition 2.4 then imply dim Fix(Σ) $= k+1$, and the result follows in the same way, since we may write $u = k(\lambda)$ away from the primary bifurcation point. This, when substituted into the equation for the second mode restricted to Fix(Σ), gives a 1-dimensional problem with one parameter to which we apply the Equivariant Branching Lemma with assumption (iii) reformulated as

$$\frac{\partial}{\partial\lambda}(dg_v)(k(\lambda), 0, \lambda)_{|\lambda=\lambda^*}(v_0) \neq 0 \text{ for } v_0 \in \text{Fix } (\Sigma)_{|V}.$$

This is to say that the branch from which the mixed-mode solutions bifurcate need not be 1-dimensional. In fact, in problems involving more than two modes, it may be a mixed-mode itself (refer to Chapters 5 and 6 for an example).

Remark One essential hypothesis implicit in both these propositions is that there is a branch from which the mixed-mode solutions bifurcate. This can be shown by direct computation in the first case and using the Equivariant Branching Lemma in the second.

We emphasize the fact that these results prove the existence of a branch involving the two modes and not that of a branch with a smaller isotropy subgroup. In fact, the latter is not always true, as we shall see when we study the $(3,5)$-mode interaction. For this problem we shall find a branch of solutions contained in $\mathbb{R}^5$ with isotropy $O(2)$ and that both Fix($SO(2)$) and Fix(D_2) contain this branch. However, Fix(D_2) does not involve the two modes and there exist no solutions with such symmetry.

As an application of this result, let us consider again the mode interaction problem with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$- symmetry defined by the equations

$$\dot{x} + \epsilon_1 x^3 + mxy^2 + \epsilon_2 \lambda x = 0$$

$$\dot{y} + nx^2 y + \epsilon_3 y^3 + \epsilon_4(\lambda - \alpha)y = 0.$$

Σ	$\text{Fix}(\Sigma)$	$\dim \text{Fix}(\Sigma)$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\{0\}$	0
$\mathbb{Z}_2(\epsilon)$	$\{(0,v) : v \in V\}$	1
$\mathbb{Z}_2(\delta)$	$\{(u,0) : u \in U\}$	1
$\boldsymbol{1}$	$U \times V$	2

Table 2.1.

Remark We referred to this problem in section 2.2. Here, we consider the unfolded equations.

In the setting of Proposition 2.4, we have $U \times V = \mathbb{R} \times \mathbb{R}$ and $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ acts on $U \times V$ as follows

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \epsilon u \\ \delta v \end{pmatrix} \quad \epsilon, \delta = \pm 1.$$

The isotropy subgroups are as in Table 2.1.

It is obvious that $\text{Fix}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)_{|V} = \{0\}$ so that hypothesis (i) holds.

Let $\Sigma = \boldsymbol{1}$. Then $\dim \text{Fix}(\boldsymbol{1})_{|V} = 1$, verifying hypothesis (ii).

Hypothesis (iii) also holds since we have

$$(dg)_\lambda(0,0)\bar{v}_0 = \begin{pmatrix} \epsilon_2 & 0 \\ 0 & \epsilon_4 \end{pmatrix} = \epsilon_4 \bar{v}_0 \quad \forall\, 0 \neq \bar{v}_0 = (0, v_0) \in \text{Fix}(\boldsymbol{1})_{|V}.$$

Finally, let $\Sigma' = \mathbb{Z}_2(\delta)$. Then $\text{Fix}(\mathbb{Z}_2(\delta)) = U$ and $\dim \text{Fix}(\mathbb{Z}_2(\delta)) = 1$. Therefore, $\{(u,v) \in U \times V\}$ defines a mixed-mode branch every time that, along $\lambda = x^2$, there exists (x_*, λ_*) such that

$$nx_*^2 + \epsilon_4(\lambda_* - \alpha) = 0. \qquad \textbf{2.4.2.}$$

Note that hypotheses (i)-(iii) also hold for the restriction to U and $\Sigma' = \mathbb{Z}_2(\epsilon)$. This accounts for the mixed-mode branch coming off the y-mode branch when, along $\lambda = y^2 + \alpha$, there exists (y_*, λ_*) such that

$$my_*^2 + \epsilon_2 \lambda_* = 0. \qquad \textbf{2.4.3.}$$

When neither 2.4.2 nor 2.4.3 are satisfied, the mixed-mode branch does not exist, as is the case in regions (2), (3) and ($3a$) when $\alpha < 0$. If both equations are satisfied, then the mixed-mode branch connects the two single-mode branches as in (2), (3) and ($3a$) for $\alpha > 0$. When only one of the equations is satisfied, and since hypotheses (i)-(iii) always hold, it determines from which single-mode solution the mixed-mode branches.

Finally, we note that, in the case of Γ acting on $\mathbb{R} \times V$, trivially on $\mathbb{R}$, there is more that can be said about the mixed-mode solutions. The following also applies when V represents more than one mode.

We are interested in knowing which results obtained in the study of the bifurcation problem in V, can be extended to the problem in $\mathbb{R} \times V$. Consider the isotropy lattice for the action on V and let Σ be a maximal isotropy subgroup such that dim Fix$(\Sigma) = 1$. Then, by Proposition 2.3 ,the branch which has existence guaranteed by the Equivariant Branching Lemma, still exists, only now as a mixed-mode branch of the type

$$\{(x, v) : x \in \mathbb{R}, v \in \text{Fix } (\Sigma)_{|V}\}.$$

Recall that the isotropy lattice is the same for the Γ-action on V and on $\mathbb{R} \times V$ (see Lemma 2.8) and suppose that, for the bifurcation problem in V, there exists an isotropy subgroup $\Sigma_1 \subset \Sigma$ such that there is a branch $B(\Sigma_1)$ with isotropy Σ_1. Then we have the following

Lemma 2.9. *In the above circumstances, there exists a branch with symmetry Σ_1 given by*

$$\{(x, v) : x \in \mathbb{R}, v \in B(\Sigma_1)\}$$

for the Γ-equivariant problem on $\mathbb{R} \times V$.

Proof As in the proof of Proposition 2.3, we use the unfolded equations on $\mathbb{R} \times V$

$$\dot{x} + a\lambda + p(x, v) = 0$$

$$\dot{v} + c(x - \alpha)v + q(x, v) = 0,$$

where $x - \alpha = \lambda_1$, acts as a second parameter. We know that there exists a solution to

$$c\lambda_1 v + q(x, v) = 0,$$

namely, values in $B(\Sigma_1)$. By replacing these values in

$$a\lambda + p(x, v) = 0,$$

we obtain an expression for λ as a function of x and v which describes a branch with Σ_1-symmetry in the subspace of $\mathbb{R} \times V$ defined by

$$\{(x, v) : x \in \mathbb{R}, v \in \text{Fix } (\Sigma_1)_{|V}\}.$$

□

This result will be very useful, especially when we study the $(1, 3, 5)$-mode interactions, since it allows us to use results previously obtained for the $(3, 5)$-mode interaction. We point out that this does not solve the problem of mode interactions on $\mathbb{R} \times V$ completely, given the results for the bifurcation problem on V. In fact, more complicated behaviour can be seen in mode interactions on $\mathbb{R} \times V$, for example the occurrence of secondary Hopf bifurcations absent on V. This happens for the $(1, 3)$-mode interaction which we study in the next chapter.

2.5. Existence of a secondary Hopf bifurcation

This section is concerned with an 'unexpected' type of behaviour which seems to occur in steady-state mode interaction problems: the existence of a secondary Hopf bifurcation along a mixed-mode branch of solutions. In the previous section, we proved the existence of these mixed-mode solutions. In this section, we show that the Hopf bifurcation is not as unexpected as it has been thought to be so far, although the existence of periodic solutions in a steady-state bifurcation problem is indeed unpredictable when considering the single-mode problems separately. We stress the fact that the proof is one of existence for **certain values of the coefficients** only and not for all problems. Hence, if we use the equations to model a particular physical problem and therefore choose and fix the coefficients, it is possible that a Hopf bifurcation never occurs. For instance, with the values chosen for the coefficients in the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-symmetric mode interaction treated in previous sections, a Hopf bifurcation never occurs since (dg) is diagonal along the mixed-mode branch. However, in a purely mathematical setting, this choice of coefficients can be made so that a Hopf bifurcation exists.

The idea behind the proof is that, if g defines a generic steady-state mode interaction such that Σ is the symmetry of a mixed-mode branch of solutions, then none of the entries of $(dg)_{|\mathrm{Fix}(\Sigma)}$ is constant and can be made so that

$$\mathrm{Tr}((dg)) = 0$$

and

$$\mathrm{Det}((dg)) > 0$$

which are characteristic of a matrix having imaginary eigenvalues.

Proposition 2.5. *Let $g \equiv (g_u, g_v)$ define a generic Γ-equivariant steady-state mode interaction problem on $U \times V$ such that there exists a 2-dimensional mixed-mode branch of solutions with symmetry Σ. Then, for certain values of the coefficients in g, $(dg)_{|Fix(\Sigma)}$ has imaginary eigenvalues, i.e., a Hopf bifurcation occurs along the mixed-mode branch.*

Proof We divide the proof in two parts, corresponding to when $U = \mathbb{R}$ and the action of Γ trivial on $\mathbb{R}$, and when U is any non-trivial representation of Γ.

In both parts, we use the fact that (u, v) is always Γ-equivariant and the norm always Γ-invariant.

(1) Let $g \equiv (g_u, g_v)$ define a mode interaction problem on $\mathbb{R} \times V$. We know the equations are of the form

$$\dot{u} + a\lambda + p(u,v) = 0$$
$$\dot{v} + c_1(u-\alpha)v + q(v) = 0.$$

Set the coefficients of certain terms to zero. Then the equations restricted to Fix(Σ) become

$$\dot{x} + a\lambda + b_1 x^2 + b_2 y^2 = 0$$
$$\dot{y} + c_1(x-\alpha)y = 0$$

and

$$(dg)_{|\text{Fix}(\Sigma)} = \begin{pmatrix} 2b_1 x & 2b_2 y \\ c_1 y & c_1(x-\alpha) \end{pmatrix}.$$

For this matrix, we have

$$\text{Tr}(dg) = 2b_1 x + c_1(x-\alpha)$$

and

$$\text{Det}(dg) = 2b_1 c_1 x(x-\alpha) - 2b_2 c_1 y^2.$$

The equation $\text{Tr}(dg) = 0$ can be solved to obtain $x \equiv x(\alpha)$ and substituting this in $\text{Det}(dg)$, we obtain a polynomial of degree 2 in y. In fact,

$$\text{Det}(dg) = F(b_1, c_1, \alpha) - 2b_2 c_1 y^2.$$

For this polynomial we have the following discriminant

$$\Delta = 8b_2 c_1 F(b_1, c_1, \alpha).$$

Choosing b_1, b_2 and c_1 so that $\Delta > 0$, $\text{Det}(dg)$ will have two roots and will be positive for values of y in a certain interval. These values of x and y determine the points at which a Hopf bifurcation occurs along the mixed-mode branch.

(2) Let Γ act on $U \times V$, two non-isomorphic spaces. Again, set coefficients to zero so that, in $\mathrm{Fix}(\Sigma)$, the equations are

$$\dot{x} + a_1\lambda x + a_2(x^2 + y^2)x = 0$$
$$\dot{y} + b_1(\lambda - \alpha)y + b_2(x^2 + y^2)y = 0,$$

and

$$(dg)_{|\mathrm{Fix}(\Sigma)} = \begin{pmatrix} a_1\lambda + a_2(x^2 + y^2) + 2a_2x^2 & 2a_2xy \\ 2b_2xy & b_1(\lambda - \alpha) + b_2(x^2 + y^2) + 2b_2y^2 \end{pmatrix}$$

for which we have,

$$\mathrm{Tr}(dg) = a_1\lambda + b_1(\lambda - \alpha) + (3a_2 + b_2)x^2 + (a_2 + 3b_2)y^2$$

and

$$\mathrm{Det}(dg) = (a_1\lambda + a_2(3x^2 + y^2))(b_1(\lambda - \alpha) + b_2(x^2 + 3y^2)) - 4a_2b_2x^2y^2.$$

Along the Σ-symmetric branch,we can write $\lambda \equiv \lambda(x, y)$ by solving

$$\begin{cases} a_1\lambda + a_2(x^2 + y^2) = 0 \\ b_1(\lambda - \alpha) + b_2(x^2 + y^2) = 0. \end{cases}$$

We can also write $x^2 \equiv x^2(y^2)$ and by substitution in the expressions for the trace and determinant, we see that

$$\mathrm{Tr}(dg) = tr(x^2, y^2) \equiv tr(y^2)$$
$$\mathrm{Det}(dg) = det(x^2, y^2) \equiv det(y^4)$$

and, generically, there exists a solution to $\mathrm{Tr}(dg) = 0$ and $\mathrm{Det}(dg) > 0$. □

With this result, we end the chapter.

Chapter 3

(1,3)-mode interaction

3.1. Introduction

In this chapter, we study the bifurcations that occur when a 1- and a 3-dimensional mode interact in a spherically symmetric problem. We identify the 1-dimensional and the 3-dimensional modes with $\mathbb{R}$ and $\mathbb{R}^3$, respectively. The action of $O(3)$ is trivial on $\mathbb{R}$ and by matrix-vector multiplication on $\mathbb{R}^3$. We prove two results which allow us to study this problem via one with $\mathbb{Z}_2$ symmetry on $\mathbb{R} \times \mathbb{R}$, the study of which has been done in Golubitsky *et al.* [16] among others. The first result concerns a simplification of the group action and the second is the proof that stability results are not lost in this simplification.

3.2. The equations

Consider the system of ODEs

$$\dot{x} + g_x(x, y, \lambda) = 0$$
$$\dot{y} + g_y(x, y, \lambda) = 0,$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ is a bifurcation parameter. Define

$$g \equiv (g_x, g_y) : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R} \times \mathbb{R}^3$$

and assume that g is $O(3)$-equivariant, i.e.,

$$\gamma.g(x, y, \lambda) = g(\gamma.(x, y), \lambda) \ \forall \gamma \in O(3) \ \forall (x, y) \in \mathbb{R} \times \mathbb{R}^3$$

under the following action of $O(3)$

$$\gamma.(x, y) = (x, \gamma.y) \ \forall \gamma \in O(3) \ \forall (x, y) \in \mathbb{R} \times \mathbb{R}^3$$

where the action on $\mathbb{R}^3$ is by matrix-vector multiplication. Assume also that

$$g(0, 0, 0) = (0, 0) \in \mathbb{R} \times \mathbb{R}^3$$

and

$$Dg(0, 0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

3.2.1. The $O(3)$-action

We can simplify the study of this mode interaction by means of a result concerning the action of $O(3)$ on $\mathbb{R}^3$. In fact, this result is true for $O(n)$ acting on $\mathbb{R}^n$ by matrix-vector multiplication for all $n \in \mathbb{N}$ and we prove it in this case. It consists of Proposition XVII, 5.1 in Golubitsky *et al.* [16] reformulated so that it applies to steady-state bifurcation.

Proposition 3.1. *Let $O(n)$ act on $\mathbb{R}^n$ by matrix-vector multiplication. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be $O(n)$-invariant and $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ be $O(n)$-equivariant. Then*
(i) $f(x) = p(|x|^2)$
and
(ii) $g(x) = p(|x|^2)x$,
where $|.|$ is the norm in $\mathbb{R}^n$ and p a polynomial function.

Proof Let $x = (x_1, x_2, \ldots, x_n)$ and define

$$V = \{x \in \mathbb{R}^n : x_2 = \cdots = x_n = 0\}.$$

We claim the following:

(1) Every group orbit of $O(n)$ on $\mathbb{R}^n$ intersects V.

(2) V is the fixed-point space of a subgroup Σ of $O(n)$.

(3) Let T be the subgroup of $O(n)$ that leaves V invariant, then $T/\Sigma \simeq \mathbb{Z}_2$.

Assume these claims are true and that $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ is $O(n)$-equivariant. By **(2)**, g maps V into V for if

$$V = \mathrm{Fix}(\Sigma) = \{v \in \mathbb{R}^n : \sigma.v = v \ , \forall \sigma \in \Sigma\}$$

and g is $O(n)$-equivariant then, for $\sigma \in \Sigma \subset O(n)$, we have

$$\sigma.g(v) = g(\sigma.v) = g(v) \ \forall v \in V \ \forall \sigma \in \Sigma,$$

that is, $g(v) \in V$. By **(3)**, $g_{|V}$ commutes with $\mathbb{Z}_2$. Assuming that the proposition holds for $n = 1$, $g_{|V}$ has the form (ii). By **(1)**, g is uniquely determined by $g_{|V}$. The obvious extension for (ii) with $n = 1$ is (ii) itself so, by uniqueness, g has the form (ii) which concludes the proof of the proposition for all n.

It remains to prove that the claims are true and that the proposition holds for $n = 1$. We begin by proving the latter.

Let $n = 1$ and let $f : \mathbb{R} \mapsto \mathbb{R}$ be $\mathbb{Z}_2$-invariant, i.e.,

$$f(-x) = f(x) \ \forall x \in \mathbb{R}.$$

So, $f(x) = p(x^2)$ where p is a polynomial, since it is an even function of x. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be $\mathbb{Z}_2$-equivariant, i.e.,

$$g(-x) = -g(x) \ \forall x \in \mathbb{R}.$$

Then, $g(x) = p(x^2)x$ because it is an odd function of x. This proves the proposition for $n = 1$.

Remark: The results concerning $\mathbb{Z}_2$ invariants and equivariants can be found in Golubitsky *et al.* [16], chapter XI, 2(b).

We now prove the claims.

(1) Let $x \in \mathbb{R}^n$ and define W to be the space spanned by $\{x\}$. Choose an orthonormal basis for $\mathbb{R}^n$, $\{w_1, w_2, \ldots, w_n\}$, such that $w_1 \in W$. Let $w_i = (w_{i1}, w_{i2}, \ldots, w_{in})$ and

$$\begin{pmatrix} w_{11} & w_{21} & \cdots & w_{n1} \\ w_{12} & w_{22} & \cdots & w_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1n} & w_{2n} & \cdots & w_{nn} \end{pmatrix}$$

be the matrix with columns w_1, w_2,...,w_n which is an element of $O(n)$ because the basis is orthonormal. Then, for $v = (v_1, 0, \ldots, 0) \in V$ we have

$$\gamma.v = v_1(w_{11}, w_{12}, \ldots, w_{1n}) = v_1.w_1 \in W.$$

Hence, γ maps V into W and therefore, for every $x \in \mathbb{R}^n$, there exists an element in $O(n)$, namely γ^{-1}, which takes it into V. This proves that every group orbit of $O(n)$ intersects V.

(2) Consider the subgroup Σ of $O(n)$ which consists of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}$$

where $\sigma \in O(n-1)$. For $(x_1, 0, \ldots, 0) \in V$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

that is, $V \subset$Fix(Σ). On the other hand, suppose that

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} x_1 \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ \sigma y \end{pmatrix} = \begin{pmatrix} x_1 \\ y \end{pmatrix}.$$

Then $\sigma y = y$ for $y \in \mathbb{R}^{n-1}$, i.e., $y \in$Fix$_{\mathbb{R}^{n-1}}(O(n-1)) = \{0\}$. So, Fix$(\Sigma) \subset V$. Hence, $V =$Fix(Σ).

(3) It is obvious that the matrices in $O(n)$ which map V into itself are those of the form

$$\begin{pmatrix} \tau & 0 \\ 0 & \sigma \end{pmatrix}$$

where $\tau \in \mathbb{Z}_2$ and $\sigma \in O(n-1)$. So, $T/\Sigma \simeq \mathbb{Z}_2$. □

The proposition above states that the study of a bifurcation problem on $\mathbb{R}^n$ with symmetry $O(n)$ is equivalent to that of a problem on $V \equiv \mathbb{R}$ with symmetry $\mathbb{Z}_2$, inasmuch as existence results hold. That is, all the solutions which exist for the $\mathbb{Z}_2$- symmetric problem also exist for $O(n)$-symmetric one, with 'added' symmetry given by $O(n-1)$. And conversely, these are the only solutions for the $O(n)$-symmetric problem.

3.2.2. Stability

A very important issue in the study of bifurcation problems is that of stability. Concerning this matter, we prove the following

Proposition 3.2. *A solution for the bifurcation problem with $O(n)$ symmetry is stable if and only if it is stable for the $\mathbb{Z}_2$-symmetric problem.*

Proof We know that each solution for an $O(n)$-equivariant vector field can be conjugated to one in V by an element of $O(n)$. Now, any perturbation within V is accounted for by the $\mathbb{Z}_2$ theory. It suffices then to prove that the group action accounts for stability in the directions transversal to V. For this we use Proposition XIII, 1.2 in Golubitsky *et al.* [16] which states that

$$\dim \Gamma x = \dim \Gamma - \dim \Sigma_x.$$

In this particular case, $\dim \Sigma_x = \dim O(n-1)$, $\Gamma = O(n)$ and therefore

$$\dim \Gamma x = \dim O(n) - \dim O(n-1) = \frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1$$

which means that perturbations in the $(n-1)$ directions transversal to V are all accounted for by the group action. Hence, the $\mathbb{Z}_2$ and $O(n)$-symmetric problems are equivalent for stability results. □

3.3. Bifurcation problems with $\mathbb{Z}_2$ symmetry

Subsections 3.2.1 and 3.2.2 guarantee that the study of the interaction of the 1- and 3-dimensional modes in a spherically symmetric problem, as described at the beginning of section 3.2, is equivalent to that of the problem defined by

$$\dot{x} + g_x(x, y, \lambda) = 0$$
$$\dot{y} + g_y(x, y, \lambda) = 0$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ is a bifurcation parameter. The function defined by

$$g \equiv (g_x, g_y) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \times \mathbb{R}$$

is $\mathbb{Z}_2$-equivariant and $\mathbb{Z}_2$ acts on $\mathbb{R} \times \mathbb{R}$ as follows

$$\kappa.(x, y) = (x, -y) \ \ \forall \kappa \in \mathbb{Z}_2 \ \ \forall (x, y) \in \mathbb{R} \times \mathbb{R}.$$

We assume that

$$g(0, 0, 0) = (0, 0) \in \mathbb{R} \times \mathbb{R}$$

and

$$Dg(0, 0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The classification of such problems has been done in Golubitsky *et al.* [16], chapter XIX, section 2 up to topological $\mathbb{Z}_2$-codimension 2. We note that the least degenerate problem is codimension 1 as previously predicted. The unfoldings and bifurcation diagrams can be found in chapter XIX, section 3 of the reference above. We remark that, even in the least degenerate problem, there are parameter values for which a secondary Hopf bifurcation occurs along the mixed-mode branch.

We end this chapter with the following words of caution:

Remark: The classification and bifurcation diagrams in Golubitsky *et al.* [16] concern the amplitude equations of a steady-state/Hopf mode interaction problem. Therefore, although the bifurcation diagrams of both problems coincide, they have distinct interpretations. Hence, periodic solutions and invariant 2-tori in Golubitsky *et al.* [16] correspond to steady-state solutions and invariant periodic orbits, respectively, in the steady-state mode interaction. However, important results such as stability and its changes remain valid.

Chapter 4

(1,5)-mode interaction

4.1. Introduction

In this chapter we study the bifurcations that occur when a 1- and a 5-dimensional mode interact in a spherically symmetric problem. The group action is trivial on the 1-dimensional mode so, this problem belongs to the category studied in chapter 2, section 2.3. We start by simplifying the group action, proving it is equivalent to that of S_3. In section 4.2, we calculate the invariants and equivariants for this problem. In the next section, we establish the equations in Birkhoff normal form and, using unfolding theory, find the modal parameters. The last section consists of the bifurcation analysis and is divided in two subsections, depending on the sign of one distinguished parameter. There we determine stability of solutions and find secondary Hopf bifurcations along mixed-mode branches, as had been predicted in section 2.5. We also show that the periodic solutions created with the Hopf bifurcation disappear in an unstable heteroclinic connection.

4.2. The group action

We want to study an $O(3)$-equivariant problem involving a 1- and a 5-dimensional space. Identifying the 5-dimensional space with that of the 3×3

real traceless symmetric matrices, which we call V, and the 1-dimensional space with that of the real scalar multiples of the 3×3 identity matrix, which we refer to as $\mathbb{R}$, the action of $O(3)$ is by similarity as follows

$$\gamma.(X, M) = (X, \gamma.M) = (X, \gamma M \gamma^{-1}) \ \ \forall (X, M) \in \mathbb{R} \times V \ \ \forall \gamma \in O(3).$$

Since the group action is trivial on $\mathbb{R}$, we shall simplify this action by dealing only with the action on V. To do so, we consider D, the 2-dimensional space of 3×3 real traceless diagonal matrices and use the following

Lemma 4.1 (1.3 in Golubitsky and Schaeffer [14]). *Let $H : V \to V$ be equivariant with respect to the action of $O(3)$. Then D is invariant under H and H is determined by its restriction to D.*

The subgroup of $O(3)$ which acts faithfully on D and preserves D is S_3, the group of permutations on 3 symbols. Golubitsky and Schaeffer [14] show in section 4 why the study of bifurcation problems H commuting with the 5-dimensional representation of $O(3)$ is equivalent to that of S_3-equivariant bifurcation problems G on D, where G is the restriction of H to D.

This allows us to study the $O(3)$-symmetric $(1,5)$-mode interaction via the study of an S_3-equivariant mode interaction, where S_3 acts as follows

$$\sigma.(X, A) = (X, \sigma.A) = (X, \sigma A \sigma^{-1}) \ \ \forall (X, A) \in \mathbb{R} \times D \ \ \forall \sigma \in S_3.$$

Note that since the action on $\mathbb{R}$ is trivial, it is irrelevant which group we consider acting on this component. We shall continue to use the notation 5-*dimensional mode* although this no longer agrees with the dimension of the space we are considering.

Furthermore, we can identify $\mathbb{C}$ with D by

$$w = y + iz \longmapsto \begin{pmatrix} y & 0 & 0 \\ 0 & \frac{-y+\sqrt{3}z}{2} & 0 \\ 0 & 0 & -\frac{y+\sqrt{3}z}{2} \end{pmatrix}$$

and hence, study the (1,5)-mode interaction by studying an S_3-equivariant problem on $\mathbb{R} \times \mathbb{C}$, where the action of S_3 is given by

$$\kappa.(x, w) = (x, \bar{w}); \quad \kappa \in \mathbb{Z}_2$$
$$\alpha.(x, w) = (x, e^{i\alpha} w); \quad \alpha = \frac{2\pi}{3}.$$

Next we find the invariant and equivariant functions for this problem.

Lemma 4.2. *The generators for the set of invariant functions for the action of S_3 on $\mathbb{R} \times \mathbb{C}$ as described above are*

$$x, \quad |w|^2 \text{ and } Re(w^3).$$

The equivariant polynomials are generated by

$$(1, 0), \quad (0, w) \text{ and } (0, \bar{w}^2).$$

Proof The invariants and equivariants for the S_3-action on $\mathbb{R}$ are obtained by straightforward computation. For the action on $\mathbb{C}$, see Golubitsky and Schaeffer [14], Proposition 1.8. □

We do one last identification, that of $\mathbb{C}$ with $\mathbb{R}^2$, i.e., $w \equiv y + iz$, so that the invariant functions are generated by

$$x, \quad u = y^2 + z^2, \quad v = y(y^2 - 3z^2)$$

and the equivariant polynomials by

$$(1, 0, 0), \quad (0, y, z) \text{ and } (0, y^2 - z^2, -2yz).$$

4.3. The equations

As usual, the mode interaction problem is defined by

$$g : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^2,$$

S_3-equivariant and such that $g(0) = 0$ and $Dg(0) = 0$.

Using lemma 4.2 together with Schwarz's and Poénaru's theorems (cf. [16], chapter XII), we may write the bifurcation equations as follows

$$\dot{x} + r(x,u,v,\lambda) = 0$$
$$\dot{y} + p(x,u,v,\lambda)y + q(x,u,v,\lambda)(y^2 - z^2) = 0$$
$$\dot{z} + p(x,u,v,\lambda)z - 2q(x,u,v,\lambda)yz = 0,$$

where $g \equiv (r, py + q(y^2 - z^2), pz - 2qyz)$.

We choose to write the equations in Birkhoff normal form for which we use Lemma 2.1. By Proposition 2.2, we know the unfolding of this problem is 2-determined, i.e., the unfolded equations are strongly equivalent to the Taylor polynomial of degree 2. The equations become

$$\dot{x} + a\lambda + b_1 x^2 + b_2(y^2 + z^2) = 0$$
$$\dot{y} + c_1 xy + c_2(y^2 - z^2) = 0$$
$$\dot{z} + c_1 xz - 2c_2 yz = 0.$$

Proposition 2.1 guarantees that the problem is codimension 1 and tells us how to unfold it, but does not provide any information about the existence or the number of modal parameters. So, and because the equations are not to a very high order, we calculate the tangent space. To this purpose, we need the generators for $\overleftrightarrow{\mathcal{E}}(S_3)$ which, by Lemma 2.4, are

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_3 = \begin{pmatrix} 0 & y^2 - z^2 & -2yz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ z & 0 & 0 \end{pmatrix}, S_5 = \begin{pmatrix} 0 & 0 & 0 \\ y^2 - z^2 & 0 & 0 \\ -2yz & 0 & 0 \end{pmatrix}, S_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$S_i = \begin{pmatrix} 0 & 0 \quad 0 \\ 0 & \\ 0 & \quad s_i \end{pmatrix}; i = 7, 8, 9,$$

where s_7, s_8 and s_9 are as given in Table XIV, 3.1 in Golubitsky *et al* [16]. The germs which span $\vec{\mathcal{M}}\ (S_3)$ are

$$X_0 = (\lambda, 0, 0),$$
$$X_1 = (x, 0, 0),\ X_2 = (0, y, z) \text{ and } X_3 = (0, y^2 - z^2, -2yz)$$

and the S_3-equivariant germs which do not vanish at the origin are of the form

$$Y_1 = (1, 0, 0).$$

Finally, we need

$$(Dg) = \begin{pmatrix} 2b_1x & 2b_2y & 2b_2z \\ c_1y & c_1x + 2c_2y & -2c_2z \\ c_1z & -2c_2z & c_1x - 2c_2y \end{pmatrix}.$$

Then $\mathrm{T}(g, S_3)$ is generated by

(1) $S_1g = (a\lambda + b_1x^2 + b_2(y^2 + z^2), 0, 0)$

(2) $S_2g = (c_1x(y^2 + z^2) + c_2y(y^2 - 3z^2), 0, 0)$

(3) $S_3g = (c_1xy(y^2 - 3z^2) + c_2(y^2 + z^2)^2, 0, 0)$

(4) $S_4g = (0, a\lambda y + b_1x^2y + b_2(y^2 + z^2)y, a\lambda z + b_1x^2z + b_2(y^2 + z^2)z)$

(5) $S_5g = (0, (a\lambda + b_1x^2 + b_2(y^2 + z^2))(y^2 - z^2), -2(a\lambda + b_1x^2 + b_2(y^2 + z^2))yz)$

(6) $S_6g = (0, c_1xy + c_2(y^2 - z^2), c_1xz - 2c_2yz)$

(7) $S_7g = (0, 2c_1x(y^2 + z^2)y + c_2y(y^2 - 3z^2)y, 2c_1x(y^2 + z^2)z + c_2y(y^2 - 3z^2)z)$

(8) $S_8g = (0, c_1x(y^2 - z^2) + c_2(y^2 + z^2)y, -2c_1xyz + c_2(y^2 + z^2)z)$

(9) $S_9g = (0, c_1xy(y^2 - 3z^2)y + 2c_2(y^2 + z^2)^2y, c_1xy(y^2 - 3z^2)z + 2c_2(y^2 + z^2)^2z)$

(10) $(Dg)X_0 = (2b_1x\lambda, c_1\lambda y, c_1\lambda z)$

(11) $(Dg)X_1 = (2b_1x^2, c_1xy, c_1xz)$

(12) $(Dg)X_2 = (2b_2(y^2 + z^2), c_1xy + 2c_2(y^2 - z^2), c_1xz - 4c_2yz)$

(13) $(Dg)X_3 = (2b_2y(y^2 - 3z^2), c_1x(y^2 - z^2) + 2c_2(y^2 + z^2)y,$
$-2c_1xyz + 2c_2(y^2 + z^2)z)$, over $\mathcal{E}\ (S_3)$

(14) $(Dg)Y_1 = (2b_1x, c_1y, c_1z)$, over $\mathbb{R}$

(15) $(Dg)_\lambda = (a, 0, 0)$, over $\mathcal{E}_\lambda$.

We know we need the unfolding term

(*) $(0, y, z)$, over $\mathbb{R}$.

However, this is not enough to obtain all the generators for $\vec{\mathcal{E}}\,(S_3)$. Note that, because terms like **(14)**, **(15)** or **(*)** do not take coefficients in the whole of $\mathcal{E}(S_3)$, we need to be able to obtain from $\mathrm{T}(g, S_3)$ the following 12 elements

(i) $(\lambda x, 0, 0)$

(ii) $(\lambda, 0, 0)$

(iii) $(x, 0, 0)$

(iv) $(x^2, 0, 0)$

(v) $(y^2 + z^2, 0, 0)$

(vi) $(y(y^2 - 3z^2), 0, 0)$

(vii) $(0, y, z)$

(viii) $(0, \lambda y, \lambda z)$

(ix) $(0, xy, xz)$

(x) $(0, (y^2 + z^2)y, (y^2 + z^2)z)$

(xi) $(0, y(y^2 - 3z^2)y, y(y^2 - 3z^2)z)$

(xii) $(0, y^2 - z^2, -2yz)$.

Because there are only 12 elements in $\mathrm{T}(g, S_3)$ involving terms of low enough order and amongst these we have

$$(1) = \lambda(15) + \frac{(12) + (11) - 2(6)}{2},$$

we shall need an additional element. Let this be

(**) $(x^2, 0, 0)$, over $\mathbb{R}$.

Then,

(11') $=$ (11)-$2b_1$(**) $\sim (0, xy, xz)$

(6') $=$ (6)-c_1(11') $\sim (0, y^2 - z^2, -2yz)$

(12') $=$ (12)-c_1(11')-$2c_2$(6') $\sim (y^2 + z^2, 0, 0)$

(1') $=$ (1)-λ(15)-b_2(12') $\sim (x^2, 0, 0)$

(8') $=$ (8)-c_1(6') $\sim (0, (y^2 + z^2)y, (y^2 + z^2)z)$

(4') $=$ (4)-$b_1 x$(11')-b_2(8') $\sim (0, \lambda y, \lambda z)$

(10') $= (10)\text{-}c_1(4') \sim (\lambda x, 0, 0)$

(2') $= (2)\text{-}c_1 x(12') \sim (y(y^2 - 3z^2), 0, 0)$

(7') $= (7)\text{-}2c_1 x(8') \sim (0, y(y^2 - 3z^2)y, y(y^2 - 3z^2)z)$

(14') $= (14)\text{-}c_1(*) \sim (x, 0, 0)$

(15') $= \lambda(15) \sim (\lambda, 0, 0)$

(*) $\sim (0, y, z)$

which are the elements we needed.

Remark In the above calculations we use the same notation as in the proof of Proposition 2.2.

Note that $(x^2, 0, 0)$ is modal. Hence, the equations become

$$\dot{x} + a\lambda + \tilde{b}_1 x^2 + b_2(y^2 + z^2) = 0$$

$$\dot{y} + c_1(x - \alpha)y + c_2(y^2 - z^2) = 0$$

$$\dot{z} + c_1(x - \alpha)z - 2c_2 yz = 0,$$

where $\tilde{b}_1$ is modal.

4.4. Bifurcations

In this section, we study the bifurcations that occur in a problem characterized by the above equations. Before attempting this, we simplify our task by restricting the domain of the parameters involved. The choice of parameter values is done based on the kind of bifurcation we are interested in. For example, if the bifurcation occurs along a stable branch, we can easily see it in a computer simulation. So, we choose parameter values which lead to this behaviour.

First of all, we note that the equations can be rescaled so that c_1 becomes either (-1) or $(+1)$. This will lead to the two subsections below.

We can also, without loss of generality, assign values to a and b_1. To draw the bifurcation diagrams, we restrict the equations to $\text{Fix}(S_3) = \mathbb{R}$. The

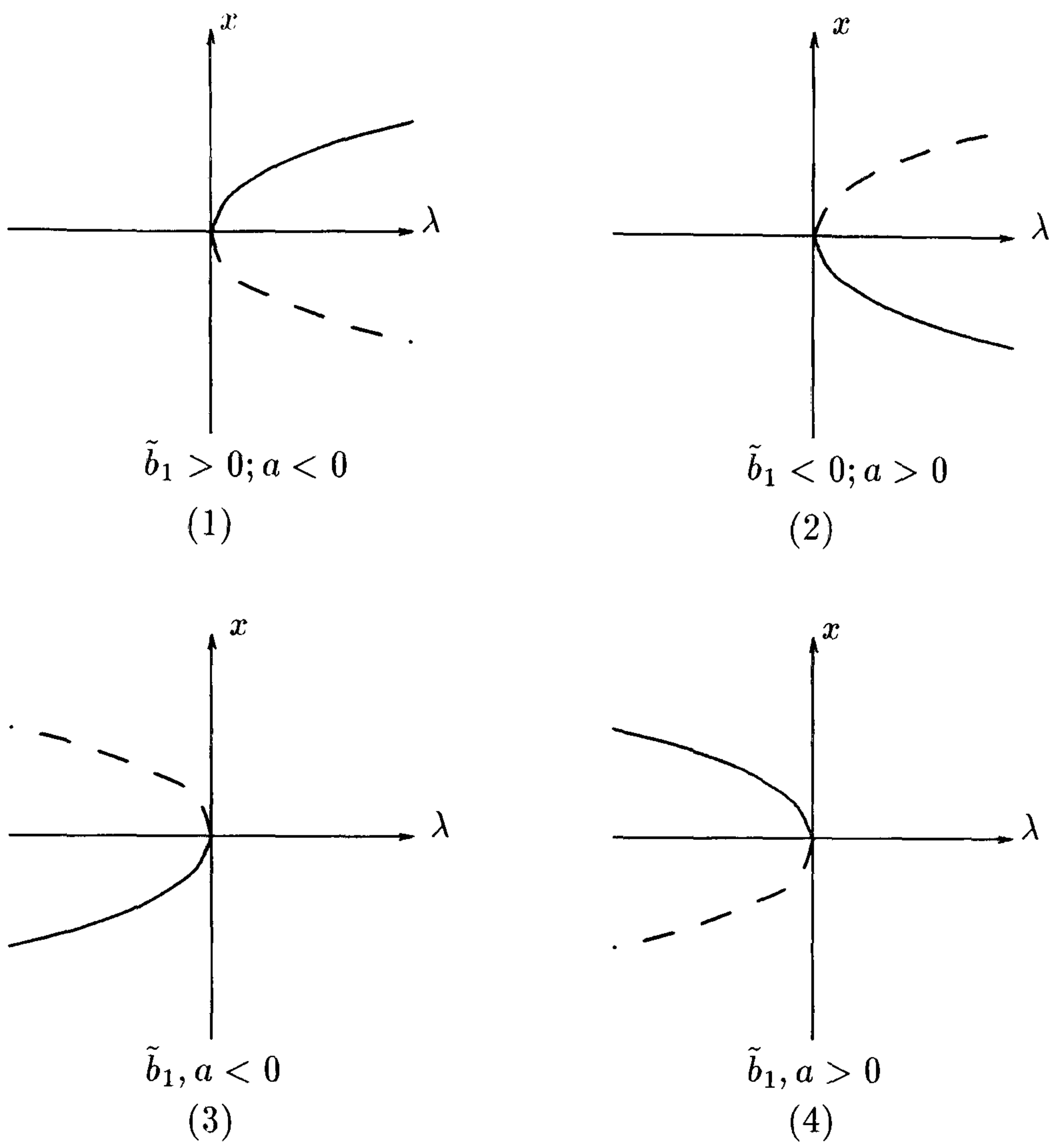


Figure 4.1.

equations in this fixed-point space are

$$\dot{x} + a\lambda + \tilde{b}_1 x^2 = 0.$$

On the (λ, x)-plane, the branching equation is

$$\lambda = -\frac{\tilde{b}_1}{a} x^2,$$

from which we see that the sign and value of a and $\tilde{b}_1$ will affect only the direction of the branch and its stability. The bifurcation diagrams are drawn in Figure 4.1.

We choose $a > 0$ and $\tilde{b}_1 < 0$ so that we have case (2), that is, there exist no solutions for $\lambda < 0$ and there are two solutions for $\lambda > 0$, one stable and one unstable. The bifurcation from this fully-symmetric branch occurs for $x = \alpha$

since

$$(Dg)_{|\mathrm{Fix}(S_3)} = \begin{pmatrix} 2\tilde{b}_1 x & 0 & 0 \\ 0 & c_1(x-\alpha) & 0 \\ 0 & 0 & c_1(x-\alpha) \end{pmatrix}.$$

So, if $\alpha < 0$, this bifurcation will take place along a stable branch. Furthermore, we choose

$$\tilde{b}_1 = -1 \text{ and } a = +1.$$

This affects only the size of the fully-symmetric branch. We shall refer to this branch as the trivial solution and bifurcations from it as primary. The solutions which determine the $(1,5)$-mode interaction have become

$$\begin{aligned} &\dot{x} + \lambda - x^2 + b_2(y^2 + z^2) = 0 \\ &\dot{y} + c_1(x-\alpha)y + c_2(y^2 - z^2) = 0 \\ &\dot{z} + c_1(x-\alpha)z - 2c_2 yz = 0; \quad c_1 = \pm 1 \end{aligned}$$

and we study them in the next two subsections.

4.4.1. Bifurcation with $c_1 = +1$

Let

$$g(x,y,z,\lambda,\alpha) = (\lambda - x^2 + b_2(y^2+z^2), (x-\alpha)y + c_2(y^2 - z^2), (x-\alpha)z - 2c_2 yz).$$

Solutions to $g \equiv 0$ are

(i) $\lambda = x^2; \quad y = 0; \quad z = 0$

(ii) $\lambda = x^2 - b_2 y^2; \quad x = \alpha - c_2 y; \quad z = 0$

(iii) $\lambda = x^2 - b_2(y^2 + z^2); \quad x = \alpha + 2c_2 y; \quad z^2 = 3y^2.$

Solution (i) is the trivial solution corresponding to a branch with full symmetry. Solution (ii) is $\mathbb{Z}_2$-symmetric. Equations (iii) correspond to two a-symmetric solutions conjugated to (ii) by the group action. In fact, any

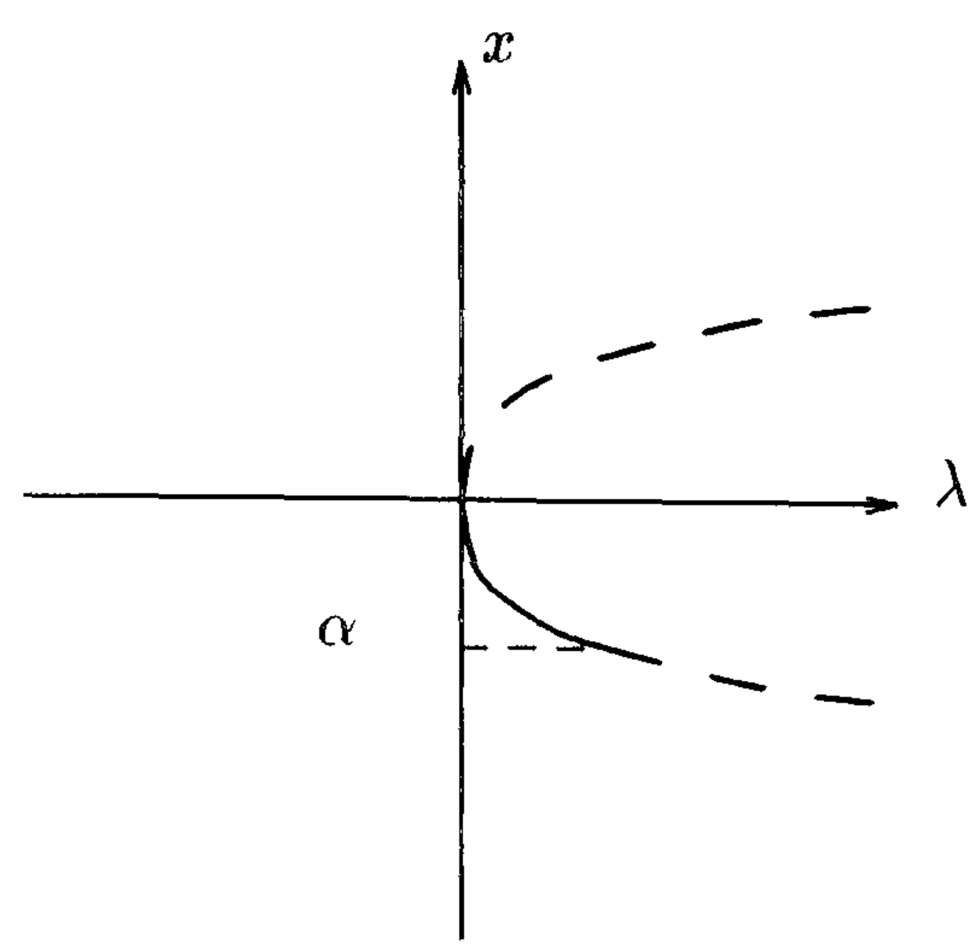


Figure 4.2.

element of $\mathbb{R}^2$ of the form $(y, \pm\sqrt{3}\,y)$can be transformed into one in $\mathbb{R} \times 0$ by the group action. Let $z = \sqrt{3}\,y$; then

$$\alpha.(y,z) = (y\cos\alpha - z\sin\alpha, y\sin\alpha + z\cos\alpha) = (-2y, 0).$$

Let $z = -\sqrt{3}\,y$; then

$$(\alpha^2).(y,z) = (y\cos 2\alpha - z\sin 2\alpha, y\sin 2\alpha + z\cos 2\alpha) = (-2y, 0).$$

Now, let $y_1 = -2y$ and $x_1 = x$. We have

$$x_1 = x = \alpha + 2c_2 y = \alpha - c_2 y_1,$$

as required by (ii).

The stability of solution (i) is determined by the eigenvalues of the matrix

$$(Dg)_{|(i)} = \begin{pmatrix} -2x & 0 & 0 \\ 0 & x-\alpha & 0 \\ 0 & 0 & x-\alpha \end{pmatrix},$$

which are all positive if $\alpha < x < 0$. From now on, we assume that $\alpha < 0$. The bifurcation diagram in $\mathrm{Fix}(S_3)$ is given in Figure 4.2 where the full line indicates stability and the dotted lines, instability. This will be our convention for all bifurcation diagrams.

The stability of the $\mathbb{Z}_2$-symmetric solution is given by the eigenvalues of

$$(Dg)_{|(ii)} = \begin{pmatrix} -2x & \frac{2b_2}{c_2}(\alpha - x) & 0 \\ \frac{\alpha - x}{c_2} & \alpha - x & 0 \\ 0 & 0 & 3(x-\alpha) \end{pmatrix},$$

which are

$$X_1(x) = 3(x-\alpha)$$

and those determined by

$$\begin{aligned} \mathrm{Tr}(x) &= \alpha - 3x \\ \mathrm{Det}(x) &= 2(x-\alpha)[(1-\frac{b_2}{c_2^2})x + \frac{b_2}{c_2^2}\alpha]. \end{aligned}$$

Remark The stability and secondary bifurcations along the branches in (iii), can be obtained from those in (ii) by performing the group transformations needed to change one into the other. This means that secondary bifurcations occur in sets of three at a time.

Along branch (ii), there is a double zero eigenvalue for

$$x_{1,2} = \alpha$$

and a simple zero eigenvalue at

$$x_3 = \frac{b_2}{b_2 - c_2^2}\alpha.$$

According to Proposition 2.5, there are coefficient values for which a Hopf bifurcation occurs along this branch. This happens when $\mathrm{Tr}(x) = 0$ and $\mathrm{Det}(x) > 0$, i.e., $(Dg)_{|(ii)}$ has a pair of imaginary eigenvalues. We have

$$\begin{aligned} \mathrm{Tr}(x_c) = 0 &\Leftrightarrow x_c = \frac{\alpha}{3} \\ \mathrm{Det}(x_c) = -\frac{4}{9}(1 + 2\frac{b_2}{c_2^2})\alpha^2 > 0 &\Leftrightarrow 1 + 2\frac{b_2}{c_2^2} < 0. \end{aligned}$$

From now on, we assume that

$$b_2 < -\frac{c_2^2}{2},$$

so that a secondary Hopf bifurcation occurs along the $\mathbb{Z}_2$-symmetric branch at x_c. This implies that

$$x_{1,2} = \alpha < x_3 < \frac{\alpha}{3} = x_c.$$

Next we are interested in knowing whether this Hopf bifurcation occurs along a stable or an unstable branch. For this purpose, we need to know the signs of the eigenvalues of $Dg)_{|(ii)}$ determined by $\mathrm{Tr}(x)$ and $\mathrm{Det}(x)$. These are

$$X_2(x) = \frac{(\alpha - 3x) + \sqrt{(\alpha - 3x)^2 - 8(x - \alpha)[(1 - \frac{b_2}{c_2^2})x + \frac{b_2}{c_2^2}\alpha]}}{2}$$

and

$$X_3(x) = \frac{(\alpha - 3x) - \sqrt{(\alpha - 3x)^2 - 8(x - \alpha)[(1 - \frac{b_2}{c_2^2})x + \frac{b_2}{c_2^2}\alpha]}}{2}.$$

Note that for $x > \alpha$, $X_1(x) > 0$ and therefore, does not make the branch change stability. For the other two eigenvalues, we have

$$X_2 \neq 0 \quad \forall x \in \mathbb{R}$$

$$X_3(x) = 0 \Leftrightarrow x = \alpha \wedge x = x_3.$$

Since X_2 and X_3 are continuous functions of x and real for $x < x_3$, they can only become imaginary by first becoming complex with non-zero real part. The sign of the real part determines the stability. We have

$$X_2(x) > 0 \quad \text{for } \alpha < x < x_3$$

and

$$X_3(x) < 0 \quad \text{for } \alpha < x < x_3$$

so that, the $\mathbb{Z}_2$-symmetric branch is unstable for such values of x. For $x > x_3$, both X_2 and X_3 are positive so the branch becomes stable until $x = x_c$. At this point, the Hopf bifurcation occurs and both eigenvalues have negative real parts beyond this point, that is, the $\mathbb{Z}_2$-symmetric branch becomes unstable again.

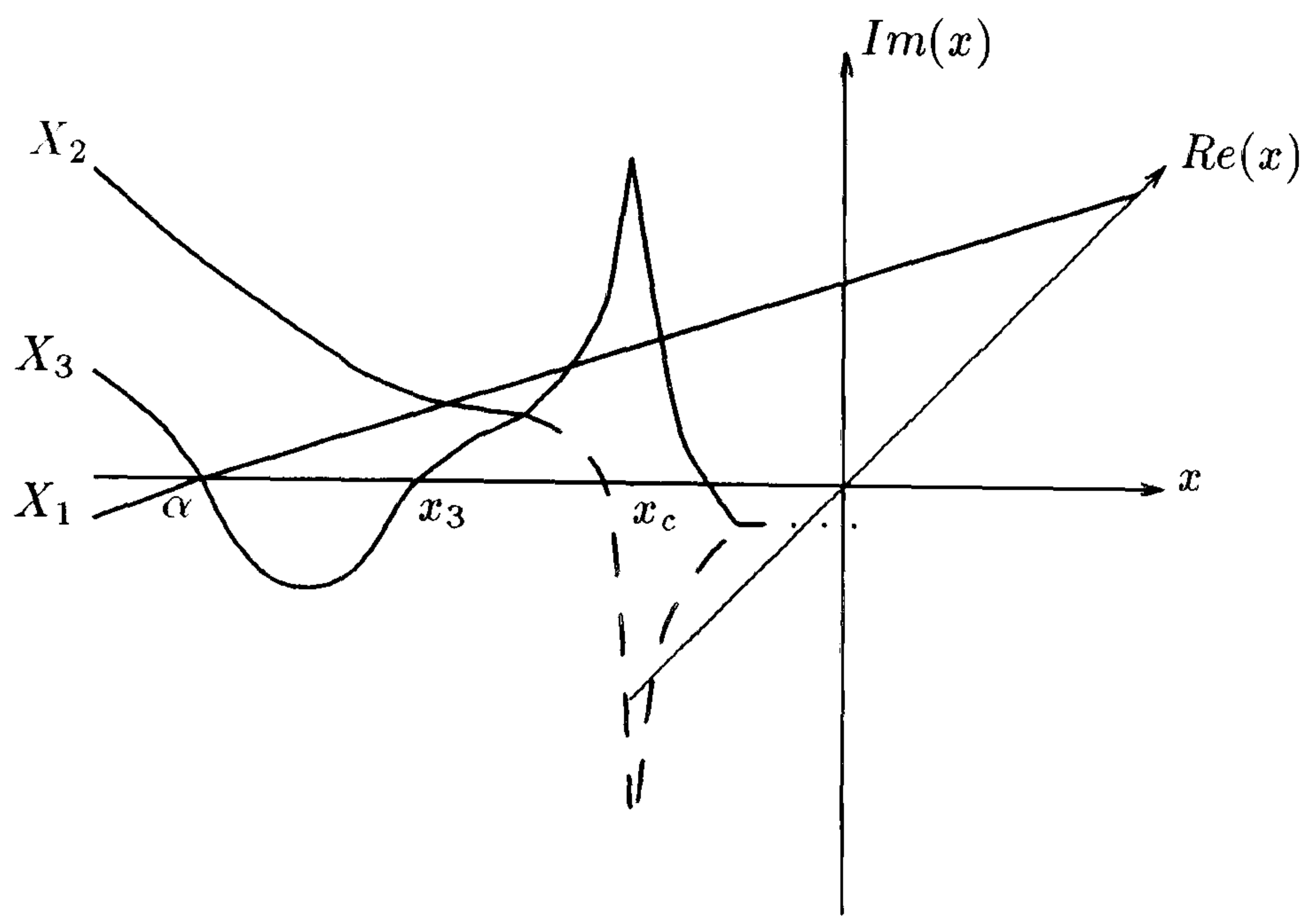


Figure 4.3.

Figure 4.3 is a good reference for these changes in stability.

The change of stability at $x = x_3$ corresponds to the vertex of the curve defined by

$$\begin{cases} \lambda = x^2 - b_2 y^2 \\ y = \frac{\alpha - x}{c_2} \end{cases}.$$

For $c_2 > 0$, we have $y(x_3) < 0$ and the reverse when $c_2 < 0$. Without loss of generality, we choose $c_2 > 0$. The bifurcation diagram in the (x, y, λ)-plane is drawn in Figure 4.4, where the circles indicate the occurrence of the Hopf bifurcation.

Note that the $\mathbb{Z}_2$-symmetric branch is in the plane defined by

$$y = \frac{\alpha - x}{c_2}.$$

At this stage, we don't know whether the limit cycles originated by the secondary bifurcation are stable or not. To find out about the stability of the limit cycles, we solve the equations numerically using KAOS [17]. In Appendix A, we give a brief and elementary explanation of this program.

In KAOS, we use the equations restricted to $\text{Fix}(\mathbb{Z}_2)$, bearing in mind that any changes along the $\mathbb{Z}_2$-symmetric branch correspond to changes in the two

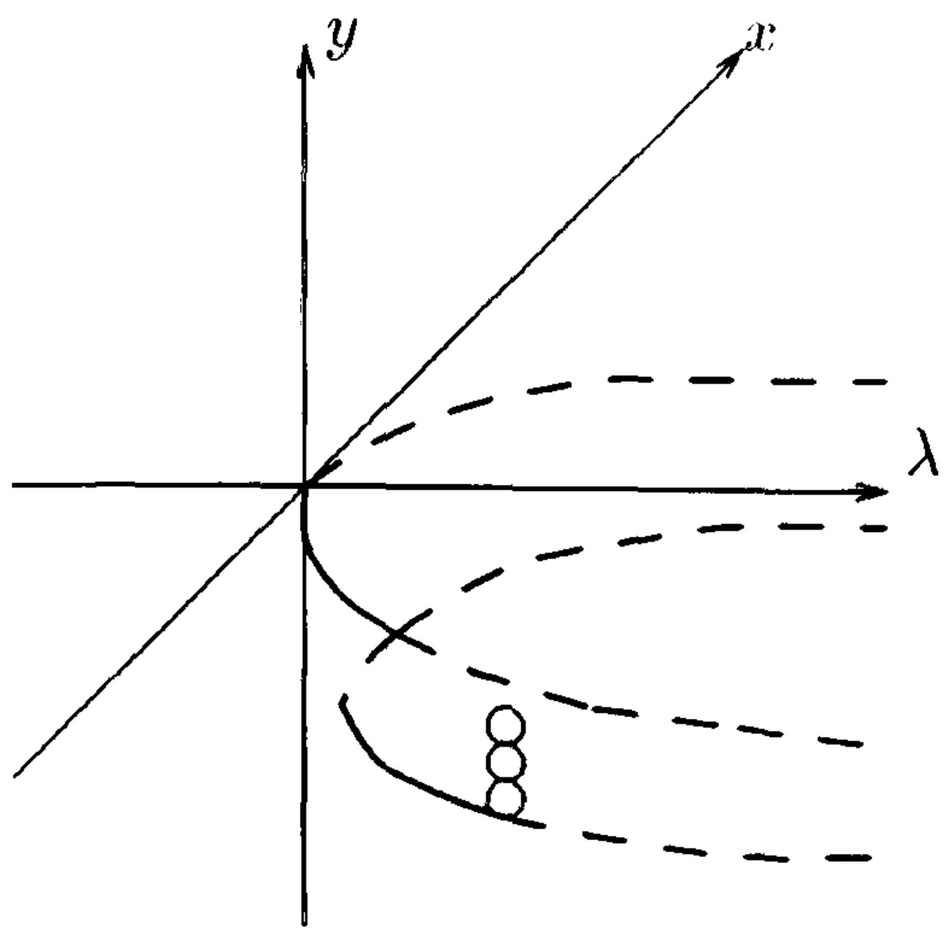


Figure 4.4.

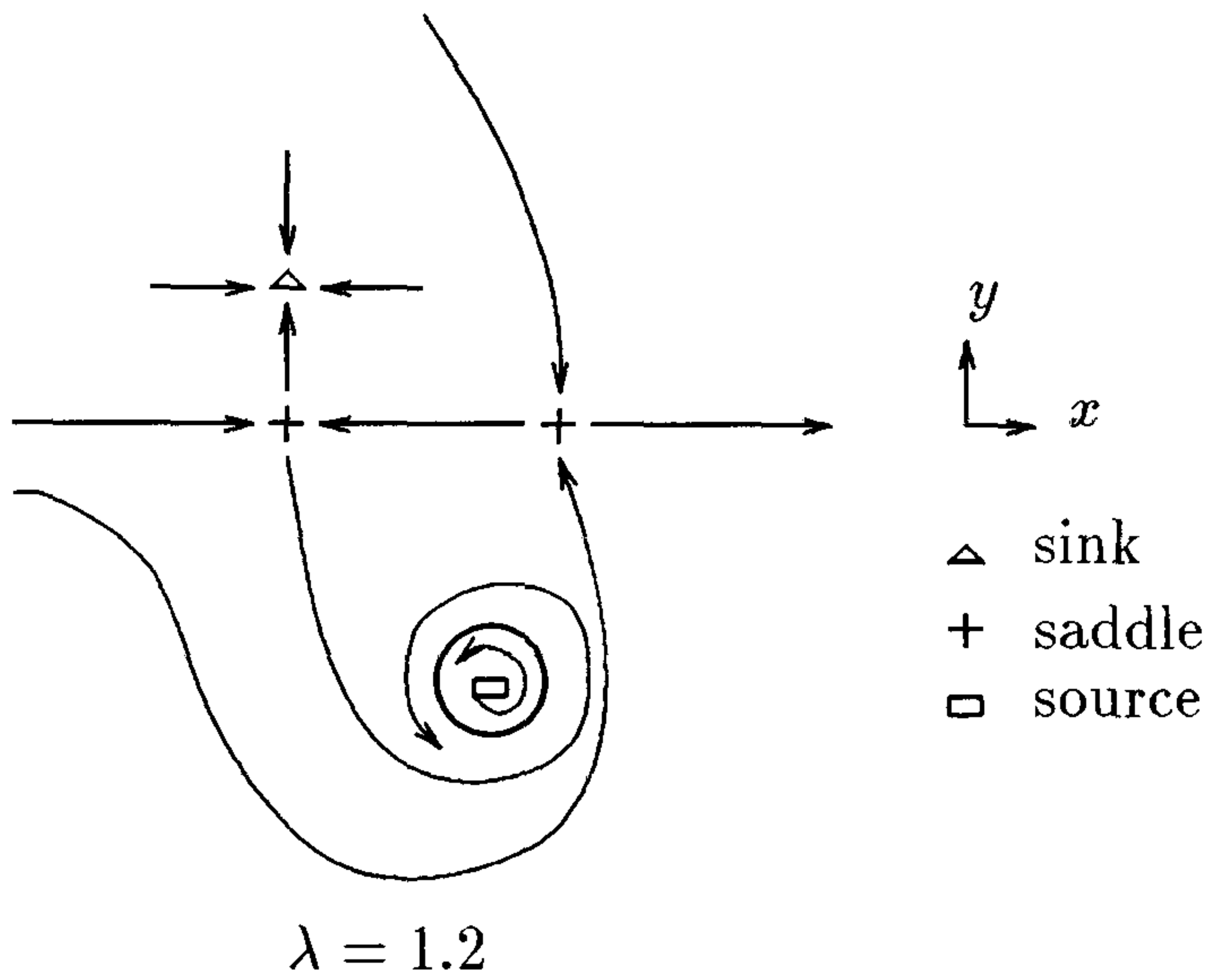


Figure 4.5.

conjugated branches. Also, restricting the equations in this way implies that the eigenvalue in the z-direction is not taken into account. We shall see that, in the pictures we obtain, there exists a stable equilibrium which is, in fact, a saddle point if we take all eigenvalues into account. However, the pictures obtained in a 2-dimensional system are much clearer. Tentatively varying parameters, we find that for

$$\lambda = 1.2, \quad \alpha = -1, \quad b_2 = -.5 \text{ and } c_2 = .5,$$

there exists a stable limit cycle as shown in Appendix B and drawn in Figure 4.5.

This guarantees that the Hopf bifurcation is supercritical. So, now we fix the values of

$$\alpha = -1, \quad b_2 = -.5, \text{ and } c_2 = .5$$

and do some more calculations before using KAOS again.

We have two branches defined by

$$\begin{cases} \lambda = x^2 \\ y = 0 \end{cases} \quad \text{and} \quad \begin{cases} \lambda = x^2 + \frac{y^2}{2} \\ y = -2(x+1) \end{cases}.$$

The points at which there is a change of stability are

$$\begin{aligned} x_{1,2} &= -1, \\ x_3 &= -\frac{2}{3} \\ \text{and} \quad x_c &= -\frac{1}{3}. \end{aligned}$$

At the Hopf bifurcation point, we have

$$\lambda = 1.$$

Remark The fact that this value is the same as that of λ for which the $\mathbb{Z}_2$-symmetric branch bifurcates from the trivial one, is simply a coincidence. It does not restrict the results in any way. In fact, for any values of b_2 and c_2 such that

$$\frac{b_2}{c_2^2} = -2,$$

the two bifurcations occur at the same time but at different solutions.

For $\lambda < 1$, there is no limit cycle and the phase portrait is as in Figure 4.6.

If we increase λ, we come to a point where the cycles disappear in a heteroclinic connection. This point is in the interval $(1.3, 1.4)$. We prove the existence of the heteroclinic connection with numerical evidence. For $\lambda = 1.3$,

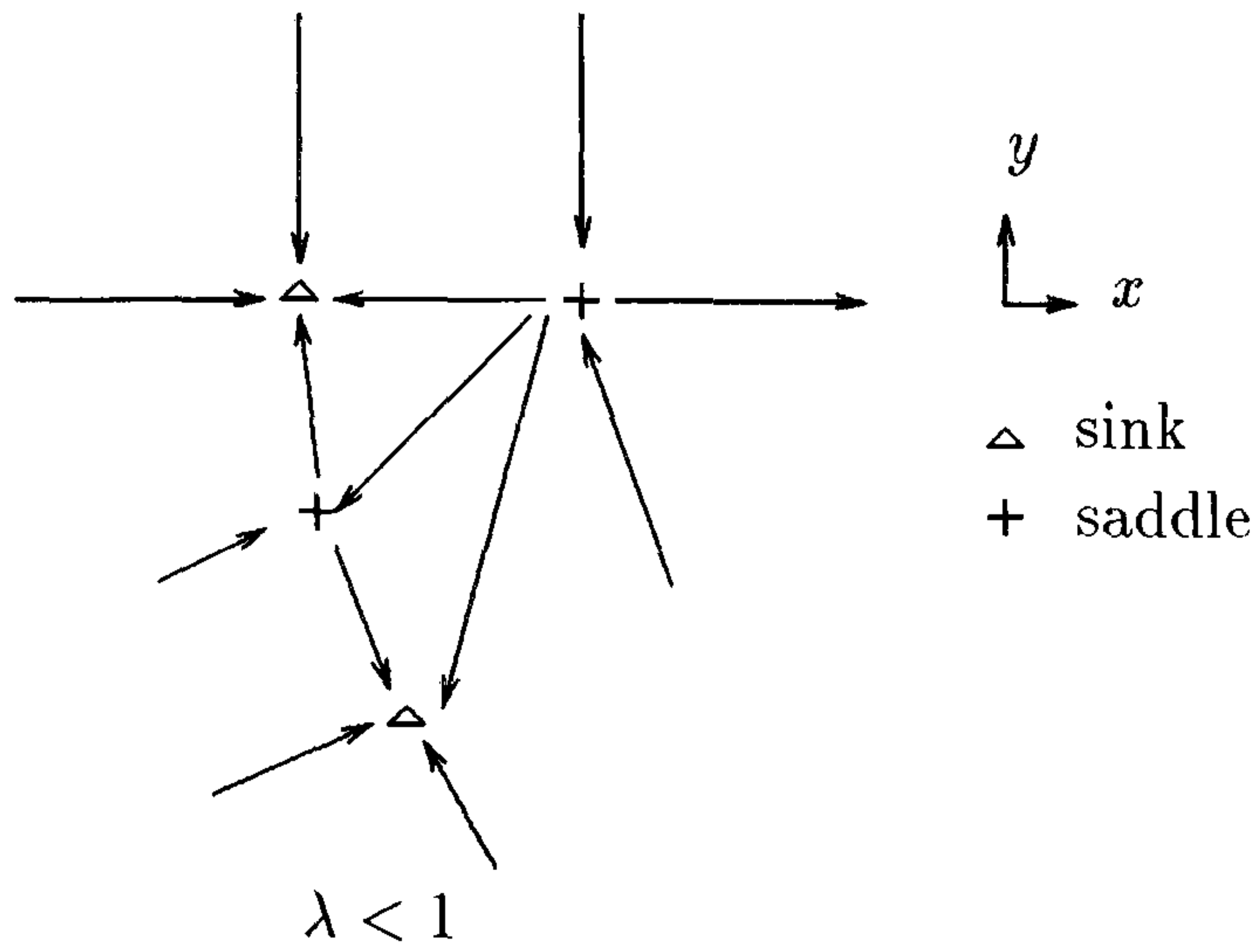


Figure 4.6.

the phase portrait is as in Figure 4.7(a), where the unstable manifold of $(x_-, 0)$ is closer to the x-axis than the stable manifold of $(x_+, 0)$. For $\lambda = 1.4$, we have the reverse situation as can be seen in Figure 4.7(b). For easier reference, the unstable manifold of $(x_-, 0)$ is represented by a dotted line and the stable manifold of $(x_+, 0)$ by a full line.

Note that in Figure 4.7(b), the limit cycle no longer exists. The only way the two manifolds can change their order in the phase space is if they are joined together in a heteroclinic connection between $(x_-, 0)$ and $(x_+, 0)$, by which the limit cycle disappears. The heteroclinic connection is shown in Figure 4.7(c).

Remark The subject of heteroclinic and homoclinic cycles in problems with symmetry is considered in Melbourne *et al* [21]. There, the definition of homoclinic cycle is extended to include orbits connecting points in the same isotropy subgroup. That is, points in the same group orbit are treated as one single point. In this setting, the connecting orbit which occurs in the $(1, 5)$-mode interaction is homoclinic. However, we prefer to call it heteroclinic to stress the fact that it connects two separate points.

Finally, we can complete the bifurcation diagram in Figure 4.4 to obtain the one in Figure 4.8.

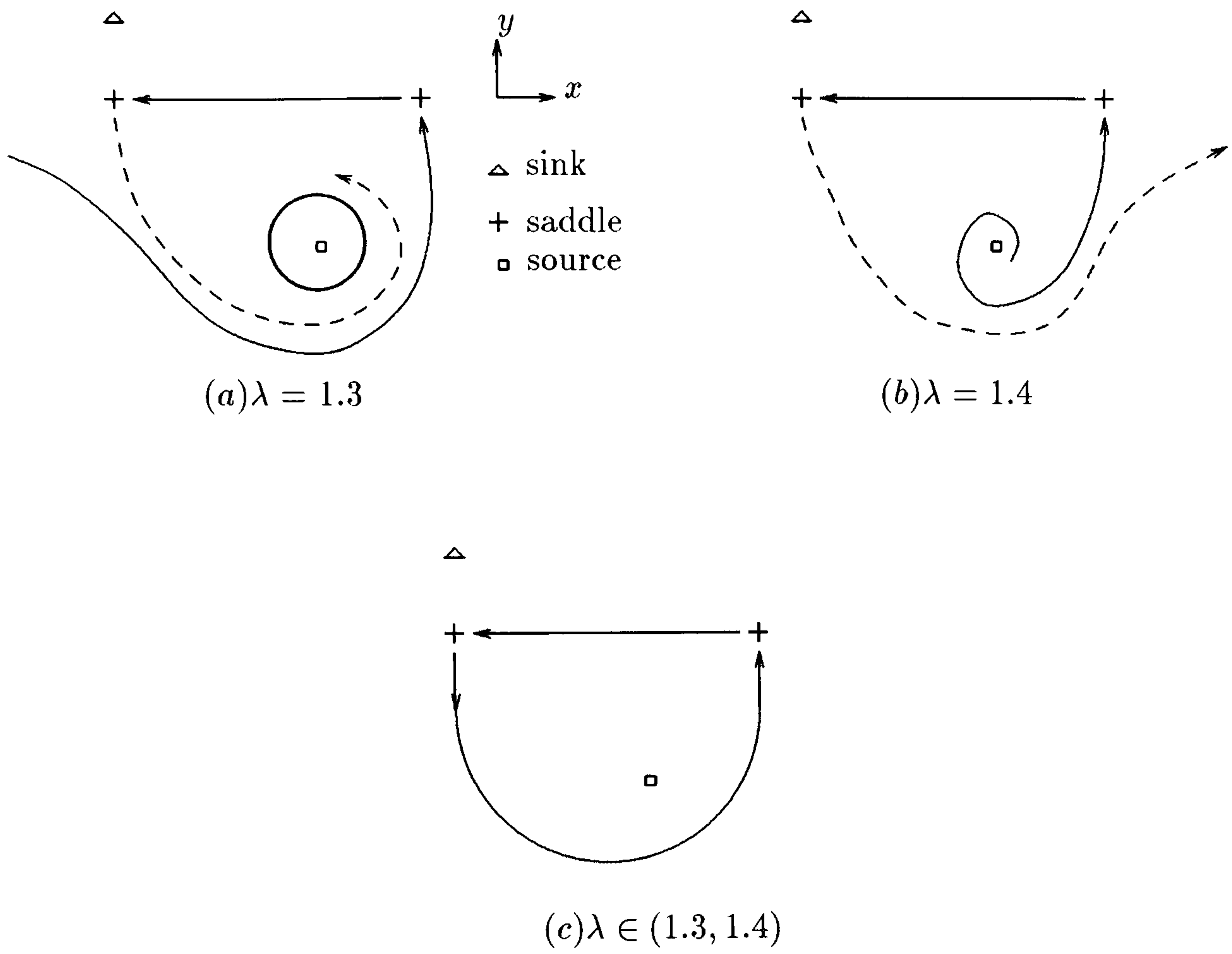


Figure 4.7.

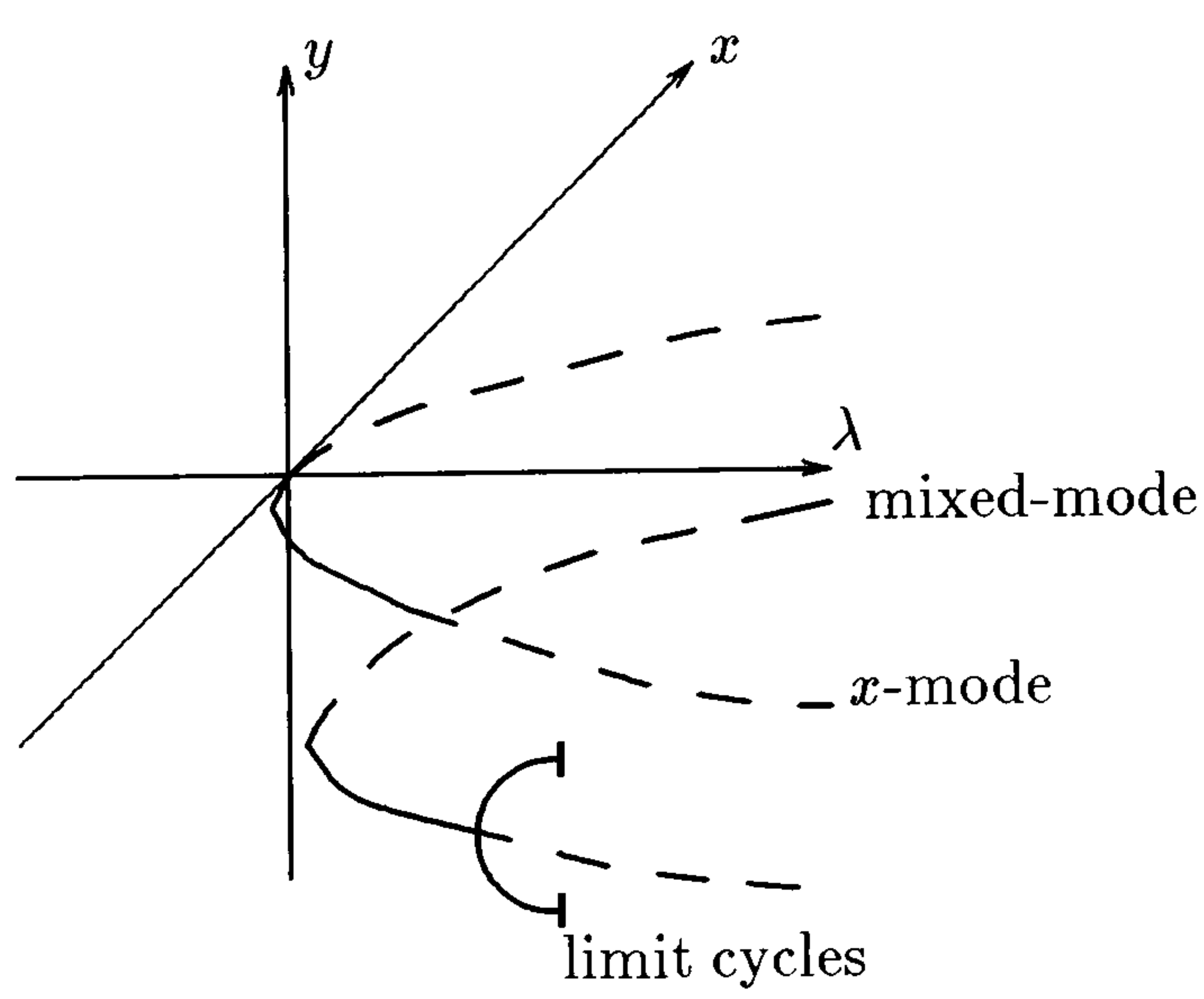


Figure 4.8.

For other parameter values, the bifurcation diagram remains essentially the same. If $c_2 < 0$, for example, the plane in which the $\mathbb{Z}_2$- symmetric branch exists has slope of the opposite sign. However, the stability changes, the Hopf bifurcation and the heteroclinic connection still exist, possibly for different parameter values.

Note that the solution with $\mathbb{Z}_2$-symmetry is a mixed-mode branch. As seems to be the case when a mode interaction involves a trivial mode, the only single-mode branch is the trivial one. All other solutions are mixed-mode.

4.4.2. Bifurcation with $c_1 = -1$

The study of the $(1, 5)$-mode interaction when $c_1 = -1$, follows the same steps as those of the study made in the previous subsection. Let

$$g(x, y, z, \lambda, \alpha) = (\lambda - x^2 + b_2(y^2 + z^2), (\alpha - x)y + c_2(y^2 - z^2), (\alpha - x)z - 2c_2yz).$$

Solutions to $g \equiv 0$ are

(i) $\lambda = x^2, \quad y = 0, \quad z = 0$

(ii) $\lambda = x^2 - b_2y^2, \quad x = \alpha + c_2y, \quad z = 0$

(iii) $\lambda = x^2 - b_2(y^2 + z^2), \quad x = \alpha - 2c_2y, \quad z^2 = 3y^2.$

As before, solution (i) corresponds to a fully symmetric branch, solution (ii) has $\mathbb{Z}_2$ symmetry and the two a-symmetric solutions (iii) can be obtained from (ii) by group transformations.

The stability of the trivial solution is determined by the signs of the eigenvalues of

$$(Dg)_{|(i)} = \begin{pmatrix} -2x & 0 & 0 \\ 0 & \alpha - x & 0 \\ 0 & 0 & \alpha - x \end{pmatrix}$$

which are all positive if $x < \alpha < 0$. Again, we assume $\alpha < 0$ so that this branch is stable for $x < \alpha$.

Solution (ii) has its stability determined by the eigenvalues of

$$(dg)_{|(ii)} = \begin{pmatrix} -2x & 2\frac{b_2}{c_2}(x-\alpha) & 0 \\ \frac{\alpha - x}{c_2} & x-\alpha & 0 \\ 0 & 0 & 3(\alpha - x) \end{pmatrix},$$

which are

$$X_1(x) = 3(\alpha - x)$$

and those determined by

$$\begin{aligned} \mathrm{Tr}(x) &= -(x+\alpha) \\ \mathrm{Det}(x) &= 2(x-\alpha)[(\frac{b_2}{c_2^2} - 1)x - \frac{b_2}{c_2^2}\alpha]. \end{aligned}$$

There is a double zero eigenvalue for

$$x_{1,2} = \alpha$$

and a simple zero eigenvalue for

$$x_3 = \frac{b_2}{b_2 - c_2^2}\alpha.$$

Now, we look for parameter values for which a secondary Hopf bifurcation occurs along this branch. These are values for which $\mathrm{Tr}(x) = 0$ and $\mathrm{Det}(x) > 0$. We have

$$\begin{aligned} \mathrm{Tr}(x) = 0 &\Leftrightarrow x_c = -\alpha \\ \mathrm{Det}(x) = 4\alpha^2(2\frac{b_2}{c_2^2} - 1) > 0 &\Leftrightarrow b_2 > \frac{c_2^2}{2}. \end{aligned}$$

From now on, we assume that $b_2 > \frac{c_2^2}{2}$.

The stability along this branch varies according to the relative position of $x_{1,2}$, x_3 and x_c. We have to consider two cases.

Case 1. $b_2 > c_2^2 > \frac{c_2^2}{2} \Rightarrow x_3 < x_{1,2}$

The changes of stability can be read from Table 4.1 and we see that the $\mathbb{Z}_2$-symmetric branch is stable only for $x < x_3$.

		x_3		$x_{1,2}$		x_c	
Tr(x)	+		+		+		−
Det(x)	+		−		+		+
$X_1(x)$	+		+		−		−

Table 4.1.

		$x_{1,2}$		x_c		x_3	
Tr(x)	+		+		−		−
Det(x)	−		+		+		−
$X_1(x)$	+		−		−		−

Table 4.2.

Case 2. $\frac{c_2^2}{2} < b_2 < c_2^2 \Rightarrow x_3 > x_c$

In this case, the sign of the eigenvalues can be determined from the data in Table 4.2. We see that the branch is always unstable.

However, restricting the equations to Fix($\mathbb{Z}_2$), as we do when using KAOS, we disregard the effect of $X_1(x)$. So, in case 1, we see a stable branch for $x < x_3$ and $x_{1,2} < x < x_c$, whereas in case 2, the branch appears to be stable for $x_{1,2} < x < x_c$.

In Fix($\mathbb{Z}_2$), we have two branches defined by

$$\begin{cases} \lambda = x^2 \\ y = 0 \end{cases} \quad \text{and} \quad \begin{cases} \lambda = x^2 - b_2 y^2 \\ y = \frac{x-\alpha}{c_2} \end{cases}.$$

We note that for $x_c = -\alpha$, we have

$$y(x_c) = -2\frac{\alpha}{c_2}$$
$$\lambda(x_c) = (1 - 4\frac{b_2}{c_2^2})\alpha^2.$$

Since $b_2 > \frac{c_2^2}{2}$, $\lambda < 0$ and so, the Hopf bifurcation occurs at a point where the trivial branch does not exist.

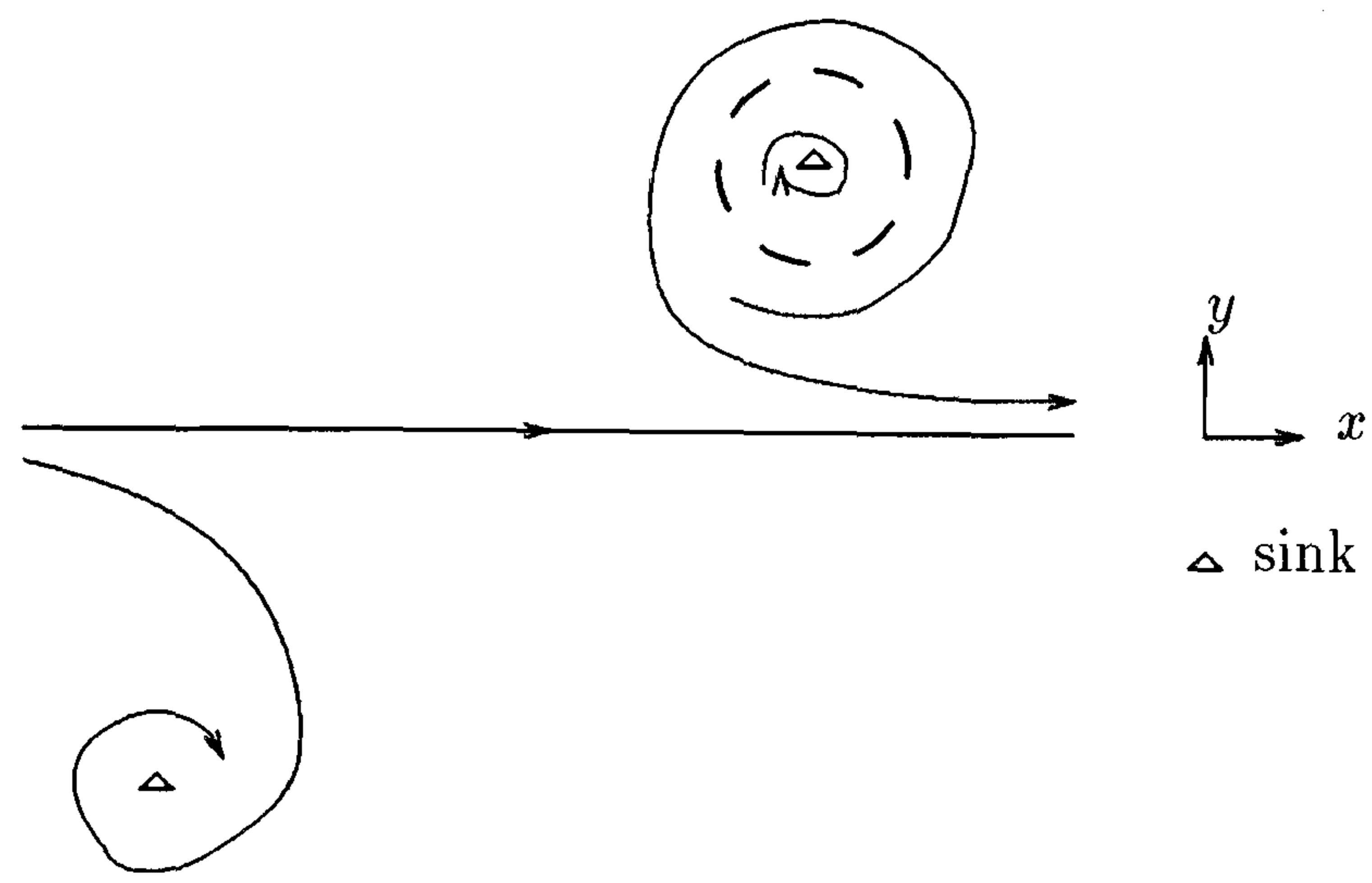


Figure 4.9.

We assume $\alpha = -1$.

In case 1, we fix $b_2 = 1.1$ and $c_2 = 1$ so that the Hopf bifurcation occurs for

$$x_c = 1 \Rightarrow \lambda(x_c) = -3.4$$

and the vertex of the $\mathbb{Z}_2$-symmetric branch is at

$$x_3 = -11 \Rightarrow \lambda(x_3) = 11.$$

Using KAOS and varying λ , we see that the limit cycles created at the Hopf bifurcation are unstable and disappear for a value of $\lambda > -3.3$.

Remark The way to spot an unstable cycle using KAOS is by noticing that trajectories spiral in different directions as in Figure 4.9. Therefore a limit cycle must exist between the two regions of opposite spirals. The limit cycle represented in Figure 4.9 grows in amplitude and eventually collides with the trajectory $y = 0$, when it disappears. Afterwards, the bifurcation diagram consists of two stable equilibria, one in the upper and one in the lower half-plane, until $\lambda = 0$.

For $\lambda > 0$, we have four equilibria until $\lambda = 11$ and then only the two trivial ones. There seem to be several heteroclinic connections in this parameter region. For $\lambda > 11$, the heteroclinic connection is between the two trivial equilibria, as shown in Figure 4.10(b). It is homoclinic in the sense of Melbourne

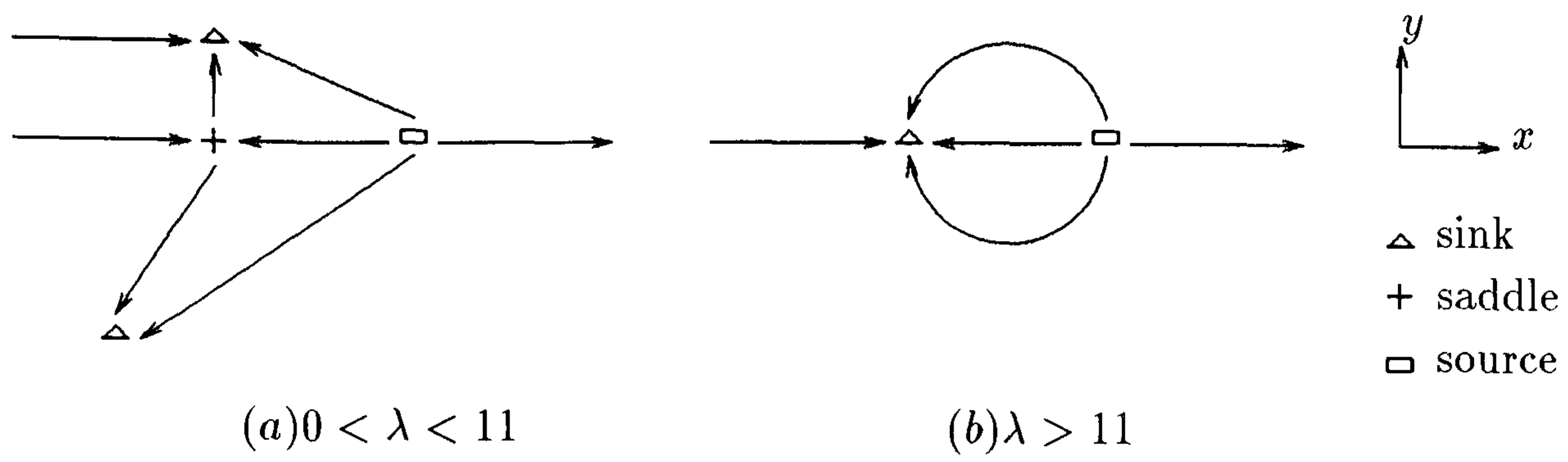


Figure 4.10.

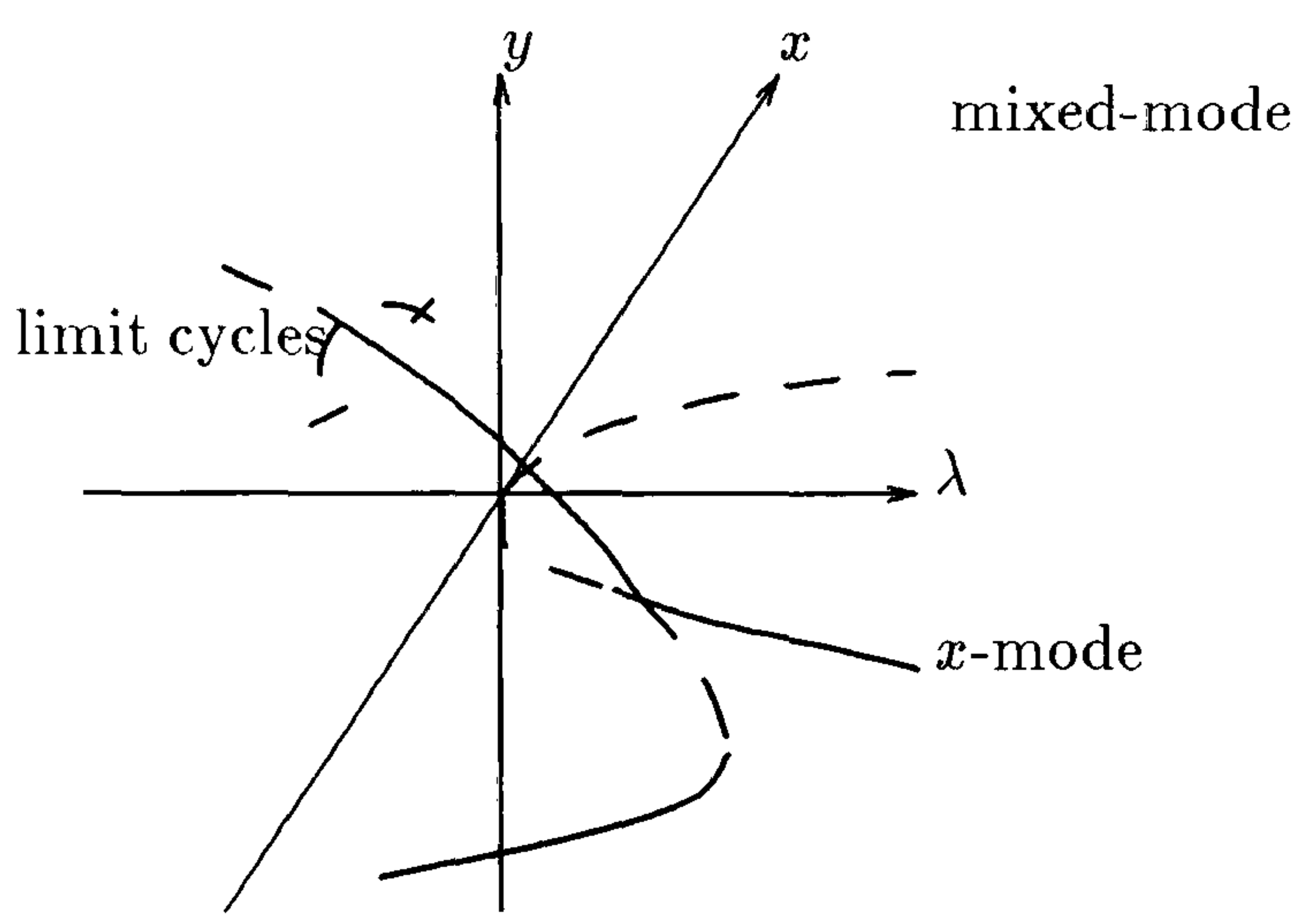


Figure 4.11.

et al [21]. For $0 < \lambda < 11$, some connections are genuinely heteroclinic and are represented in Figure 4.10(a).

The bifurcation diagram is drawn in Figure 4.11.

In case 2, we choose $b_2 = .6$. For this choice of parameter values, the Hopf bifurcation occurs for

$$x_c = 1 \Rightarrow \lambda(x_c) = -1.4$$

and the vertex of the $\mathbb{Z}_2$-symmetric branch is at

$$x_3 = 1.5 \Rightarrow \lambda(x_3) = -1.5.$$

The dynamics in this case are very similar to those of the previous case. This time, there are no equilibria for $\lambda < -1.5$. For $-1.5 < \lambda < -1.4$, there are two equilibria in the upper half-plane, one saddle and one source. At $\lambda = -1.4$, the

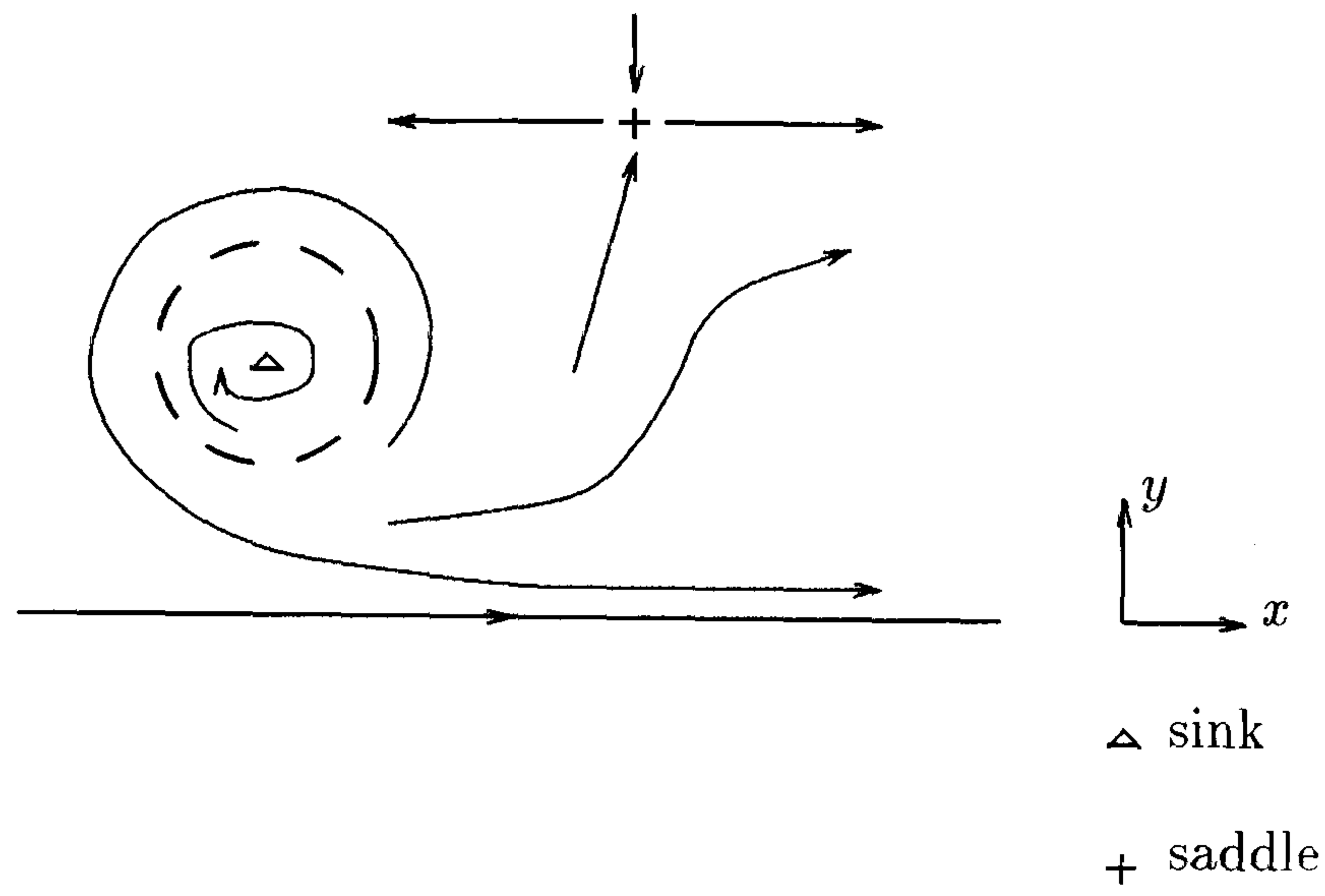


Figure 4.12.

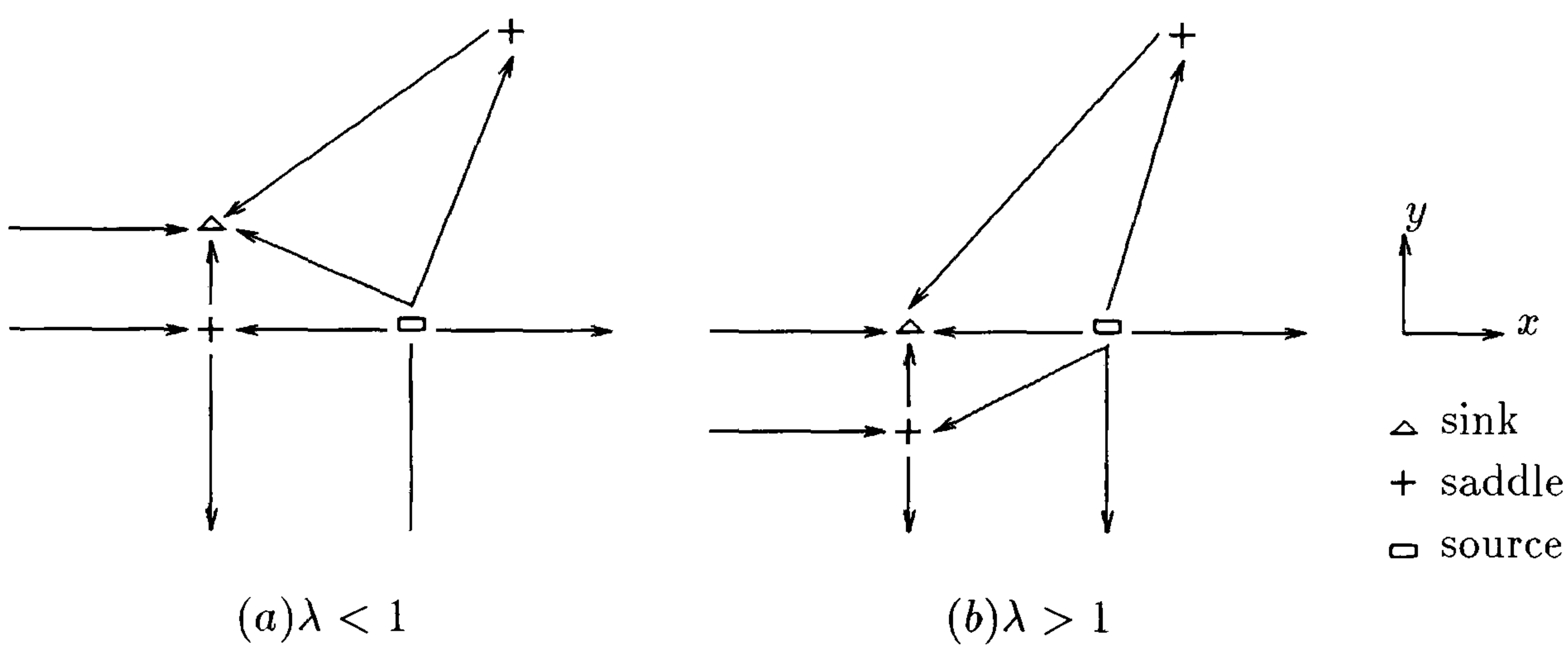


Figure 4.13.

source undergoes a Hopf bifurcation becoming a sink and originating a family of unstable limit cycles, as shown in figure 4.12 . The limit cycle disappears by becoming homoclinic to the saddle. For $\lambda > 0$, we have two other equilibria, the trivial ones. Again, we find some heteroclinic connections, as drawn in Figure 4.13. The bifurcation diagram for this case is shown in Figure 4.14.

Remark We note that in any case of the $(1, 5)$-mode interaction, the $\mathbb{Z}_2$-symmetric branch bifurcates transcritically from the trivial one. This analogy between the mode interaction and the single-mode bifurcation (cf. Golubitsky and Schaeffer [14]), stresses the idea that a trivial mode does not change drastically the primary bifurcation.

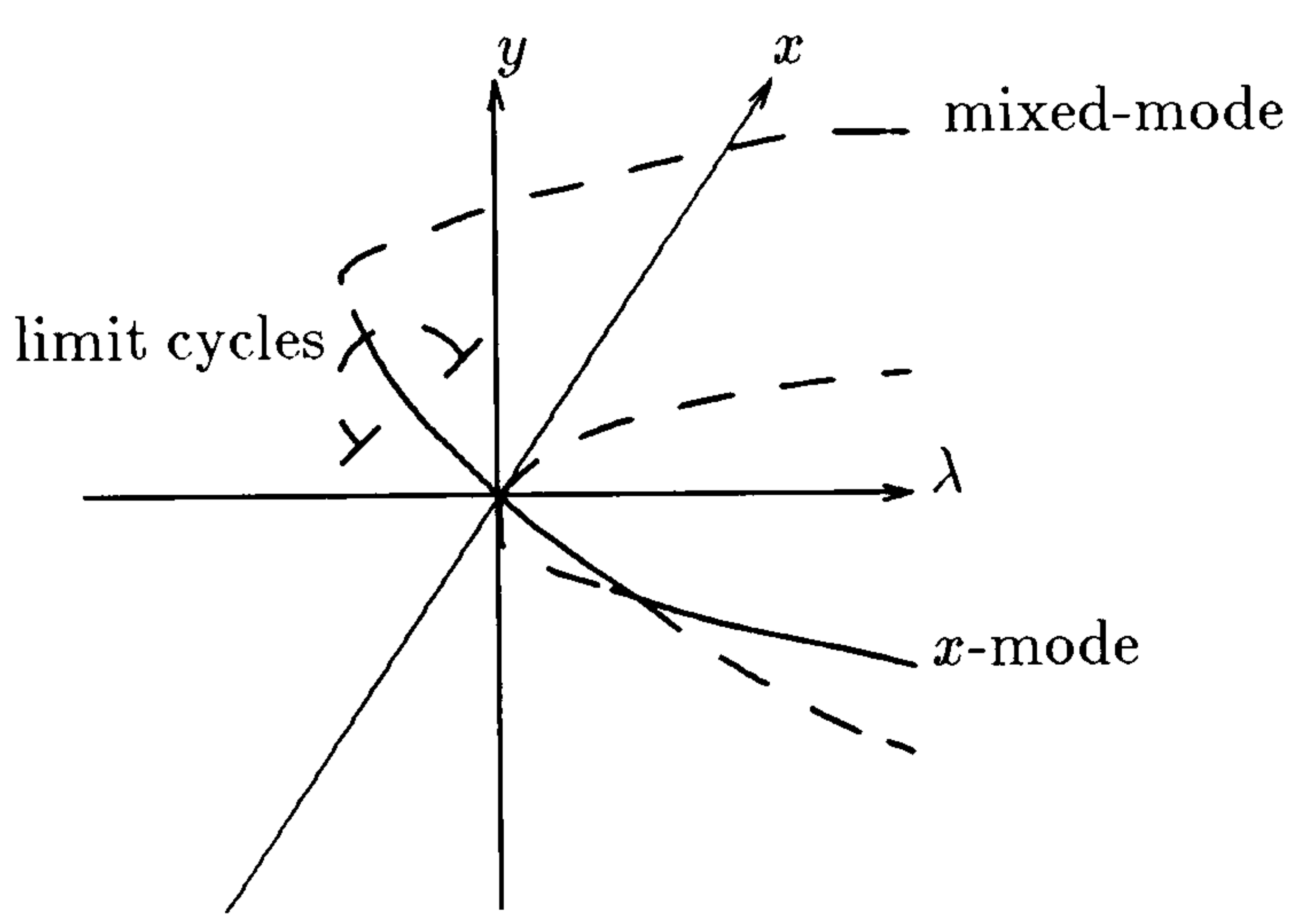


Figure 4.14.

Chapter 5

Mode interactions under the action of $SO(3)$

5.1. Introduction

After having studied the $(1,3)$- and the $(1,5)$-mode interactions, it is natural to think about the $(3,5)$- and eventually the $(1,3,5)$- mode interaction problems. The interaction of the 3- and 5-dimensional modes in a spherically invariant system has been analyzed by Friedrich and Haken [13], by Chossat [10] and, more recently, by Armbruster and Chossat [1]. Our study differs from theirs from the very beginning since we consider a different representation for the group action. This representation is defined in section 5.2 and has the advantage of being suitable both for the $(3,5)$- and the $(1,3,5)$-mode interaction. However, we discover that it forces the element of order 2 in $O(3)$ to act trivially and hence, it corresponds to a representation of $SO(3)$ rather than $O(3)$. In section 5.2, we also calculate the isotropy subgroups and fixed-point subspaces for this representation of $SO(3)$. The following section is dedicated to computing the invariants and equivariants for the problem, which is done using results from Spencer [27] and Spencer and Rivlin [28]. Section 5.4 is concerned with the interaction of the 3- and the 5-dimensional modes. We use the results from the previous sections to establish the equations and our bifurcation analysis relies heavily on results from Armbruster and Chossat [1]. There are interesting similarities, as well as differences, between the two problems. In the last section, we study the simultaneous interaction of the three modes.

5.2. The group action

Our first aim is to define the representation for the action of $O(3)$ which suits the problem best. We begin by noting that any $n \times n$ matrix can be written as the sum of a symmetric and a skew-symmetric one. We can make the symmetric matrix traceless by subtracting from it a scalar multiple of the identity, the scalar being $1/n$ of its trace. We can then write a matrix as the sum of the aforementioned three matrices, thus obtaining a decomposition for $\mathcal{M}_n$, the space of real $n \times n$ matrices, into the sum of three subspaces $U \oplus V \oplus W$, respectively, the 1-dimensional space of real scalar multiples of the identity, the $n(n-1)/2$-dimensional space of skew-symmetric matrices and the $[n(n+1)/2-1]$-dimensional space of traceless symmetric matrices. When $n = 3$, we have a decomposition of $\mathcal{M}_3$ into a 1-, a 3- and a 5-dimensional space which constitutes a suitable representation for the group acting by similarity on the space of matrices. Recall the action of $O(3)$ on $\mathbb{R}^3$ by matrix multiplication

$$\gamma.v = \gamma v \quad \forall v \in \mathbb{R}^3 \quad \forall \gamma \in O(3),$$

and consider the natural action of $O(3)$ on $\mathbb{R}^3 \otimes \mathbb{R}^3$ which follows from the above: given $v, w \in \mathbb{R}^3$, the element in $\mathbb{R}^3 \otimes \mathbb{R}^3$ which comes from these is $v \otimes w^t$. Hence, we have

$$(\gamma v) \otimes (\gamma w)^t = (\gamma v) \otimes (w^t \gamma^t) = \gamma(v \otimes w^t)\gamma^t \quad \forall v, w \in \mathbb{R}^3 \quad \forall \gamma \in O(3)$$

and therefore, the natural action of $O(3)$ on $\mathbb{R}^3 \otimes \mathbb{R}^3$ is by conjugation. We point out that $\mathbb{R}^3 \otimes \mathbb{R}^3 \simeq \mathcal{M}_3$.

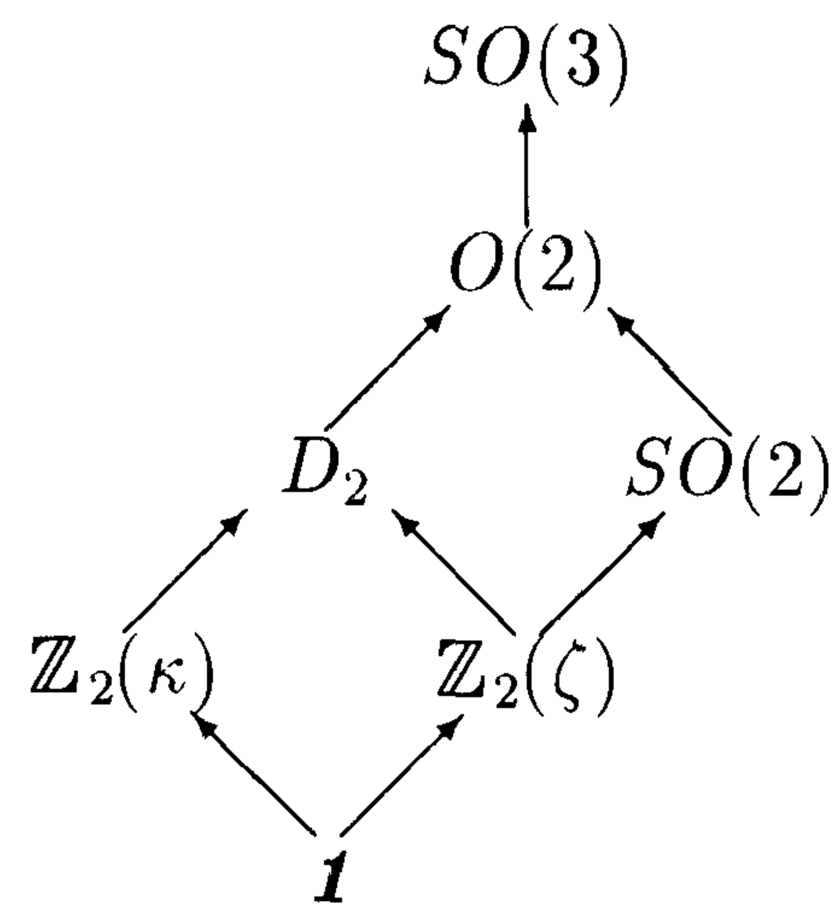


Figure 5.1.

With this representation however, the subgroup $\mathbb{Z}_2^c$ of $O(3)$ acts trivially and so, this representation is, in fact, one of the action of $SO(3)$. From now on we shall be concerned with the $SO(3)$-symmetric problem.

We note that we can use this representation both for the $(3,5)$- and the $(1,3,5)$- mode interaction. To obtain a representation for the $(3,5)$-mode interaction problem, we simply disregard the 1-dimensional space of real scalar multiples of the identity by ensuring that all matrices are traceless. The isotropy subgroups are the same in both cases and the fixed-point subspaces differ only by scalar multiples of the identity which increase their dimension by 1, in the case of the $(1,3,5)$-mode interaction.

In the following proposition, we calculate the isotropy lattice and the fixed-point subspaces for the $(3,5)$-mode interaction. To obtain a similar result for the interaction of all three modes, it suffices to use Lemma 2.8, which proves that the isotropy lattice remains the same and to add one to the dimension of every fixed-point subspace, by adding a scalar multiple of the identity matrix to each fixed-point subspace.

Proposition 5.1. *The isotropy subgroups and fixed-point subspaces for the action of $O(3)$ on $V \oplus W$ as described above are given in Table 5.1 and the isotropy lattice is as in Figure 5.1.*

Isotropy subgroup Σ	*Fix*(Σ)	*dim Fix*(Σ)
$SO(3)$	$\{ 0 \}$	0
$O(2)$	$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} ; a \in \mathbb{R} \right\}$	1
$SO(2)$	$\left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix} ; a, b \in \mathbb{R} \right\}$	2
D_2	$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -(a+b) \end{pmatrix} ; a, b \in \mathbb{R} \right\}$	2
$\mathbb{Z}_2(\zeta)$	$\left\{ \begin{pmatrix} a & p & 0 \\ q & b & 0 \\ 0 & 0 & -(a+b) \end{pmatrix} ; a, b, p, q \in \mathbb{R} \right\}$	4
$\mathbb{Z}_2(\kappa)$	$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & r \\ 0 & s & -(a+b) \end{pmatrix} ; a, b, r, s \in \mathbb{R} \right\}$	4
$\mathbf{1}$	$V \oplus W$	8

Table 5.1.

Proof The subgroups of $SO(3)$ and the containment relations are from Golubitsky *et al* [16], chapter XIII. We use Proposition XX, 2.3 in [16] to determine the isotropy subgroups and the fixed-point spaces, for which we need the isotropy subgroups and fixed-point subspaces for the action of $SO(3)$ on V, given in Table 5.2.

From Theorem XIII, 8.1 in [16], we obtain the dimension of the fixed-point subspaces of the closed subgroups of $SO(3)$ acting on a 5-dimensional space corresponding to $l = 2$ in the space of spherical harmonics, restricting the options for the isotropy subgroups. These are as in Table 5.3.

Isotropy subgroup Σ	$\mathrm{Fix}(\Sigma)$
$SO(3)$	$\{\,0\,\}$
$SO(2)$	$\left\{\begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; a \in \mathbb{R}\right\}$
$\mathbf{1}$	V

Table 5.2.

Subgroup	Dimension
$\mathbb{Z}_2$	3
D_2	2
$SO(2)$	1
$O(2)$	1
$\mathbb{T}$	0

Table 5.3.

(i) $O(2) \subset SO(3)$ acts by the following elements

$$\kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\theta . \begin{pmatrix} a & p & q \\ p & b & r \\ q & r & -(a+b) \end{pmatrix} =$$

$$\begin{pmatrix} a\cos^2\theta + b\sin^2\theta + p\sin 2\theta & (b-a)\sin\theta\cos\theta + p\cos 2\theta & q\cos\theta + r\sin\theta \\ (b-a)\sin\theta\cos\theta + p\cos 2\theta & a\sin^2\theta + b\cos^2\theta - p\sin 2\theta & -q\sin\theta + r\cos\theta \\ q\cos\theta + r\sin\theta & -q\sin\theta + r\cos\theta & -(a+b) \end{pmatrix}$$

so, the element in W is fixed if and only if

$$\begin{cases} a\cos^2\theta + b\sin^2\theta + p\sin 2\theta = a \\ a\sin^2\theta + b\cos^2\theta - p\sin 2\theta = b \\ (b-a)\sin\theta\cos\theta + p\cos 2\theta = p \\ q\cos\theta + r\sin\theta = q \\ -q\sin\theta + r\cos\theta = r \end{cases} \Leftrightarrow \begin{cases} a = b \\ p = q = r = 0 \end{cases}.$$

Then,

$$\kappa. \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix},$$

justifying the second line of Table 5.1 since no other matrices are fixed by $O(2)$.

(ii) From (i) we know which matrices in W are fixed by $SO(2)$ which acts uniquely by θ. We also have

$$\theta. \begin{pmatrix} 0 & c & d \\ -c & 0 & e \\ -d & -e & 0 \end{pmatrix} = \begin{pmatrix} 0 & c & d\cos\theta + e\sin\theta \\ -c & 0 & -d\sin\theta + e\cos\theta \\ d\cos\theta + e\sin\theta & -d\sin\theta + e\cos\theta & 0 \end{pmatrix}$$

$$\Leftrightarrow d = e = 0$$

and so,

$$\mathrm{Fix}(SO(2)) = \left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix}; a, b \in \mathbb{R} \right\}.$$

(iii) $D_2 \subset SO(3)$ acts by κ and

$$\zeta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\zeta \cdot \begin{pmatrix} a & p & q \\ p & b & r \\ q & r & -(a+b) \end{pmatrix} = \begin{pmatrix} a & p & -q \\ p & b & -r \\ -q & -r & -(a+b) \end{pmatrix} \Leftrightarrow q = r = 0$$

and

$$\kappa \cdot \begin{pmatrix} a & p & 0 \\ p & b & 0 \\ 0 & 0 & -(a+b) \end{pmatrix} = \begin{pmatrix} a & -p & 0 \\ -p & b & 0 \\ 0 & 0 & -(a+b) \end{pmatrix} \Leftrightarrow p = 0$$

hence, D_2 fixes the space given in line 4 of Table 5.1.

(iv) D_2 contains two elements of order 2, namely, ζ and κ. These generate two subgroups which we denote by $\mathbb{Z}_2(\zeta)$ and $\mathbb{Z}_2(\kappa)$, respectively.

(a) From (iii), we know which elements are fixed by ζ in W and

$$\zeta \cdot \begin{pmatrix} 0 & c & d \\ -c & 0 & e \\ -d & -e & 0 \end{pmatrix} = \begin{pmatrix} 0 & c & -d \\ -c & 0 & -e \\ d & e & 0 \end{pmatrix} \Leftrightarrow d = e = 0.$$

Therefore, $\mathbb{Z}_2(\zeta)$ fixes matrices of the form given in line 5 of Table 5.1 and no others.

(b) We know that κ fixes the diagonal. So, it suffices to investigate

$$\kappa \cdot \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -b \\ -c & 0 & d \\ -e & f & 0 \end{pmatrix} \Leftrightarrow a = b = c = e = 0$$

which ends the proof of the proposition. □

In the case of the $(3,5)$-mode interaction, this proposition is sufficient to guarantee the existence of a branch with $O(2)$ symmetry since the hypotheses of the Equivariant Branching Lemma are satisfied. We shall return to this subject.

5.3. Invariant theory

In this section, we compute the invariants and equivariants for $SO(3)$ acting on $U \oplus V \oplus W$ as defined above. The invariants are obtained from Spencer [27], sections 2.5 and 2.6 and the equivariants are obtained from Spencer and Rivlin [28], section 4. Their results can be simplified for our problem since U is the space of diagonal matrices , not just symmetric ones. Let $X \in U$, $Y \in V$ and $Z \in W$. In addition to all results used in the aforementioned papers, we use the following

Lemma 5.1. *All matrix products commute with X^n, for all n, i.e.,*

$$MX^n = X^nM \;\; \forall n \in \mathbb{N} \;\; \forall M \in U \oplus V \oplus W.$$

Proof Let $X = xI$, $x \in \mathbb{R}$. This implies that

$$X^n = x^nI \;\; \forall n \in \mathbb{N}.$$

Therefore,

$$M.X^n = M.x^nI = x^n.MI = x^nI.M = X^n.M \;\; \forall M \in U \oplus V \oplus W.$$

□

Lemma 5.2. *There is only one factor involving powers of X in any matrix polynomial and its degree is less than or equal to 2.*

Proof By Lemma 5.1 , we have

$$X^nMX^m = X^{n+m}M \;\; \forall n, m \in \mathbb{N} \;\; \forall M \in U \oplus V \oplus W.$$

If $n + m \geq 3$ then we use the Hamilton-Cayley theorem to obtain X^{n+m} as a function of powers of X of degree ≤ 2. □

Proposition 5.2. *A basis for the invariant functions $U \oplus V \oplus W \to \mathbb{R}$ is given by the traces of the following matrix polynomials*

degree 1	X
degree 2	X^2, Y^2, Z^2, XZ, YZ
degree 3	$XZ^2, X^2Z, XY^2, Y^2Z, XYZ, X^3, Y^3, Z^3$
degree 4	$XYZ^2, X^2YZ, XY^2Z, X^2Y^2, Y^2Z^2, X^2Z^2$
degree 5	X^2Y^2Z, XY^2Z^2, XY^2ZY
degree 6	$X^2Y^2Z^2, X^2Y^2ZY.$

Proposition 5.3. *A basis for the equivariant matrix polynomials $U \oplus V \oplus W \to U \oplus V \oplus W$ is given by the following matrix polynomials*

degree 0	I
degree 1	X, Y, Z
degree 2	$X^2, Y^2, Z^2, XZ, XY, YZ, ZY$
degree 3	$XYZ, X^2Y, XY^2, X^2Z, XZ^2, Y^2Z, Z^2Y, YZ^2, ZY^2$
degree 4	$X^2Y^2, X^2Z^2, Y^2Z^2, Z^2Y^2, ZYZ^2, YZY^2, X^2YZ$
degree 5	$YZ^2Y^2, ZY^2Z^2, X^2Y^2Z, X^2Z^2Y.$

We note that, because X is in fact a scalar multiple of the identity, we can write all the equivariants involving second degree matrices in X as an invariant in X multiplied by the other terms of the equivariant polynomial in question. More precisely, suppose

$$X^2.M$$

is an equivariant polynomial. Then,

$$X^2.M = \frac{1}{3}\mathrm{tr}(X^2).M,$$

meaning that we don't need to consider the equivariants involving second degree powers of X. This reduces the number of equivariants considerably.

We do not give explicit proofs for these propositions. A list of the invariants and equivariants without restrictions can be found in Spencer [27] and Spencer and Rivlin [28] as stated before. To obtain a basis, we use Lemmas 5.1 and 5.2 above to prove that some matrix polynomials are redundant.

These two propositions are enough to write the Birkhoff normal form equations both for the (3, 5)- and the (1, 3, 5)-mode interaction. For the latter, we just have to introduce λ into these matrix polynomials. For the (3, 5)-mode interaction, it suffices to make $X = 0$ and subtract $\mathrm{tr}(M)/3.I$ to all other matrix polynomials M. We shall study the (3, 5)-mode interaction first and then use Lemma 2.7 to write the equations for the (1, 3, 5)-mode interaction problem.

5.4. (3, 5)-mode interaction

The study of this mode interaction is done by comparison with that done by Armbruster and Chossat [1]. We shall see that some of the features exist for both the $O(3)$- and the $SO(3)$-invariant bifurcation. In order to be able to compare the two sets of equations, we need to use equivariants up to third order. However, existence or non-existence of bifurcating branches can be proved using second order equations only and we start with these.

Let us write any matrix M as

$$M = (M)_Y + (M)_Z \in V \oplus W.$$

The unfolded bifurcation equations up to second order are

$$\begin{aligned} &\dot{Y} + a_1(\lambda - \alpha)Y + a_2(YZ)_Y = 0 \\ &\dot{Z} + b_1\lambda Z + b_2(Z^2 - \frac{1}{3}\mathrm{tr}(Z^2).I) + b_3(Y^2 - \frac{1}{3}\mathrm{tr}(Y^2).I) + b_4(YZ)_Z = 0. \end{aligned}$$

There are a few remarks concerning these equations:

(1) We have

$$ZY = Z^t(-Y^t) = -(YZ)^t = -(YZ)^t_Y - (YZ)^t_Z = (YZ)_Y - (YZ)_Z,$$

hence, we do not need the equivariant term (ZY).

(2) We claim that we don't need any higher-order terms in λ for the problem to be generic. In fact, if we split the equations into two as in 2.2.3 and 2.2.4 and use Lemma 2.6, we see that any other terms in λ are higher-order terms for each single-mode problem.

(3) We cannot claim that this problem is generic. We can however guarantee that if these equations show that a given bifurcation branch does not occur, then it does not occur for any higher order set of equations containing these. This is because, if these equations give incompatible conditions for a given branch, any other equations containing these shall do the same.

Using the Equivariant Branching Lemma, we can prove the existence of a branch with $O(2)$-symmetry. This branch is 1-dimensional, contained in W and bifurcates from the origin at $\lambda = 0$. It is a single-mode solution. By Proposition 2.4, there exists a mixed-mode branch with $SO(2)$-symmetry, branching off the $O(2)$-symmetric branch at $\lambda = \alpha$. To find out about the existence of any other branches, we must calculate the branching equations in each fixed-point subspace. We compute these using Maple (see Appendix D) and find that there are no other branches, except for two branches in $\mathrm{Fix}(D_2)$ which are conjugated to the $O(2)$-symmetric branch. These are referred to in Armbruster and Chossat [1], p.162 as L' and L''. Recall that a similar situation occurs for the $(1, 5)$-mode interaction with respect to the same 5-dimensional mode.

Next, we write the bifurcation equations up to third order and compare the $SO(3)$-symmetric bifurcation to the $O(3)$-symmetric studied in Armbruster and Chossat [1]. The fact that, for our $SO(3)$-symmetric problem only two non-trivial branches exist, restricts the behaviour of the system considerably, in that we cannot expect any of the features of the $O(3)$-symmetric problem which involve more than these two branches.

The third order equations are as follows

$$\dot{Y} + a_1(\lambda - \alpha)Y + a_2(YZ)_Y + a_3\mathrm{tr}(Y^2)Y + a_4\mathrm{tr}(Z^2)Y +$$
$$+ a_5(Y^2Z)_Y + a_6(YZ^2)_Y = 0$$

$$\dot{Z} + b_1\lambda Z + b_2(Z^2 - \frac{1}{3}\mathrm{tr}(Z^2).I) + b_3(Y^2 - \frac{1}{3}\mathrm{tr}(Y^2).I) + b_4(YZ)_Z +$$
$$+ b_5\mathrm{tr}(Y^2)Z + b_6\mathrm{tr}(Z^2)Z + b_7(Y^2Z)_Z + b_8(YZ^2)_Z = 0.$$

Since there are only the two $O(2)$- and $SO(2)$-symmetric branches and $\mathrm{Fix}(O(2)) \subset \mathrm{Fix}(SO(2))$, we restrict the equations to $\mathrm{Fix}(SO(2))$. This simplifies the study of the problem in that we reduce the study of an 8-dimensional system to that of a 2-dimensional one. We are aware of the disadvantage of losing all the information in directions not in $\mathrm{Fix}(SO(2))$. So, we shall not be concerned with stability results outside $\mathrm{Fix}(SO(2))$. In this fixed-point subspace, we write the equations in coordinates by identifying $V \oplus W$ with

$$\{(y_1, y_2, y_3, z_1, z_2, z_3, z_4, z_5) : \; y_i, z_j \in \mathbb{R}\}$$

as follows

$$Y = \begin{pmatrix} 0 & y_1 & y_2 \\ -y_1 & 0 & y_3 \\ -y_2 & -y_3 & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_1 & z_3 & z_4 \\ z_3 & z_2 & z_5 \\ z_4 & z_5 & -(z_1 + z_2) \end{pmatrix}.$$

The equations in $\mathrm{Fix}(SO(2))$ are

$$\dot{y_1} + [a_1(\lambda - \alpha) + a_2 z_1 + (6a_4 + \frac{a_5}{2})z_1^2 - 2a_3 y_1^2]y_1 = 0$$
$$\dot{z_1} + [b_1\lambda - b_2 z_1 + 6b_6 z_1^2]z_1 - [\frac{1}{3}b_3 + (2b_5 + \frac{b_7}{3})z_1]y_1^2 = 0.$$

We do not prove explicitly that these equations are generic. Instead, we use the results in Armbruster and Chossat [1]. In fact, the equations in [1], p.162 are exactly the same as the ones we obtained by means of the following correspondence

$$\begin{array}{ll} a_1(\lambda - \alpha) = -\lambda_1 & b_1\lambda = -\lambda_2 \\ 2a_3 = \gamma & b_2 = c \\ 6a_4 + \frac{a_5}{2} = -\delta & 6b_6 = -d \\ a_2 = 1 & \frac{b_3}{3} = 1 \\ & 2b_5 + \frac{b_7}{3} = f - f'. \end{array}$$

Hence, every result in Fix($O(2)^-$) for the $O(3)$-symmetric problem is valid in Fix($SO(2)$) for our problem. We choose $b_1 < 0$ and $a_1 < 0$ so that the bifurcation parameters vary proportionally.

Although surprising, this coincidence between the two problems is not totally unexpected if we consider the similarities between the isotropy lattices. Using the notation in Golubitsky *et al*, [16], chapter XIII, we see that $O(3)$, $O(2) \oplus \mathbb{Z}_2^c$ and $D_2 \oplus \mathbb{Z}_2^c$ are the class (II) subgroups of $O(3)$ corresponding, respectively, to $SO(3)$, $O(2)$ and D_2. As for $O(2)^-$, it is a class (III) subgroup such that $O(2)^- \cap SO(3) = SO(2)$.

Having done this identification between the two problems, there is no bifurcation analysis to be done since we can find it in Armbruster and Chossat [1]. However, we include it here for completeness.

There are two types of equilibria: those in Fix($O(2)$), which we call type 1 and those in Fix($SO(2)$), which we call type 2. The type 1 equilibria occur on a 1-dimensional space and are defined by

$$-\lambda_2 \equiv b_1\lambda = b_2 z_1 - 6b_6 z_1^2.$$

We assume $|b_2| \ll 1$ so that there is a 'bending back' of the transcritical branch of type 1 solutions. In Fix($SO(2)$), we study the limit case $b_2 = 0$, although for b_2 small, the picture of the phase portrait does not change qualitatively. The type 2 equilibria occur in the plane $\{y_1, z_1\}$. They branch off the trivial solution and off the type 1 in a secondary bifurcation.

To study the dynamics and connections in the invariant subspaces, we make the following, physically relevant, assumptions

$$b_6 > 0 \quad \text{and} \quad a_3 < 0.$$

Armbruster and Chossat prove that for $b_2 = 0$ and either $a_1(\alpha - \lambda) < 0$ or $|a_1(\alpha - \lambda)|$ small and $-b_1\lambda + \frac{a_1(\lambda-\alpha)(2b_5+\frac{b_7}{3})}{2a_3} > 0$, there exists a heteroclinic connection in Fix($SO(2)$) between the two type 1 equilibria on the axis $y_1 = 0$. Let β_1 and β_2 be the type 1 equilibria when $z_1 < 0$ and $z_1 > 0$, respectively.

For $\lambda > 0$, the bifurcation is as follows and illustrated in Figure 5.2

(1) at $\lambda = 0$, the two type 1 equilibria bifurcate from the origin.

(2) At $a_1(\lambda - \alpha) = \sqrt{-b_1\lambda}$ ($\lambda_1 = -\sqrt{\lambda_2}$), the type two equilibria bifurcate from β_1. The type 2 equilibria are sinks until

(3) at $a_1(\lambda - \alpha) = \frac{\sqrt{-3b_1\lambda}}{2}$ ($\lambda_1 = -\frac{\sqrt{3\lambda_2}}{2}$), a Hopf bifurcation occurs creating attracting limit cycles. These limit cycles persist and grow in amplitude until

(4) at $a_1(\lambda - \alpha) = \frac{\sqrt{-b_1\lambda}}{2}$ ($\lambda_1 = -\frac{\sqrt{\lambda_2}}{2}$), they meet the stable manifold of the origin. Then, a saddle-sink connection is established between β_1 and β_2. The type 2 solutions exist as spiral sources until

(5) at $a_1(\lambda - \alpha) = 0$ they die off at the origin.

(6) At $-a_1(\lambda - \alpha) = \sqrt{-b_1\lambda}$ ($\lambda_1 = \sqrt{\lambda_2}$), there is another branch of type 2 solutions bifurcating from β_2 and the $\beta_1 - \beta_2$ connection is destroyed.

If we extend the domain to $\text{Fix}(D_2)$, although there aren't any branches with D_2-symmetry, we see connections between the β_1 and β_2 types of equilibria as in Figure 5.3.

Remark The values given for the bifurcation in the description above are not exact. The results derive from an asymptotic analysis as described in Armbruster *et al* [2] .

5.5. (1, 3, 5)-mode interaction

In this section, we consider the simultaneous interaction of all three modes. Hence,we consider a bifurcation problem defined by

$$g \equiv (g_X, g_Y, g_Z) : U \oplus V \oplus W \to U \oplus V \oplus W,$$

an $SO(3)$-equivariant function.

By Lemma 2.7 and Proposition 2.1, we can write the bifurcation equations as follows

$$\dot{X} + g_X(X, Y, Z) + a\lambda = 0$$

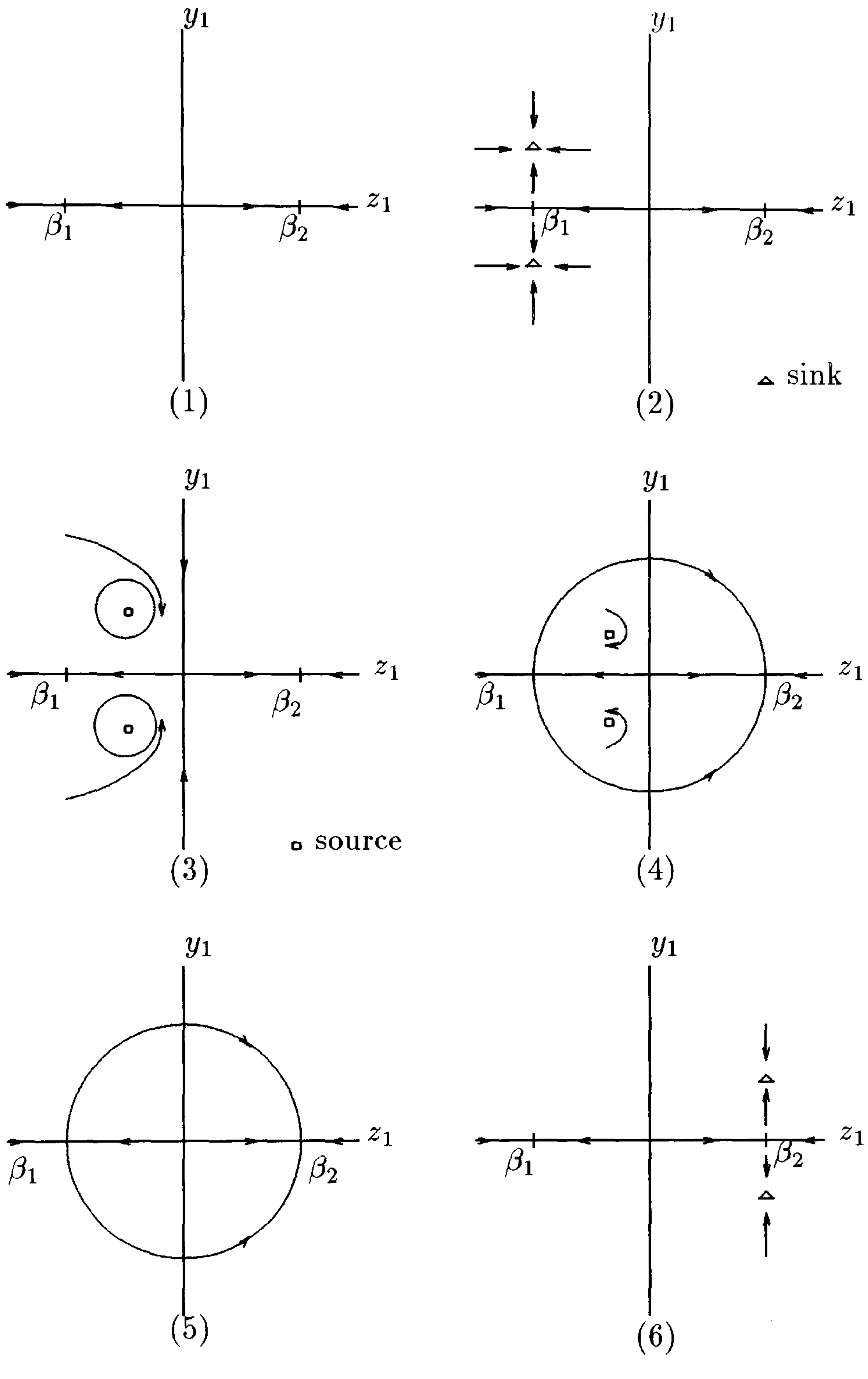


Figure 5.2.

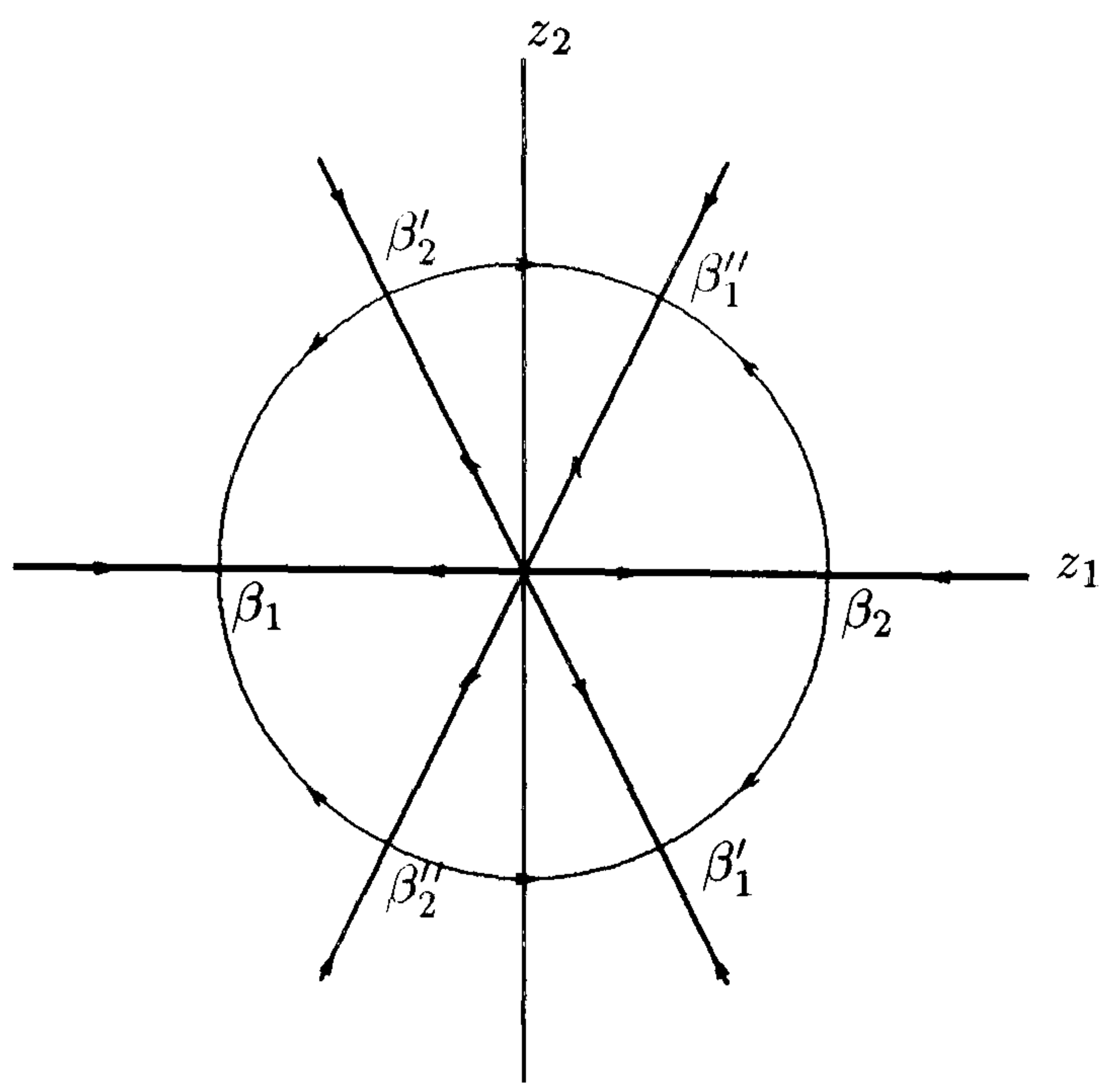


Figure 5.3.

$$\dot{Y} - \alpha Y + g_Y(X, Y, Z) = 0$$
$$\dot{Z} - \beta Z + g_Z(X, Y, Z) =,$$

where $(X - \alpha)$ and $(X - \beta)$ replace $(\lambda - \alpha)$ and λ, respectively in the equations for the $(3, 5)$-mode interaction. In matrix form, the equations are

$$\dot{X} + \lambda - X^2 + c_1 \mathrm{tr}(Y^2) + c_2 \mathrm{tr}(Z^2) + c_3 \mathrm{tr}(Y^2)X + c_4 \mathrm{tr}(Z^2)X + $$
$$+ c_5 \mathrm{tr}(Y^2 Z) + c_6 \mathrm{tr}(Z^3) = 0$$
$$\dot{Y} + a_1(X - \alpha)Y + a_2(YZ)_Y + a_3 \mathrm{tr}(Y^2)Y + a_4 \mathrm{tr}(Z^2)Y +$$
$$+ a_5(Y^2 Z)_Y + a_6(YZ^2)_Y = 0$$
$$\dot{Z} + b_1(X - \beta)Z + b_2 Z^2 + b_3 Y^2 + b_4(YZ)_Z +$$
$$+ b_5 \mathrm{tr}(Y^2)Z + b_6 \mathrm{tr}(Z^2)Z + b_7(Y^2 Z)_Z + b_8(YZ^2)_Z = 0.$$

Note that we have rescaled λ and X. Again, we write the equations in coordinates identifying $U \oplus V \oplus W$ with

$$\{(x, y_1, y_2, y_3, z_1, z_2, z_3, z_4, z_5);\ x, y_i, z_j \in \mathbb{R}\}.$$

We have $\mathrm{Fix}(SO(3)) = \mathbb{R}$ and therefore, we cannot use the Equivariant Branching Lemma. However, the isotropy lattice is the same as for the $(3, 5)$-mode interaction (see Lemma 2.8) and the similarity between the equations for the two problems guarantee that, if any non-trivial branches exist, they can only be $O(2)$- or $SO(2)$-symmetric. So, we shall consider the equations restricted to $\mathrm{Fix}(SO(2))$, which are

$$\dot{x} + \lambda - x^2 - 2c_1y_1^2 + 6c_2z_1^2 - 2c_3y_1^2x + 6c_4z_1^2x - 2c_5y_1^2z_1 - 6c_6z_1^3 = 0$$
$$\dot{y_1} + a_1(x-\alpha)y_1 + a_2y_1z_1 - 2a_3y_1^3 + 6a_4z_1^2y_1 + \frac{1}{2}a_6z_1^2y_1 = 0$$
$$\dot{z}_1 + b_1(x-\beta)z_1 - b_2z_1^2 - \frac{1}{3}b_3y_1^2 - 2b_5y_1^2z_1 + 6b_6z_1^3 - b_7y_1^2z_1 = 0.$$

The fully symmetric branch is defined by

$$\lambda = x^2.$$

Let us choose $\beta < \alpha$ so that the $O(2)$-symmetric branch bifurcates from the fully symmetric one at $x = \beta$. We know this $O(2)$-symmetric branch exists by Proposition 2.3. Then, because the $O(2)$-symmetric branch exists, we can use Proposition 2.4 to prove the existence of the $SO(2)$- symmetric mode branching off the $O(2)$-symmetric branch.

A more detailed study of this problem, would lead to tedious calculations given the number of parameters which can be varied.

Chapter 6

$(1, 3, 5)$-mode interaction with $O(3)$ symmetry

6.1. Introduction

In this chapter, we study the simultaneous interaction of the 1-, the 3- and the 5-dimensional modes under the action of the group $O(3)$. The representation of $O(3)$ is that in Armbruster and Chossat [1] and our bifurcation analysis is entirely based on this reference. In section 6.2, we use the aforementioned reference to obtain the isotropy lattice and the fixed-point subspaces, referring also to results in chapter 2. In the following section, we write the equations in Birkhoff normal form to convenient order, again using results from chapter 2. In the final section, we study the bifurcation. Given the number of parameters, we consider the possibility of existence or absence of heteroclinic connections. Our arguments are mainly of a geometric nature. We leave the complete study of this problem for anyone interested in an application with concrete values for the parameters. For convenience, in this chapter, we use the same notation as Armbruster and Chossat [1].

6.2. The group action

As in [1], we assume that $O(3)$ acts on an 8-dimensional space $V = V_1 \oplus V_2$, where each V_l is invariant by the absolutely irreducible natural representation of $O(3)$ of dimension $2l+1$ (cf. Golubitsky *et al* [16] for a clear explanation of this action). To obtain a representation for the $(1,3,5)$-mode interaction, we simply consider the action of $O(3)$ on $\mathbb{R} \times V$ as follows

$$\gamma.(x,v) = (x, \gamma v) \ \ \forall \gamma \in O(3) \ \ \forall (x,v) \in \mathbb{R} \times V,$$

where the action on V is the one mentioned above. We may realize V as the span of spherical harmonics of order 1 and of order 2. In this space, we define coordinates

$$\underline{x} = (x_{-1}, x_0, x_1)$$

for the 3-dimensional mode and

$$\underline{y} = (y_{-2}, y_{-1}, y_0, y_1, y_2)$$

for the 5-dimensional mode, as in Armbruster and Chossat, p.158 [1]. Using Lemma 2.8, we know that the isotropy lattice remains the same and is as in Figure 6.1.

If x denotes the coordinate on $\mathbb{R}$, the trivial mode, then the fixed-point spaces associated to the isotropy groups are as indicated in Figure 6.1. These are just the fixed-point spaces found by Armbruster and Chossat in [1] with the extra dimension corresponding to the trivial mode.

6.3. The equations

To write the equations in Birkhoff normal form, we use Lemma 2.7 and equations (4) and (5) in [1] , p.159. Let

$$\dot{X} = F(X, \lambda, \alpha_1, \alpha_2),$$

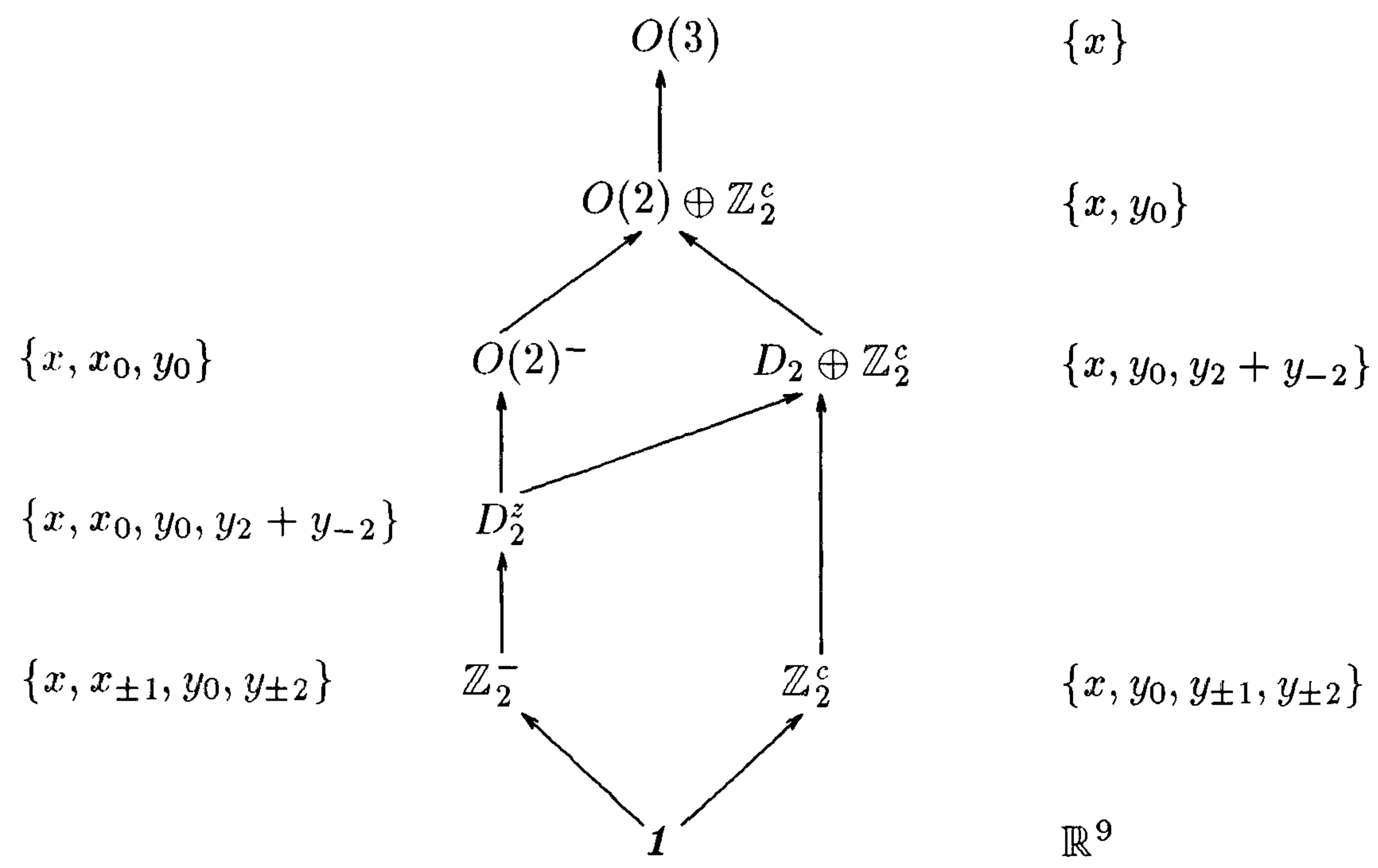


Figure 6.1.

where $X = (x, x_{-1}, x_0, x_1, y_{-2}, y_{-1}, y_0, y_1, y_2)$, be $O(3)$-equivariant under the action described in the previous section. Let $F \equiv (f_0, g_j, h_k)$ for $j = -1, 0, 1$ and $k = -2, -1, 0, 1, 2$. Up to third order and using the notation in [1] the bifurcation equations are defined by

$$
\begin{aligned}
f_0 &= a\lambda + a_1 x^2 + a_2 \|\underline{x}\|^2 + a_3 \|\underline{y}\|^2 + x(a_4 \|\underline{x}\|^2 + a_5 \|\underline{y}\|^2) + a_6 x^3 \\
g_j &= x_j[(x - \alpha_1) + \gamma \|\underline{x}\|^2 + \delta \|\underline{y}\|^2] + \beta K_j(\underline{x}, \underline{y}) + \delta' \Gamma_i(\underline{x}, \underline{y}) \\
h_k &= y_k[(x - \alpha_2) + f \|\underline{x}\|^2 + d \|\underline{y}\|^2] + b B_k(\underline{x}, \underline{x}) + c C_k(\underline{y}, \underline{y}) + f' D_k(\underline{x}, \underline{y}),
\end{aligned}
$$

where

$$
\begin{aligned}
K_0(\underline{x}, \underline{y}) &= x_0 y_0 - \frac{\sqrt{3}}{2}(x_1 y_{-1} + x_{-1} y_1), \\
K_1(\underline{x}, \underline{y}) &= \frac{1}{2}(-x_1 y_0 + \sqrt{3} x_0 y_1 - \sqrt{6} x_{-1} y_2); \\
B_0(\underline{x}, \underline{x}) &= x_0^2 + x_1 x_{-1}, \\
B_1(\underline{x}, \underline{x}) &= \sqrt{3} x_0 x_1, \\
B_2(\underline{x}, \underline{x}) &= \sqrt{\frac{3}{2}} x_1^2; \\
C_0(\underline{y}, \underline{y}) &= y_0^2 - y_1 y_{-1} - 2 y_2 y_{-2}, \\
C_1(\underline{y}, \underline{y}) &= y_0 y_1 - \sqrt{6} y_2 y_{-1},
\end{aligned}
$$

$$C_2(\underline{y}, \underline{y}) = -2y_0y_2 + \frac{\sqrt{6}}{2}y_1^2;$$

$$\Gamma_0(\underline{x}, \underline{y}) = \frac{3}{2}x_0y_0^2 - 2x_0y_1y_{-1} - x_0y_2y_{-2} - \frac{\sqrt{3}}{2}(x_1y_0y_{-1} + x_{-1}y_0y_1) + \frac{3\sqrt{2}}{2}(x_1y_1y_2 + x_{-1}y_0y_2),$$

$$\Gamma_1(\underline{x}, \underline{y}) = \frac{\sqrt{3}}{2}x_0y_0y_1 - \frac{3\sqrt{2}}{2}x_0y_2y_{-1} - \frac{1}{2}x_1y_1y_{-1} + 2x_1y_2y_{-2} + \sqrt{6}x_{-1}y_0y_2 - \frac{3}{2}x_{-1}y_1^2;$$

$$D_0(\underline{x}, \underline{y}) = -x_0^2y_0 - 4x_1x_{-1}y_0 + \sqrt{6}(x_{-1}^2y_2 + x_1^2y_{-2}) + \sqrt{3}(x_0x_1y_{-1} + x_0x_{-1}y_1),$$

$$D_1(\underline{x}, \underline{y}) = -\sqrt{3}x_0x_1y_0 - 3x_1x_{-1}y_1 + 3x_1^2y_{-1} + 3\sqrt{2}x_0x_{-1}y_2,$$

$$D_2(\underline{x}, \underline{y}) = \sqrt{6}x_1^2 - 3\sqrt{2}x_0x_1y_1 + 3x_0^2y_2;$$

$$||\underline{x}||^2 = x_0^2 - 2x_1x_{-1},$$

$$||\underline{y}||^2 = y_0^2 - 2y_1y_{-1} + 2y_2y_{-2}$$

and $g_{-j} = (-1)^j\bar{g}_j$, $h_{-k} = (-1)^k\bar{h}_k$. Note that all parameters are real and

$$\lambda_1 = x - \alpha_1 \text{ and } \lambda_2 = x - \alpha_2$$

correspond to unfolding parameters.

We do not however work with the full equations, but restrict them to a fixed-point subspace where interesting behaviour is likely to occur, namely,

$$\text{Fix}\ (D_2^z) = \{x, x_0, y_0, y_2 + y_{-2}\}.$$

Define y_{2r} =Re (y_2). Then, up to third order, the bifurcation equations are

$$\dot{x} = a\lambda + a_1x^2 + a_2x_0^2 + a_3(y_0^2 + y_{2r}^2) + x(a_4x_0^2 + a_5(y_0^2 + y_{2r}^2)) + a_6x^3$$

$$\dot{x_0} = x_0[(x - \alpha_1) + \gamma x_0^2 - y_0 + (\delta + 3\delta')y_0^2 + (\frac{1}{2}\delta - 2\delta')y_{2r}^2]$$

$$\dot{y_0} = y_0[(x - \alpha_2) + d(y_0^2 + y_{2r}^2)] + c(y_0^2 - y_{2r}^2) + x_0^2[1 + (f - f')y_0]$$

$$\dot{y_{2r}} = y_{2r}(x - \alpha_2) + d(y_0^2 + y_{2r}^2)] - 2cy_0y_{2r} + (f + 3f')y_{2r}x_0^2.$$

6.4. Bifurcations

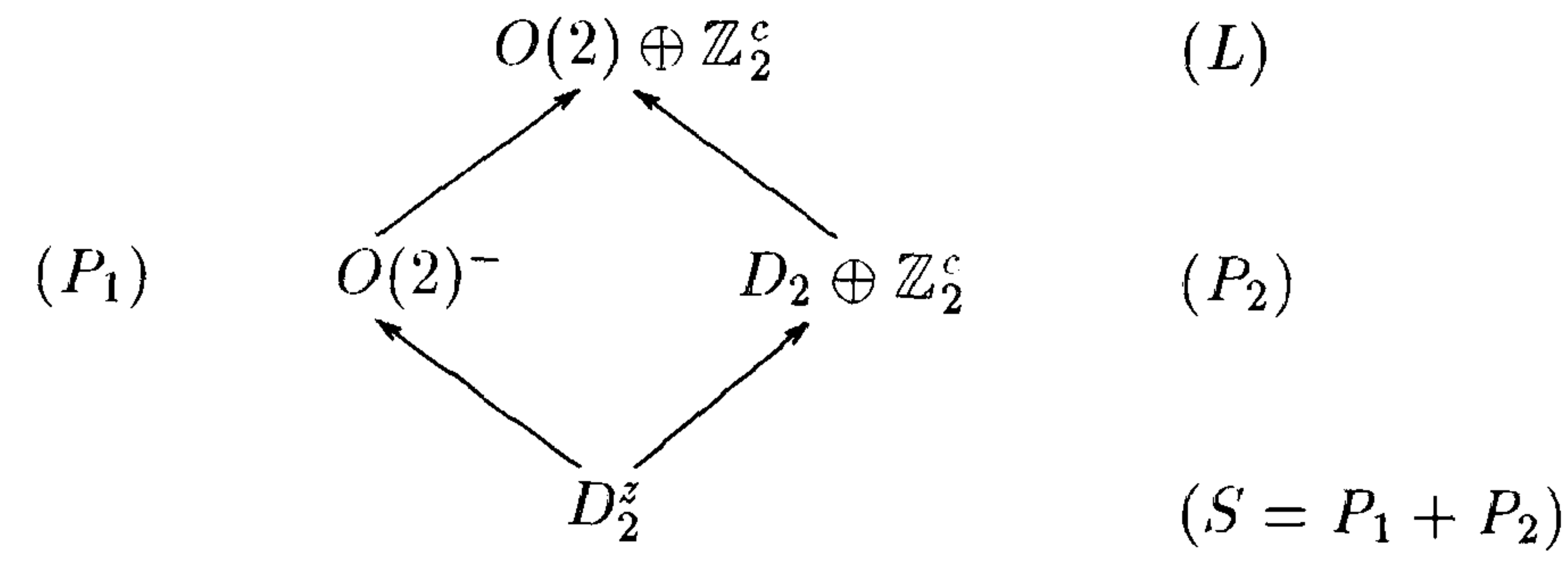


Figure 6.2.

In this final section, we proceed with the bifurcation analysis of the problem using the equations in $\mathrm{Fix}(D_2^z)$ as in the previous section. We may choose

$$a = +1,$$

without loss of generality, so that the fully symmetric trivial mode branch is stable in $\mathrm{Fix}(O(3))$ for values of x such that

$$\begin{cases} x > 0 \\ x > -\frac{2a_1}{3a_6} \end{cases} \text{or} \begin{cases} x < 0 \\ x < -\frac{2a_1}{3a_6} \end{cases}$$

We also note that the origin is not a trivial solution; instead, there is a whole branch of trivial equilibria

$$\lambda \equiv \lambda(x).$$

Lemma 2.9 guarantees that all the branches of equilibria that existed in the $(3,5)$-mode interaction remain present with an added dimension, that of the trivial mode. We shall now see how this, apparently small detail, can change bifurcation features which are not in strict connection with the fixed-point space, namely, the existence or absence of heteroclinic connections and secondary Hopf bifurcations.

Armbruster and Chossat [1], have proved that considering the section of the isotropy lattice represented in Figure 6.2 (in parentheses we denote the respective fixed-point spaces), there are several types of heteroclinic connections.

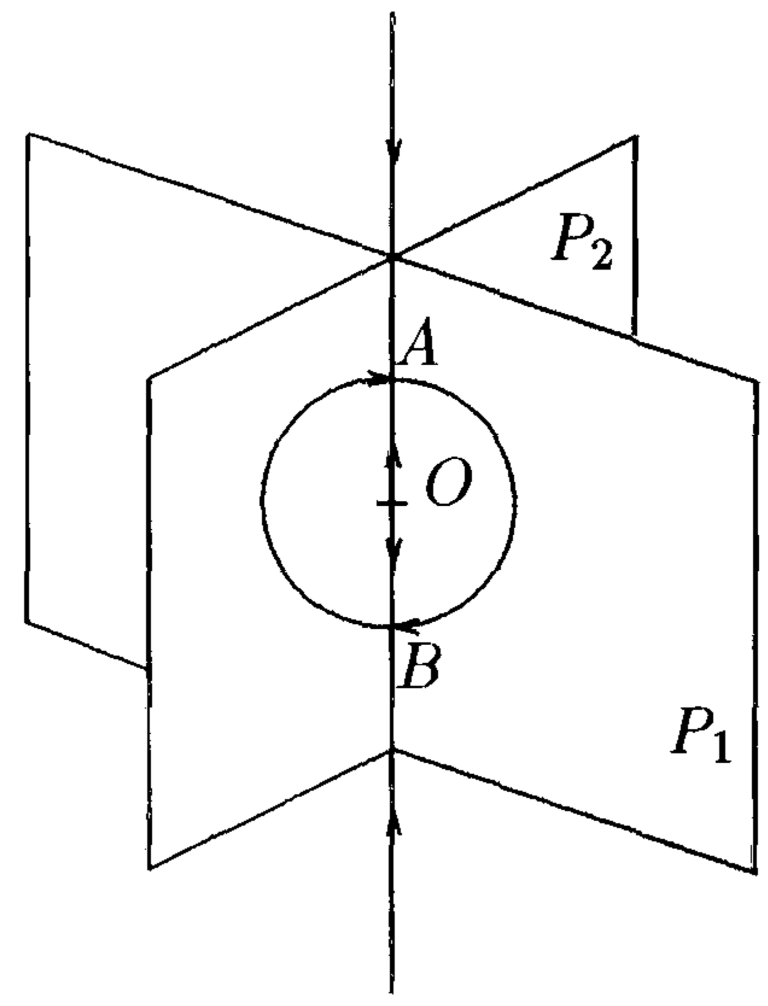
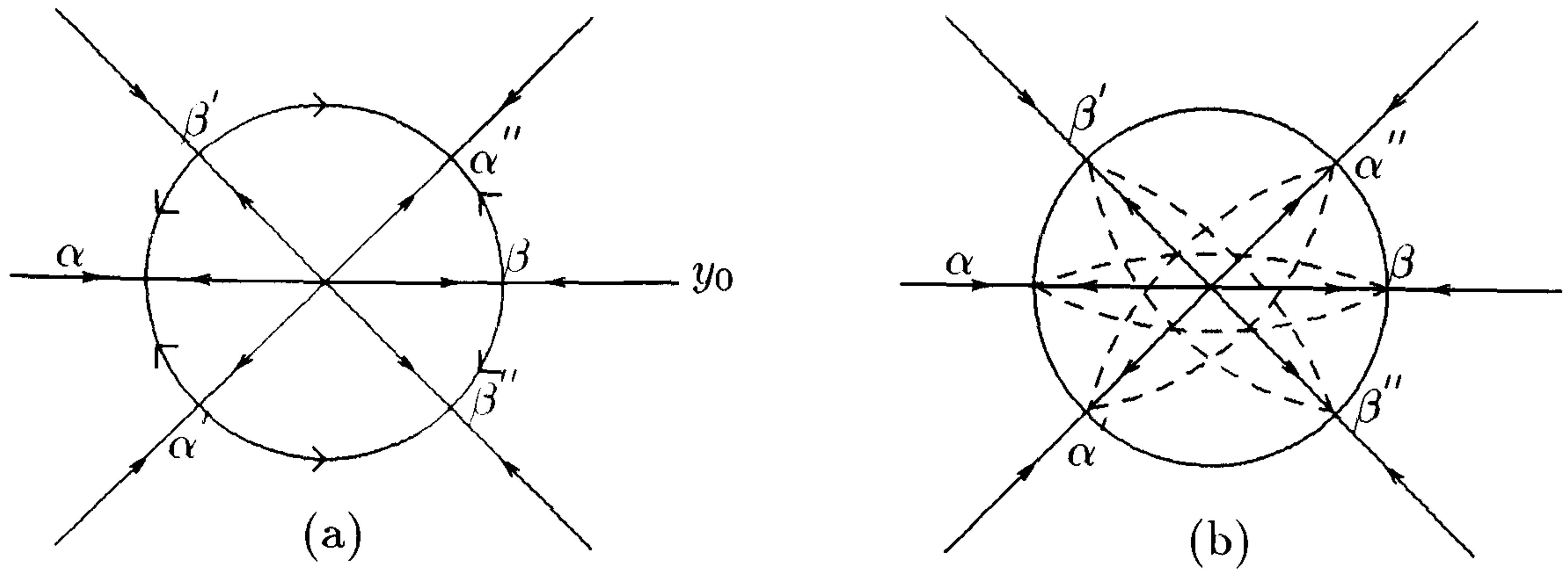


Figure 6.3.

A heteroclinic connection can occur, for example in S, as in Figure 6.4 (cf. Melbourne *et al* [21], figure 2.1).

A heteroclinic connection of the type indicated in Figure 6.4 occurs for the $(3,5)$-mode interaction between the two equilibria (α and β) in L. It is as follows: in P_1, there is the line L and its two conjugates L' and L''. If $\lambda_2 > 0$, $d < 0$ and $c < 0$, there are two equilibria on L ($\alpha < 0$ and $\beta > 0$) and their conjugates in L' and L''. The phase portrait is as in Figure 6.4 (a). Then, we consider P_2 which intersects P_1 along L and its conjugates which intersect P_1 along L' and L''. In $S = P_1 \oplus P_2$, there are heteroclinic connections between α and β, involving also their conjugates as indicated in Figure 6.4 (b). This heteroclinic cycle is present when $\lambda_2 > 0$ and either $\lambda_1 < 0$ (close to 0) or $0 < \lambda_1 < \sqrt{\lambda_2}$.

Consider now the $(1,3,5)$-mode interaction problem. All the fixed-point spaces mentioned above exist but now have an extra dimension, x, and the equilibria an extra coordinate. Suppose that the phase portrait in the x-direction is attracted to the fixed-point spaces which already existed for the $(3,5)$-mode interaction problem. In the case of Figure 6.4, for example, trajectories in the x-direction would be attracted to the paper. Then, it is easy to believe, although harder to prove, that the connections persist in exactly



Dotted lines indicate trajectories in P_2 and their conjugates. The flow is from α to β.

Figure 6.4.

the same way. Note that the stability in the x-direction does not depend on the parameters that determine the existence of the connections.

However, if the fixed-point space containing the heteroclinic connection is unstable in the x-direction, the connection could be destroyed. In fact, trajectories which are bounded in the fixed-point space restricted to the two original modes, are now unbounded and actually attracted away from the fixed-point space.

The most interesting feature of this mode interaction is the possibility of the simultaneous occurrence of heteroclinic connections and limit cycles. We illustrate this using Figure 6.4 (a) since it has a suitably low dimension. For higher dimensions, the explanation is the same but the visualization much harder. Suppose then that the fixed-point space P_1 is stable in the x-direction. It is clear that, for the $(1,3,5)$-mode interaction, both equilibria α and β are in a mixed-mode branch. According to Proposition 2.5, there are parameter values for which a Hopf bifurcation occurs, say at α, originating a limit cycle. This limit cycle, although intersecting P_1 twice, does not generically intersect the heteroclinic connection. This can be seen in Figure 6.5. The stability of the limit cycle is determined by the parameters in the x-direction and hence, does not affect the existence of the heteroclinic connection.

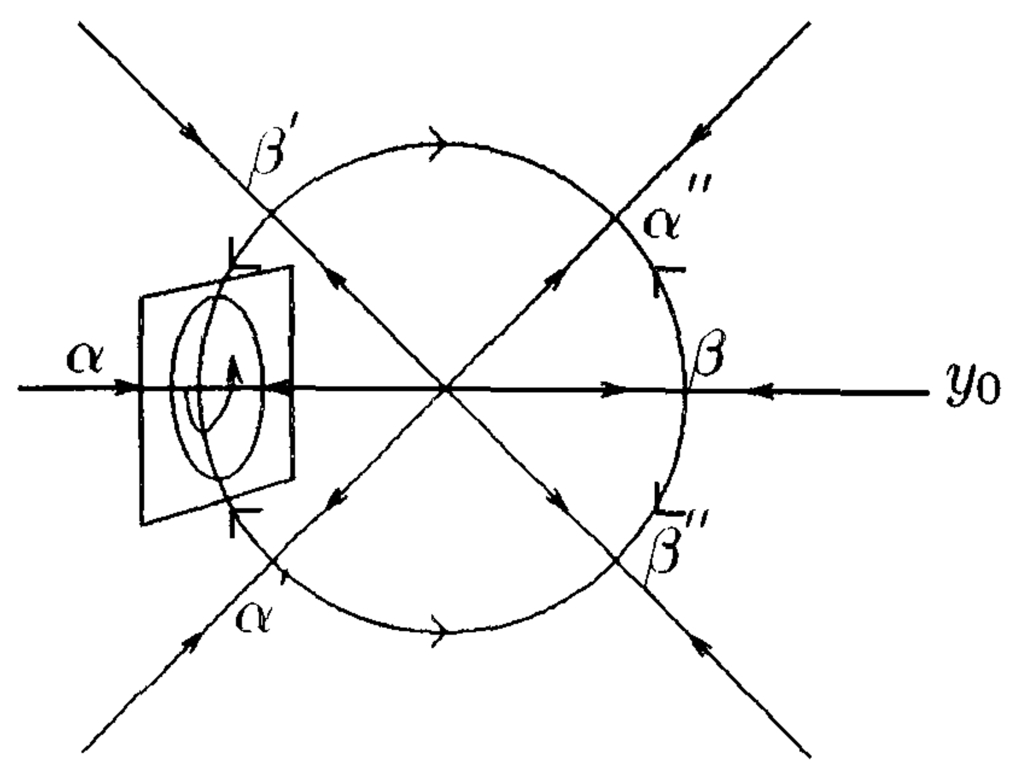


Figure 6.5.

We are aware of the fact that this study of the $(1,3,5)$-mode interaction is intuitive rather than analytical but the number of parameters involved does not even allow an appropriate numerical simulation, Maple [9] being also inadequate. Therefore, we chose to simply point out the different possibilities for the dynamics and leave it to anyone with a concrete application, i.e., concrete values for most parameters, to find regions in the parameter space where each event takes place.

Appendix A

This appendix gives a brief description of KAOS. KAOS is a program developped by J. Guckenheimer and S. Kim. It can be run in any Sun3 or Sun4 workstation. Given a set of differential equations defining a dynamical system

$$\dot{x} = f(x, \lambda),$$

the aim of KAOS is to solve them. Options in KAOS allow us to calculate equilibria and to plot trajectories in a 2-dimensional phase-space, as well as easily varying parameter values.

Our procedure in using this program has been as follows:

- after implementing the equations, we look for equilibria. The stability of the equilibria is displayed on the screen.
- we plot trajectories we consider relevant, for example, to determine stable and unstable manifolds of saddle points.
- we vary the bifurcation parameter and start again.

This way, it is easy to notice the change of stability of equilibria and the existence of limit cycles. The stability of the equilibria is indicated by three different symbols in the out-put. Crosses denote saddle points, squares denote sources and triangles, sinks.

We include an example of a dynamical system implemented on KAOS. All others differ only in the equations and parameter values.

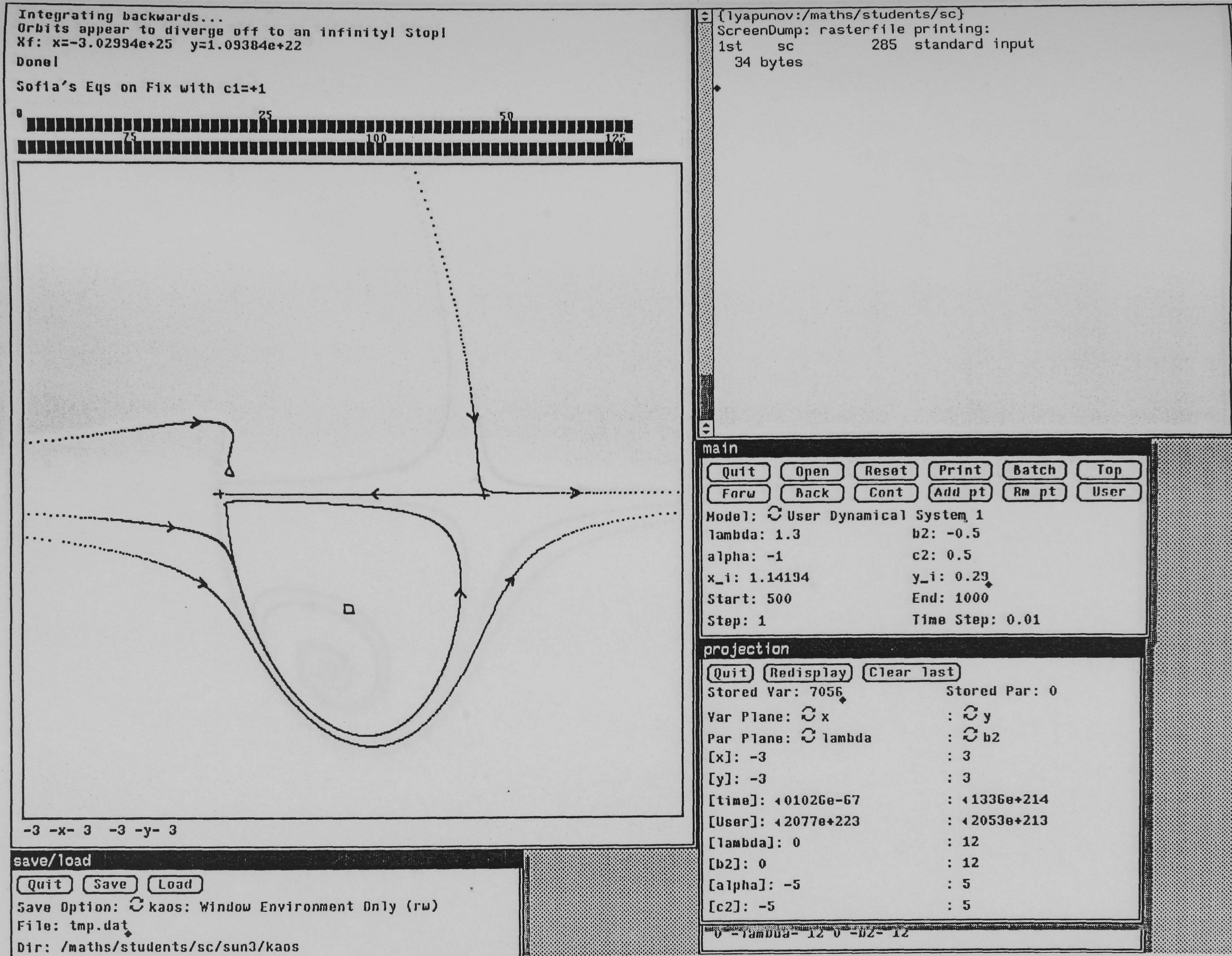
```
/*
Initialize all parameters for a given dynamical system to be installed
Parameters are assigned to the default values before this program is called.
---------------------------------------------------------------------------
This is a GENERIC subroutine. If you want, you can change
the name string "userds1" to a proper one globally in this program
but then you need to change the same strings in the header file defining
the current class of dynamical systems.
*/

/*
Example 1: vector field with no periodic variable
*/

#include "model.h"

int userds1_init()
{

        /* title label */
        title_label = "Sofia's Eqs on Fix with c1=+1";

        mapping_on = 0;
        inverse_on = 0;
        fderiv_on = 0;
        enable_polar = 0;
        enable_period = 0;

        /* phase space dimension */
        var_dim = 2;
        /* parameter space dimension */
        param_dim = 4;
        /* function space dimension */
        func_dim = 2;

        (void) malloc_init();

        /* primary phase space variable label (DIM=var_dim)*/
        var_label[0] = "x";
        var_label[1] = "y";
        /* parameter variable label (DIM=param_dim)*/
        param_label[0] = "lambda";
        param_label[1] = "b2";
        param_label[2] = "alpha";
        param_label[3] = "c2";
        /* function variable label (DIM=func_dim)*/
        func_label[0] = "time";
        func_label[1] = "User";

        /* starting parameter values (DIM=param_dim)*/
        param[0] = 1.2;
        param[1] = -.5;
        param[2] = -1;
        param[3] = .5;

        /* starting primary phase space variable values (DIM=param_dim)*/
        var_i[0] = .3;
        var_i[1] = .4;

        /* starting bounds of parameter window box */
        param_min[0]= 0; param_max[0]= 12;
        param_min[1]= 0; param_max[1]= 12;

        /* starting bounds of primary phase space window box */
        var_min[0]= -3; var_max[0]= 3;
        var_min[1]= -3; var_max[1]= 3;
```

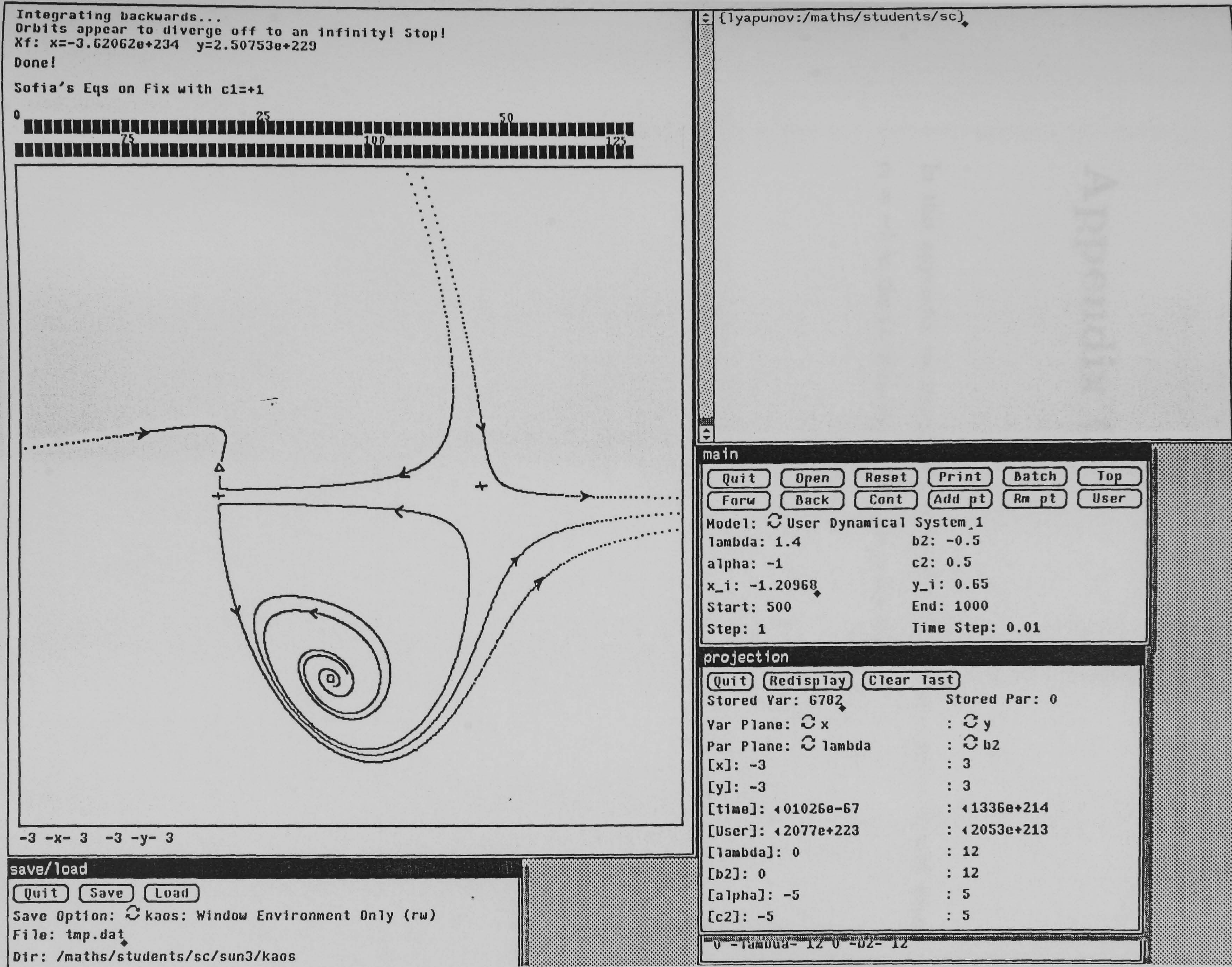
```
        /* dynamical system and function pointer assignments */
        f_p = userds1_f;
        func_p = userds1_func;
}
/*
definition of user dynamical system
*/

int userds1_f(f,index,x,p,t,dim)
int index,dim;
double f[],x[],p[],t;
{
        f[0] = x[0]*x[0]-p[0]-p[1]*x[1]*x[1];
        f[1] = (p[2]-x[0])*x[1]-p[3]*x[1]*x[1];
}
/*
user function subroutine
*/

int userds1_func(f,x,p,t,dim)
double f[],x[],p[],t;
int dim;
{
        int s,k;
        s = t/p[3];
        k = s%4;
        f[0] = t;
        f[1] = x[k] + k*p[4];
}
```

Appendix B

In this appendix we include the out-put obtained by running KAOS when $c_1 = +1$ in the $(1, 5)$-mode interaction problem.

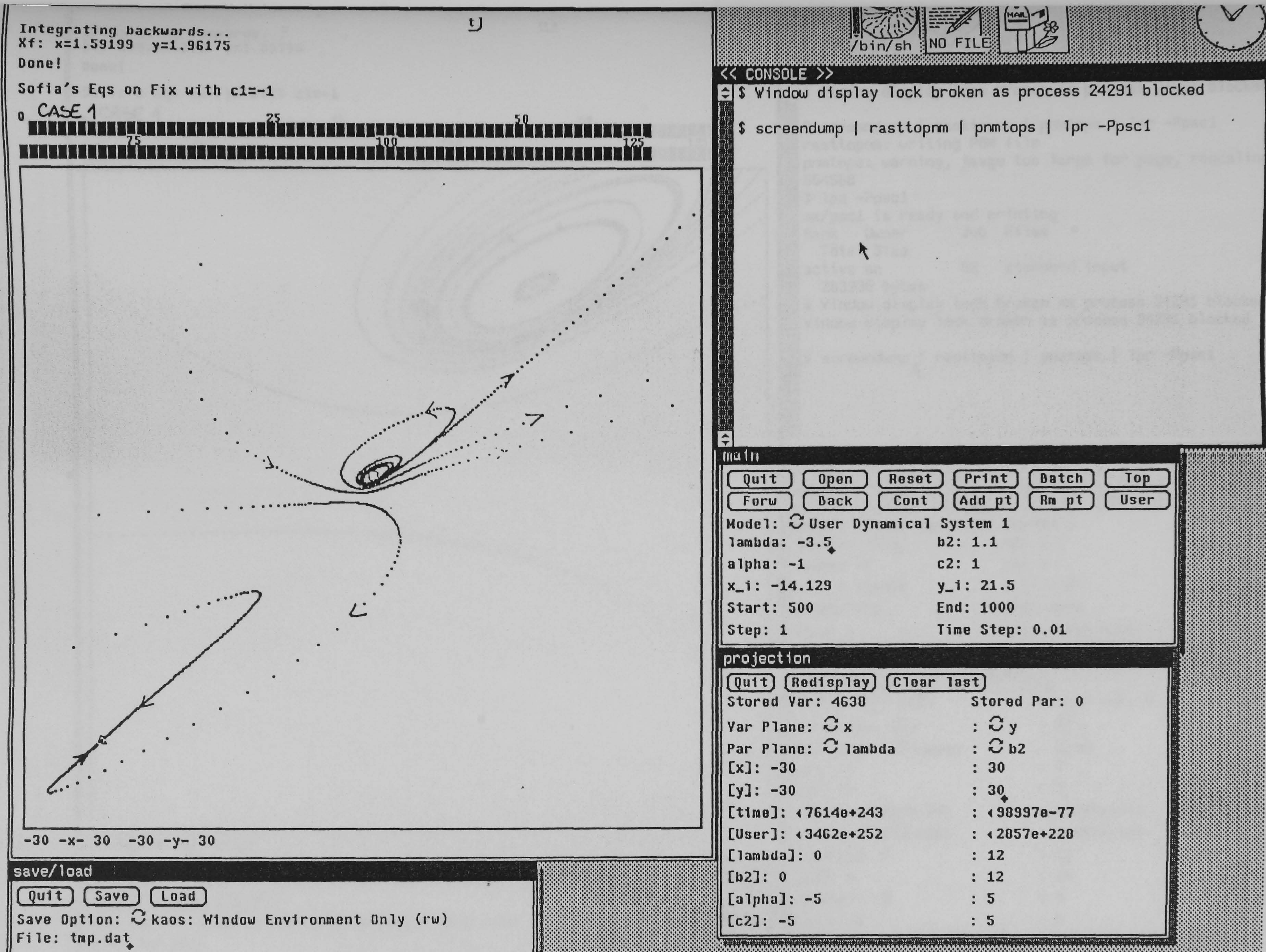

Integrating backwards...
Orbits appear to diverge off to an infinity! Stop!
Xf: x=-7.71544e+102 y=-5.51240e+97
Done!
Sofia's Eqs on Fix with c1=+1
0 25 50 75 100 125
-3 -x- 3 -3 -y- 3
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
Dir: /maths/students/sc/sun3/kaos
ScreenDump: rasterfile printing:
1st sc 283 standard input
113262 bytes
{lyapunov:/maths/students/sc}
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System 1
lambda: 0.9 b2: -0.5
alpha: -1 c2: 0.5
x_1: 0.967742 y_1: -0.3
Start: 500 End: 1000
Step: 1 Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 6459 Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -3 : 3
[y]: -3 : 3
[time]: ◂01026e-67 : ◂1336e+214
[User]: ◂2077e+223 : ◂2053e+213
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5

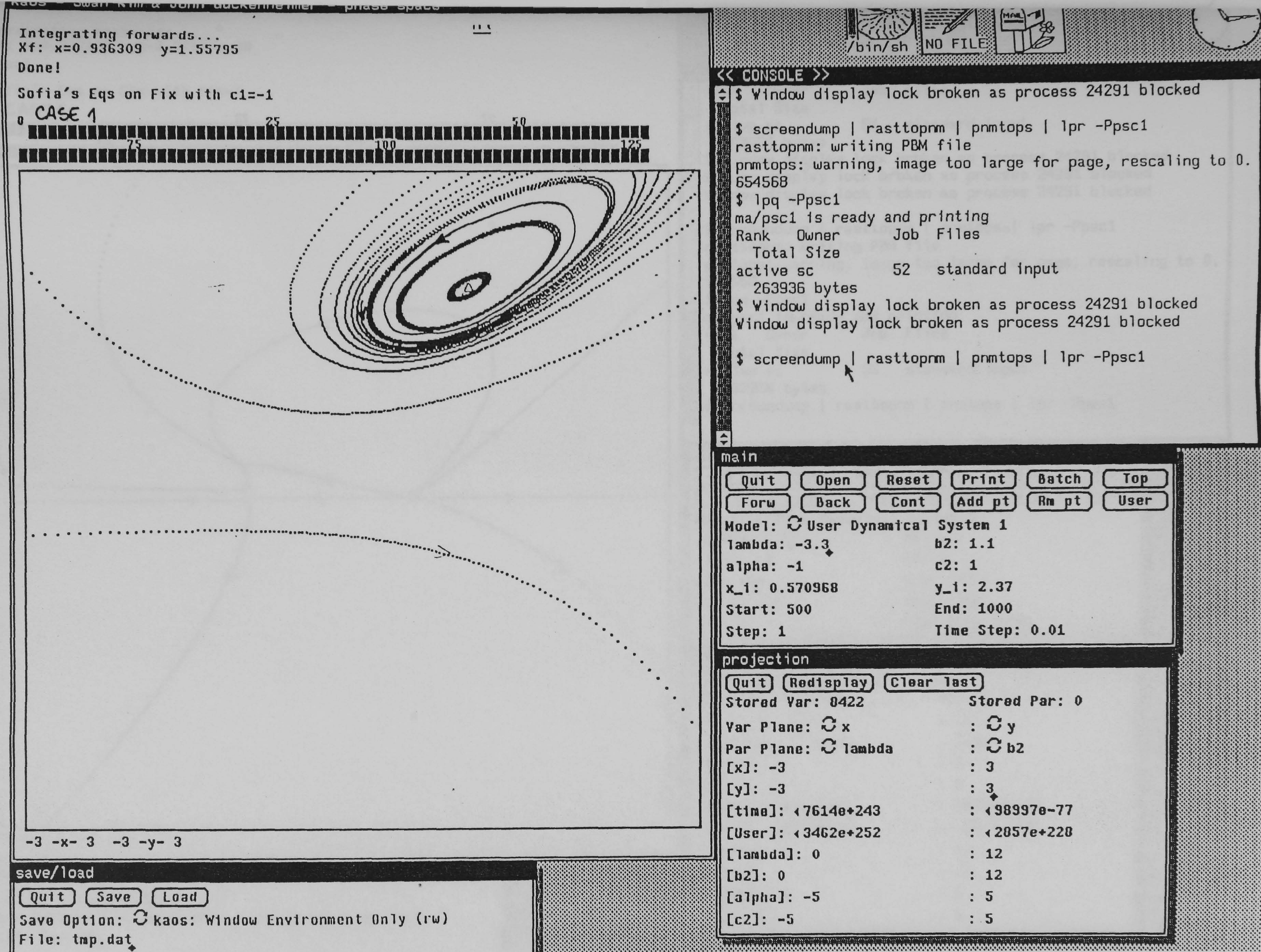

Integrating backwards...
Orbits appear to diverge off to an infinity! Stop!
Xf: x=-3.02994e+25 y=1.09384e+22
Done!
Sofia's Eqs on Fix with c1=+1
0 25 50
75 100 125
-3 -x- 3 -3 -y- 3
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
Dir: /maths/students/sc/sun3/kaos
{lyapunov:/maths/students/sc}
ScreenDump: rasterfile printing:
1st sc 285 standard input
34 bytes
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System 1
lambda: 1.3 b2: -0.5
alpha: -1 c2: 0.5
x_i: 1.14194 y_i: 0.23
Start: 500 End: 1000
Step: 1 Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 7056 Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -3 : 3
[y]: -3 : 3
[time]: 40102Ge-67 : 41336e+214
[User]: 42077e+223 : 42053e+213
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5
0 -lambda- 12 0 -b2- 12

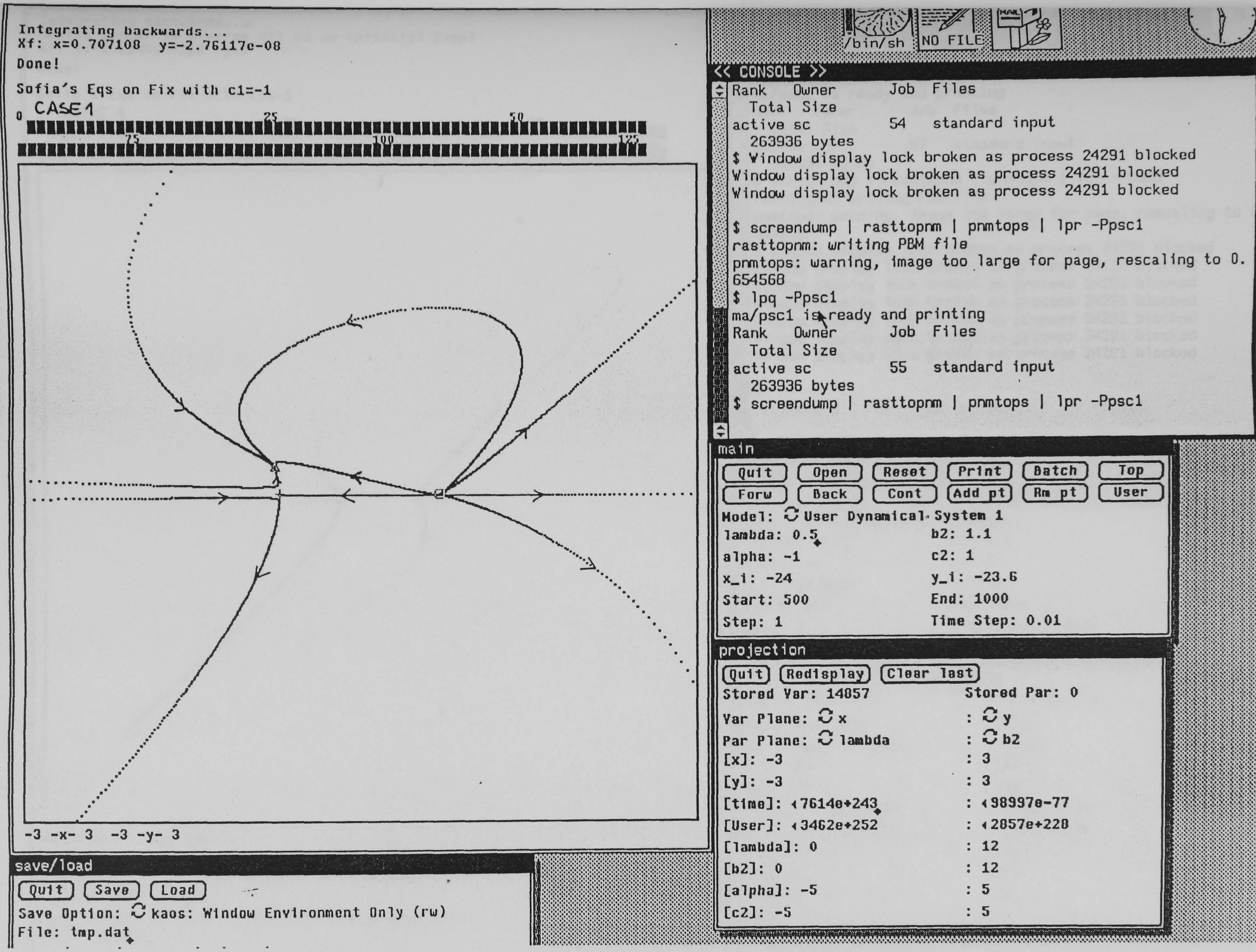

Integrating backwards...
Orbits appear to diverge off to an infinity! Stop!
Xf: x=-3.62062e+234 y=2.50753e+229
Done!
Sofia's Eqs on Fix with c1=+1
0 25 50
75 100 125
-3 -x- 3 -3 -y- 3
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
Dir: /maths/students/sc/sun3/kaos
{lyapunov:/maths/students/sc}
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System_1
lambda: 1.4 b2: -0.5
alpha: -1 c2: 0.5
x_i: -1.20968 y_i: 0.65
Start: 500 End: 1000
Step: 1 Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 6702 Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -3 : 3
[y]: -3 : 3
[time]: 401026e-67 : 41336e+214
[User]: 42077e+223 : 42053e+213
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5
0 -lambda- 12 0 -b2- 12

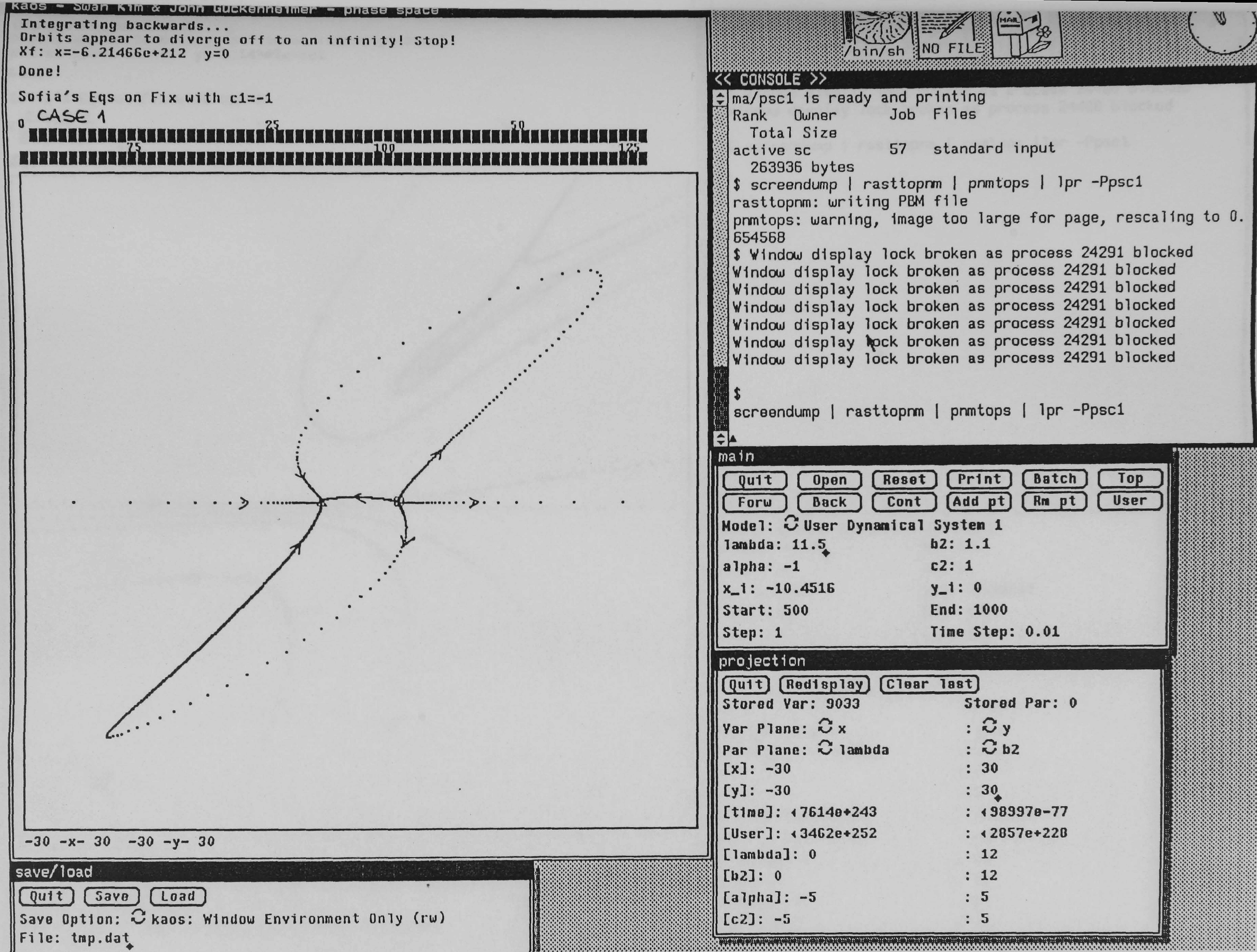
Appendix C

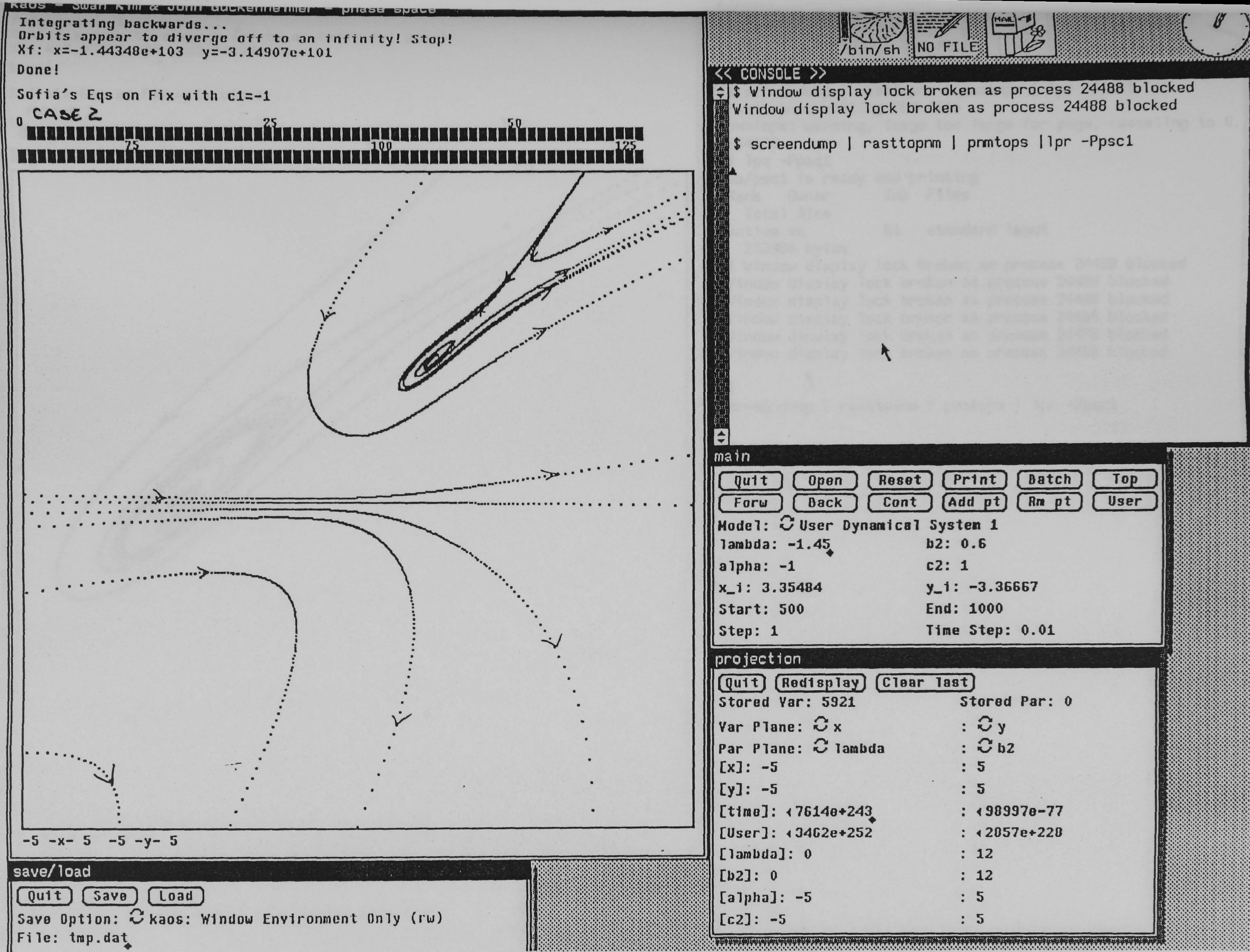
In this appendix we include the out-put obtained by running KAOS when $c_1 = -1$ in the $(1, 5)$-mode interaction problem.

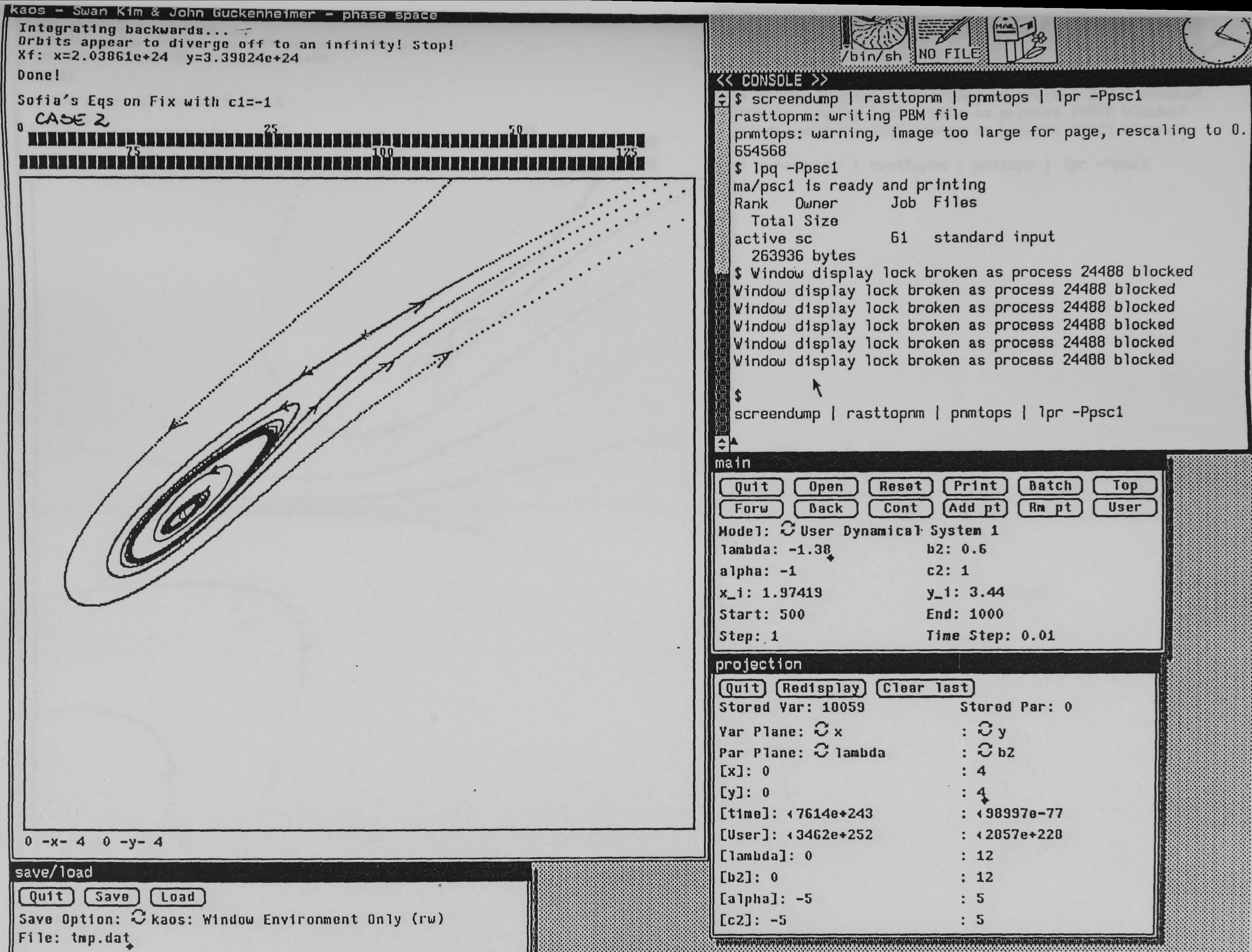

Integrating backwards...
Xf: x=1.59199 y=1.96175
Done!
Sofia's Eqs on Fix with c1=-1
CASE 1
-30 -x- 30 -30 -y- 30
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
/bin/sh
NO FILE
<< CONSOLE >>
$ Window display lock broken as process 24291 blocked
$ screendump | rasttoprm | pnmtops | lpr -Ppsc1
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System 1
lambda: -3.5 b2: 1.1
alpha: -1 c2: 1
x_i: -14.129 y_i: 21.5
Start: 500 End: 1000
Step: 1 Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 4630 Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -30 : 30
[y]: -30 : 30
[time]: ‹7614e+243 : ‹98997e-77
[User]: ‹3462e+252 : ‹2057e+228
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5

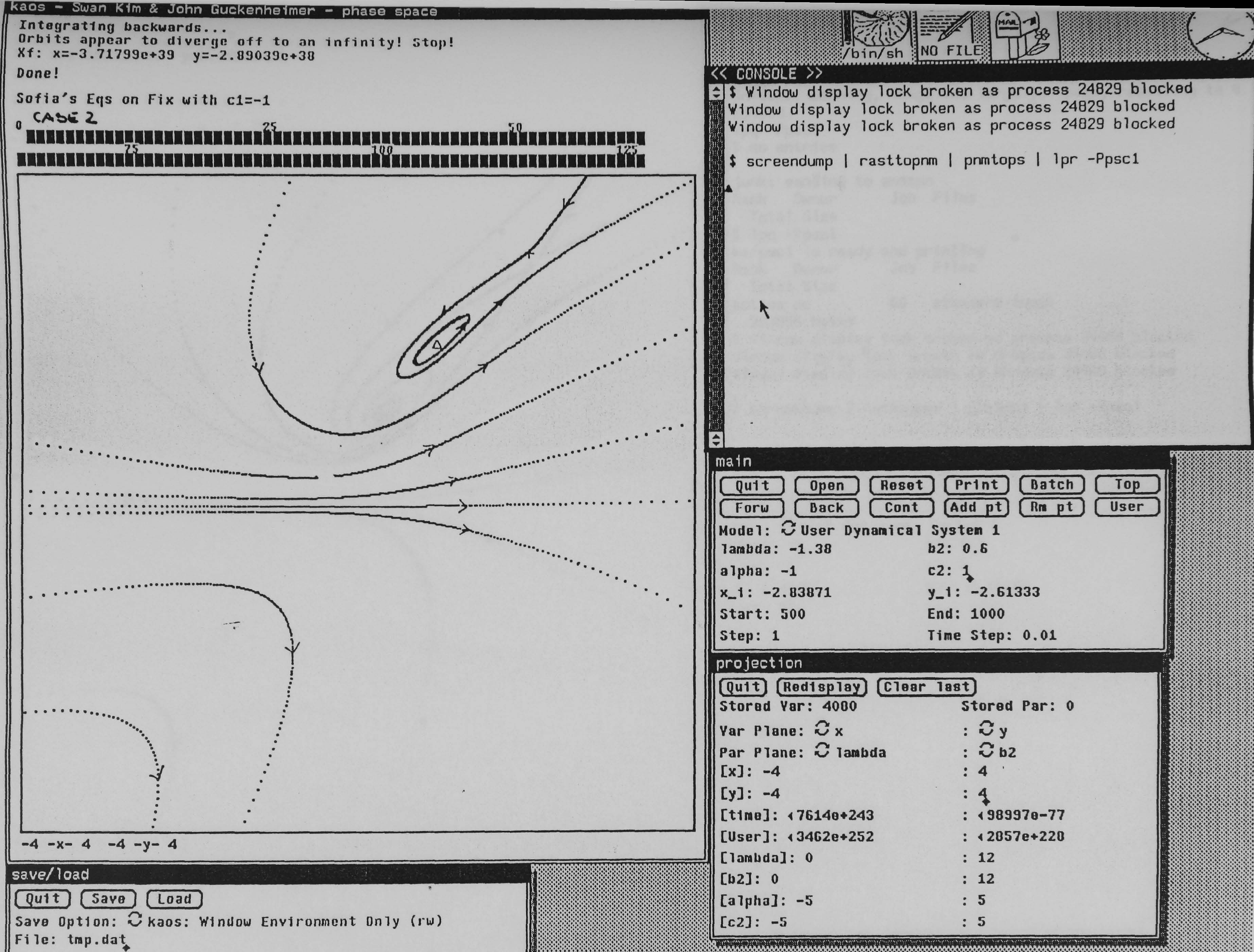

Integrating forwards...
Xf: x=0.936309 y=1.55795
Done!
Sofia's Eqs on Fix with c1=-1
CASE 1
0
25
50
75
100
125
-3 -x- 3 -3 -y- 3
save/load
Quit
Save
Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
/bin/sh
NO FILE
<< CONSOLE >>
$ Window display lock broken as process 24291 blocked
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
rasttopnm: writing PBM file
pnmtops: warning, image too large for page, rescaling to 0.
654568
$ lpq -Ppsc1
ma/psc1 is ready and printing
Rank Owner Job Files
Total Size
active sc 52 standard input
263936 bytes
$ Window display lock broken as process 24291 blocked
Window display lock broken as process 24291 blocked
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
main
Quit
Open
Reset
Print
Batch
Top
Forw
Back
Cont
Add pt
Rm pt
User
Model: User Dynamical System 1
lambda: -3.3
b2: 1.1
alpha: -1
c2: 1
x_i: 0.570968
y_i: 2.37
Start: 500
End: 1000
Step: 1
Time Step: 0.01
projection
Quit
Redisplay
Clear last
Stored Var: 0422
Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -3 : 3
[y]: -3 : 3
[time]: ‹7614e+243 : ‹98997e-77
[User]: ‹3462e+252 : ‹2857e+228
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5

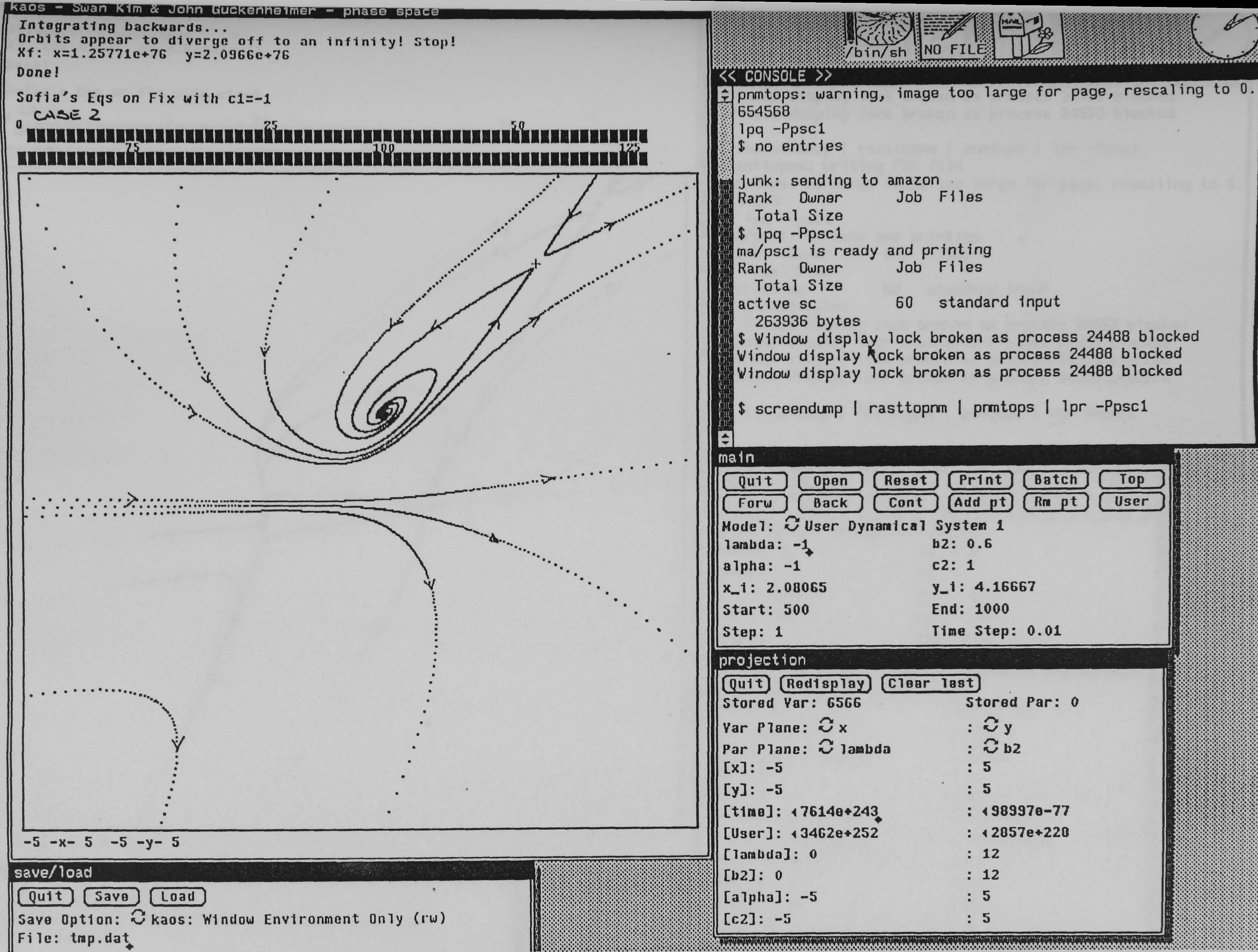

Integrating backwards...
Xf: x=0.707108 y=-2.76117e-08
Done!
Sofia's Eqs on Fix with c1=-1
CASE 1
0 25 50
75 100 125
-3 -x- 3 -3 -y- 3
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
/bin/sh NO FILE
<< CONSOLE >>
Rank Owner Job Files
Total Size
active sc 54 standard input
263936 bytes
$ Window display lock broken as process 24291 blocked
Window display lock broken as process 24291 blocked
Window display lock broken as process 24291 blocked
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
rasttopnm: writing PBM file
pnmtops: warning, image too large for page, rescaling to 0.
654568
$ lpq -Ppsc1
ma/psc1 is ready and printing
Rank Owner Job Files
Total Size
active sc 55 standard input
263936 bytes
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System 1
lambda: 0.5 b2: 1.1
alpha: -1 c2: 1
x_i: -24 y_i: -23.6
Start: 500 End: 1000
Step: 1 Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 14857 Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -3 : 3
[y]: -3 : 3
[time]: ‹7614e+243 : ‹98997e-77
[User]: ‹3462e+252 : ‹2857e+228
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5

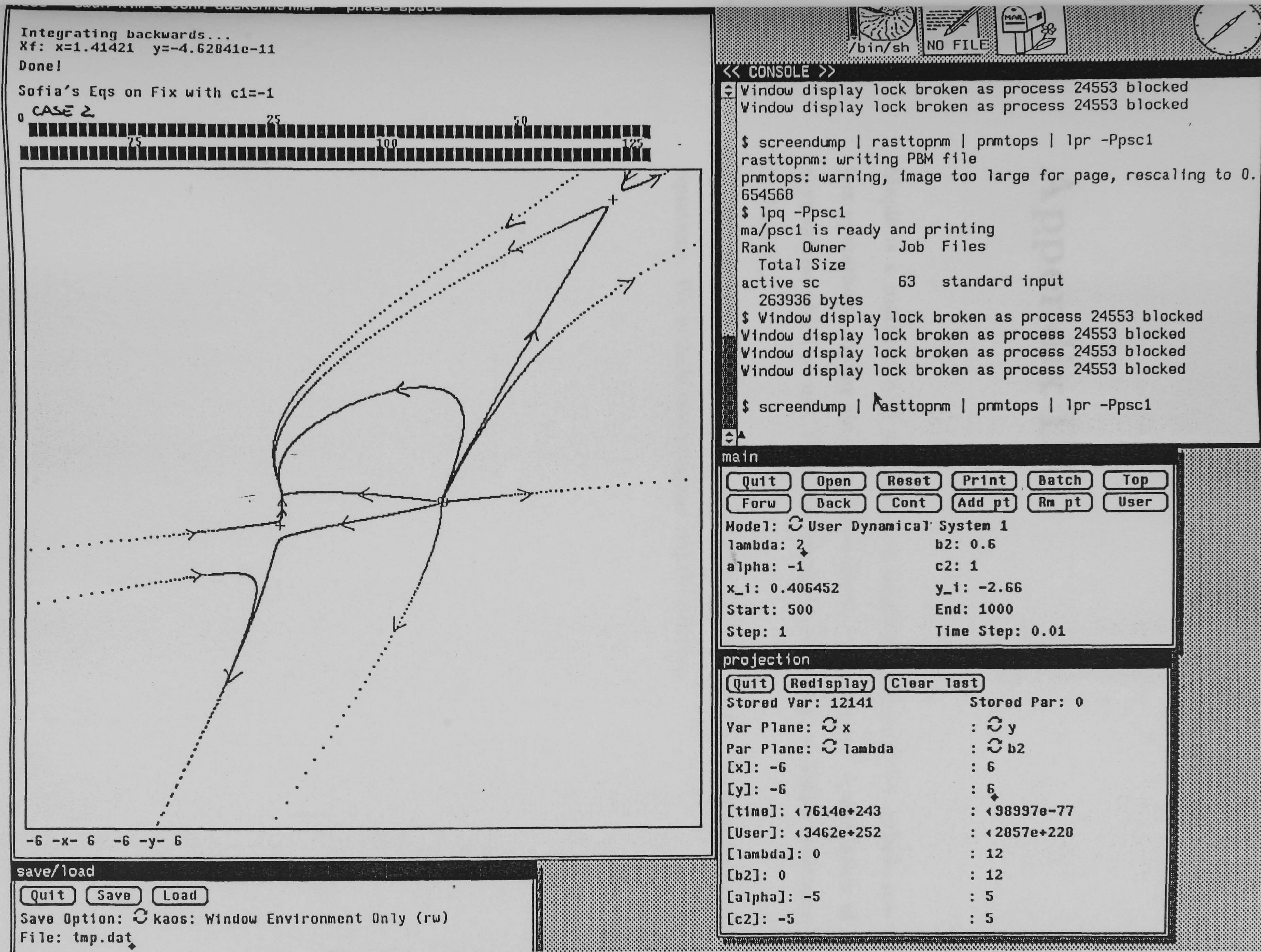

kaos - Swan Kim & John Guckenheimer - phase space
Integrating backwards...
Orbits appear to diverge off to an infinity! Stop!
Xf: x=-6.21466e+212 y=0
Done!
Sofia's Eqs on Fix with c1=-1
CASE 1
-30 -x- 30 -30 -y- 30
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
/bin/sh
NO FILE
<< CONSOLE >>
ma/psc1 is ready and printing
Rank Owner Job Files
Total Size
active sc 57 standard input
263936 bytes
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
rasttopnm: writing PBM file
pnmtops: warning, image too large for page, rescaling to 0.654568
$ Window display lock broken as process 24291 blocked
Window display lock broken as process 24291 blocked
Window display lock broken as process 24291 blocked
Window display lock broken as process 24291 blocked
Window display lock broken as process 24291 blocked
Window display lock broken as process 24291 blocked
Window display lock broken as process 24291 blocked
$
screendump | rasttopnm | pnmtops | lpr -Ppsc1
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System 1
lambda: 11.5 b2: 1.1
alpha: -1 c2: 1
x_1: -10.4516 y_1: 0
Start: 500 End: 1000
Step: 1 Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 9033 Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -30 : 30
[y]: -30 : 30
[time]: ‹7614e+243 : ‹98997e-77
[User]: ‹3462e+252 : ‹2857e+228
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5


Integrating backwards...
Orbits appear to diverge off to an infinity! Stop!
Xf: x=-1.44348e+103 y=-3.14907e+101
Done!
Sofia's Eqs on Fix with c1=-1
CASE 2
0 25 50 75 100 125
-5 -x- 5 -5 -y- 5
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
/bin/sh
NO FILE
<< CONSOLE >>
$ Window display lock broken as process 24488 blocked
Window display lock broken as process 24488 blocked
$ screendump | rasttopnm | pnmtops |lpr -Ppsc1
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System 1
lambda: -1.45 b2: 0.6
alpha: -1 c2: 1
x_i: 3.35484 y_i: -3.36667
Start: 500 End: 1000
Step: 1 Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 5921 Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -5 : 5
[y]: -5 : 5
[time]: ‹7614e+243 : ‹98997e-77
[User]: ‹3462e+252 : ‹2057e+228
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5


kaos - Swan Kim & John Guckenheimer - phase space
Integrating backwards...
Orbits appear to diverge off to an infinity! Stop!
Xf: x=2.03861e+24 y=3.39824e+24
Done!
Sofia's Eqs on Fix with c1=-1
CASE 2
0 25 50
75 100 125
0 -x- 4 0 -y- 4
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
/bin/sh
NO FILE
<< CONSOLE >>
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
rasttopnm: writing PBM file
pnmtops: warning, image too large for page, rescaling to 0.654568
$ lpq -Ppsc1
ma/psc1 is ready and printing
Rank Owner Job Files Total Size
active sc 61 standard input 263936 bytes
$ Window display lock broken as process 24488 blocked
Window display lock broken as process 24488 blocked
Window display lock broken as process 24488 blocked
Window display lock broken as process 24488 blocked
Window display lock broken as process 24488 blocked
Window display lock broken as process 24488 blocked
$
screendump | rasttopnm | pnmtops | lpr -Ppsc1
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System 1
lambda: -1.38 b2: 0.6
alpha: -1 c2: 1
x_i: 1.97419 y_i: 3.44
Start: 500 End: 1000
Step: 1 Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 10059 Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: 0 : 4
[y]: 0 : 4
[time]: ‹7614e+243 : ‹98997e-77
[User]: ‹3462e+252 : ‹2057e+228
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5


kaos - Swan Kim & John Guckenheimer - phase space
Integrating backwards...
Orbits appear to diverge off to an infinity! Stop!
Xf: x=-3.71799e+39 y=-2.89039e+38
Done!
Sofia's Eqs on Fix with c1=-1
CASE 2
0 25 50
75 100 125
-4 -x- 4 -4 -y- 4
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
/bin/sh
NO FILE
<< CONSOLE >>
$ Window display lock broken as process 24829 blocked
Window display lock broken as process 24829 blocked
Window display lock broken as process 24829 blocked
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System 1
lambda: -1.38 b2: 0.6
alpha: -1 c2: 1
x_1: -2.83871 y_1: -2.61333
Start: 500 End: 1000
Step: 1 Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 4000 Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -4 : 4
[y]: -4 : 4
[time]: ‹7614e+243 : ‹98997e-77
[User]: ‹3462e+252 : ‹2057e+220
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5


kaos - Swan Kim & John Guckenheimer - phase space
Integrating backwards...
Orbits appear to diverge off to an infinity! Stop!
Xf: x=1.25771e+76 y=2.0966e+76
Done!
Sofia's Eqs on Fix with c1=-1
CASE 2
0
25
50
75
100
125
-5 -x- 5 -5 -y- 5
save/load
Quit
Save
Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
/bin/sh
NO FILE
<< CONSOLE >>
pnmtops: warning, image too large for page, rescaling to 0.
654568
lpq -Ppsc1
$ no entries
junk: sending to amazon
Rank Owner Job Files
Total Size
$ lpq -Ppsc1
ma/psc1 is ready and printing
Rank Owner Job Files
Total Size
active sc 60 standard input
263936 bytes
$ Window display lock broken as process 24488 blocked
Window display lock broken as process 24488 blocked
Window display lock broken as process 24488 blocked
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
main
Quit
Open
Reset
Print
Batch
Top
Forw
Back
Cont
Add pt
Rm pt
User
Model: User Dynamical System 1
lambda: -1
b2: 0.6
alpha: -1
c2: 1
x_1: 2.08065
y_1: 4.16667
Start: 500
End: 1000
Step: 1
Time Step: 0.01
projection
Quit
Redisplay
Clear last
Stored Var: 6566
Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -5 : 5
[y]: -5 : 5
[time]: ‹7614e+243 : ‹98997e-77
[User]: ‹3462e+252 : ‹2857e+228
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5


Integrating backwards...
Xf: x=1.41421 y=-4.62841e-11
Done!
Sofia's Eqs on Fix with c1=-1
CASE 2
-6 -x- 6 -6 -y- 6
save/load
Quit Save Load
Save Option: kaos: Window Environment Only (rw)
File: tmp.dat
/bin/sh
NO FILE
<< CONSOLE >>
Window display lock broken as process 24553 blocked
Window display lock broken as process 24553 blocked
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
rasttopnm: writing PBM file
pnmtops: warning, image too large for page, rescaling to 0.654568
$ lpq -Ppsc1
ma/psc1 is ready and printing
Rank Owner Job Files Total Size
active sc 63 standard input 263936 bytes
$ Window display lock broken as process 24553 blocked
Window display lock broken as process 24553 blocked
Window display lock broken as process 24553 blocked
Window display lock broken as process 24553 blocked
$ screendump | rasttopnm | pnmtops | lpr -Ppsc1
main
Quit Open Reset Print Batch Top
Forw Back Cont Add pt Rm pt User
Model: User Dynamical System 1
lambda: 2
b2: 0.6
alpha: -1
c2: 1
x_i: 0.406452
y_i: -2.66
Start: 500
End: 1000
Step: 1
Time Step: 0.01
projection
Quit Redisplay Clear last
Stored Var: 12141
Stored Par: 0
Var Plane: x : y
Par Plane: lambda : b2
[x]: -6 : 6
[y]: -6 : 6
[time]: ‹7614e+243 : ‹98997e-77
[User]: ‹3462e+252 : ‹2857e+220
[lambda]: 0 : 12
[b2]: 0 : 12
[alpha]: -5 : 5
[c2]: -5 : 5

Appendix D

Maple is a mathematical manipulation language. Its facilities include matrix operations, solving linear and non-linear equations and computation of eigenvalues, which we use. For more detailed information see Maple Reference Manual [9].

We use Maple to do operations with matrices and to solve systems of equations. We include one program and the results.

```
with(linalg):
#
#UNFOLDING FOR THE (3,5) MODE INTERACTION TO ORDER 2
#This program determines the branching equations
#in each fixed-point subspace.
#
y1 := 0:
y2 := 0:
y3 := 0:
#z1 := z2:
z3 := 0:
z4 := 0:
z5 := 0:
dy1 := a1*(lambda-alpha)*y1+a2*(y1*z2+y2*z5+y1*z1-y3*z4)/2:
dy2 := a1*(lambda-alpha)*y2+a2*(y1*z5-y2*z2+y3*z3)/2:
dy3 := a1*(lambda-alpha)*y3+a2*(y2*z3-y1*z4-y3*z1)/2:
dz1 := b1*lambda*z1+b2*(z1*z1+z3*z3+z4*z4-2*(z2*z2+z5*z5+z1*z2))/3
+b3*(2*y3*y3-(y1*y1+y2*y2))+b4*(y1*z3+y2*z4):
dz2 := b1*lambda*z2+b2*(z2*z2+z3*z3+z5*z5-2*(z1*z1+z4*z4+z1*z2))/3
+b3*(2*y2*y2-(y1*y1+y3*y3))+b4*(y3*z5-y1*z3):
dz3 := b1*lambda*z3+b2*(z1*z3+z2*z3+z4*z5)-b3*y2*y3
+b4*(y1*z2+y2*z5-y1*z1+y3*z4)/2:
dz4 := b1*lambda*z4+b2*(z5*z3-z2*z4)+b3*y1*y3
+b4*(y1*z5-y2*(z1+z2)-y2*z1+y3*z3)/2:
dz5 := b1*lambda*z5+b2*(z4*z3-z1*z5)-b3*y2*y1
+b4*(y1*z4+y3*(z1+z2)+y2*z3+y3*z2)/2:
solve( {dz1,dz2,dz3,dz4,dz5,dy1,dy2,dy3},{z1,z2,lambda,alpha} );
quit;
```

```
#
#UNFOLDING FOR THE (3,5) MODE INTERACTION TO ORDER 2
#THE D2-BRANCH
#
> y1 := 0:
> y2 := 0:
> y3 := 0:
#z1 := z2:
> z3 := 0:
> z4 := 0:
> z5 := 0:
> dy1 := a1*(lambda-alpha)*y1+a2*(y1*z2+y2*z5+y1*z1-y3*z4)/2:
> dy2 := a1*(lambda-alpha)*y2+a2*(y1*z5-y2*z2+y3*z3)/2:
> dy3 := a1*(lambda-alpha)*y3+a2*(y2*z3-y1*z4-y3*z1)/2:
> dz1 := b1*lambda*z1+b2*(z1*z1+z3*z3+z4*z4-2*(z2*z2+z5*z5+z1*z2))/3
> +b3*(2*y3*y3-(y1*y1+y2*y2))+b4*(y1*z3+y2*z4):
> dz2 := b1*lambda*z2+b2*(z2*z2+z3*z3+z5*z5-2*(z1*z1+z4*z4+z1*z2))/3
> +b3*(2*y2*y2-(y1*y1+y3*y3))+b4*(y3*z5-y1*z3):
> dz3 := b1*lambda*z3+b2*(z1*z3+z2*z3+z4*z5)-b3*y2*y3
> +b4*(y1*z2+y2*z5-y1*z1+y3*z4)/2:
> dz4 := b1*lambda*z4+b2*(z5*z3-z2*z4)+b3*y1*y3
> +b4*(y1*z5-y2*(z1+z2)-y2*z1+y3*z3)/2:
> dz5 := b1*lambda*z5+b2*(z4*z3-z1*z5)-b3*y2*y1
> +b4*(y1*z4+y3*(z1+z2)+y2*z3+y3*z2)/2:
> solve( {dz1,dz2,dz3,dz4,dz5,dy1,dy2,dy3},{z1,z2,lambda,alpha} );

       {z2 = 0, z1 = 0, alpha = alpha, lambda = lambda},

                                  b2 z2
           {z1 = - 2 z2, lambda = -----, alpha = alpha, z2 = z2},
                                    b1

                              b2 z2
           {z1 = z2, lambda = -----, alpha = alpha, z2 = z2},
                                b1

                                               b2 z2
           {z1 = - 1/2 z2, lambda = - 1/2 -----, alpha = alpha, z2 = z2}
                                                 b1

> quit;
bytes used=642696, alloc=524192, time=8.066
```

```
#
#UNFOLDING FOR THE (3,5) MODE INTERACTION TO ORDER 2
#THE Z2(z)-BRANCH
#
#y1 := 0:
> y2 := 0:
> y3 := 0:
#z1 := z2:
#z3 := 0:
> z4 := 0:
> z5 := 0:
> dy1 := a1*(lambda-alpha)*y1+a2*(y1*z2+y2*z5+y1*z1-y3*z4)/2:
> dy2 := a1*(lambda-alpha)*y2+a2*(y1*z5-y2*z2+y3*z3)/2:
> dy3 := a1*(lambda-alpha)*y3+a2*(y2*z3-y1*z4-y3*z1)/2:
> dz1 := b1*lambda*z1+b2*(z1*z1+z3*z3+z4*z4-2*(z2*z2+z5*z5+z1*z2))/3
> +b3*(2*y3*y3-(y1*y1+y2*y2))+b4*(y1*z3+y2*z4):
> dz2 := b1*lambda*z2+b2*(z2*z2+z3*z3+z5*z5-2*(z1*z1+z4*z4+z1*z2))/3
> +b3*(2*y2*y2-(y1*y1+y3*y3))+b4*(y3*z5-y1*z3):
> dz3 := b1*lambda*z3+b2*(z1*z3+z2*z3+z4*z5)-b3*y2*y3
> +b4*(y1*z2+y2*z5-y1*z1+y3*z4)/2:
> dz4 := b1*lambda*z4+b2*(z5*z3-z2*z4)+b3*y1*y3
> +b4*(y1*z5-y2*(z1+z2)-y2*z1+y3*z3)/2:
> dz5 := b1*lambda*z5+b2*(z4*z3-z1*z5)-b3*y2*y1
> +b4*(y1*z4+y3*(z1+z2)+y2*z3+y3*z2)/2:
> solve( {dz1,dz2,dz3,dz4,dz5,dy1,dy2,dy3},{y1,z1,z2,z3,lambda,alpha} );

                                       2          2         2
                               a1 b2 z2  + a1 b3 y1  + a2 z2  b1
{y1 = y1, z2 = z2, alpha = --------------------------------, z3 = 0,
                                        b1 z2 a1

                    2         2
                b2 z2  + b3 y1
    lambda = ---------------, z1 = z2},
                  b1 z2

{alpha = alpha, lambda = lambda, y1 = 0, z3 = 0, z2 = 0, z1 = 0},

                                                b2 z2
{alpha = alpha, z2 = z2, y1 = 0, lambda = -----, z3 = 0, z1 = - 2 z2},
                                                 b1

                                                b2 z2
{alpha = alpha, z2 = z2, y1 = 0, lambda = -----, z3 = 0, z1 = z2},
                                                 b1

                                                       b2 z2
{alpha = alpha, z2 = z2, y1 = 0, lambda = - 1/2 -----, z3 = 0, z1 = - 1/2 z2},
                                                        b1

                                                       2                     2 1/2
{z1 = z1, alpha = alpha, z2 = z2, y1 = 0, z3 = (2 z1  + 5 z1 z2 + 2 z2 )    ,

                 b2 (z1 + z2)
    lambda = - ------------},
                      b1

                                                         2                     2 1/2
{z1 = z1, alpha = alpha, z2 = z2, y1 = 0, z3 = - (2 z1  + 5 z1 z2 + 2 z2 )    ,

                 b2 (z1 + z2)
    lambda = - ------------}
                      b1

> quit;
bytes used=7446424, alloc=1441528, time=150.766
```

```
#
#UNFOLDING FOR THE (3,5) MODE INTERACTION TO ORDER 2
#THE Z2(k)-BRANCH
#
> y1 := 0:
> y2 := 0:
#y3 := 0:
#z1 := z2:
> z3 := 0:
> z4 := 0:
#z5 := 0:
> dy1 := a1*(lambda-alpha)*y1+a2*(y1*z2+y2*z5+y1*z1-y3*z4)/2:
> dy2 := a1*(lambda-alpha)*y2+a2*(y1*z5-y2*z2+y3*z3)/2:
> dy3 := a1*(lambda-alpha)*y3+a2*(y2*z3-y1*z4-y3*z1)/2:
> dz1 := b1*lambda*z1+b2*(z1*z1+z3*z3+z4*z4-2*(z2*z2+z5*z5+z1*z2))/3
> +b3*(2*y3*y3-(y1*y1+y2*y2))+b4*(y1*z3+y2*z4):
> dz2 := b1*lambda*z2+b2*(z2*z2+z3*z3+z5*z5-2*(z1*z1+z4*z4+z1*z2))/3
> +b3*(2*y2*y2-(y1*y1+y3*y3))+b4*(y3*z5-y1*z3):
> dz3 := b1*lambda*z3+b2*(z1*z3+z2*z3+z4*z5)-b3*y2*y3
> +b4*(y1*z2+y2*z5-y1*z1+y3*z4)/2:
> dz4 := b1*lambda*z4+b2*(z5*z3-z2*z4)+b3*y1*y3
> +b4*(y1*z5-y2*(z1+z2)-y2*z1+y3*z3)/2:
> dz5 := b1*lambda*z5+b2*(z4*z3-z1*z5)-b3*y2*y1
> +b4*(y1*z4+y3*(z1+z2)+y2*z3+y3*z2)/2:
> solve( {dz1,dz2,dz3,dz4,dz5,dy1,dy2,dy3},{y3,z1,z2,z5,lambda,alpha} );
bytes used=1000028, alloc=786288, time=15.866
bytes used=2000180, alloc=1244956, time=35.533

{y3 = 0, z5 = 0, z2 = 0, z1 = 0, alpha = alpha, lambda = lambda},

                                  b2 z2
{y3 = 0, z5 = 0, lambda = -----, z1 = - 2 z2, z2 = z2, alpha = alpha},
                                   b1

                                  b2 z2
{y3 = 0, z5 = 0, lambda = -----, z1 = z2, z2 = z2, alpha = alpha},
                                   b1

                                         b2 z2
{y3 = 0, z5 = 0, lambda = - 1/2 -----, z1 = - 1/2 z2, z2 = z2, alpha = alpha},
                                          b1

                             2         2 1/2                z1 b2
{y3 = 0, z5 = (- z1 z2 - z2  + 2 z1 )    , lambda = -----, z2 = z2, z1 = z1,
                                                             b1

    alpha = alpha},

                               2         2 1/2                z1 b2
{y3 = 0, z5 = - (- z1 z2 - z2  + 2 z1 )    , lambda = -----, z2 = z2, z1 = z1,
                                                               b1

    alpha = alpha},

{z5 = 0,

alpha =

                                                  2                      2
       2 a1 b2 z1 + 2 a1 b4 RootOf(3 b2 z1  + 2 b4 _Z z1 + 4 b3 _Z ) - a2 z1 b1
   1/2 ------------------------------------------------------------------------
                                        b1 a1

    ,

                                                     2                      2
```

D4

$$z2 = -\frac{1}{2} z1,\ \lambda = \frac{b2\ z1 + b4\ \mathrm{RootOf}(3\ b2\ z1 + 2\ b4\ \_Z\ z1 + 4\ b3\ \_Z)}{b1},$$

$$z1 = z1,\ y3 = \mathrm{RootOf}(3\ b2\ z1^{2} + 2\ b4\ \_Z\ z1 + 4\ b3\ \_Z^{2})\},$$

$$\{z5 = 0,\ \alpha = -\frac{1}{2}\ \frac{a1\ b2\ z1^{2} + 4\ a1\ b3\ y3^{2} + a2\ z1^{2}\ b1}{b1\ z1\ a1},\ z2 = -\frac{1}{2} z1,$$

$$z1 = z1,\ \lambda = -\frac{1}{2}\ \frac{b2\ z1^{2} + 4\ b3\ y3^{2}}{b1\ z1},\ y3 = y3\},$$

$$\{$$

$$\alpha =$$

$$\frac{1}{2}\ \frac{2\ a1\ b2\ z1 - 2\ a1\ b4\ \mathrm{RootOf}(3\ b2\ z1^{2} + 4\ b3\ \_Z^{2} - 2\ b4\ \_Z\ z1) - a2\ z1\ b1}{b1\ a1}$$

$$,$$

$$\lambda = \frac{b2\ z1 - b4\ \mathrm{RootOf}(3\ b2\ z1^{2} + 4\ b3\ \_Z^{2} - 2\ b4\ \_Z\ z1)}{b1},\ z5 = 0,$$

$$z2 = -\frac{1}{2} z1,\ y3 = \mathrm{RootOf}(3\ b2\ z1^{2} + 4\ b3\ \_Z^{2} - 2\ b4\ \_Z\ z1),\ z1 = z1\},$$

$$\{\alpha = \frac{1}{2}\ \frac{2\ a1\ b2\ z1 + 2\ a1\ b4\ y3 - a2\ z1\ b1}{b1\ a1},$$

$$z2 = -\frac{1}{4}\ \frac{3^{1/2}\ 2^{1/2}\ \%2^{1/2} + 2\ z1\ b2^{1/2}}{b2^{1/2}},\ z5 = \frac{1}{4}\ \frac{3^{1/2}\ 2^{1/2}\ \%2^{1/2}}{b2^{1/2}},$$

$$\lambda = \frac{b2\ z1 + b4\ y3}{b1},\ z1 = z1,\ y3 = y3\},$$

$$\{z2 = -\frac{1}{4}\ \frac{-3^{1/2}\ 2^{1/2}\ \%2^{1/2} + 2\ z1\ b2^{1/2}}{b2^{1/2}},$$

$$\alpha = \frac{1}{2}\ \frac{2\ a1\ b2\ z1 + 2\ a1\ b4\ y3 - a2\ z1\ b1}{b1\ a1},$$

$$z5 = -\frac{1}{4}\ \frac{3^{1/2}\ 2^{1/2}\ \%2^{1/2}}{b2^{1/2}},\ \lambda = \frac{b2\ z1 + b4\ y3}{b1},\ z1 = z1,\ y3 = y3\},$$

```
          1/2  1/2   1/2
         3    2    %1                       b2 z1 - b4 y3
{z5 = 1/4 ---------------, lambda = -------------, z1 = z1,
                1/2                             b1
              b2

              1/2  1/2   1/2              1/2
             3    2    %1     - 2 z1 b2
    z2 = 1/4 ---------------------------,
                             1/2
                           b2

                   2 a1 b2 z1 - 2 a1 b4 y3 - a2 z1 b1
    alpha = 1/2 ----------------------------------, y3 = y3},
                              b1 a1

             1/2  1/2   1/2
            3    2    %1                       b2 z1 - b4 y3
{z5 = - 1/4 ---------------, lambda = -------------, z1 = z1,
                  1/2                             b1
                b2

                1/2  1/2   1/2              1/2
               3    2    %1     + 2 z1 b2
    z2 = - 1/4 ---------------------------,
                               1/2
                             b2

                   2 a1 b2 z1 - 2 a1 b4 y3 - a2 z1 b1
    alpha = 1/2 ----------------------------------, y3 = y3}
                              b1 a1

                                 2            2
%1 :=                  3 b2 z1  + 4 b3 y3  - 2 b4 y3 z1

                                 2                           2
%2 :=                  3 b2 z1  + 2 b4 y3 z1 + 4 b3 y3

> quit;
bytes used=2335180, alloc=1244956, time=42.016
```

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