

Quantified Reflection Calculus with one modality

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Abstract

This paper presents the logic QRC_1 , which is a strictly positive fragment of quantified modal logic. The intended reading of the diamond modality is that of consistency of a formal theory. Predicate symbols are interpreted as parametrized axiomatizations. We prove arithmetical soundness of the logic QRC_1 with respect to this arithmetical interpretation.

Quantified provability logic is known to be undecidable. However, the undecidability proof cannot be performed in our signature and arithmetical reading. We conjecture the logic QRC_1 to be arithmetically complete. This paper takes the first steps towards arithmetical completeness by providing relational semantics for QRC_1 with a corresponding completeness proof. We further show the finite model property with respect to domains and number of worlds, which implies decidability.

Keywords: Provability logic, strictly positive logics, quantified modal logic, arithmetic interpretations, feasible fragments, decidable logics, finite model property.

1 Introduction

We present a new provability logic QRC_1 , standing for Quantified Reflection Calculus with one modality. The best known provability logic is perhaps GL [7]. Recall that GL is a PSPACE decidable propositional modal logic where the modality \Box is used to model formal provability in some base theory such as Peano Arithmetic (PA). Likewise, the dual modality \Diamond is used to model consistency over the base theory. By Solovay's celebrated completeness result [20] we know that, in a sense, the logic GL generates exactly the provable-in-PA structural behavior of formal provability.

Let us make this slightly more precise. By a *realization* \star we mean a map from propositional variables to sentences in the language of Peano Arithmetic.

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The realization is extended to all propositional modal formulas by defining $(\varphi \wedge \psi)^* := \varphi^* \wedge \psi^*$ and likewise for other Boolean connectives. Finally, we define $(\Box \varphi)^* := \Box_{\text{PA}} \varphi^*$, where \Box_{PA} is a formula in the language of PA satisfying the Hilbert-Bernays-Löb derivability conditions (see [7]) that arithmetizes formal provability in PA in the sense that $\text{PA} \vdash \chi$ if and only if $\mathbb{N} \models \Box_{\text{PA}} \chi$.³ We can now paraphrase Solovay's result as $\text{GL} = \{\varphi \mid \forall * \text{PA} \vdash \varphi^*\}$.

After Solovay's completeness theorem, it was natural to ask whether one could find a logic that generates exactly the provable-in-PA structural behavior of formal provability for (relational) *quantified modal logic*. The main difference with GL is that we now understand a realization $*$ as a map from relation symbols to formulas in the language of Peano Arithmetic such that the free variables match the arity of the relation symbol. Vardanyan showed in [21] that this situation is completely different: now $\{\varphi \mid \forall * \text{PA} \vdash \varphi^*\}$ is Π_2^0 -complete, a big jump from the PSPACE decidability of GL.

Visser and de Jonge showed that Vardanyan's result can be extended to a wide range of arithmetical theories and called their paper *No escape from Vardanyan's theorem* [22]. Here we shall take some first steps to indeed find an escape to Vardanyan's theorem. We do so by making two adaptations to the standard setting. First, we resort to a very small fragment of relational predicate modal logic called the *strictly positive fragment*. Second, we slightly change the realizations so that we interpret relation symbols not directly as formulas, but as axiomatizations of theories. As such our study follows a recent development of *strictly positive logics* in general (such as [18]) and *reflection calculi* in particular (see [8], [4], and [9]).

Japaridze [17] generalized the logic GL to a polymodal version called GLP, and Beklemishev [2] generalized this further to a transfinite setting yielding GLP_Λ , where for each ordinal $\xi < \Lambda$ there is a provability modality $[\xi]$, and larger ordinals refer to stronger provability notions. The logic GLP_ω has been successfully used in performing a modular ordinal analysis of PA and related systems (see [1], and more recently [6]). A key feature in the ordinal analysis is that consistency operators $\langle n \rangle$ can be interpreted as reflection principles, which are finitely axiomatizable.

However, an interpretation of limit modalities like $\langle \omega \rangle$ would require non finitely axiomatizable reflection schemata. One way to overcome this problem is by resorting to what was coined the *Reflection Calculus* [8], [3] and its transfinite version RC_Λ [11]. Reflection calculi only allow strictly positive formulas, which are based solely on propositional variables, a verum constant, consistency operators, and conjunctions. As such, the arithmetical realizations as above can be taken to be *arithmetical theories* instead of arithmetical formulas.

The logic QRC_1 we present in this paper follows this set-up: we will work with sequents of the form $\varphi \vdash \psi$ where both φ and ψ are strictly positive formulas built up from \top , predicate symbols, conjunction, universal quantification and the \diamond modality. The latter will refer to the usual notion of formal

³ We refrain from distinguishing a formula φ from its Gödel number $\ulcorner \varphi \urcorner$.

consistency and predicate symbols are interpreted as theories parametrized by the free variables.

Independently of the reflection calculi, other strictly positive modal logics were studied because of their computational desirable properties when compared to their non-strict counterparts (see [18] for an example). In this line, the logic RC can be seen as a PTIME decidable fragment of the PSPACE complete logic GLP (shown in [8]). If indeed the logic QRC_1 we present in this paper turns out to be arithmetically complete, this would yield, in a sense, a shift from undecidability (Π_2^0 -completeness) to decidability when resorting to a strictly positive fragment.

2 Quantified Reflection Calculus with one modality

The *Quantified Reflection Calculus with one modality*, or QRC_1 , is a sequent logic in a strictly positive predicate modal language.

Towards describing the language of QRC_1 , we fix a countable set of variables x_0, x_1, \dots (also referred to as x, y, z , etc.) and define a signature Σ as a set of constants and a set of relation symbols with corresponding arity (we have no function symbols). We use the letters c, c_i, \dots to refer to constants and the letters S, S_i, \dots to refer to relation symbols.

Given a signature, a term t is either a variable or a constant of that signature. Both \top and any n -ary relation symbol applied to n terms are atomic formulas. The set of formulas is the closure of the atomic formulas under the binary connective \wedge , the unary modal operator \diamond , and the quantifier $\forall x$, where x is a variable. Formulas are represented by Greek letters such as φ, ψ, χ , etc.

The free variables of a formula are defined as usual. The expression $\varphi[x \leftarrow t]$ denotes the formula φ with all free occurrences of the variable x simultaneously replaced by the term t . We say that t is free for x in φ if no occurrence of a free variable in t becomes bound in $\varphi[x \leftarrow t]$.

Definition 2.1 Let Σ be a signature and φ, ψ , and χ be any formulas in that language. The axioms and rules of QRC_1 are the following:

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|--|--|
| (i) $\varphi \vdash \top$ and $\varphi \vdash \varphi$; | (viii) if $\varphi \vdash \psi$, then $\varphi \vdash \forall x \psi$
($x \notin \text{fv}(\varphi)$); |
| (ii) $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$; | (ix) if $\varphi[x \leftarrow t] \vdash \psi$ then $\forall x \varphi \vdash \psi$
(t free for x in φ); |
| (iii) if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then
$\varphi \vdash \psi \wedge \chi$; | (x) if $\varphi \vdash \psi$, then $\varphi[x \leftarrow t] \vdash \psi[x \leftarrow t]$
(t free for x in φ and ψ); |
| (iv) if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$; | (xi) if $\varphi[x \leftarrow c] \vdash \psi[x \leftarrow c]$, then $\varphi \vdash \psi$
(c not in φ nor ψ). |
| (v) if $\varphi \vdash \psi$, then $\diamond \varphi \vdash \diamond \psi$; | |
| (vi) $\diamond \diamond \varphi \vdash \diamond \varphi$; | |
| (vii) $\diamond \forall x \varphi \vdash \forall x \diamond \varphi$; | |

If $\varphi \vdash \psi$, we say that ψ follows from φ in QRC_1 . When the signature is not clear from the context, we write $\varphi \vdash_\Sigma \psi$ instead.

The following easy lemma presents a number of consequences of QRC_1 .

Lemma 2.2 *The following are theorems (or derivable rules) of QRC_1 :*

- (i) $\forall x \forall y \varphi \vdash \forall y \forall x \varphi$;
- (ii) $\forall x \varphi \vdash \varphi[x \leftarrow t]$ (t free for x in φ);
- (iii) $\forall x \varphi \vdash \forall y \varphi[x \leftarrow y]$ (y free for x in φ and $y \notin \text{fv}(\varphi)$);
- (iv) if $\varphi \vdash \psi$, then $\varphi \vdash \psi[x \leftarrow t]$ (x not free in φ and t free for x in ψ);
- (v) if $\varphi \vdash \psi[x \leftarrow c]$, then $\varphi \vdash \forall x \psi$ (x not free in φ and c not in φ nor ψ).

In order to analyze various aspects of our calculus we define two complexity measures on formulas.

Definition 2.3 Given a formula φ , its *modal depth* $d_\diamond(\varphi)$ is defined inductively as follows:

- $d_\diamond(\top) := d_\diamond(S(x_0, \dots, x_{n-1})) := 0$;
- $d_\diamond(\psi \wedge \chi) := \max\{d_\diamond(\psi), d_\diamond(\chi)\}$;
- $d_\diamond(\forall x \psi) := d_\diamond(\psi)$;
- $d_\diamond(\diamond\psi) := d_\diamond(\psi) + 1$.

Given a finite set of formulas Γ , its modal depth is $d_\diamond(\Gamma) := \max_{\varphi \in \Gamma} \{d_\diamond(\varphi)\}$.

The definition of quantifier depth d_\forall is analogous except for:

- $d_\forall(\forall x \psi) = d_\forall(\psi) + 1$; and
- $d_\forall(\diamond\psi) = d_\forall(\psi)$.

The modal depth provides a necessary condition for derivability, proven by a straightforward induction on $\varphi \vdash \psi$.

Lemma 2.4 *If $\varphi \vdash \psi$, then $d_\diamond(\varphi) \geq d_\diamond(\psi)$.*

In particular, we get irreflexivity for free as stated in the next result. For other calculi this usually requires hard work via either modal or arithmetical semantics, as can be seen in [5], [10], and [12].

Corollary 2.5 *For any formula φ , we have $\varphi \not\vdash \diamond\varphi$.*

The following lemma tells us that adding new constants to our signature yields a conservative extension of the calculus.

Lemma 2.6 *Let Σ be a signature and let C be a collection of new constants not yet occurring in Σ . By Σ_C we denote the signature obtained by including these new constants C in Σ . Let φ, ψ be formulas in the language of Σ . Then, if $\varphi \vdash_{\Sigma_C} \psi$, so does $\varphi \vdash_\Sigma \psi$.*

Proof. This is a standard result and a proof for a calculus similar to ours can be found in Section 1.8 of [14]. The idea is to replace every constant from C appearing in the proof of $\varphi \vdash_{\Sigma_C} \psi$ by a fresh variable. It can easily be seen that axioms are mapped to axioms under this replacement, and that the rules are also mapped correctly. The most interesting case is that of the generalization of constants rule, because replacing new constants by variables in the premise

$\varphi[x \leftarrow c] \vdash_{\Sigma_C} \psi[x \leftarrow c]$ may leave us unable to apply the same rule. Fortunately the term instantiation rule (Rule 2.1.(x)) suffices to complete the proof. \square

3 Arithmetical semantics

In this section we look at the intended arithmetical reading of the logic QRC_1 . We consider first-order theories in the language $\{0, 1, +, \times, \leq, =\}$ of arithmetic. We refer the reader to [15] for details and definitions. We recall that bounded formulas are those formulas where each quantifier occurs bounded as in $\forall y \leq t$, where y does not occur in t . The Σ_1 formulas are those that arise by existential quantification of bounded formulas. Sets of numbers that can be defined by a Σ_1 formula are called *c.e.* for *computably enumerable*.

The theory IS_1 contains the defining axioms for our constants and function symbols, say as in Robinson's arithmetic, and moreover allows induction for Σ_1 formulas. It is well-known that IS_1 proves Σ_1 -collection, that is, $\forall x < z \exists y \varphi(x, y) \rightarrow \exists y_0 \forall x < z \exists y < y_0 \varphi(x, y)$ where φ is a Σ_1 formula. For the sake of an easy exposition we shall assume that all the theories we work with extend IS_1 . By $\tau(x)$ we denote the elementary formula that presents the standard axiomatization of IS_1 . In particular, $\mathbb{N} \models \tau(n)$ if and only if n is the Gödel number of an axiom of IS_1 .

In the arithmetical interpretation of the propositional logic RC, the propositional variables are mapped to (axiomatizations of) theories, and the conjunction of two theories is interpreted as the union of both theories (corresponding to a disjunction in the sense of either being an axiom of the one or of the other). The arithmetical interpretation of each diamond modality is a consistency notion.

We will fix a provability predicate $\Box_\alpha \varphi$ formalizing the existence of a Hilbert-style proof, which is a sequence of formulas the last of which is φ and such that each element of the sequence is either a logical axiom, an axiom in the sense of α , or the result of applying a rule to earlier elements in the sequence. We denote the dual consistency notion by $\text{Con}_\alpha(\psi)$ and sometimes write Con_α instead of $\text{Con}_\alpha(\top)$.

If we now interpret relation symbols from QRC_1 as theories (parametrized by the free variables), then a universal quantification (which can be conceived of as an infinite conjunction) will be interpreted as an infinite union/disjunction, that is, an existential quantifier. These observations are reflected in Definition 3.1 below.

In this section, we reserve the variables x_i for variables in QRC_1 , and the variables y_i, z_i and u are reserved for the arithmetic language with the understanding that the y_i interpret the QRC_1 -constants c_i and the z_i interpret the QRC_1 -variables x_i . The variable u is reserved for (Gödel numbers of) axioms of the theories that we denote.

Definition 3.1 A realization $*$ takes n -ary predicate symbols in the language of QRC_1 to $(n + 1)$ -ary Σ_1 -formulas in the language of arithmetic, each representing a set of axioms of theories indexed by n parameters. In particular, a realization $*$ is such that $S(\mathbf{c}, \mathbf{x})^* = \sigma(\mathbf{y}, \mathbf{z}, u)$ for some Σ_1 formula σ such that

for each concrete numerical values for \mathbf{y}, \mathbf{z} we have that $\mathbb{N} \models \sigma(\mathbf{y}, \mathbf{z}, u)$ if and only if u is the Gödel number of an axiom of the intended corresponding theory. When we use the vector notation in $S(\mathbf{c}, \mathbf{x})^* = \sigma(\mathbf{y}, \mathbf{z}, u)$ we understand that \mathbf{y} matches with \mathbf{c} and \mathbf{z} matches with \mathbf{x} , and thus if, say, y_i occurs in σ , then c_i occurs in $S(\mathbf{c}, \mathbf{x})$.

We extend a given realization $*$ to $(\)^*$ on any formula of QRC_1 as follows:

- $(\top)^* := \tau(u)$;
- $(S(\mathbf{c}, \mathbf{x}))^* := S(\mathbf{c}, \mathbf{x})^* \vee \tau(u)$;
- $(\psi(\mathbf{c}, \mathbf{x}) \wedge \delta(\mathbf{c}, \mathbf{x}))^* := (\psi(\mathbf{c}, \mathbf{x}))^* \vee (\delta(\mathbf{c}, \mathbf{x}))^*$;
- $(\diamond\psi(\mathbf{c}, \mathbf{x}))^* := \tau(u) \vee (u = \ulcorner \text{Con}_{(\psi(\mathbf{c}, \mathbf{x}))^*} \urcorner)$;
- $(\forall x_i \psi(\mathbf{c}, \mathbf{x}))^* := \exists z_i (\psi(\mathbf{c}, \mathbf{x}))^*$.

From now on we omit outer brackets, using the same notation for $*$ and $(\)^*$. This may lead to confusion for predicate symbols, but the context should tell us which reading to use. We fix the notation $\psi(\mathbf{c}, \mathbf{x})^* = \psi^*(\mathbf{y}, \mathbf{z})$ suppressing mention of u when convenient.

Let T be a c.e. theory in the language of arithmetic which extends IS_1 . We define (recall that χ^* will in general depend on \mathbf{y} and \mathbf{z}):

$$\mathcal{QRC}_1(T) := \{\varphi(\mathbf{c}, \mathbf{x}) \vdash \psi(\mathbf{c}, \mathbf{x}) \mid \forall^* T \vdash \forall \theta \forall \mathbf{y} \forall \mathbf{z} (\Box_{\psi^*} \theta \rightarrow \Box_{\varphi^*} \theta)\}.$$

We defer the question of whether $\text{QRC}_1 = \mathcal{QRC}_1(T)$ for any sound c.e. T containing IS_1 to a future paper and prove here only the soundness inclusion.

Theorem 3.2 (Arithmetical soundness) $\text{QRC}_1 \subseteq \mathcal{QRC}_1(\text{IS}_1)$.

Proof. By induction on $\varphi \vdash \psi$. The details can be found in Appendix A. Here we just observe the arithmetical content expressed by of some of the axioms. For example, $\varphi \vdash \top$ is sound because all of our theories extend IS_1 . The $\diamond\varphi \vdash \varphi$ axiom reflects provable Σ_1 completeness (see [7]). The $\diamond\forall x \varphi \vdash \forall x \diamond\varphi$ axiom reflects that if a sum of theories is consistent, then each of the summands is consistent too. \square

4 Relational semantics

There have been several proposals for relational semantics for modal predicate logics, from Kripke [19] to many others. Overviews can be found in [16], [13] and [14]. We essentially have first-order models glued together by an accessibility relation. Our interpretation of the universal quantifiers is actualist, which means that $\forall x \varphi$ is true at a world w if and only if $\varphi[x \leftarrow d]$ is true at w for every d in the domain of w , i.e., for every entity d that exists in that world. It might happen, however, that some other world u has a different domain, and thus that it falsifies $\varphi[x \leftarrow e]$ for some specific e .

We proceed by defining frames and relational models.

Definition 4.1 A frame \mathcal{F} is a tuple $\langle W, R, \{M_w\}_{w \in W} \rangle$ where:

- W is a non-empty set (the set of worlds, where individual worlds are referred to as w, u, v , etc);
- R is a binary relation on W (the accessibility relation); and
- each M_w is a finite set (the domain of the world w , whose elements are referred to as d, d_0, d_1 , etc).

The domain of the frame is $M := \bigcup_{w \in W} M_w$. We say that a frame is *finite* if its set of worlds W is finite. In that case both the relation R and the domain M will be finite as well.

Definition 4.2 A *relational model* \mathcal{M} in a signature Σ is a tuple $\langle \mathcal{F}, \{I_w\}_{w \in W}, \{J_w\}_{w \in W} \rangle$ where:

- $\mathcal{F} = \langle W, R, \{M_w\}_{w \in W} \rangle$ is a frame;
- for each $w \in W$, the interpretation I_w assigns an element of the domain M_w to each constant $c \in \Sigma$, written c^{I_w} ; and
- for each $w \in W$, the interpretation J_w assigns a set of tuples $S^{J_w} \subseteq \wp((M_w)^n)$ to each n -ary relation symbol $S \in \Sigma$.

Even though we interpret the universal quantifiers in the actualist way, we cannot allow the domains of each world to be completely unrelated to each other. This is because we want statements such as the axiom $\diamond \forall x \varphi \vdash \forall x \diamond \varphi$ to be sound. This axiom forces us to have *inclusive* frames, which means that if w sees a world u , then the domain of w is included or at least embedded in the domain of u . We also require that our frames be transitive, for we want the axiom $\diamond \diamond \varphi \vdash \diamond \varphi$ to be sound. Finally, the interpretation of a constant should indeed be constant throughout (the relevant part of) any useful model. Thus, we introduce the notion of adequate frames and models.

Definition 4.3 A frame \mathcal{F} is *adequate* if the accessibility relation R is:

- inclusive: if wRu , then $M_w \subseteq M_u$; and
- transitive: if wRu and uRv , then wRv .

A model is *adequate* if it is based on an adequate frame and it is:

- concordant: if wRu , then $c^{I_w} = c^{I_u}$ for every constant c .

Note that in an adequate and rooted model the interpretation of the constants is the same at every world.

In order to define truth at a world in a first-order model, we use assignments. A w -assignment g is a function assigning a member of the domain M_w to each variable in the language. In an adequate frame, any w -assignment can be seen as a v -assignment as long as wRv , because $M_w \subseteq M_v$ and hence there is a trivial inclusion (or coercion) $\iota_{w,v} : M_w \rightarrow M_v$. If g is such a w -assignment, we represent the corresponding v -assignment $\iota_{w,v} \circ g$ by g^v when w and v are clear from the context.

Two w -assignments g and h are Γ -*alternative*, denoted by $g \sim_\Gamma h$, if they coincide on all variables other than the ones in Γ . If $\Gamma = \{x\}$, then we write

simply x -alternative and $g \sim_x h$.

We extend a given w -assignment g to terms by defining $g(c) := c^{I_w}$ where c is any constant.

Definition 4.4 Let $\mathcal{M} = \langle W, R, \{M_w\}_{w \in W}, \{I_w\}_{w \in W}, \{J_w\}_{w \in W} \rangle$ be a relational model in some signature Σ , and let $w \in W$ be a world, g be a w -assignment, S be an n -ary relation symbol, and φ, ψ be formulas in the language of Σ .

We define $\mathcal{M}, w \Vdash^g \varphi$ (φ is true at w under g) by induction on φ as follows.

- $\mathcal{M}, w \Vdash^g \top$;
- $\mathcal{M}, w \Vdash^g S(t_0, \dots, t_{n-1})$ iff $\langle g(t_0), \dots, g(t_{n-1}) \rangle \in S^{J_w}$;
- $\mathcal{M}, w \Vdash^g \varphi \wedge \psi$ iff both $\mathcal{M}, w \Vdash^g \varphi$ and $\mathcal{M}, w \Vdash^g \psi$;
- $\mathcal{M}, w \Vdash^g \diamond \varphi$ iff there is $v \in W$ such that wRv and $\mathcal{M}, v \Vdash^{g^t} \varphi$;
- $\mathcal{M}, w \Vdash^g \forall x \varphi$ iff for all w -assignments h such that $h \sim_x g$, we have $\mathcal{M}, w \Vdash^h \varphi$.

We now present a number of simple results needed to prove the relational soundness of QRC_1 . These are standard observations about either first-order models or Kripke models that we adapted to our case.

Remark 4.5 Let \mathcal{M} be an adequate model, w be any world, g, h be any Γ -alternative w -assignments, and φ be a formula with no free variables in Γ . Then:

$$\mathcal{M}, w \Vdash^g \varphi \iff \mathcal{M}, w \Vdash^h \varphi.$$

Lemma 4.6 (Substitution in formula) *Let \mathcal{M} be an adequate model, w be a world, and g, \tilde{g} be x -alternative w -assignments such that $\tilde{g}(x) = g(t)$. Then for every formula φ with t free for x :*

$$\mathcal{M}, w \Vdash^{\tilde{g}} \varphi \iff \mathcal{M}, w \Vdash^g \varphi[x \leftarrow t].$$

Proof. By induction on φ . We only present the cases of the diamond and of the universal quantifier; the remaining cases are straightforward. We assume without loss of generality that x is a free variable of φ , since otherwise we could use Remark 4.5.

Suppose that φ is $\diamond \psi$ and assume that $\mathcal{M}, w \Vdash^{\tilde{g}} \diamond \psi$. Then there is a world v such that wRv and $\mathcal{M}, v \Vdash^{\tilde{g}^t} \psi$. Note that $g^t \sim_x \tilde{g}^t$ and $\tilde{g}^t(x) = g^t(t)$ (either t is a variable and this is a consequence of $\tilde{g}(x) = g(t)$, or t is a constant and this follows from $t^{I_w} = t^{I_v}$) and thus by the induction hypothesis $\mathcal{M}, v \Vdash^{g^t} \psi[x \leftarrow t]$. This gives us $\mathcal{M}, w \Vdash^g \diamond \psi[x \leftarrow t]$, as desired. The other direction is analogous.

Suppose now that $\varphi = \forall z \psi$ and assume that $\mathcal{M}, w \Vdash^{\tilde{g}} \forall z \psi$. Note that x and z are different variables, for otherwise x would not be free in φ . Let h be any w -assignment such that $h \sim_z g$. We wish to show $\mathcal{M}, w \Vdash^h \psi[x \leftarrow t]$. Define \tilde{h} such that $\tilde{h} \sim_x h$ and $\tilde{h}(x) := h(t)$. Then by the induction hypothesis we can reduce our goal to $\mathcal{M}, w \Vdash^{\tilde{h}} \psi$. By our assumption, it is enough to check that $\tilde{h} \sim_z \tilde{g}$.

In order to see this, note first that $\tilde{h} \sim_{\{x,z\}} h$ (because $\tilde{h} \sim_x h$). Similarly, $h \sim_{\{x,z\}} g$ and $g \sim_{\{x,z\}} \tilde{g}$. Then $\tilde{h} \sim_{\{x,z\}} \tilde{g}$ by transitivity of $\sim_{\{x,z\}}$. But $\tilde{g}(x) = g(t)$ by assumption; $g(t) = h(t)$ because $g \sim_z h$ (z and t are not the same variable because otherwise t would not be free for x in φ); and $h(t) = \tilde{h}(x)$ by construction of \tilde{h} . Thus $\tilde{g}(x) = \tilde{h}(x)$, and $\tilde{h} \sim_z \tilde{g}$.

Towards the other direction, assume that $\mathcal{M}, w \Vdash^g (\forall z \psi)[x \leftarrow t]$ and that x and z are not the same variable. Let $\tilde{h} \sim_z \tilde{g}$ be a w -assignment. We wish to show $\mathcal{M}, w \Vdash^{\tilde{h}} \psi$. Define $h \sim_x \tilde{h}$ such that $h(x) := g(x)$. Note that $h \sim_z g$ by the transitivity of $\sim_{x,z}$ (using a similar argument to the one above). Thus we know that $\mathcal{M}, w \Vdash^h \psi[x \leftarrow t]$ by assumption. It only remains to show that $\tilde{h}(x) = h(t)$, as we can then finally use the induction hypothesis to finish. If t is x there is nothing to show, and t cannot be z , because z is not free for x in $\forall z \psi$. Thus, $h(t) = g(t) = \tilde{g}(x) = \tilde{h}(x)$. \square

We need some more work to prove the soundness of Rule 2.1.(xi), which we postpone to Appendix B. Otherwise we are ready to prove that QRC_1 is sound with respect to the relational semantics presented above.

Theorem 4.7 (Relational soundness) *If $\varphi \vdash \psi$, then for any adequate model \mathcal{M} , for any world $w \in W$, and for any w -assignment g :*

$$\mathcal{M}, w \Vdash^g \varphi \implies \mathcal{M}, w \Vdash^g \psi.$$

Proof. By induction on the proof of $\varphi \vdash \psi$.

In the case of the axioms $\varphi \vdash \top$ and $\varphi \vdash \varphi$, the result is clear, as it is for the conjunction elimination axioms. The conjunction introduction and cut rules follow easily from the definitions.

For the necessitation rule assume the result for $\varphi \vdash \psi$ and further assume that $\mathcal{M}, w \Vdash^g \Diamond \varphi$. Then there is a world v such that wRv and $\mathcal{M}, v \Vdash^{g^t} \varphi$. We wish to see $\mathcal{M}, w \Vdash^g \Diamond \psi$. Taking v as a suitable witness, our goal changes to $\mathcal{M}, v \Vdash^{g^t} \psi$. Thus the induction hypothesis for v and g^t finishes the proof.

For the transitivity axiom, $\Diamond \varphi \vdash \Diamond \psi$, assume that $\mathcal{M}, w \Vdash^g \Diamond \varphi$. Then there is a world v such that wRv and $\mathcal{M}, v \Vdash^{\iota_{w,v} \circ g} \varphi$, and also a subsequent world u such that vRu and $\mathcal{M}, u \Vdash^{\iota_{v,u} \circ (\iota_{w,v} \circ g)} \varphi$. Observing that $\iota_{v,u} \circ (\iota_{w,v} \circ g)$ is the same as $\iota_{w,u} \circ g$, we get $\mathcal{M}, u \Vdash^{\iota_{w,u} \circ g} \varphi$, and the transitivity of R provides wRu , which is enough to see $\mathcal{M}, w \Vdash^g \Diamond \varphi$, as desired.

In the case of $\Diamond \forall x \varphi \vdash \forall x \Diamond \varphi$, assume that $\mathcal{M}, w \Vdash^g \Diamond \forall x \varphi$. Then there is $v \in W$ such that wRv and for every v -assignment h with $h \sim_x g^t$ we have $\mathcal{M}, v \Vdash^h \varphi$. Let f be any w -assignment such that $f \sim_x g$. Taking v as a suitable world seen by w , we wish to check that $\mathcal{M}, v \Vdash^{f^t} \varphi$. By assumption, it is enough to see $f^t \sim_x g^t$, and this follows from $f \sim_x g$.

For the \forall -introduction rule on the right, assume the result for $\varphi \vdash \psi$ towards showing the soundness of $\varphi \vdash \forall x \psi$ with $x \notin \text{fv}(\varphi)$. Assume further that $\mathcal{M}, w \Vdash^g \varphi$. Let h be a w -assignment such that $h \sim_x g$. We wish to see that $\mathcal{M}, w \Vdash^h \psi$. Since x is not a free variable in φ , we know that $\mathcal{M}, w \Vdash^h \varphi$ by Remark 4.5. The result follows from the induction hypothesis with w -assignment h .

Consider now the \forall -introduction rule on the left. Assume the result for $\varphi[x \leftarrow t] \vdash \psi$ with t free for x in φ and assume further that $\mathcal{M}, w \Vdash^g \forall x \varphi$. Then for every w -assignment h such that $h \sim_x g$ we have $\mathcal{M}, w \Vdash^h \varphi$. Define $h \sim_x g$ such that $h(x) = g(t)$. We obtain $\mathcal{M}, w \Vdash^g \psi$ by the induction hypothesis and Lemma 4.6.

The term instantiation rule, Rule 2.1.(x), is sound by Lemma 4.6, and the generalization on constants rule, Rule 2.1.(xi), is sound by Lemma B.6. \square

5 Relational completeness

We now wish to prove the relational completeness of QRC_1 . For every undervivable sequent we provide a model that doesn't satisfy it. These models are term models where the worlds are akin to maximal consistent sets. However, since we have no way to express negative formulas, each world is a pair of sets of formulas instead: the set of positive formulas at that world and the set of negative ones.

We start by defining some notions about pairs of formulas, and we write p, q, \dots to refer to generic pairs that may not have all the necessary properties to be a world in a term model. Given a pair of sets p , the first set is the positive set, or p^+ , and the second one is the negative set, or p^- .

Definition 5.1 Given a set of formulas Γ and a formula φ , we say that φ follows from Γ , and write $\Gamma \vdash \varphi$ (overloading the existing notation), if there are formulas $\gamma_0, \dots, \gamma_n \in \Gamma$ such that $\gamma_0 \wedge \dots \wedge \gamma_n \vdash \varphi$.

Definition 5.2 Let Φ be a set of formulas.

- A Φ -extension of a pair $p = \langle p^+, p^- \rangle$ is a pair $q = \langle q^+, q^- \rangle$ such that $p^+ \subseteq q^+ \subseteq \Phi$ and $p^- \subseteq q^- \subseteq \Phi$. In that case we write $p \subseteq q \subseteq \Phi$.
- A pair p is consistent if for every $\delta \in p^-$ we have $p^+ \not\vdash \delta$.
- A pair $p \subseteq \Phi$ is Φ -maximal consistent if it is consistent and there is no consistent Φ -extension of p .
- A pair p is fully witnessed if for every formula $\forall x \varphi \in p^-$ there is a constant c such that $\varphi[x \leftarrow c] \in p^-$.
- A pair p is Φ -MCW if it is Φ -maximal consistent and fully witnessed.

Lemma 5.3 A pair p is Φ -maximal consistent if and only if it is consistent and for every $\varphi \in \Phi$ either $\varphi \in p^+$ or $\varphi \in p^-$.

Proof. The right-to-left implication is obvious. To check the other one assume that p is Φ -maximal consistent and let $\varphi \in \Phi$. If $p^+ \vdash \varphi$, then $\langle p^+ \cup \{\varphi\}, p^- \rangle$ is still consistent, and thus by maximality it must be that $\varphi \in p^+$. If on the other hand $p^+ \not\vdash \varphi$, then $\langle p^+, p^- \cup \{\varphi\} \rangle$ is consistent, and thus we may conclude $\varphi \in p^-$. \square

Definition 5.4 Given a set of constants C , the closure of a formula φ under C , written $\mathcal{Cl}_C(\varphi)$, is defined by induction on the formula as such: $\mathcal{Cl}_C(\top) := \{\top\}$; $\mathcal{Cl}_C(S(t_0, \dots, t_{n-1})) := \{S(t_0, \dots, t_{n-1}), \top\}$; $\mathcal{Cl}_C(\varphi \wedge \psi) := \{\varphi \wedge \psi\} \cup \mathcal{Cl}_C(\varphi) \cup \mathcal{Cl}_C(\psi)$.

$\mathcal{C}l_C(\psi); \mathcal{C}l_C(\diamond\varphi) := \{\diamond\varphi\} \cup \mathcal{C}l_C(\varphi)$; and

$$\mathcal{C}l_C(\forall x \varphi) := \{\forall x \varphi\} \cup \bigcup_{c \in C} \mathcal{C}l_C(\varphi[x \leftarrow c]).$$

The closure under C of a set of formulas Γ is the union of the closures under C of each of the formulas in Γ :

$$\mathcal{C}l_C(\Gamma) := \bigcup_{\gamma \in \Gamma} \mathcal{C}l_C(\gamma).$$

The closure of a pair p is defined as the closure of $p^+ \cup p^-$.

Note that the closure of a set of closed formulas is itself a set of closed formulas. We often use the concept of closure under a set of constants on an already Φ -maximal pair when we wish to extend the signature of the formulas in Φ with a new set of constants.

Given a consistent pair p , we wish to generate a Φ -maximal consistent and fully witnessed extension of p , for some set of formulas Φ . In the usual Henkin construction this is traditionally accomplished in two steps: first extend the signature to include a constant for each existential statement and add every closed formula of the form $\exists x \varphi \rightarrow \varphi[x \leftarrow c_\varphi]$ to your set, proving that this didn't break consistency. Then prove a Lindenbaum Lemma to the effect that consistent sets can be extended to maximal consistent sets. The resulting sets will be maximal, consistent, and fully witnessed. However we can not do this because we cannot express implications. Thus if we were to add a witness for every existential formula in our original pair p (read: universal formula in p^-) and then use a Lindenbaum lemma to make it maximal, there could be new existential formulas without witnesses. We might have to iterate the process over and over again, or at least a proof of termination would be non-trivial. Fortunately, this isn't needed. We can manage with a finite set of witnesses, as is shown by the following lemma.

Lemma 5.5 *Given a finite signature Σ with constants C , a finite set of closed formulas Φ in the language of Σ and a consistent pair $p \subseteq \mathcal{C}l_C(\Phi)$, there is a finite set of constants $D \supseteq C$ and a pair $q \supseteq p$ in the language of Σ extended by D such that q is $\mathcal{C}l_D(\Phi)$ -MCW and $d_\diamond(q^+) = d_\diamond(p^+)$.*

Proof. Let $N := \{c_0, \dots, c_{d_\forall(\Phi)-1}\}$ and $D := C \cup N$.

Let $q_0 := p$. For every formula φ_i in $\mathcal{C}l_D(\Phi)$, if $p^+ \vdash \varphi_i$, define $q_{i+1} = \langle q_i^+ \cup \{\varphi_i\}, q_i^- \rangle$; otherwise define $q_{i+1} = \langle q_i^+, q_i^- \cup \{\varphi_i\} \rangle$. Let $q := q_n$, where n is the size of $\mathcal{C}l_D(\Phi)$, i.e., q is what we have at the final iteration of this process.

Now assume by way of contradiction that q is not consistent, and let $\psi \in q^-$ be such that $q^+ \vdash \psi$. Note that for every $\chi \in q^+$ we know that $p^+ \vdash \chi$, because this was the required condition to add χ to q^+ in the first place. Thus, it must be that $p^+ \vdash \psi$. But then the algorithm would have placed ψ in q^+ instead of q^- and we reach a contradiction. We conclude that q is consistent.

Lemma 5.3 tells us that q is $\mathcal{C}l_D(\Phi)$ -maximal consistent, because every formula of $\mathcal{C}l_D(\Phi)$ is either in q^+ or q^- .

On the other hand, we know by Lemma 2.4 that $d_\diamond(q^+) \leq d_\diamond(p^+)$ because every formula in q^+ is a consequence of p^+ . We obtain the equality by observing that $p^+ \subseteq q^+$.

It remains to show that q is fully witnessed. Let $\forall x \psi$ be a formula in q^- . We claim that there is $c_i \in N$ such that c_i does not appear in $\forall x \psi$. The constants in N are new, so the only way to have a formula $\chi \in \mathcal{C}l_D(\Phi)$ with a constant $c_j \in N$ is if the formula $\forall y \chi[c_j \leftarrow y]$ is also in $\mathcal{C}l_D(\Phi)$, for some variable y that does not appear (free) in χ . Assume then that all the constants in N appear in ψ , and let m be the size of N . Then the formula $\forall y_0 \cdots \forall y_{m-1} \forall x \psi[c_{s_{m-1}} \leftarrow y_{m-1}] \cdots [c_{s_0} \leftarrow y_0]$ must be in $\mathcal{C}l_D(\Phi)$ for some variables y_i and permutation s of the numbers between 0 and $m-1$. But this formula has quantifier depth $m+1$, which is a contradiction because the closure under any set of constants doesn't change the depth of a set of formulas.

Let then $\forall x \psi \in q^-$ and $c_i \in N$ be a constant that does not appear in $\forall x \psi$. Then we claim that $\psi[x \leftarrow c_i] \in q^-$. Assume it is not the case. Then it must be that $p^+ \vdash \psi[x \leftarrow c_i]$. Note that c_i does not appear in p^+ and that x is not a free variable of p^+ due to it being a set of closed formulas. Then by Lemma 2.2.(v) we obtain that $p^+ \vdash \forall x \psi$, which is a contradiction. \square

The next step is to link maximal consistent and fully witnessed pairs through a relation that respects the diamond formulas in the pair. To that end we define \hat{R} and prove some properties about it.

Definition 5.6 The relation \hat{R} between pairs is such that $p \hat{R} q$ if and only if both of following hold:

- (i) for any formula $\diamond \varphi \in p^-$ we have $\varphi, \diamond \varphi \in q^-$; and
- (ii) there is some formula $\diamond \psi \in p^+ \cap q^-$.

Lemma 5.7 *The relation \hat{R} restricted to consistent pairs is transitive and irreflexive.*

Proof. In order to see that \hat{R} is transitive, assume that $p \hat{R} q \hat{R} r$. We wish to see that $p \hat{R} r$. Let $\diamond \varphi \in p^-$ be arbitrary. Then $\diamond \varphi \in q^-$ because $p \hat{R} q$, and then $\varphi, \diamond \varphi \in r^-$ because $q \hat{R} r$. Let now $\diamond \psi \in p^+ \cap q^-$. Since $q \hat{R} r$ we know that $\diamond \psi \in r^-$. Then $\diamond \psi \in p^+ \cap r^-$.

Regarding irreflexivity, suppose that there is a pair p such that $p \hat{R} p$. Then there must be $\diamond \psi \in p^+ \cap p^-$, which contradicts the consistency of p . \square

There is an equivalent formulation of \hat{R} by looking at the positive sets.

Lemma 5.8 *Given a set of formulas Φ , two sets of constants $C \subseteq D$, and pairs p, q such that p is $\mathcal{C}l_C(\Phi)$ -maximal consistent and q is $\mathcal{C}l_D(\Phi)$ -maximal consistent, we have that $p \hat{R} q$ if and only if both of the following hold:*

- (i) for every formula $\diamond \varphi \in \mathcal{C}l_C(\Phi)$, if either $\varphi \in q^+$ or $\diamond \varphi \in q^+$, then $\diamond \varphi \in p^+$; and
- (ii) there is some formula $\diamond \psi \in p^+ \cap q^-$.

Proof. Assume that $p\hat{R}q$ and let $\diamond\varphi \in \mathcal{C}l_C(\Phi)$ be such that either $\varphi \in q^+$ or $\diamond\varphi \in q^+$. Assume by contradiction that $\diamond\varphi \notin p^+$. Then by Lemma 5.3 we know that $\diamond\varphi \in p^-$. Thus since $p\hat{R}q$, we obtain both $\varphi \in q^-$ and $\diamond\varphi \in q^-$. But this contradicts the consistency of q . The last condition holds by the definition of \hat{R} .

Assume now that these conditions hold, towards checking that $p\hat{R}q$. Only the first condition is in question. Let $\diamond\varphi \in p^-$ and assume that $\varphi \notin q^-$. By Lemma 5.3, it must be that $\varphi \in q^+$. Then $\diamond\varphi \in p^+$, which contradicts the consistency of p . Assume now that $\diamond\varphi \notin q^-$. By the same token, $\diamond\varphi$ must be in q^+ . Then $\diamond\varphi \in p^+$, reaching a contradiction again. \square

The following lemma states that, given a suitable pair p where $\diamond\varphi$ holds, we can find a second suitable pair q where φ holds and $p\hat{R}q$.

Lemma 5.9 (Pair existence) *Let Σ be a signature with a finite set of constants C , and Φ be a finite set of closed formulas in the language of Σ . If p is a $\mathcal{C}l_C(\Phi)$ -MCW pair and $\diamond\varphi \in p^+$, then there is a finite set of constants $D \supseteq C$ and a $\mathcal{C}l_D(\Phi)$ -MCW pair q such that $p\hat{R}q$, $\varphi \in q^+$, and $d_\diamond(q^+) < d_\diamond(p^+)$.*

Proof. Consider the pair $r = (\{\varphi\}, \{\delta, \diamond\delta \mid \diamond\delta \in p^-\} \cup \{\diamond\varphi\})$. Assume that r is not consistent, and thus that there is a formula $\psi \in r^-$ such that $\varphi \vdash \psi$. It cannot be that ψ is $\diamond\varphi$ due to Lemma 2.5. Thus there is $\diamond\delta \in p^-$ such that either $\varphi \vdash \delta$ or $\varphi \vdash \diamond\delta$. By Rule 2.1.(v) we get either $\diamond\varphi \vdash \diamond\delta$ or $\diamond\varphi \vdash \diamond\diamond\delta$, which also implies $\diamond\varphi \vdash \diamond\delta$ by Axiom 2.1.(vi). This contradicts the consistency of p , which leads us to conclude that r is consistent.

We can now use Lemma 5.5 to obtain a finite set of constants $D \supseteq C$ and a $\mathcal{C}l_D(\Phi)$ -MCW pair $q \supseteq r$ such that $d_\diamond(q^+) = d_\diamond(r^+) = d_\diamond(\varphi) < d_\diamond(p^+)$.

It remains to show that $p\hat{R}q$, but this is clear by the definition of r : for every $\diamond\delta \in p^-$, the formulas δ and $\diamond\delta$ are in r^- (and hence in q^-), and the formula $\diamond\varphi$ is both in p^+ and in q^- . \square

We are now ready to define an adequate model $\mathcal{M}[p]$ from any given finite and consistent pair p such that $\mathcal{M}[p]$ satisfies the formulas in p^+ and doesn't satisfy the formulas in p^- . The idea is to build a term model where each world w is a $\mathcal{C}l_{M_w}(p)$ -MCW pair, and the worlds are related by (a sub-relation of) \hat{R} . The worlds in this model will be pairs of formulas in different signatures, as we will add new constants every time we create a new world. However, the model is intended to satisfy only formulas in the original signature of p .

Definition 5.10 Given a finite consistent pair p of closed formulas with constants in a finite set C , we define an adequate model $\mathcal{M}[p]$.

We start by defining the underlying frame in an iterative manner. The root is given by Lemma 5.5 applied to C and p , obtaining D and q . Frame \mathcal{F}^0 is then defined such that its set of worlds is $W^0 := \{q\}$, its relation R^0 is empty, and the domain of q is $M_q^0 := D$.

Assume now that we already have a frame \mathcal{F}^i , and we set out to define \mathcal{F}^{i+1} as an extension of \mathcal{F}^i . For each leaf w of \mathcal{F}^i , i.e., each world such that there is no world $v \in \mathcal{F}^i$ with $wR^i v$, and for each formula $\diamond\varphi \in w^+$, use Lemma 5.9 to

obtain a finite set $E \supseteq M_w^i$ and a $\mathcal{C}l_E(w)$ -MCW pair v such that $w\hat{R}v$, $\varphi \in v^+$, and $d_\diamond(v^+) < d_\diamond(w^+)$. Now add v to W^{i+1} , add $\langle w, v \rangle$ to R^{i+1} , and define M_v^{i+1} as E .

The process described above terminates because each pair is finite and the modal depth of p^+ (and consequently of $\mathcal{C}l_X(p)$ for any set X) is also finite. Thus there is a final frame $\mathcal{F}^{d_\diamond(p^+)}$. This frame is inclusive by construction, but not transitive. We obtain $\mathcal{F}[p]$ as the transitive closure of $\mathcal{F}^{d_\diamond(p^+)}$, which can be easily seen to still be inclusive. Thus the frame $\mathcal{F}[p]$ is adequate.

In order to obtain the model $\mathcal{M}[p]$ based on the frame $\mathcal{F}[p]$, let I_q take constants in C to their corresponding version as domain elements and if w is any other world, let I_w coincide with I_q . This is necessary to make sure that the model is concordant, because q sees every other world, and is sufficient to see that $\mathcal{M}[p]$ is adequate. Finally, given an n -ary predicate letter S and a world w , define S^{J_w} as the set of n -tuples $\langle d_0, \dots, d_{n-1} \rangle \subseteq (M_w)^n$ such that $S(d_0, \dots, d_{n-1}) \in w^+$.

Lemma 5.11 *Let p be as above. The following are properties of $\mathcal{F}[p] = \langle W, R, \{M_w\}_{w \in W} \rangle$ and $\mathcal{M}[p] = \langle \mathcal{F}[p], \{I_w\}_{w \in W}, \{J_w\}_{w \in W} \rangle$:*

- (i) *For every world w , its domain M_w is finite.*
- (ii) *The set of worlds W is finite.*
- (iii) *Every world $w \in W$ is $\mathcal{C}l_{M_w}(p)$ -maximal consistent and fully witnessed.*
- (iv) *For every world $w \in W$, we have $\top \in w^+$.*
- (v) *For any two worlds $w, u \in W$, if wRu , then $w\hat{R}u$.*

Proof. These are simple consequences of the definition of $\mathcal{M}[p]$. The finiteness of the domains is achieved in Lemma 5.5, while the finiteness of W is proved in Definition 5.10. For the last property, note that R is the transitive closure of $R^{d_\diamond(p^+)}$. If $wR^{d_\diamond(p^+)}u$, then $w\hat{R}u$ by construction. The result then follows by the transitivity of \hat{R} (Lemma 5.7). \square

We are almost ready to state the truth lemma, which roughly states that provability at a world w of $\mathcal{M}[p]$ is the same as membership in w^+ . However, the signatures of the worlds of $\mathcal{M}[p]$ are more expressive than the signature of the formulas we care about. Furthermore, all the formulas in the worlds of $\mathcal{M}[p]$ are closed, while formulas in general may have free variables. In order to deal with this, we replace the free variables of a formula with constants in the appropriate signature first.

Definition 5.12 Given a formula φ in a signature Σ and a function g from the set of variables to a set of constants in some signature $\Sigma' \supseteq \Sigma$, we define the formula φ^g in the signature Σ' as φ with each free variable x simultaneously replaced by $g(x)$.

Given a variable x , the formula $\varphi^{g \setminus x}$ is as above, except that the variable x doesn't get replaced even if it is free in φ .

Lemma 5.13 (Truth lemma) *Let Σ be a signature with a finite set of constants C . For any finite non-empty consistent pair p of closed formulas in the*

language of Σ , world $w \in \mathcal{M}[p]$, w -assignment g , and formula φ in the language of Σ such that $\varphi^g \in \mathcal{C}\ell_{M_w}(p)$, we have that

$$\mathcal{M}[p], w \Vdash^g \varphi \iff \varphi^g \in w^+.$$

Proof. By induction on φ . The cases of \top and conjunction are straightforward, so we focus on the other ones.

In the case of the relational symbols, we can take $\varphi = S(x, c)$ without loss of generality, where $c \in C$. Note that $\mathcal{M}[p], w \Vdash^g S(x, c)$ if and only if $\langle g(x), c^{I_w} \rangle \in S^{J_w}$, if and only if $S(g(x), c^{I_w}) \in w^+$. Since $c \in C$, we know by the definition of $\mathcal{M}[p]$ that $c^{I_w} = c$. Thus, we conclude that $\mathcal{M}[p], w \Vdash^g S(x, c)$ if and only if $S(g(x), c) \in w^+$, as desired.

Consider now the case of the universal quantifier. For the left to right implication, suppose that $\mathcal{M}[p], w \Vdash^g \forall x \varphi$. Then for every w -assignment $h \sim_x g$ we have $\mathcal{M}[p], w \Vdash^h \varphi$. Thus for each such h we know that $\varphi^h \in w^+$ by the induction hypothesis ($\varphi^h \in \mathcal{C}\ell_{M_w}(p)$ because $(\forall x \varphi)^g \in \mathcal{C}\ell_{M_w}(p)$). We want to show that $(\forall x \varphi)^g \in w^+$, i.e., that $\forall x \varphi^{g \setminus x} \in w^+$. Assume by contradiction that this is not the case. Then, since w is $\mathcal{C}\ell_{M_w}(p)$ -maximal consistent, it must be that $\forall x \varphi^{g \setminus x} \in w^-$. Let $c \in M_w$ be a witness such that $\varphi^{g \setminus x}[x \leftarrow c] \in w^-$, which exists because w is fully witnessed. Let h be the w -assignment that coincides with g everywhere except at x , where $h(x) = c$. Then $g \sim_x h$ and $\varphi^{g \setminus x}[x \leftarrow c] = \varphi^h$. But this contradicts our earlier observation that for every such h the formula φ^h is in w^+ .

For the right to left implication, let $\forall x \varphi^{g \setminus x} \in w^+$, and let $h \sim_x g$ be any w -assignment. We want to show that $\mathcal{M}[p], w \Vdash^h \varphi$. By the induction hypothesis this is the same as showing that $\varphi^h \in w^+$. But $\varphi^h = \varphi^{g \setminus x}[x \leftarrow h(x)]$, and this is in w^+ by the completeness and consistency of w .

Finally, consider the case of the diamond. For the left to right implication, assume that $\mathcal{M}[p], w \Vdash^g \diamond \varphi$. Then there is some world u such that wRu and $\mathcal{M}[p], u \Vdash^{g^t} \varphi$.⁴ By the induction hypothesis we obtain $\varphi^{g^t} \in u^+$, and consequently $\varphi^g \in u^+$. Now, since wRu , we also know that $w\hat{R}u$ by Lemma 5.11.(v), and thus by Lemma 5.8 we obtain $\diamond \varphi^g \in w^+$ as desired.

For the right to left implication, assume that $(\diamond \varphi)^g \in w^+$. By the construction of $\mathcal{M}[p]$, there is a world u such that $\varphi^g \in u^+$ (and hence $\varphi^{g^t} \in u^+$) and wRu , and then $\mathcal{M}[p], u \Vdash^{g^t} \varphi$ by the induction hypothesis, from which we finally conclude $\mathcal{M}[p], w \Vdash^g \diamond \varphi$. \square

Theorem 5.14 (Completeness) *If $\varphi \not\vdash \psi$, then there are an adequate finite model \mathcal{M} , a world $w \in W$, and a w -assignment g such that*

$$\mathcal{M}, w \Vdash^g \varphi \quad \text{and} \quad \mathcal{M}, w \not\vdash^g \psi.$$

Proof. Define a set of new constants $C := \{c_{x_i} \mid x_i \in \text{fv}(p)\}$ and let g be a map from the set of variables to C that assigns c_{x_i} to x_i for each i . Define p

⁴ Recall that if g is a w -assignment and wRu , we write g^t to refer to the u -assignment that behaves like g .

as $\langle \varphi^g, \psi^g \rangle$, and assume it is not consistent, i.e., that $\varphi^g \vdash \psi^g$. Then by (a generalization of) Rule 2.1.(xi) and Lemma 2.6 we would get that $\varphi \vdash \psi$. Thus p is consistent. Let $\mathcal{M}[p]$ be the model generated from p as in Definition 5.10 and let w be the root of this model, which is an extension of p . Lemma 5.13 tells us that $\mathcal{M}[p], w \Vdash^g \varphi$ and $\mathcal{M}[p], w \not\Vdash^g \psi$ because $\varphi^g \in w^+$ and $\psi^g \notin w^+$. \square

We observe that QRC_1 has the finite model property with respect to domains and number of worlds (as a consequence of Lemma 5.11 and Theorem 5.14). It is interesting that QRC_1 is this well behaved while, say, predicate intuitionistic logic doesn't enjoy the finite model property with respect to either of these.

Theorem 5.15 *QRC_1 is decidable.*

Proof. Since QRC_1 is recursively axiomatized, has the finite model property, and it is easy to check whether a given finite model is adequate, Post's Theorem allows us to decide whether $\varphi \vdash \psi$. \square

Appendix

A Arithmetical soundness

Here we finally prove the soundness of QRC_1 regarding the arithmetical semantics. The following lemma is standard for Σ_1 axiomatizations α and the reader can consult [7] for details.

Lemma A.1 *For any Σ_1 formula α , we have that*

- (i) $\text{I}\Sigma_1 \vdash \text{Con}_\alpha(\text{Con}_\alpha) \rightarrow \text{Con}_\alpha$;
- (ii) $\text{I}\Sigma_1 \vdash \exists z \Box_\alpha \varphi \rightarrow \Box_\alpha \exists z \varphi$.

Recall that we define $\text{QRC}_1(T)$ for a c.e. theory T in the language of arithmetic which extends $\text{I}\Sigma_1$ as follows.

$$\text{QRC}_1(T) = \{ \varphi(\mathbf{c}, \mathbf{x}) \vdash \psi(\mathbf{c}, \mathbf{x}) \mid \forall * T \vdash \forall \theta \forall \mathbf{y} \forall \mathbf{z} (\Box_{\psi^*} \theta \rightarrow \Box_{\varphi^*} \theta) \}.$$

In the above we assume that all the free variables other than u in $\psi^* \wedge \varphi^*$ are among the \mathbf{y} and \mathbf{z} . The θ are sentences without free variables. Furthermore, we stress that all realizations map to Σ_1 formulas (modulo provable equivalence).

Theorem A.2 (Arithmetical soundness) $\text{QRC}_1 \subseteq \text{QRC}_1(\text{I}\Sigma_1)$.

Proof. We proceed by (an external) induction on the proof of $\varphi \vdash \psi$. We shall briefly comment on some of the cases. The case of the axiom $\varphi \vdash \top$ is clear since by an easy induction on φ we can prove that over predicate logic $\varphi^*(\mathbf{y}, \mathbf{z}, u) \leftrightarrow \tau(u) \vee \varphi'(\mathbf{y}, \mathbf{z}, u)$ for some formula φ' . The axioms $\varphi \wedge \psi \vdash \varphi$ are easily seen to be sound since $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$, that is, the formula that defines the union of two axiom sets.

The rule that if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$ is straightforward but the rule that if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$ is slightly more tricky. To see the soundness, we fix a particular realization $*$ and reason in $\text{I}\Sigma_1$. Inside $\text{I}\Sigma_1$ we fix arbitrary \mathbf{y} , \mathbf{z} and θ and assume $\Box_{(\psi \wedge \chi)^*(\mathbf{y}, \mathbf{z})} \theta$, that is, $\Box_{\psi^*(\mathbf{y}, \mathbf{z}) \vee \chi^*(\mathbf{y}, \mathbf{z})} \theta$.

Thus, $\Box_{\psi^*(\mathbf{y}, \mathbf{z})}(\forall i < n \ \xi_i \rightarrow \theta)$ for some collection of axioms $\{\xi_i\}_{i < n}$ satisfying $\chi^*(\mathbf{y}, \mathbf{z})$. By the induction hypothesis on $\varphi \vdash \psi$ we obtain $\Box_{\varphi^*(\mathbf{y}, \mathbf{z})}(\forall i < n \ \xi_i \rightarrow \theta)$, so that $\Box_{\tau}(\forall i < n \ \xi_i \rightarrow (\forall j < m \ \varphi_j \rightarrow \theta))$ for some collection of axioms $\{\varphi_j\}_{j < m}$ satisfying $\varphi^*(\mathbf{y}, \mathbf{z})$. Since all the ξ_i satisfy $\chi^*(\mathbf{y}, \mathbf{z})$ we conclude $\Box_{\chi^*(\mathbf{y}, \mathbf{z})}(\forall j < m \ \varphi_j \rightarrow \theta)$. Using now the induction hypothesis on $\varphi \vdash \chi$ we conclude $\Box_{\varphi^*(\mathbf{y}, \mathbf{z})}(\forall j < m \ \varphi_j \rightarrow \theta)$ whence $\Box_{\varphi^*(\mathbf{y}, \mathbf{z})}\theta$ as was to be shown.

We will now see the soundness of the necessitation rule, that is, if $\varphi \vdash \psi$, then $\Diamond\varphi \vdash \Diamond\psi$. We fix some realization $*$. The induction hypothesis for $\varphi \vdash \psi$ applied to the formula \perp gives us $\mathbb{I}\Sigma_1 \vdash \forall \mathbf{y}, \mathbf{z} (\Box_{\psi^*(\mathbf{y}, \mathbf{z})}\perp \rightarrow \Box_{\varphi^*(\mathbf{y}, \mathbf{z})}\perp)$, whence

$$\mathbb{I}\Sigma_1 \vdash \forall \mathbf{y}, \mathbf{z} (\text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})} \rightarrow \text{Con}_{\psi^*(\mathbf{y}, \mathbf{z})}). \quad (\text{A.1})$$

Let π be the standard proof of this. We reason in $\mathbb{I}\Sigma_1$, fixing parameters $\mathbf{y}, \mathbf{z}, \theta$ and assuming $\Box_{(\Diamond\psi)^*(\mathbf{y}, \mathbf{z})}\theta$. Since $(\Diamond\psi)^* := \tau(u) \vee (u = \ulcorner \text{Con}_{\psi^*(\mathbf{y}, \mathbf{z})} \urcorner)$, we conclude $\Box_{\tau}(\text{Con}_{\psi^*(\mathbf{y}, \mathbf{z})} \rightarrow \theta)$. We combine this proof with the proof π of (A.1) to conclude $\Box_{\tau}(\text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})} \rightarrow \theta)$, whence $\Box_{(\Diamond\varphi)^*(\mathbf{y}, \mathbf{z})}\theta$.

The soundness of the axiom $\Diamond\Diamond\varphi \vdash \Diamond\varphi$ is similar, now using Lemma A.1.(i) instead of (A.1).

To see the soundness of the axiom $\Diamond\forall x_i \varphi \vdash \forall x_i \Diamond\varphi$ we start by proving a First Claim:

$$\mathbb{I}\Sigma_1 \vdash \text{Con}_{(\forall x_i \varphi)^*(\mathbf{y}, \mathbf{z})} \rightarrow \forall z_i \text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})}. \quad (\text{A.2})$$

To prove this, we reason in $\mathbb{I}\Sigma_1$ and assume $\exists z_i \Box_{\varphi^*(\mathbf{y}, \mathbf{z})}\perp$, whence for some number ζ we have that $\Box_{\varphi^*(\mathbf{y}, \mathbf{z})[z_i \leftarrow \zeta]}\perp$. Then a slight variation of Lemma A.1.(ii) allows us to see that $\Box_{\exists z_i \varphi^*(\mathbf{y}, \mathbf{z})}\perp$, and thus $\Box_{(\forall x_i \varphi)^*(\mathbf{y}, \mathbf{z})}\perp$.

We now prove a Second Claim:

$$\mathbb{I}\Sigma_1 \vdash \Box_{(\forall x_i \Diamond\varphi)^*(\mathbf{y}, \mathbf{z})}\delta \rightarrow \Box_{\tau(u) \vee (u = \ulcorner \forall x_i \text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})} \urcorner)} \delta. \quad (\text{A.3})$$

We observe $(\forall x_i \Diamond\varphi)^*(\mathbf{y}, \mathbf{z}) = \exists z_i (\Diamond\varphi)^*(\mathbf{y}, \mathbf{z}) = \exists z_i (\tau(u) \vee u = \ulcorner \text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})} \urcorner)$, the latter being provably equivalent to $\tau(u) \vee \exists z_i (u = \ulcorner \text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})} \urcorner)$. To prove the Second Claim, we reason in $\mathbb{I}\Sigma_1$ and assume the antecedent $\Box_{(\forall x_i \Diamond\varphi)^*(\mathbf{y}, \mathbf{z})}\delta$ fixing some $\mathbf{y}, \mathbf{z}, \delta$. Thus, we find a collection of numbers $\{\zeta_j\}_{j < m}$ such that

$$\Box_{\tau}(\forall j < m \ \text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})[z_j \leftarrow \zeta_j]} \rightarrow \delta).$$

Clearly, $\Box_{\tau}(\forall z_i \text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})} \rightarrow \forall j < m \ \text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})[z_j \leftarrow \zeta_j]})$, which suffices to prove the Second Claim.

Let us now go back the soundness of the axiom $\Diamond\forall x_i \varphi \vdash \forall x_i \Diamond\varphi$. We fix $*$, reason in $\mathbb{I}\Sigma_1$, fix $\mathbf{y}, \mathbf{z}, \theta$, and assume $\Box_{(\forall x_i \Diamond\varphi)^*(\mathbf{y}, \mathbf{z})}\theta$. By the Second Claim and the formalized deduction theorem we get $\Box_{\tau}(\forall z_i \text{Con}_{\varphi^*(\mathbf{y}, \mathbf{z})} \rightarrow \theta)$. The First Claim now gives us $\Box_{(\Diamond\forall x_i \varphi)^*(\mathbf{y}, \mathbf{z})}\theta$ as was to be shown.

The soundness of the \forall -introduction rule on the right, that if $\varphi \vdash \psi$, then $\varphi \vdash \forall x_i \psi$ ($x_i \notin \text{fv}(\varphi)$), is not hard but contains a subtlety. To prove it we fix

, reason in $\text{I}\Sigma_1$, fix $\mathbf{y}, \mathbf{z}, \theta$ and assume $\Box_{(\forall x_i \psi)^(\mathbf{y}, \mathbf{z})} \theta$. Since $(\forall x_i \psi)^*(\mathbf{y}, \mathbf{z}) = \exists z_i \psi^*(\mathbf{y}, \mathbf{z})$ we can find numbers $\{\zeta_j\}_{j < m}$ such that

$$\Box_\tau (\forall j < m \ \psi^*(\mathbf{y}, \mathbf{z})[z_i \leftarrow \zeta_j] \rightarrow \theta).$$

Now by the induction hypothesis we get

$$\forall j < m \ \Box_{\varphi^*(\mathbf{y}, \mathbf{z})[z_i \leftarrow \zeta_j]} \psi^*(\mathbf{y}, \mathbf{z})[z_i \leftarrow \zeta_j].$$

Since $x_i \notin \text{fv}(\varphi)$ we have $\forall j < m \ \Box_{\varphi^*(\mathbf{y}, \mathbf{z})} \psi^*(\mathbf{y}, \mathbf{z})[z_i \leftarrow \zeta_j]$. Using Σ_1 -collection we obtain $\Box_{\varphi^*(\mathbf{y}, \mathbf{z})} \forall j < m \ \psi^*(\mathbf{y}, \mathbf{z})[z_i \leftarrow \zeta_j]$, from which the required $\Box_{\varphi^*(\mathbf{y}, \mathbf{z})} \theta$ follows.

The soundness of the remaining axioms and rules is straightforward and boils down to interchanging universal quantifiers. \square

B Relational soundness of the generalization on constants rule

The generalization on constants rule, that if $\varphi[x \leftarrow c] \vdash \psi[x \leftarrow c]$ then also $\varphi \vdash \psi$ as long as c does not appear in φ nor ψ , is indeed sound with respect to Kripke models but the proof needs a couple of extra definitions and results.

We wish to provide counterparts to Remark 4.5 and Lemma 4.6 for when the change happens in the interpretation of a constant instead of a variable. It is straightforward to check that the interpretation of constants not appearing in a formula is not relevant for the truth of that formula:

Remark B.1 Let \mathcal{M} and \mathcal{M}' be adequate models differing only in their constant interpretations $\{I_w\}_{w \in W}$ and $\{I'_w\}_{w \in W}$. Let w be any world, g be any w -assignment, and φ be a formula whose constants are interpreted in the same way by both \mathcal{M} and \mathcal{M}' . Then

$$\mathcal{M}, w \Vdash^g \varphi \iff \mathcal{M}', w \Vdash^g \varphi.$$

However, we need a bit of work to be able to state a counterpart of Lemma 4.6 for constants. We want to be able to replace the interpretation of a constant by an element of the domain of some world w , but this element may not exist in the domains of the worlds below w . Thus we need to first get rid of that part of the model and keep only the sub-graph rooted at w .

Definition B.2 Given a frame $\mathcal{F} = \langle W, R, \{M_w\}_{w \in W} \rangle$ and a world $r \in W$, the *frame restricted at r* , written $\mathcal{F}|_r = \langle W|_r, R|_r, \{M_w\}_{w \in W|_r} \rangle$, is defined as the restriction of \mathcal{F} to the world r and all the worlds accessible from r by R . Thus, $W|_r := \{r\} \cup \{w \in W \mid rRw\}$, and the relation $R|_r$ is R restricted to $W|_r$.

If $\mathcal{M} = \langle \mathcal{F}, \{I_w\}_{w \in W}, \{J_w\}_{w \in W} \rangle$ is a model, then $\mathcal{M}|_r$ is defined as $\langle \mathcal{F}|_r, \{I_w\}_{w \in W|_r}, \{J_w\}_{w \in W|_r} \rangle$.

Remark B.3 If \mathcal{F} is an adequate frame, then so is $\mathcal{F}|_r$ for any $r \in W$. Furthermore, if \mathcal{M} is an adequate model, then so is $\mathcal{M}|_r$.

Remark B.4 Given an adequate model \mathcal{M} and a world $r \in W$, we have that for any formula φ , any world $w \in W_r$ and any w -assignment g :

$$\mathcal{M}, w \Vdash^g \varphi \iff \mathcal{M}|_r, w \Vdash^g \varphi.$$

Definition B.5 Given an adequate model $\mathcal{M} = \langle \mathcal{F}, \{I_w\}_{w \in W}, \{J_w\}_{w \in W} \rangle$, a world $r \in W$, a constant c , and an element of the domain $d \in M_r$, we define $\mathcal{M}|_r[c \leftarrow d] := \langle \mathcal{F}|_r, \{I'_w\}_{w \in W|_r}, \{J_w\}_{w \in W|_r} \rangle$ such that its frame is \mathcal{F} truncated at r , the relational symbols interpretation and the interpretation of all constants except for c coincides with that of $\mathcal{M}|_r$, and the interpretation $c^{I'_w}$ of the constant c is d for every $w \in W|_r$.

Lemma B.6 *Given a constant c , a formula φ where c does not appear, an adequate model \mathcal{M} , a world w , and a w -assignment g , we have:*

$$\mathcal{M}, w \Vdash^g \varphi \iff \mathcal{M}|_w[c \leftarrow g(x)], w \Vdash^g \varphi[x \leftarrow c].$$

Proof. We proceed by induction on the formula φ . The cases of \top , relational symbols, and conjunction are trivial. We assume that x is free in φ , for otherwise we could use Remarks B.1 and B.4.

Consider the diamond case. If $\mathcal{M}, w \Vdash^g \diamond \psi$, then there is a world v such that wRv and $\mathcal{M}, v \Vdash^{g^t} \psi$. By the induction hypothesis we obtain $\mathcal{M}|_v[c \leftarrow g^t(x)], v \Vdash^{g^t} \psi[x \leftarrow c]$. Observe that $\mathcal{M}|_v[c \leftarrow g^t(x)]$ is the same model as $(\mathcal{M}|_w[c \leftarrow g(x)])|_v$, since they share the same frame, the same constant interpretation (because $g(x) = g^t(x)$) and the same relational symbol interpretation. Then by Remark B.4 we get $\mathcal{M}|_w[c \leftarrow g(x)], v \Vdash^{g^t} \psi[x \leftarrow c]$ and consequently $\mathcal{M}|_w[c \leftarrow g(x)] \Vdash^g \diamond \psi[x \leftarrow c]$, as desired. The other implication is analogous.

Finally, let $\varphi = \forall z \psi$ and assume that $\mathcal{M}, w \Vdash^g \forall z \psi$. Let $h \sim_z g$ be a w -assignment, and set out to prove $\mathcal{M}|_w[c \leftarrow g(x)], w \Vdash^h \psi[x \leftarrow c]$ (note that z and x are not the same variable for otherwise x would not be free in φ). Since $h \sim_z g$, we know that $g(x) = h(x)$, so by the induction hypothesis it is enough to show $\mathcal{M}, w \Vdash^h \psi$, which follows from our assumption. The other implication is analogous. \square

The above lemma is now enough to show the soundness of Rule 2.1.(xi) with respect to Kripke models.

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