Algorithmic correspondence and analytic rules

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Abstract

We introduce the algorithm MASSA which takes classical modal formulas in input, and, when successful, effectively generates: (a) (analytic) geometric rules of the labelled calculus G3K, and (b) cut-free derivations (of a certain 'canonical' shape) of each given input formula in the geometric labelled calculus obtained by adding the rule in output to G3K. We show that MASSA successfully terminates whenever its input formula is a (definite) analytic inductive formula, in which case, the geometric axiom corresponding to the output rule is, modulo logical equivalence, the first-order correspondent of the input formula.

Keywords: Structural proof theory of modal logic, labelled calculi, analytic extensions of labelled calculi, automatic rule-generation, algorithmic correspondence theory.

1 Introduction

The labelled calculus G3K was presented by Sara Negri in [17] as a basic G3-style sequent calculus for the normal modal logic K (see [19, Chapter 3] and [20, Chapter 11] for the genesis of this calculus). The calculus G3K shares many of the characteristic properties of Gentzen's original sequent calculus G3 for classical logic; for instance, all its rules are invertible, and the basic structural rules (weakening, contraction and cut) are admissible. Moreover, in [17], Negri introduces a general method for extending G3K so as to capture a large class of axiomatic extensions of K; namely, all those axiomatic extensions of K which define *elementary* (i.e. first-order definable) classes of Kripke frames, and such that their defining first-order conditions are, modulo logical equivalence, geometric implications. The rules generated by Negri's method for capturing these axiomatic extensions of K are defined on the basis of their corresponding geometric implications, and are referred to as *geometric rules.* Negri uniformly

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shows that the structural rules (and cut in particular) are admissible in the calculi obtained by extending G3K with geometric rules.

One important subclass of geometric implications is given, modulo logical equivalence, by the first-order correspondents of the class of *analytic inductive formulas* in classical modal logic. General (i.e. not necessarily analytic) inductive formulas have been introduced by Goranko and Vakarelov in [11], and have been shown to have (local) first-order correspondents, which can be effectively computed via an algorithmic correspondence procedure introduced in [5].

In the present paper, we refine Negri's method for extending G3K, and introduce the algorithm MASSA for generating analytic labelled rules uniformly and equivalently capturing the *analytic inductive* axiomatic extensions of K. An important difference between the algorithmic rule-generation method introduced in this paper and Negri's method is that the present method takes *modal formulas* in input, and, if the input formula is analytic inductive (cf. Section 2.2), it computes its equivalent analytic rule *directly* from the input formula, via a computation which incorporates the effective generation of its first-order correspondent, whereas Negri's method starts from geometric implications in the first-order frame correspondence language, and generates rules which are equivalent to those modal formulas which are assumed to have a first-order correspondent which is (logically equivalent to) a geometric implication.

This paper is structured as follows. In Section 2, we collect basic definitions and results on G3K and analytic inductive formulas in classical modal logic; in Section 3, we introduce the algorithm MASSA and provide intuitive motivation for some of its key steps. In Section 4, we illustrate how MASSA works, by running it on some well known modal axioms; in Section 5, we discuss how the present results embed in a wider research context in structural proof theory, which provides motivations for further research directions.

2 Preliminaries

2.1 The labelled calculus G3K

In what follows, we adopt the usual conventions: $p, q, \ldots$ denote proposition variables, $x, y, z, \ldots$ are labels (corresponding to world-variables in the intended interpretation on Kripke frames), given a label x and a modal formula A, well-formed formulas are of the type $x : A$, while $\varphi, \psi, \ldots$ are meta-variables for well-formed formulas. $\Gamma, \Delta, \ldots$ are meta-variables for sets of wffs, and a sequent is an expression of the form $\Gamma \vdash \Delta$. Given a well formed sequent $S = \Gamma \vdash \Delta$, if the formula $\varphi \in \Gamma$ (resp. $\varphi \in \Delta$), we say that φ occurs in *precedent* (resp. *succedent*) position in S.

Below, we list the rules of the labelled relational sequent calculus G3K for the basic normal modal logic K, where cut, weakening, contraction, and necessitation are admissible rules (see for instance [17]). In the list below, we explicitly mention the cut rule and we do not include the rules for negation. The propositional and modal rules are all invertible.

Initial rules and cut rule

$$\bot_L \frac{}{\Gamma, x : \bot \vdash \Delta} \qquad \frac{}{\Gamma, x : p \vdash x : p, \Delta}\,\mathrm{Id}_{x:p} \qquad \frac{\Gamma \vdash x : p, \Delta \qquad \Gamma', x : p \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

Invertible propositional rules

$$\wedge_L \frac{\Gamma, x : A, x : B \vdash \Delta}{\Gamma, x : A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash x : A, \Delta \qquad \Gamma \vdash x : B, \Delta}{\Gamma \vdash x : A \wedge B, \Delta}\,\wedge_R$$

$$\vee_L \frac{\Gamma, x : A \vdash \Delta \qquad \Gamma, x : B \vdash \Delta}{\Gamma, x : A \vee B \vdash \Delta} \qquad \frac{\Gamma \vdash x : A, x : B, \Delta}{\Gamma \vdash x : A \vee B, \Delta}\,\vee_R$$

$$\rightarrow_L \frac{\Gamma \vdash x : A, \Delta \qquad \Gamma, x : B \vdash \Delta}{\Gamma, x : A \rightarrow B \vdash \Delta} \qquad \frac{\Gamma, x : A \vdash x : B, \Delta}{\Gamma \vdash x : A \rightarrow B, \Delta}\,\rightarrow_R$$

Invertible modal rules*

$$\Box_L \frac{xRy, \Gamma, x : \Box A, y : A \vdash \Delta}{xRy, \Gamma, x : \Box A \vdash \Delta} \qquad \frac{xRy, \Gamma \vdash y : A, \Delta}{\Gamma \vdash x : \Box A, \Delta}\,\Box_R$$

$$\Diamond_L \frac{xRy, \Gamma, y : A \vdash \Delta}{\Gamma, x : \Diamond A \vdash \Delta} \qquad \frac{xRy, \Gamma \vdash y : A, x : \Diamond A, \Delta}{xRy, \Gamma \vdash x : \Diamond A, \Delta}\,\Diamond_R$$

Equality rules

$$\text{Eq-Ref} \frac{x = x, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad \text{Eq-Trans} \frac{y = z, x = y, x = z, \Gamma \vdash \Delta}{x = y, x = z, \Gamma \vdash \Delta}$$

$$\text{Repl}_{R1} \frac{yRz, x = y, xRz, \Gamma \vdash \Delta}{x = y, xRz, \Gamma \vdash \Delta} \qquad \text{Repl}_{R2} \frac{xRz, y = z, xRy, \Gamma \vdash \Delta}{y = z, xRy, \Gamma \vdash \Delta}$$

$$\text{Repl} \frac{x = y, y : A, x : A, \Gamma \vdash \Delta}{x = y, x : A, \Gamma \vdash \Delta}$$

*Side condition: the label y must not occur in the conclusion of $\Box_R$ and $\Diamond_L$.

Remark 2.1 The logical rules above (namely Propositional and Modal rules) reflect the semantic clauses of each connective in the intended Kripke semantics. Logical rules can be grouped together as *tonicity rules* $(\wedge_R, \vee_L, \rightarrow_L, \Box_L, \Diamond_R)$ versus *translation rules* $(\wedge_L, \vee_R, \rightarrow_R, \Box_R, \Diamond_L)$. Tonicity rules specify the arity of a connective (i.e. a connective of arity n is introduced by a tonicity rule with n premises) and its tonicity (i.e. if the connective is positive or negative in each coordinate). The translation rules convert a proxy occurring in the premise (either the comma or a relational atom) into a logical connective (namely, the main connective of the principal formula occurring in the conclusion).

Below we list the non-invertible versions of the tonicity logical rules. We sometimes refer to them as multiplicative rules.

Non-invertible tonicity propositional rules

$$\vee_L \frac{\Gamma, x : A \vdash \Delta \qquad \Gamma', x : B \vdash \Delta'}{\Gamma, \Gamma', x : A \vee B \vdash \Delta, \Delta'} \qquad \frac{\Gamma \vdash x : A, \Delta \qquad \Gamma' \vdash x : B, \Delta'}{\Gamma, \Gamma', \vdash x : A \wedge B, \Delta, \Delta'} \wedge_R$$

$$\rightarrow_L \frac{\Gamma \vdash x : A, \Delta \qquad \Gamma', x : B \vdash \Delta'}{\Gamma, \Gamma', x : A \rightarrow B \vdash \Delta, \Delta'}$$

Non-invertible tonicity modal rules*

$$\Box_L \frac{xRy, \Gamma, y : A \vdash \Delta}{xRy, \Gamma, x : \Box A \vdash \Delta} \qquad \frac{xRy, \Gamma \vdash y : A, \Delta}{xRy, \Gamma \vdash x : \Diamond A, \Delta} \Diamond_R$$

Lemma 2.2 *For any modal formula A, the sequent $\Gamma, x : A \vdash x : A, \Delta$ is derivable in G3K.*

Proof. By induction on A. The cases of $A := \bot$ and $A := p \in \mathsf{Prop}$ are immediate. If $A := *(\overline{A'})$ where $* \in \{\Box, \rightarrow, \vee\}$, then the required proof is obtained by applying, from bottom to top, $*_R$ to the occurrence of A in succedent position, followed by a bottom-up application of $*_L$ to the occurrence of A in precedent position, and then using the induction hypothesis on each A' in $\overline{A'}$. Similarly, the required proof if $A := *(\overline{A'})$ where $* \in \{\Diamond, \wedge\}$, is obtained by applying, from bottom to top, $*_L$ followed by $*_R$. $\Box$

Notice that the derivation generated in the proof of the lemma above introduces *every* subformula of each occurrence of φ via a logical rule, and, modulo renaming variables, we can assume w.l.o.g. that every new label introduced proceeding bottom-up be fresh in the entire derivation (and not just in every branch, as already required by the side conditions of the rule $\Box_R$ and $\Diamond_L$). Below we recall the definition of a geometric implication.

Definition 2.3 (cf. [16, Section 3]) A *geometric implication* is a first-order sentence of the form

$$\forall \overline{x}(s \rightarrow t),$$

where both s and t are *geometric formulas*, i.e. first-order formulas not containing $\rightarrow$ or $\forall$. Geometric implications can be equivalently rewritten as *geometric axioms*, namely, sentences of the type

$$\forall \overline{x}(P_1 \wedge ... \wedge P_m \rightarrow \overline{\exists y_1} M_1 \vee ... \vee \overline{\exists y_n} M_n)$$

where each P_i is an atomic formula with no free occurrences of any variable y in $\overline{y}$, and M_j is a conjunction of atomic formulas $Q_{j_1} \wedge ... \wedge Q_{j_{k_j}}$. The rule scheme corresponding to geometric axioms takes the form

$$\frac{\overline{Q_1}[\overline{y_1}/\overline{z_1}], \overline{P}, \Gamma \vdash \Delta \qquad ... \qquad \overline{Q_n}[\overline{y_n}/\overline{z_n}], \overline{P}, \Gamma \vdash \Delta}{\overline{P}, \Gamma \vdash \Delta} GR$$

where $\overline{Q_i}[\overline{y_i}/\overline{z_i}]$ denotes the simultaneous replacement of each z in $\overline{z_i}$ with the corresponding y in $\overline{y_i}$, in every Q in $\overline{Q_i}$. In this scheme, the eigenvariables in $\overline{y_i}$

are not free in $\overline{P}, \Delta, \Gamma$. Rules corresponding to geometric axioms are referred to as *geometric (labelled) rules.*

A *geometric labelled calculus* is any extension of G3K with geometric labelled rules.

Theorem 2.4 *(cf. [17, Theorem 4.13]) Any geometric labelled calculus preserves cut admissibility.*

2.2 Analytic inductive formulas

In this section, we specialize and adapt the definition of analytic inductive inequality (cf. [12, Definition 55], [9, Definition 2.14], [1, Section 2.3]) to the language and properties of classical modal logic.

The language of the basic normal modal logic K is recursively defined from a set Prop of proposition variables as follows:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Diamond\varphi \mid \Box\varphi,$$

where p ranges over Prop. In what follows, we will need to keep track of the multiplicity of occurrences of proposition variables in formulas, as well as the order-theoretic properties of the various coordinates of the term-functions associated with formulas. Therefore, we will write e.g. $\psi(!\overline{x})$ to signify that each variable in the vector $\overline{x}$ of placeholder variables occurs exactly once in ψ. Moreover, we will write e.g. $\psi(!\overline{x}, !\overline{y})$ to mean that ψ (resp. the term-function $\psi^{\mathbb{A}}$ in a modal algebra $\mathbb{A}$) is positive (resp. monotone) in each x-coordinate and negative (resp. antitone) in each y-coordinate. In other contexts, we will sometimes need to group coordinates according to different criteria. In each context in which this is the case, we will specifically indicate these criteria. Negative (resp. positive) *Skeleton* formulas $\psi(!\overline{x}, !\overline{y})$ (resp. $\varphi(!\overline{x}, !\overline{y})$) are defined by simultaneous recursion as follows:

$$\begin{array}{lcl}\psi(!\overline{x}, !\overline{y}) & ::= & x \mid \neg\varphi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \varphi \to \psi \mid \Box\psi,\\ \varphi(!\overline{x}, !\overline{y}) & ::= & x \mid \neg\psi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi.\end{array}$$

Positive Skeleton formulas will sometimes be referred to as *negative PIA* formulas. *Definite* negative Skeleton (resp. PIA) formulas are defined by simultaneous recursion as follows:

$$\begin{array}{lcl}\psi(!\overline{x}, !\overline{y}) & ::= & x \mid \neg\varphi \mid \psi \vee \psi \mid \varphi \to \psi \mid \Box\psi,\\ \varphi(!\overline{x}, !\overline{y}) & ::= & x \mid \neg\psi \mid \varphi \wedge \varphi \mid \Diamond\varphi.\end{array}$$

Modulo exhaustively distributing all the other connectives over $\vee$ and $\wedge$, any negative Skeleton (resp. PIA) formula can be equivalently rewritten as a conjunction (resp. disjunction) of definite negative Skeleton (resp. PIA) formulas (cf. [1, Lemma 2.9]).

Definition 2.5 A modal formula $\psi'(\overline{p})$ is (negative) *analytic inductive* if its negative normal form (NNF) is $\psi(\overline{\beta}/!\overline{x}, \overline{\delta}/!\overline{y})$ such that:

(i) $\psi(!\overline{x}, !\overline{y})$ (which we refer to as the *Skeleton* of ψ') is a negative Skeleton formula, and is monotone *both* in its x-coordinates and in its y-coordinates;

(ii) each β in $\overline{\beta}$ and δ in $\overline{\delta}$ is a negative PIA formula;

(iii) the term-function $\delta^{\mathbb{A}}(!\overline{x})$ associated with each $\delta(\overline{p}/!\overline{x})$ in $\overline{\delta}$ is monotone in each coordinate;

(iv) the term-function $\beta^{\mathbb{A}}(!\overline{x}, !\overline{y})$ associated with each β in $\overline{\beta}$ is monotone in each x-coordinate and antitone in each y-coordinate;

(v) the transitive closure $<_\Omega$ of the relation Ω (defined below) is a well-founded strict order on $\overline{p}$, where for all p, p' in $\overline{p}$, $(p, p') \in \Omega$ iff some $\beta \in \overline{\beta}$ exists s.t. $\beta = \beta(\overline{p_1}/!\overline{x}, \overline{p_2}/!\overline{y})$ and p' occurs in $\overline{p_1}$ and p occurs in $\overline{p_2}$, and the lowest common node in the branches ending in p' and p in the generation tree of β is a $\wedge$-node.

In an analytic inductive formula ψ' as above, the variable occurrences in the y-coordinates of each β in $\overline{\beta}$ are referred to as the *critical* occurrences [2] in ψ'. All the other variable occurrences are *non-critical*. An analytic inductive formula is *Sahlqvist* if the relation Ω is empty, and is *definite* if its Skeleton is definite.

As discussed above, for any analytic inductive formula $\psi' := \psi(\overline{\beta}/!\overline{x}, \overline{\delta}/!\overline{y})$, any negative PIA subformula β and δ of ψ' can be equivalently rewritten as a disjunction of definite negative PIA formulas (cf. [1, Lemma 2.9]). Hence, once these $\vee$-nodes have reached the root of β by distributing all the other connectives over them, they can all be considered part of the Skeleton of ψ'. Hence, when representing an analytic inductive formula ψ' as $\psi(\overline{\beta}/!\overline{x}, \overline{\delta}/!\overline{y})$, we can assume w.l.o.g. that each β and δ is a *definite* negative PIA formula, and that there is *exactly one* critical occurrence of a proposition variable in each β in $\overline{\beta}$. To emphasise this, we sometimes write β as β_p.

Example 2.6 (i) The formula $\psi'(p) := \Diamond p \to \Box p$ can be rewritten in NNF as $\psi(\beta/x, \delta/y)$ where $\psi(x, y) := \Box x \vee \Box y$, and $\beta(p) := \neg p$, and $\delta(p) := p$, and is hence (negative) analytic Sahlqvist.

(ii) The formula $\psi'(p) := \Box p \to \Diamond p$ can be rewritten in NNF as $\psi(\beta/x, \delta/y)$ where $\psi(x, y) := x \vee y$, and $\beta(p) := \Diamond\neg p$ and $\delta(p) := \Diamond p$, and is hence (negative) analytic Sahlqvist.

(iii) The formula $\psi'(p) := \Diamond\Box p \to \Box\Diamond p$ can be rewritten in NNF as $\psi(\beta/x, \delta/y)$ where $\psi(x, y) := \Box x \vee \Box y$, and $\beta(p) := \Diamond\neg p$ and $\delta(p) := \Diamond p$, and is hence (negative) analytic Sahlqvist.

(iv) The formula $\psi'(p_1, p_2) := \Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2)$ can be rewritten in NNF as $\psi(\beta_1/x_1, \beta_2/x_2, \delta/y)$ where $\psi(x_1, x_2, y) := x_1 \vee (x_2 \vee \Box_t y)$, and $\beta_1(p_1, p_2) := \Diamond_y(p_1 \wedge \neg p_2)$ and $\beta_2(p_1) := \Diamond\neg p_1$ and $\delta(p_2) := p_2$, and is hence (negative) analytic inductive with $p_1 <_\Omega p_2$.

(v) The formula $\psi'(p_1, p_2) := \Box(\Box p_1 \to p_2) \vee \Box(\Box p_2 \to p_1)$ can be rewritten

[2] In the more general setting of (D)LE-logics (see e.g. [7][6]), inductive and Sahlqvist formulas/inequalities are defined parametrically in every order-type ε on the proposition variables occurring in the given formula/inequality. However, in the Boolean setting this is not needed, and the definition given here corresponds to the general definition relative to the order-type $\varepsilon(p) = 1$ for each $p \in \mathsf{Prop}$.

in NNF as $\psi(\beta_1/x_1, \beta_2/x_2, \delta_1/y_1, \delta_2/y_2)$ where $\psi(x_1, x_2, y_1, y_2) := \Box(x_1 \vee y_2) \vee \Box(x_2 \vee y_1)$, and $\beta_1(p_1) := \Diamond\neg p_1$ and $\beta_2(p_2) := \Diamond\neg p_2$, and $\delta_1(p_1) := p_1$ and $\delta_(p_2) := p_2$, and is hence (negative) analytic Sahlqvist.

Theorem 2.7 *(cf. [11, Theorem 37]) Every (analytic) inductive formula has a first-order correspondent.*

3 The algorithm MASSA

In this section, we describe the algorithm MASSA. The steps (i)-(iv) generate the analytic labelled rule r associated with the input modal formula φ. Step (v) describes how to read off the geometric implication from the rule r.

(i) **Logical rules.** For any modal formula φ, consider the identity end-sequent $x : \varphi \vdash x : \varphi$ where the formula in precedent position is coloured red and the formula in succedent position is coloured blue. Let π'_φ be a derivation of $x : \varphi \vdash x : \varphi$ obtained by applying the procedure described in the proof of Lemma 2.2, where, as discussed early on, each subformula of φ and φ has been introduced in the proof via a logical rule, and every new label introduced proceeding bottom-up must be fresh in the entire proof (and not just in every branch).[3] At each rule application in π'_φ, propagate the colour of the principal formula to the auxiliary formulas. Prune the proof-tree π'_φ, thereby generating a new proof-tree π_φ with the same structure as π'_φ, but such that each tonicity rule is applied in multiplicative form (cf. Section 2.1).

(ii) **Atomic cuts + PIA parts.** Consider the leaves of π_φ and perform all possible cuts on atomic red-coloured formulas $x : p$ occurring in π_φ. These cuts generate new axioms of the form $\Gamma, y = z, y : p \vdash z : p, \Delta$ in which the new relational atom $y = z$ appears in the conclusion of each cut with cut formulas $y : p$ and $z : p$. If a proposition variable $x : p$ occurs only positively or only negatively in φ, then cut either with an atomic initial rule of the form $x : \bot \vdash x : p$ or with $x : p \vdash \top$. Collect all the conclusions of these cut-applications, and use them as leaves in a (cut-free) forward-chaining proof-search with goal $\vdash x : \varphi$, where only tonicity rules are used.[4] Collect all the attempts π^i_φ generated in this way.

(iii) **Skeleton parts.** Perform a backward-chaining proof search on $\vdash x : \varphi$ in which only translation rules are used.[5]

(iv) **Skeleton-PIA merging.** A *merging point* is a tuple of sequents

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[3] The latter requirement guarantees that all the relevant information contained in the end-sequent is maintained (and exploited in rule form) in π'_φ.

[4] Notice that we are compositionally constructing all the *maximal PIA subformulas*, that here coincide with those subformulas that can be constructed using only tonicity rule. Notice that whenever an atomic subformula of φ is uniform, it is substituted with $\bot$ (resp. $\top$) if in succedent (resp. precedent) position, so it is not strictly speaking a subformula of φ.

[5] Here we are compositionally destroying all the Skeleton connectives namely $\Diamond$ and $\wedge$ if occurring in precedent position, and $\Box, \vee$ and $\rightarrow$ if occurring in succedent position.

$(S_1, ..., S_n, S)$, of which $S_1, ..., S_n$ are the *premises* and S the *conclusion*, and such that the set of labelled formulae of S is the union of the sets of labelled formulae of $S_1, ..., S_n$. Verify whether $(S_1, ..., S_n, S)$ is a merging point, where the S_i are the endsequents of all the proof-trees π^i_φ generated in item (ii), and S is the uppermost sequent of the proof-section generated in item (iii). If $(S_1, ..., S_n, S)$ is a merging point, then it is an application of the rule r in output, which provides the missing step in proof of $\vdash x : \varphi$.

Let R_i and R be the relational parts of S_i and S respectively. The rule r associated with the merging point is:

$$r\ \frac{R_1, \Gamma_1 \vdash \Delta_1 \qquad ... \qquad R_n, \Gamma_n \vdash \Delta_n}{R, \Gamma_1, \ldots, \Gamma_n \vdash \Delta_1, \ldots, \Delta_n}$$

(v) **Reading off the geometric axiom from the rule.** Let F_i be defined as the conjunction of the relational atoms in R_i in case in S_i there are no occurrences of $y : \bot$ in precedent position or of $y : \top$ in succedent position (an empty conjunction will be regarded as $\top$). Otherwise, let F_i be $\bot$. If in S there are formulas $y : \bot$ (resp. $y : \top$) in precedent (resp. succedent) position, then the required geometric formula is $\top$. Otherwise, the geometric axiom which we can read off from the rule r is:

$$\forall \overline{x}[\bigwedge R(\overline{x}) \to \bigvee_i \overline{\exists y_i} F_i(\overline{x}, \overline{y_i}))].$$

Steps (i) and (ii) can be intuitively justified as follows. Whenever φ is a theorem of K, the calculus G3K derives $\vdash x : \varphi$ without any additional rule. Otherwise, we need to identify some assumptions Γ which allow us to derive $\Gamma \vdash x : \varphi$. Clearly, the *minimal* set of assumptions Γ under which φ is derivable is $\Gamma = \{x : \varphi\}$. Then, at step (i), we equivalently transform the additional assumption $x : \varphi$ into pure relational information and also information stored in the atomic propositions of the form $x : p$. The cuts performed in step (ii) extract additional pure relational information from these atomic propositions.

Theorem 3.1 *The algorithm MASSA successfully terminates whenever it receives a definite analytic inductive formula of classical modal logic in input, in which case, the geometric axiom read off from the output rule is, modulo logical equivalence, the first-order correspondent of the input formula.*

Proof. see Appendix A. □

4 Examples

In the present section, we illustrate the algorithm MASSA by running it on some definite analytic inductive formulas. Let us start with the Church-Rosser axiom (cf. Example 2.6 (iii)).

Step (i). We build the proof π'_φ using (invertible) additive rules. Below we split the derivation tree into three proof sections: the numbers assigned to each sequent allow to uniquely reconstruct the original tree.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(7.1)\ xRy, yRt, y:\Box A_1, t:A_1 \vdash t:A_3, x:\Diamond\Box A_3, x:\Box\Diamond A_2}\ \mathrm{Id}_{t:A}}{(6.1)\ xRy, yRt, y:\Box A_1 \vdash t:A_3, x:\Diamond\Box A_3, x:\Box\Diamond A_2}\ \Box_L}{(5.1)\ xRy, y:\Box A_1 \vdash y:\Box A_3, x:\Diamond\Box A_3, x:\Box\Diamond A_2}\ \Box_R}{(4.1)\ xRy, y:\Box A_1 \vdash x:\Diamond\Box A_3, x:\Box\Diamond A_2}\ \Diamond_R}{x:\Diamond\Box A_1 \vdash x:\Diamond\Box A_3, x:\Box\Diamond A_2}\ \Diamond_L
$$

(3.1)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(7.2)\ xRz, zRw, x:\Diamond\Box A_1, x:\Box\Diamond A_4, w:A_4 \vdash w:A_2, z:\Diamond}\ \mathrm{Id}_{w:A}}{(6.2)\ xRz, zRw, x:\Diamond\Box A_1, x:\Box\Diamond A_4, w:A_4 \vdash z:\Diamond A_2}\ \Diamond_R}{(5.2)\ xRz, x:\Diamond\Box A_1, x:\Box\Diamond A_4, z:\Diamond A_4 \vdash z:\Diamond A_2}\ \Diamond_L}{(4.2)\ xRz, x:\Diamond\Box A_1, x:\Box\Diamond A_4 \vdash z:\Diamond A_2}\ \Box_L}{x:\Diamond\Box A_1, x:\Box\Diamond A_4 \vdash x:\Box\Diamond A_2}\ \Box_R
$$

(3.2)

$$
\dfrac{\dfrac{\begin{matrix}(3.1) \\ x:\Diamond\Box A_1 \vdash x:\Diamond\Box A_3, x:\Box\Diamond A_2\end{matrix} \qquad \begin{matrix}(3.2) \\ x:\Diamond\Box A_1, x:\Box\Diamond A_4 \vdash x:\Box\Diamond A_2\end{matrix}}{(2)\ x:\Diamond\Box A_1, x:\Diamond\Box A_3 \to \Box\Diamond A_4 \vdash x:\Box\Diamond A_2}\ \to_L}{(1)\ x:\Diamond\Box A_3 \to \Box\Diamond A_4 \vdash x:\Diamond\Box A_1 \to \Box\Diamond A_2}\ \to_R
$$

We prune the proof tree π'_φ obtaining the "multiplicative" proof tree π_φ.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(7.1)\ xRy, yRt, t:A_1 \vdash t:A_3}\ \mathrm{Id}_{t:A}}{(6.1)\ xRy, yRt, y:\Box A_1 \vdash t:A_3}\ \Box_L}{(5.1)\ xRy, y:\Box A_1 \vdash y:\Box A_3}\ \Box_R}{(4.1)\ xRy, y:\Box A_1 \vdash x:\Diamond\Box A_3}\ \Diamond_R}{(3.1)\ x:\Diamond\Box A_1 \vdash x:\Diamond\Box A_3}\ \Diamond_L \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{(7.2)\ xRz, zRw, w:A_4 \vdash w:A_2}\ \mathrm{Id}_{w:A}}{(6.2)\ xRz, zRw, w:A_4 \vdash z:\Diamond A_2}\ \Diamond_R}{(5.2)\ xRz, z:\Diamond A_4 \vdash z:\Diamond A_2}\ \Diamond_L}{\dfrac{(4.2)\ xRz, x:\Box\Diamond A_4 \vdash z:\Diamond A_2}{(3.2)\ x:\Box\Diamond A_4 \vdash x:\Box\Diamond A_2}\ \Box_R}\ \Box_L}{\dfrac{(2)\ x:\Diamond\Box A_1, x:\Diamond\Box A_3 \to \Box\Diamond A_4 \vdash x:\Box\Diamond A_2}{(1)\ x:\Diamond\Box A_3 \to \Box\Diamond A_4 \vdash x:\Diamond\Box A_1 \to \Box\Diamond A_2}\ \to_R}\ \to_L
$$

Step (ii). We consider the leaves (7.1) and (7.2) and perform all the atomic cuts on red coloured formulas.

$$
\dfrac{\dfrac{}{(7.1)\ xRy, yRt, t:A_1 \vdash t:A_3}\ \mathrm{Id}_{t:A} \qquad \dfrac{}{(7.2)\ xRz, zRw, w:A_4 \vdash w:A_2}\ \mathrm{Id}_{w:A}}{xRy, yRt, xRz, zRw, t=w; t:A_1 \vdash w:A_2}\ \mathrm{Cut}(A_3, A_4)
$$

We now construct the upper portion of the proof π^1_φ.[6] In this step, we build up the PIA sub-formulas of φ.

$$
\begin{matrix}\pi^1_\varphi \\ \dfrac{\dfrac{\dfrac{}{xRy, yRt, xRz, zRw, t=w, t:A \vdash w:A}}{xRy, yRt, xRz, zRw, t=w, t:A \vdash z:\Diamond A}\ \Diamond_R}{xRy, yRt, xRz, zRw, t=w, y:\Box A \vdash z:\Diamond A}\ \Box_L \\ -\end{matrix}
$$

[6] Notice that we could also construct a proof with a different order of rule applications (e.g. in this case, proceeding top down, first we apply $\Box_L$ and then $\Diamond_R$). Such trivial permutations of rules generate, strictly speaking, different syntactic proofs but do not change the merging point. So, it is enough to pick one of those proofs.

Step (iii). In this step, we work on the Skeleton of φ.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{xRz, xRy, y : \Box A \vdash z : \Diamond A}}{xRz, x : \Diamond\Box A \vdash z : \Diamond A}\,\Diamond_L}{x : \Diamond\Box A \vdash x : \Box\Diamond A}\,\Box_R}{\vdash x : \Diamond\Box A \to \Box\Diamond A}\,\to_R$$

Step (iv). We now reach a merging point, and hence generate the rule Dir:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy, yRt, xRz, zRw, t = w, t : A \vdash w : A}}{xRy, yRt, xRz, zRw, t = w, t : A \vdash z : \Diamond A}\,\Diamond_R}{xRy, yRt, xRz, zRw, t = w, y : \Box A \vdash z : \Diamond A}\,\Box_L}{xRy, xRz, y : \Box A \vdash z : \Diamond A}\,\mathrm{Dir}}{xRz, x : \Diamond\Box A \vdash z : \Diamond A}\,\Diamond_L}{x : \Diamond\Box A \vdash x : \Box\Diamond A}\,\Box_R}{\vdash x : \Diamond\Box A \to \Box\Diamond A}\,\to_R$$

Step (v). Finally, the FO-correspondent reads

$$\forall x\forall y\forall z[xRy \wedge xRz \to \exists t\exists w(yRt \wedge zRw \wedge t = w)],$$

which is equivalent to directedness.

Let us execute MASSA on the 'functionality' axiom (cf. Example 2.6 (i)). The pruned proof-tree generated in the first step is the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy, y : A \vdash y : A}\,Id_{y:A}}{xRy, y : A \vdash x : \Diamond A}\,\Diamond_R}{x : \Diamond A \vdash x : \Diamond A}\,\Diamond_L \quad \dfrac{\dfrac{\dfrac{}{xRz, z : A \vdash z : A}\,Id_{z:A}}{xRz, x : \Box A \vdash z : A}\,\Box_L}{x : \Box A \vdash x : \Box A}\,\Box_R}{x : \Diamond A \to \Box A, x : \Diamond A \vdash x : \Box A}\,\to_L}{x : \Diamond A \to \Box A \vdash x : \Diamond A \to \Box A}\,\to_R$$

The leaves on which we perform the only possible cut are written below:

$$xRy, y : A \vdash y : A \quad xRz, z : A \vdash z : A.$$

After performing step (ii) and (iii), the merging point is reached, which generates the following derivation and rule (step (iv)):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y = z, xRy, xRz, y : A \vdash z : A}}{xRy, xRz, y : A \vdash z : A}\,\mathrm{Fun}}{xRz, x : \Diamond A \vdash z : A}\,\Diamond_L}{x : \Diamond A \vdash x : \Box A}\,\Box_R}{\vdash x : \Diamond A \to \Box A}\,\to_R$$

from which the first-order correspondent (step (v)) below can be read off:

$$\forall x\forall y\forall z(xRy \wedge xRz \to y = z).$$

Merging points do not need to be unary. To see this, let us consider the formula $\Box(\Box A \to B) \vee \Box(\Box B \to A)$ (cf. Example 2.6 (v)). After performing step (i), the leaves of π are as follows:

$xRy, yRz, z : A \vdash z : A$ $xRt, t : A \vdash t : A$ $\quad xRt, tRw, w : B \vdash w : B$ $xRy, y :$

$$B \vdash y : B$$

After performing steps (ii) and (iii), we generate a binary merging point and we provide the following derivation (step (iv)):

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$xRy, yRz, xRt, z = t, z : A \vdash t : A$}
\UnaryInfC{$xRy, yRz, xRt, z = t, y : \Box A \vdash t : A$}
\AxiomC{}
\UnaryInfC{$xRt, tRw, xRy, y = w, w : B \vdash y : B$}
\UnaryInfC{$xRt, tRw, xRy, y = w, t : \Box B \vdash y : B$}
\dashedLine
\BinaryInfC{$xRy, xRt, y : \Box A, t : \Box B \vdash y : B, t : A$}
\UnaryInfC{$xRy, xRt, y : \Box A \vdash y : B, t : \Box B \to A$}
\UnaryInfC{$xRy, xRt \vdash y : \Box A \to B, t : \Box B \to A$}
\UnaryInfC{$xRy \vdash y : \Box A \to B, x : \Box(\Box B \to A)$}
\UnaryInfC{$\vdash x : \Box(\Box A \to B), x : \Box(\Box B \to A)$}
\UnaryInfC{$\vdash x : \Box(\Box A \to B) \vee \Box(\Box B \to A)$}
\end{prooftree}
$$

The first order correspondent (step (v)) reads

$$\forall x \forall y \forall t(xRy \wedge xRt \to \exists z(yRz \wedge z = t) \vee \exists w(tRw \wedge y = w))$$

which is equivalent to

$$\forall x \forall y \forall t(xRy \wedge xRt \to yRt \vee tRy).$$

The examples discussed so far are all Sahlqvist. However, MASSA is successful on (definite analytic) formulas which are *properly inductive*, such as the axiom $K := \Box(A \to B) \to (\Box A \to \Box B)$. After performing step (i), the leaves of π_K are as follows:

$xRz, z : A \vdash z : A \quad xRy, y : A \vdash y : A \quad xRy, y : B \vdash y : B \quad xRt, t : B \vdash t : B$

After performing steps (ii) and (iii), we reach a merging point and hence the rule deriving K as follows (step (iv)):

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$xRz, zRy, y = z, z : A \vdash y : A$}
\UnaryInfC{$xRz, zRy, y = z, x : \Box A \vdash y : A$}
\AxiomC{}
\UnaryInfC{$xRy, xRt, t = y, y : B \vdash t : B$}
\BinaryInfC{$xRt, xRy, xRz, t = y, t = z, y : A \to B, x : \Box A \vdash t : B$}
\UnaryInfC{$xRt, xRy, xRz, t = y, t = z, x : \Box(A \to B), x : \Box A \vdash t : B$}
\dashedLine
\LeftLabel{K}
\UnaryInfC{$xRt, x : \Box(A \to B), x : \Box A \vdash t : B$}
\UnaryInfC{$x : \Box(A \to B), x : \Box A \vdash x : \Box B$}
\UnaryInfC{$x : \Box(A \to B) \vdash x : \Box A \to \Box B$}
\UnaryInfC{$\vdash x : \Box(A \to B) \to (\Box A \to \Box B)$}
\end{prooftree}
$$

The first-order correspondent (step (v)) reads

$$\forall x \forall t(xRt \to \exists y \exists z(xRy \wedge xRz \wedge t = y \wedge t = z))$$

which is equivalent to $\top$ as expected, since the input formula K is derivable in G3K, i.e. is valid in every Kripke frame.

Let us finish this section by discussing a couple of unsuccessful MASSA runs; let us try and run MASSA on the (non inductive and famously non elementary, see [22]) McKinsey formula $\Box\Diamond A \to \Diamond\Box A$. The first step produces the leaves

$$xRy, yRz, z : A \vdash z : A \quad xRw, wRt; t : A \vdash t : A,$$

but after performing the cut, at step (ii) and (iii) we get stuck:

$$\cfrac{\cfrac{xRy, yRz, xRw, wRt, z = t, z : A \vdash t : A}{???}}{\cfrac{x : \Box\Diamond A \vdash x : \Diamond\Box A}{\vdash x : \Box\Diamond A \rightarrow \Diamond\Box A}}$$

We cannot proceed bottom-up since we do not have the necessary relational information, and we cannot proceed top-down without violating the side conditions of G3K.

When we take as input the (Sahlqvist but not analytic) formula $A \rightarrow \Diamond\Box A$, the first step produces the leaves

$$x : A \vdash x : A \quad xRw, wRt; t : A \vdash t : A.$$

Again, after performing the cut, we cannot proceed further:

$$\cfrac{\cfrac{xRw, wRt, x = t, x : A \vdash t : A}{???}}{\cfrac{x : A \vdash x : \Diamond\Box A}{\vdash x : A \rightarrow \Diamond\Box A}}$$

5 Conclusions

Related work. The results in the present paper pertain to a larger line of research in structural proof theory focusing on the uniform generation of analytic rules for classes of axiomatic extensions in different (nonclassical) logics, which includes e.g., [21,23,18,16,17] in the context of sequent and labelled calculi, [2,14,15] in the context of sequent and hypersequent calculi, and [13,3,12] in the context of (proper) display calculi. We refer to [1] for an overview of this literature.

Range of applicability. We conjecture that the present approach extends to the analytic inductive axiomatic extensions of the basic normal and regular LE-logics (cf. [8]), so, in particular, to the case of intermediate logics [10], and also to a large class of (substructural) non-normal modal logics. In future work, we plan to explore this direction. Moreover, we plan to define (invertible) translations between proofs of different calculi modulo intermediate translations into a suitable calculus in the language of ALBA (as we have done in Appendix A). We also intend to extend the present approach to capture *inductive formulas* [12, Definition 16], a class of formulas strictly larger than analytic inductive formulas. [7]

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A Proof of Theorem 3.1

Main goal. In the present section, we show that, if the algorithm MASSA receives a definite analytic inductive formula ψ' in input, it successfully reaches a merging point in step (iv), and hence it outputs a geometric rule r which derives ψ' when added to G3K , and from which the first-order correspondent of ψ' (which exists, cf. [11]) can be read off. In fact, we will prove even more; namely, that a cut-free derivation of ψ' can be effectively generated in G3K+r, and this derivation has a specific shape.

Our proof will make use of the fact that the first-order correspondent of a generic analytic inductive formula can be represented in the language of the algorithm ALBA [4]. To prove the required statement, it is enough to show that the merging point is reached, and the geometric axiom that we read off from r is *effectively recognizable* as the first-order correspondent of ψ'. To guarantee this effective recognizability, we also translate G3K in a format set in the language of ALBA (cf. [9, Section 2.5]).

Following the conventions and notation of Section 2.2, we represent ψ' as $\psi(\overline{\beta}, \overline{\delta})$. The algorithm ALBA is guaranteed to succeed in computing the first-order correspondent of ψ' (cf. [7, Theorem 8.8]), e.g. via the following run, which, for the sake of simplicity, we execute under the assumption that $\psi'(\overline{p})$ is Sahlqvist, and each $p \in \overline{p}$ occurs both positively and negatively. In what follows, (vectors of) variables $\mathbf{i}$, $\mathbf{j}$, $\mathbf{h}$, and $\mathbf{k}$, referred to as *nominal* variables, are interpreted in Kripke frames as possible worlds, or equivalently as atoms (i.e. completely join-irreducible elements) of the complex algebra of any Kripke frame, while (vectors of) variables $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ and $\mathbf{o}$, referred to as *conominal* variables, are interpreted in Kripke frames as complements of possible worlds, or equivalently as co-atoms (i.e. completely meet-irreducible elements) of the complex algebra of any Kripke frame. Moreover, for every definite negative PIA formula β_p (where the subscript indicates the single critical occurrence of a proposition variable p), the term $RA(\beta_p)(!u)$ is a formula in the language of ALBA the associated term function of which on perfect algebras $\mathbb{A}$ (i.e. on complex algebras of Kripke frames) is characterized by the following equivalence: $\beta^{\mathbb{A}}(b/!p) \leq a$ iff $(RA(\beta))^{\mathbb{A}}(a) \leq b$ for every $a, b \in \mathbb{A}$ (cf. [1, Definition 2.15]).

$$
\begin{array}{ll}
& \forall \overline{p}[\top \leq \psi(\overline{\beta}, \overline{\delta})] \\
\text{iff} & \forall \overline{p} \forall \overline{\mathbf{n}} \forall \overline{\mathbf{o}}[(\overline{\beta_p \leq \mathbf{n}} \ \&\ \overline{\delta(\overline{p}) \leq \mathbf{o}}) \Rightarrow \top \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})] \\
\text{iff} & \forall \overline{p} \forall \overline{\mathbf{n}} \forall \overline{\mathbf{o}}[(\overline{RA(\beta_p)(\mathbf{n}) \leq p} \ \&\ \overline{\delta(\overline{p}) \leq \mathbf{o}}) \Rightarrow \top \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})] \\
\text{iff} & \forall \overline{\mathbf{n}} \forall \overline{\mathbf{o}}[\overline{\delta(\overline{\bigvee RA(\beta_p)(\mathbf{n})}/\overline{p}) \leq \mathbf{o}} \Rightarrow \top \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})] \\
\text{iff} & \forall \overline{\mathbf{n}} \forall \overline{\mathbf{o}}[\overline{\bigvee \delta(\overline{RA(\beta_p)(\mathbf{n})}/\overline{p}) \leq \mathbf{o}} \Rightarrow \top \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})] \\
\text{iff} & \forall \overline{\mathbf{n}} \forall \overline{\mathbf{o}}[\mathop{\&}\overline{(\delta(\overline{RA(\beta_p)(\mathbf{n})}/\overline{p}) \leq \mathbf{o})} \Rightarrow \top \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})] \\
\text{iff} & \forall \mathbf{j} \forall \overline{\mathbf{n}} \forall \overline{\mathbf{o}}[\mathop{\&}\overline{(\delta(\overline{RA(\beta_p)(\mathbf{n})}/\overline{p}) \leq \mathbf{o})} \Rightarrow \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})] \\
\text{iff} & \forall \mathbf{j} \forall \overline{\mathbf{n}} \forall \overline{\mathbf{o}}[\psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}) \leq \neg \mathbf{j} \Rightarrow \parr(\neg \mathbf{o} \leq \delta(\overline{RA(\beta_p)(\mathbf{n})}/\overline{p}))]
\end{array}
$$

Once the proposition variables have been eliminated, any of the conditions above can be translated in the first-order frame correspondence language of Kripke frames (see [6] for details). A moment's reflection will convince the reader that the ensuing implication is geometric. Our strategy will hinge on representing the run generating r so as to read off the last line in the computation above.

Labelled calculus in the language of ALBA. In the present section, we introduce rules for a labelled calculus in which the relational information is captured via *pure inequalities* (i.e. inequalities in which the only variables occurring in formulas are nominal and conominals) in the language of ALBA. In order to match the level of generality used to describe ALBA runs on generic definite analytic inductive (Salhqvist) formulas, we find it convenient to define this calculus via left- and right-introduction rules for definite Skeleton and PIA formulas.

In what follows, $\psi(!\overline{x}, !\overline{y})$ (resp. $\varphi(!\overline{x}, !\overline{y})$) is a definite negative (resp. positive) Skeleton formula which is monotone in its x-coordinates and antitone in its y-coordinates. To make notation lighter, we will write e.g. $\psi(\overline{\mathbf{n}}, \overline{\mathbf{h}})$ for $\psi(\overline{\mathbf{n}}/!\overline{x}, \overline{\mathbf{h}}/!\overline{y})$.

$$\frac{\Gamma, \overline{\mathbf{h} \leq A}, \overline{B \leq \mathbf{n}} \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{h}}), \Delta}{\Gamma \vdash \mathbf{j} \leq \psi(\overline{B}, \overline{A}), \Delta}\,\psi_R \qquad \frac{\Gamma, \overline{\mathbf{h} \leq A}, \overline{B \leq \mathbf{n}} \vdash \varphi(\overline{\mathbf{h}}, \overline{\mathbf{n}}) \leq \mathbf{m}, \Delta}{\Gamma \vdash \varphi(\overline{A}, \overline{B}) \leq \mathbf{m}, \Delta}\,\varphi_L$$

$$\frac{\Big(\Gamma, \mathbf{j} \leq \psi(\overline{B}, \overline{A}) \vdash \mathbf{h}_j \leq A_j, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{h}}), \Delta\Big)_j \qquad \Big(\Gamma, \mathbf{j} \leq \psi(\overline{B}, \overline{A}) \vdash B_i \leq \mathbf{n}_i, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{h}}), \Delta\Big)_i}{\Gamma, \mathbf{j} \leq \psi(\overline{B}, \overline{A}) \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{h}}), \Delta}\,\psi_L$$

$$\frac{\Big(\Gamma, \varphi(\overline{A}, \overline{B}) \leq \mathbf{m} \vdash \mathbf{h}_i \leq A_i, \varphi(\overline{\mathbf{h}}, \overline{\mathbf{n}}) \leq \mathbf{m}, \Delta\Big)_i \qquad \Big(\Gamma, \varphi(\overline{A}, \overline{B}) \leq \mathbf{m} \vdash B_j \leq \mathbf{n}_j, \varphi(\overline{\mathbf{h}}, \overline{\mathbf{n}}) \leq \mathbf{m}, \Delta\Big)_j}{\Gamma, \varphi(\overline{A}, \overline{B}) \leq \mathbf{m} \vdash \varphi(\overline{\mathbf{h}}, \overline{\mathbf{n}}) \leq \mathbf{m}, \Delta}\,\varphi_R$$

where the index i (resp. j) ranges over the length of the vector $\overline{x}$ (resp. $\overline{y}$), and no variable in $\overline{\mathbf{n}}$ or in $\overline{\mathbf{h}}$ occurs in the conclusion of ψ_R or φ_l. The soundness of ψ_R hinges on the fact that, on perfect (complex) modal algebras $\mathbb{A}$, the term-function $\psi^{\mathbb{A}}(!\overline{x}, !\overline{y})$ associated with $\psi(!\overline{x}, !\overline{y})$ is completely meet-preserving (resp. join-reversing), hence monotone (resp. antitone), in each x-coordinate (resp. y-coordinate), and moreover, perfect algebras are completely join-generated (resp. meet-generated) by their completely join-irreducible (resp. meet-irreducible) elements. Thus,

$$\psi^{\mathbb{A}}(\overline{B^{\mathbb{A}}}, \overline{A^{\mathbb{A}}}) = \bigwedge\{\psi^{\mathbb{A}}(\overline{n}, \overline{h}) \mid h \in J^{\infty}(\mathbb{A}), n \in M^{\infty}(\mathbb{A}), \overline{h \leq A^{\mathbb{A}}}, \overline{B^{\mathbb{A}} \leq n}\},$$

and hence, for any $j \in J^{\infty}(\mathbb{A})$, the inequality $j \leq \psi^{\mathbb{A}}(\overline{B^{\mathbb{A}}}, \overline{A^{\mathbb{A}}})$ holds iff $j \leq \psi^{\mathbb{A}}(\overline{n}, \overline{h})$ for all $h \in J^{\infty}(\mathbb{A})$ and $n \in M^{\infty}(\mathbb{A})$ such that $h \leq A^{\mathbb{A}}$ and $B^{\mathbb{A}} \leq n$. Similarly, the soundness of φ_L hinges on the fact that the term-function $\varphi^{\mathbb{A}}(!\overline{x}, !\overline{y})$ associated with $\varphi(!\overline{x}, !\overline{y})$ is completely join-preserving (resp. meet-reversing), hence monotone (resp. antitone), in each x-coordinate (resp. y-coordinate). The soundness of ψ_L (resp. φ_R) follows from the coordinate-wise monotonicity/antitonicity of the term-functions $\psi^{\mathbb{A}}$ and $\varphi^{\mathbb{A}}$.

The rule ψ_R can be regarded as a right-introduction rule, which, in particular, can be instantiated to the counterparts of the rules $\Box_R$, $\vee_R$, $\rightarrow_R$ of

G3K when $\psi(\overline{B},\overline{A}) := \Box B$, $\psi(\overline{B},\overline{A}) := B_1 \vee B_2$, $\psi(\overline{B},\overline{A}) := A \to B$, respectively. In this context, the pure inequality $\mathbf{j} \leq \psi^{\mathbb{A}}(\overline{\mathbf{n}},\overline{\mathbf{h}})$ captures the relational information. For instance, if $\psi(\overline{B},\overline{A}) := \Box B$, then $\mathbf{j} \leq \psi^{\mathbb{A}}(\overline{\mathbf{n}},\overline{\mathbf{h}})$ is $\mathbf{j} \leq \Box\mathbf{n}$, which translates on Kripke frames as $\{x\} \subseteq (R^{-1}[\{y\}^{cc}])^c$, i.e. $x \notin R^{-1}[y]$, i.e. $\neg(xRy)$. If $\psi(\overline{B},\overline{A}) := A \to B$, then $\mathbf{j} \leq \psi^{\mathbb{A}}(\overline{\mathbf{n}},\overline{\mathbf{h}})$ is $\mathbf{j} \leq \mathbf{h} \to \mathbf{n}$, which is equivalent to $\mathbf{j} \wedge \mathbf{h} \leq \mathbf{n}$, which translates on Kripke frames as $\{x\} \cap \{y\} \subseteq \{z\}^c$, i.e. $x \neq y$ or $x \neq z$. Similarly, φ_L can be regarded as a left-introduction rule and can be instantiated to the counterparts of the rules $\Diamond_L$, $\wedge_L$ of G3K.

Step (i) + cuts. In the present section, we execute the first phase of MASSA as indicated in Section 3, and derive the axiom $\mathbf{j} \leq \psi' \vdash \mathbf{j} \leq \psi'$ for an arbitrary definite negative analytic Sahlqvist formula. For simplicity, we assume that $\psi' := \psi(\overline{\beta},\overline{\delta})$ is in NNF with $\psi(!\overline{x},!\overline{y})$ positive in each x in $\overline{x}$ and each y in $\overline{y}$, and for any $q, q_1 \in \mathsf{Prop}$, we let $p := \neg q$, $p_1 := \neg q_1$, etc.

$$\dfrac{\dfrac{\left(\overline{\beta \leq \mathbf{n}}, \overline{\delta \leq \mathbf{o}}, \mathbf{j} \leq \psi(\overline{\beta},\overline{\delta}) \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}}), \beta_n \leq \mathbf{n}_n\right)_n \qquad \left(\overline{\beta \leq \mathbf{n}}, \overline{\delta \leq \mathbf{o}}, \mathbf{j} \leq \psi(\overline{\beta},\overline{\delta}) \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}}), \delta_o \leq \mathbf{o}_o\right)_o}{\overline{\beta \leq \mathbf{n}}, \overline{\delta \leq \mathbf{o}}, \mathbf{j} \leq \psi(\overline{\beta},\overline{\delta}) \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}})}\,\psi_L}{\mathbf{j} \leq \psi(\overline{\beta},\overline{\delta}) \vdash \mathbf{j} \leq \psi(\overline{\beta},\overline{\delta})}\,\psi_R$$

where n (resp. o) ranges over the length of $\overline{x}$ (resp. $\overline{y}$). Before proceeding on each branch, we prune the proof-tree as follows:

$$\dfrac{\dfrac{\left(\beta_n \leq \mathbf{n}_n \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}}), \beta_n \leq \mathbf{n}_n\right)_n \qquad \left(\delta_o \leq \mathbf{o}_o \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}}), \delta_o \leq \mathbf{o}_o\right)_o}{\overline{\beta \leq \mathbf{n}}, \overline{\delta \leq \mathbf{o}}, \mathbf{j} \leq \psi(\overline{\beta},\overline{\delta}) \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}})}\,\psi_L}{\mathbf{j} \leq \psi(\overline{\beta},\overline{\delta}) \vdash \mathbf{j} \leq \psi(\overline{\beta},\overline{\delta})}\,\psi_R$$

Now we proceed on each branch separately. The assumption that the input formula ψ' is Sahlqvist entails that there are no non-critical occurrences of proposition variables in each β in $\overline{\beta}$ (which, as discussed in Section 2.2, can be assumed w.l.o.g. to be a definite negative PIA formula containing exactly one critical occurrence of a proposition variable). Likewise, each δ in $\overline{\delta}$ can be assumed w.l.o.g. to be a definite negative PIA formula which only contains non-critical occurrences. Thus, the branches of the proof-tree continue as follows for each β_n and δ_o up to the leaves:

$$\dfrac{\beta_{nR}\,\dfrac{p \leq \mathbf{l} \vdash p \leq \mathbf{l}, \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}}), \beta_n(\varnothing,\mathbf{l}) \leq \mathbf{n}_n}{\beta_n(\varnothing,p) \leq \mathbf{n}_n, p \leq \mathbf{l} \vdash \beta_n(\varnothing,\mathbf{l}) \leq \mathbf{n}_n, \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}})}}{\beta_n(\varnothing,p) \leq \mathbf{n}_n \vdash \beta_n(\varnothing,p) \leq \mathbf{n}_n, \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}})}\,\beta_{nL}$$

$$\dfrac{\delta_{oR}\,\dfrac{\left(\mathbf{k}_k \leq q_k \vdash \mathbf{k}_k \leq q_k, \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}}), \delta_o(\overline{\mathbf{k}},\varnothing) \leq \mathbf{o}_o\right)_k}{\delta_o(\overline{q},\varnothing) \leq \mathbf{o}_o, \overline{\mathbf{k} \leq q} \vdash \delta_o(\overline{\mathbf{k}},\varnothing) \leq \mathbf{o}_o, \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}})}}{\delta_o(\overline{q},\varnothing) \leq \mathbf{o}_o \vdash \delta_o(\overline{q},\varnothing) \leq \mathbf{o}_o, \mathbf{j} \leq \psi(\overline{\mathbf{n}},\overline{\mathbf{o}})}\,\delta_{oL}$$

Next, we perform all the possible cuts between the leaves of the proof-tree described above. The cut formulas involved in each of these cuts will necessarily be one critical and one non-critical occurrence of the same proposition variable. Thus, these cuts can be executed as follows:

$$\dfrac{\dfrac{p \leq \mathbf{l} \vdash p \leq \mathbf{l}, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}), \beta_n(\varnothing, \mathbf{l}) \leq \mathbf{n}_n}{\vdash \neg\mathbf{l} \leq p, p \leq \mathbf{l}, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}), \beta_n(\varnothing, \mathbf{l}) \leq \mathbf{n}_n} \quad \dfrac{\mathbf{k}_k \leq q_k \vdash \mathbf{k}_k \leq q_k, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}), \delta_o(\overline{\mathbf{k}}, \varnothing) \leq \mathbf{o}_o}{\vdash q_k \leq \neg\mathbf{k}_k, \mathbf{k}_k \leq q_k, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}), \delta_o(\overline{\mathbf{k}}, \varnothing) \leq \mathbf{o}_o}}{\dfrac{\dfrac{\vdash \neg\mathbf{l} \leq \neg\mathbf{k}_k, \mathbf{k}_k \leq q, p \leq \mathbf{l}, \beta_n(\varnothing, \mathbf{l}) \leq \mathbf{n}_n, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}), \delta_o(\overline{\mathbf{k}}, \varnothing) \leq \mathbf{o}_o}{\neg(\neg\mathbf{l} \leq \neg\mathbf{k}_k) \vdash \mathbf{k}_k \leq q, p \leq \mathbf{l}, \beta_n(\varnothing, \mathbf{l}) \leq \mathbf{n}_n, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}), \delta_o(\overline{\mathbf{k}}, \varnothing) \leq \mathbf{o}_o}}{\mathbf{k}_k \leq \neg\mathbf{l} \vdash \mathbf{k}_k \leq q, p \leq \mathbf{l}, \beta_n(\varnothing, \mathbf{l}) \leq \mathbf{n}_n, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}), \delta_o(\overline{\mathbf{k}}, \varnothing) \leq \mathbf{o}_o}}$$

for any n, o and k, where, for each proposition variable, the index k ranges over the multiplicity of that variable in δ_o, and q_k and p are a negative and a positive occurrence of that variable, while each $\mathbf{k}_k$ is a different nominal variable; however, $\mathbf{k}_k \leq \neg\mathbf{l}$ translates into $\{x\} \subseteq \{y\}^{cc}$, i.e. $x = y$, that is, the cuts above give rise to new axioms.

Steps (ii) - (iv) Next, we start generating the rule corresponding to $\psi(\overline{\beta}, \overline{\delta})$ by applying the right-introduction rule bottom-up to its Skeleton $\psi(!\overline{x}, !\overline{y})$:

$$\dfrac{\overline{\overline{\beta \leq \mathbf{n}}, \overline{\delta \leq \mathbf{o}} \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})}}{\vdash \mathbf{j} \leq \psi(\overline{\beta}, \overline{\delta})} \ \psi_R$$

Writing it contrapositively, the relational information generated by the bottom-up application of ψ_R is exactly the antecedent of the last line in the ALBA run executed in Section A, which we report here for the reader's convenience.

$$\forall \mathbf{j} \forall \overline{\mathbf{n}} \forall \overline{\mathbf{o}} [\psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}) \leq \neg\mathbf{j} \Rightarrow \invamp_{n,o} \overline{(\neg\mathbf{o} \leq \delta_o(\overline{RA(\beta_n)(\mathbf{n})}/\overline{p})}].$$

We claim that every inequality $\neg\mathbf{o} \leq \delta_o(\overline{RA(\beta_n)(\mathbf{n})}/\overline{p})$ in the disjunction of the succedent of the implication above provides the relational information of a premise in the rule generated by the algorithm. Indeed, the starting point for proving this claim is the observation that each such disjunct corresponds to a certain subset of the axioms generated by the cuts executed at the end of the first phase. Accordingly, in what follows, we proceed on one such disjunct $\neg\mathbf{o} \leq \delta_o(\overline{RA(\beta_n)(\mathbf{n})})$, by identifying the corresponding axioms, and using them as the leaves of a derivation of the corresponding premise, which we will generate by successive applications of right-introduction rules on the βs and δs. Below, the index k ranges over the length of $\overline{x}$ in $\delta_o(!\overline{x})$.

$$\delta_{oR} \ \dfrac{\Big(\mathbf{k}_k \leq \neg\mathbf{l} \vdash \mathbf{k}_k \leq q_k, p_k \leq \mathbf{l}_k, \beta_k(\mathbf{l}_k) \leq \mathbf{n}_k, \delta_o(\overline{\mathbf{k}}) \leq \mathbf{o}, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})\Big)_k}{\dfrac{\dfrac{\delta_o(\overline{q}) \leq \mathbf{o}, \overline{\mathbf{k} \leq \neg\mathbf{l}} \vdash \overline{q \leq \mathbf{l}}, \overline{\beta(\mathbf{l}) \leq \mathbf{n}}, \delta_o(\overline{\mathbf{k}}) \leq \mathbf{o}, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})}{\vdots} \ \beta_R}{\overline{\beta(p) \leq \mathbf{n}}, \delta_o(\overline{q}) \leq \mathbf{o}, \overline{\mathbf{k} \leq \neg\mathbf{l}} \vdash \overline{\beta(\mathbf{l}) \leq \mathbf{n}}, \delta_o(\overline{\mathbf{k}}) \leq \mathbf{o}, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})} \ \beta_R}$$

To see that we have reached a merging point (cf. Section 3), observe that, by construction, the set of the non-pure inequalities of $S := \overline{\beta \leq \mathbf{n}}, \overline{\delta \leq \mathbf{o}} \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})$ is the union of the non-pure inequalities of the $S_{(o,n)} := \overline{\beta(p) \leq \mathbf{n}}, \delta_o(\overline{q}) \leq \mathbf{o}, \overline{\mathbf{k} \leq \neg\mathbf{l}} \vdash \overline{\beta(\mathbf{l}) \leq \mathbf{n}}, \delta_o(\overline{\mathbf{k}}) \leq \mathbf{o}, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}})$ associated with every disjunct, and the derivations $\pi^{(o,n)}_{\psi'}$ also corresponding to all these disjuncts. Hence, the rule r generated by the algorithm corresponding to $\psi(\overline{\beta}, \overline{\delta})$ is

$$\frac{\Big(\Gamma, \overline{\mathbf{k} \leq \neg \mathbf{l}} \vdash \overline{\beta_n(\mathbf{l}) \leq \mathbf{n}}, \delta_o(\overline{\mathbf{k}}) \leq \mathbf{o}, \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}), \Delta\Big)_{o,n}}{\Gamma \vdash \mathbf{j} \leq \psi(\overline{\mathbf{n}}, \overline{\mathbf{o}}), \Delta}$$

where o as before ranges over the number of δs, while the index n ranges over all possible combinations of βs whose critical proposition variables occurs in δ_o. The last line in the derivation above is the premise of the rule r corresponding to the disjunct $\neg \mathbf{o} \leq \delta_o(\overline{RA(\beta_n)})$. To complete the proof, let us show that the relational information in this premise is equivalent to $\neg \mathbf{o} \leq \delta_o(\overline{RA(\beta_n)})$:

Lemma A.1 *The following are equivalent for every perfect distributive modal algebra $\mathbb{A}$, and all formulas $\delta(!\overline{x})$ and $\overline{\beta(!y)}$ such that δ is monotone in each x-coordinate and each β in $\overline{\beta}$ is antitone in y:*

(i) $\mathbb{A} \models \forall \overline{\mathbf{k}} \forall \overline{\mathbf{l}} \forall \overline{\mathbf{n}} \forall \mathbf{o} \big(\overline{\mathbf{k} \leq \neg \mathbf{l}} \vdash \overline{\beta(\mathbf{l}) \leq \mathbf{n}}, \delta(\overline{\mathbf{k}}) \leq \mathbf{o}\big)$;

(ii) $\mathbb{A} \models \forall \overline{\mathbf{n}} \forall \mathbf{o} \big(\neg \mathbf{o} \leq \delta(\overline{RA(\beta)(\mathbf{n})}) \vdash \ \big)$.

Proof. From (ii) to (i), it is enough to show that if $\overline{k} \in J^{\infty}(\mathbb{A})^k$ and $\overline{l} \in M^{\infty}(\mathbb{A})^l$ and $\overline{n} \in M^{\infty}(\mathbb{A})^n$ and $o \in M^{\infty}(\mathbb{A})$, such that $\overline{k \leq \neg l}$ and $\overline{\neg n \leq \beta^{\mathbb{A}}(l)}$ and $\neg o \leq \delta^{\mathbb{A}}(\overline{k})$, then $\neg o \leq \delta^{\mathbb{A}}(\overline{RA(\beta^{\mathbb{A}})(n)})$.

The assumption $\neg o \leq \delta^{\mathbb{A}}(\overline{k})$ and the monotonicity of $\delta^{\mathbb{A}}$ imply that it is enough to show $k \leq RA(\beta^{\mathbb{A}})(n)$ for each coordinate of $\delta^{\mathbb{A}}$. The assumption $\neg n \leq \beta^{\mathbb{A}}(l)$ is equivalent to $\beta^{\mathbb{A}}(l) \not\leq n$ which by adjunction is equivalent to $(RA(\beta))^{\mathbb{A}}(n) \not\leq l$, i.e. $\neg l \leq (RA(\beta))^{\mathbb{A}}(n)$. The required inequality then follows by transitivity, combining the latter inequality with the assumption $k \leq \neg l$.

Conversely, let $\overline{n} \in M^{\infty}(\mathbb{A})^n$ and $o \in M^{\infty}(\mathbb{A})$ such that $\neg o \leq \delta^{\mathbb{A}}(\overline{(RA(\beta))^{\mathbb{A}}(n)})$, and let us find $\overline{k} \in J^{\infty}(\mathbb{A})^k$ and $\overline{l} \in M^{\infty}(\mathbb{A})^l$ such that $\overline{k \leq \neg l}$ and $\overline{\neg n \leq \beta^{\mathbb{A}}(l)}$ and $\neg o \leq \delta^{\mathbb{A}}(\overline{k})$.

Since $\delta(!\overline{x})$ is a definite negative PIA formula which is positive in each coordinate, $\delta^{\mathbb{A}}(!\overline{x})$ is completely join-preserving in each coordinate; thus, $\neg o \leq \delta^{\mathbb{A}}(\overline{(RA(\beta))^{\mathbb{A}}(n)})$ can be equivalently rewritten as $\neg o \leq \bigvee\{\delta^{\mathbb{A}}(\overline{k}) \mid \overline{k} \in J^{\infty}(\mathbb{A}), \overline{k \leq (RA(\beta))^{\mathbb{A}}(n)}\}$. Since $\neg o \in J^{\infty}(\mathbb{A})$ and is hence completely join-prime, the latter inequality is equivalent to $\neg o \leq \delta^{\mathbb{A}}(\overline{k})$ for some $\overline{k} \in J^{\infty}(\mathbb{A})^l$ such that $\overline{k \leq (RA(\beta))^{\mathbb{A}}(n)}$. Let $l := \neg k \in M^{\infty}(\mathbb{A})$ for each k in $\overline{k}$. Then $k \leq (RA(\beta))^{\mathbb{A}}(n)$ iff $(RA(\beta))^{\mathbb{A}}(n) \not\leq l$ iff $\beta^{\mathbb{A}}(l) \not\leq n$ iff $\neg n \leq \beta^{\mathbb{A}}(l)$, as required. □

The proof of correctness when $\psi(\overline{\beta}, \overline{\delta})$ is properly inductive w.r.t. some strict order Ω is similar. Consider for instance a formula $\psi'(p_1, p_2) := \psi(\beta_1(\varnothing, p_1), \beta_2(q_1, p_2), \delta(q_2, \varnothing))$, with $p_1 <_{\Omega} p_2$. Running ALBA on $\psi'(p_1, p_2)$ yields

$$\forall \mathbf{n}_1 \forall \mathbf{n}_2 \forall \mathbf{o} \forall \mathbf{j} [\psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}) \leq \neg \mathbf{j} \Rightarrow \neg \mathbf{o} \leq \delta(RA(\beta_2)(RA(\beta_1)(\mathbf{n}_1), \mathbf{n}_2))]$$

The first phase proceeds as described in section 3, and produces the following cuts:

$$\dfrac{\dfrac{\dfrac{p_1 \leq \mathbf{l}_1 \ \vdash \ p_1 \leq \mathbf{l}_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_1(\varnothing, \mathbf{l}_1) \leq \mathbf{n}_1}{\vdash \ \neg \mathbf{l}_1 \leq p_1, p_1 \leq \mathbf{l}_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_1(\varnothing, \mathbf{l}_1) \leq \mathbf{n}_1} \qquad \dfrac{\mathbf{i} \leq q_1 \ \vdash \ \mathbf{i} \leq q_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2}{\vdash \ q_1 \leq \neg \mathbf{i}, \mathbf{i} \leq q_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2}}{\vdash \ \neg \mathbf{l}_1 \leq \neg \mathbf{i}, \mathbf{i} \leq q_1, p_1 \leq \mathbf{l}_1, \beta_1(\varnothing, \mathbf{l}_1) \leq \mathbf{n}_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2}}{\mathbf{i} \leq \neg \mathbf{l}_1 \ \vdash \ \mathbf{i} \leq q_1, p_1 \leq \mathbf{l}_1, \beta_1(\varnothing, \mathbf{l}_1) \leq \mathbf{n}_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2}$$

$$\dfrac{\dfrac{\dfrac{\mathbf{k} \leq q_2 \ \vdash \ \mathbf{k} \leq q_2, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \delta(\mathbf{k}, \varnothing) \leq \mathbf{o}}{\vdash \ q_2 \leq \neg \mathbf{k}, \mathbf{k} \leq q_2, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \delta(\mathbf{k}, \varnothing) \leq \mathbf{o}} \qquad \dfrac{p_2 \leq \mathbf{l}_2 \ \vdash \ p_2 \leq \mathbf{l}_2, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2}{\vdash \ \neg \mathbf{l}_2 \leq p_2, p_2 \leq \mathbf{l}_2, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2}}{\vdash \ \neg \mathbf{l}_2 \leq \neg \mathbf{k}, \mathbf{k} \leq q_2, p_2 \leq \mathbf{l}_2, \delta(\mathbf{k}, \varnothing) \leq \mathbf{o}, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2}}{\mathbf{k} \leq \neg \mathbf{l}_2 \ \vdash \ \mathbf{k} \leq q_2, p_2 \leq \mathbf{l}_2, \delta(\mathbf{k}, \varnothing) \leq \mathbf{o}, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2}$$

Using the new axioms, let us complete the second phase as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{i} \leq \neg \mathbf{l}_1 \ \vdash \ \mathbf{i} \leq q_1, p_1 \leq \mathbf{l}_1, \beta_1(\varnothing, \mathbf{l}_1) \leq \mathbf{n}_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2}{\mathbf{i} \leq \neg \mathbf{l}_1, \beta_1(\varnothing, p_1) \leq \mathbf{n}_1 \ \vdash \ \mathbf{i} \leq q_1, \beta_1(\varnothing, \mathbf{l}_1) \leq \mathbf{n}_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2} \qquad \dfrac{\mathbf{k} \leq \neg \mathbf{l}_2 \ \vdash \ \mathbf{k} \leq q_2, p_2 \leq \mathbf{l}_2, \delta(\mathbf{k}, \varnothing) \leq \mathbf{o}, \mathbf{j} \leq}{\mathbf{k} \leq \neg \mathbf{l}_2, \delta(q_2, \varnothing) \leq \mathbf{o} \ \vdash \ p_2 \leq \mathbf{l}_2, \delta(\mathbf{k}, \varnothing) \leq \mathbf{o}, \mathbf{j}}}{\mathbf{i} \leq \neg \mathbf{l}_1, \mathbf{k} \leq \neg \mathbf{l}_2, \beta_1(\varnothing, p_1) \leq \mathbf{n}_1, \delta(q_2, \varnothing) \leq \mathbf{o}, \beta_2(q_1, p_2) \leq \mathbf{n}_2 \ \vdash \ \beta_1(\varnothing, \mathbf{l}_1) \leq \mathbf{n}_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2, \delta}}{\beta_1(\varnothing, p_1) \leq \mathbf{n}_1, \delta(q_2, \varnothing) \leq \mathbf{o}, \beta_2(q_1, p_2) \leq \mathbf{n}_2 \ \vdash \ \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o})}}{\vdash \ \mathbf{j} \leq \psi(\beta_1(\varnothing, p_1), \beta_2(q_1, p_2), \delta(q_2, \varnothing))}$$

Notice that the dashed line above is a successful merging point where there is only one premise. Hence, the output rule is:

$$\dfrac{\Gamma, \mathbf{i} \leq \neg \mathbf{l}_1, \mathbf{k} \leq \neg \mathbf{l}_2 \ \vdash \ \beta_1(\varnothing, \mathbf{l}_1) \leq \mathbf{n}_1, \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \beta_2(\mathbf{i}, \mathbf{l}_2) \leq \mathbf{n}_2, \Delta(\mathbf{k}, \varnothing) \leq \mathbf{o}, \Gamma}{\Gamma \ \vdash \ \mathbf{j} \leq \psi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{o}), \Delta}$$

With an argument similar to the one used to prove the equivalence in Lemma A.1, it can be shown that the relational information of the premise of the rule above is equivalent to $\neg \mathbf{o} \leq \delta(RA(\beta_2)(RA(\beta_1)(\mathbf{n}_1), \mathbf{n}_2))$.