Many-valued hybrid logic

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abstract. In this paper we define a many-valued semantics for hybrid logic and we give a sound and complete tableau system which is proof-theoretically well-behaved, in particular, it gives rise to a decision procedure for the logic. This shows that many-valued hybrid logics is a natural enterprise and opens up the way for future applications.

Keywords: Modal logic, hybrid logic, many-valued logic, tableau systems.

1 Introduction

Classical hybrid logic is obtained by adding to ordinary, classical modal logic further expressive power in the form of a second sort of propositional symbols called nominals, and moreover, by adding so-called satisfaction operators. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. Thus, in hybrid logic a name is a particular sort of propositional symbol whereas in first-order logic it is an argument to a predicate. If i is a nominal and ϕ is an arbitrary formula, then a new formula $@_i\phi$ called a satisfaction statement can be formed. The part $@_i$ of $@_i\phi$ is called a satisfaction operator. The satisfaction statement $@_i\phi$ expresses that the formula ϕ is true at one particular world, namely the world at which the nominal i is true. Hybrid logic is proof-theoretically well-behaved, which is documented in the forthcoming book [7]. Hybrid-logical proof-theory includes a long line of work on tableau systems for hybrid logic, see [1, 2, 6, 4, 15, 3].

Now, classical hybrid logic can be viewed as a combination of two logics, namely classical, two-valued logic (where the standard propositional connectives are interpreted in terms of the truth-values true and false) and hybrid modal logic (where modal operators, nominals, and satisfaction operators are interpreted in terms of a set of possible worlds equipped with an accessibility relation). The present paper concerns many-valued hybrid logic, that is, hybrid logic where the two-valued logic basis has been generalized to a many-valued logic basis. To be more precise, we shall define a many-valued semantics for hybrid logic, and we shall give a tableau system that is sound and complete with respect to the semantics. Not only is the many-valued semantics a generalization of the two-valued semantics, but if we chose a two-valued version of the many-valued tableau system, then modulo minor reformulations and the deletion of superfluous rules, the tableau system obtained is identical to an already known tableau systems for hybrid logic.

Our many-valued semantics is a hybridized version of a many-valued semantics for modal logic given in the papers [11, 12, 13]. A notable feature of this semantics is that it allows the accessibility relation as well as formulas to take on many truth-values (in other many-valued modal logics it is only formulas that can take on many truth-values).

A leading idea behind our work is that we distinguish between the way of reasoning and what the reasoning is about, and in accordance with this idea, we generalize the way of reasoning from two-valued logic to many-valued logic such that we reason in a many-valued way about time, space, knowledge, states in a computer, or whatever the subject-matter is. Given our distinction between the way of reasoning and what the reasoning is about, we take it that the concerns of hybrid logic basically are orthogonal to as whether the logic basis is two-valued or many-valued. Thus, it is expectable that the already known proof-theoretically well-behaved tableau systems for two-valued hybrid logic can be generalized to proof-theoretically well-behaved tableau systems for many-valued hybrid logic. Accordingly, if we define a many-valued hybrid logic and give a tableau system that satisfies standard proof-theoretic requirements (it is cut-free, it satisfies a version of the subformula property, and it gives rise to a decision procedure), then we learn more about hybrid logic and we provide more evidence that hybrid logic and hybrid-logical proof-theory is a natural enterprise.

This paper is structured as follows. In the second section of the paper we define the many-valued semantics for hybrid logic and we make some remarks on the relation to intuitionistic hybrid logic. In the third section we introduce a tableau system, in the fourth section we prove termination, and in the fifth section we prove completeness.

2 A many-valued hybrid logic language

In this section a Many-Valued Hybrid Logic language (denoted by **MVHL**) is presented and a semantics for the language is given. We have included global modalities, one reason being that they are used in our motivation for our choice of semantics for the nominals, but our termination and completeness proofs later in the paper do not include global modalities. In the following let $\mathcal{T}$ denote a fixed finite Heyting algebra. That is, $\mathcal{T}$ is a finite lattice such that for all a and b in $\mathcal{T}$ there is a greatest element x of $\mathcal{T}$ satisfying $a \wedge x \leq b$. The element x is called the *relative pseudo-complement* of a with respect to b (denoted $a \Rightarrow b$). To avoid notational ambiguity in relation to the syntax of our hybrid logic, we will in the following use the symbol $\Rightarrow$ for relative pseudo-complement, and $\sqcup$ and $\sqcap$ for meet and join, respectively. The largest and smallest elements of $\mathcal{T}$ are denoted $\top$ and $\perp$, respectively. The elements of the Heyting algebra $\mathcal{T}$ are going to be used as truth values for our many-valued logic. Thus, in the following, we will often refer to the elements of $\mathcal{T}$ as *truth values*.[1]

[1] In order to give reasonable semantics for $\wedge$ and $\vee$ a Lattice structure is needed. A complete Lattice would be enough if the accessibility relation was only allowed to have two values, but since we also allows for the accessibility relation to take values in $\mathcal{T}$, the

2.1 Syntax for MVHL

Let a countable infinite set of propositional variables PROP and a countable infinite set of nominals NOM be given. In addition to the usual connectives of propositional model logic, we include the global modalities E and A, and for every $i \in \mathsf{NOM}$, a satisfaction operator $@_i$.

DEFINITION 1 (**MVHL**-formulas). The set of **MVHL**-formulas is given by the following grammar:

$$\varphi ::= p \mid a \mid i \mid (\psi_1 \wedge \psi_2) \mid (\psi_1 \vee \psi_2) \mid (\psi_1 \to \psi_2) \mid \Box\psi \mid \Diamond\psi \mid @_i\psi \mid E\psi \mid A\psi,$$

where $p \in \mathsf{PROP}$, $a \in \mathcal{T}$, and $i \in \mathsf{NOM}$.

In general we will use i, j, k and so on for nominals and a, b, c for elements of $\mathcal{T}$.

2.2 Semantics for MVHL

The semantics for **MVHL** is a Kripke semantics in which the accessibility relation is allowed to take values in $\mathcal{T}$. This is inspired by [13]. A model $\mathcal{M}$ is a tuple $\mathcal{M} = \langle W, R, \mathbf{n}, \nu\rangle$, where W is the set of worlds, and R a mapping $R : W \times W \to \mathcal{T}$ called the accessibility relation. $\mathbf{n}$ is a function interpreting the nominals, i.e. $\mathbf{n} : \mathsf{NOM} \to W$. Finally the valuation $\nu : W \times \mathsf{PROP} \to \mathcal{T}$ assigns truth values to the propositional variables at each world.

Now given a model $\mathcal{M} = \langle W, R, \mathbf{n}, \nu\rangle$, we can extend the valuation ν to all formulas in the following inductive way, where $w \in W$:

$$\begin{array}{rcl}
\nu(w, a) & := & a \quad \text{for } a \in \mathcal{T} \\
\nu(w, i) & := & \begin{cases} \top & \text{, if } \mathbf{n}(i) = w \\ \bot & \text{, else} \end{cases} \\
\nu(w, \varphi \wedge \psi) & := & \nu(w, \varphi) \sqcap \nu(w, \psi) \\
\nu(w, \varphi \vee \psi) & := & \nu(w, \varphi) \sqcup \nu(w, \psi) \\
\nu(w, \varphi \to \psi) & := & \nu(w, \varphi) \Rightarrow \nu(w, \psi) \\
\nu(w, \Box\varphi) & := & \bigsqcap\{R(w, v) \Rightarrow \nu(v, \varphi) \mid v \in W\} \\
\nu(w, \Diamond\varphi) & := & \bigsqcup\{R(w, v) \sqcap \nu(v, \varphi) \mid v \in W\} \\
\nu(w, @_i\varphi) & := & \nu(\mathbf{n}(i), \varphi) \\
\nu(w, A\varphi) & := & \bigsqcap\{\nu(v, \varphi) \mid v \in W\} \\
\nu(w, E\varphi) & := & \bigsqcup\{\nu(v, \varphi) \mid v \in W\}
\end{array}$$

The semantics chosen for the hybrid logical constructions is discussed in the following. The semantics for $@_i\varphi$ is obvious, its truth value is simply the truth value of φ at the world i denotes. The semantics chosen for the

structure of a Heyting algebra is needed. For further discussions of the choice of a finite Heyting algebra as the set of truth values see [12, 13].

global modalities A and E reflect the fact that these modalities are simply the global versions of the modalities $\Box$ and $\Diamond$. The choice of semantics for nominals is less obvious. In this paper we have chosen to assign each nominal i the value $\top$ in exactly one world, and $\bot$ in all other worlds. This is in agreement with the the standard semantics for hybrid logic in which a nominal "points to a unique world". It would probably also be possible to allow nominals to take values outside the set $\{\top, \bot\}$, but at least a nominal should receive the value $\top$ in one and only one world in order for the semantics to be in accordance with classical, two-valued, hybrid logic (and for nominals to be semantically different from ordinary propositional symbols). Our decision of making the semantics of nominals two-valued rests primarily on the fact that it allows us to preserve the following well-known logical equivalence from classical, two-valued, hybrid logic:

$$@_i\varphi \leftrightarrow E(i \wedge \varphi)$$
$$@_i\varphi \leftrightarrow A(i \rightarrow \varphi)$$

With the chosen semantics, these equivalences also hold in MVHL:

$$\nu(w, @_i\varphi) = \nu(\mathbf{n}(i), \varphi) = \bigsqcup\{\nu(v, i) \sqcap \nu(v, \varphi) \mid v \in W\} = \nu(w, E(i \wedge \varphi))$$
$$\nu(w, @_i\varphi) = \nu(\mathbf{n}(i), \varphi) = \bigsqcap\{\nu(v, i) \Rightarrow \nu(v, \varphi) \mid v \in W\} = \nu(w, A(i \rightarrow \varphi)).$$

Here we have been using that the following holds in a Heyting algebra: $\top \sqcap a = a$, $\bot \sqcap a = \bot$, $a \sqcup \bot = a$, $\top \Rightarrow a = a$ and $\bot \Rightarrow a = \top$. Another pleasant property resulting from the choice of semantics for nominals is the following:

$$\nu(w, @_i\Diamond j) = \nu(\mathbf{n}(i), \Diamond j) = \bigsqcup\{R(\mathbf{n}(i), v) \sqcap \nu(v, j) \mid v \in W\} = R(\mathbf{n}(i), \mathbf{n}(j)).$$

This identity expresses that the reachability of the world denoted by j from the world denoted by i is described by the formula $@_i\Diamond j$. This property also holds in classical hybrid logic. Identity between worlds denoted by nominals can also be expressed as usual, since we have:

$$\nu(w, @_i j) = \top \text{ iff } \mathbf{n}(i) = \mathbf{n}(j).$$

2.3 The relation to intuitionistic hybrid logic

As pointed out in the paper [12], there is a close relation between the many-valued modal logic given in that paper and intuitionistic modal logic. We shall in this subsection consider the relation between many-valued hybrid logic and a variant of the intuitionistic hybrid logic given in the paper [9] (which in turn is a hybridization of an intuitionistic modal logic introduced in a tense-logical version in [10]). In the present subsection we do not assume that a finite Heyting algebra has been fixed in advance, so the only atomic formulas we consider are ordinary propositional symbols, nominals, and the symbol $\bot$. We first define an appropriate notion of an intuitionistic model, which can be seen as a restricted variant of the notion of a model given in [9][2].

DEFINITION 2. A *restricted model* for intuitionistic hybrid logic is a tuple

$$(W, \leq, D, \{R_w\}_{w \in W}, \{\nu_w\}_{w \in W})$$

where

1. W is a non-empty finite set partially ordered by $\leq$;
2. D is a non-empty set;
3. for each w, R_w is a binary relation on D such that $w \leq v$ implies $R_w \subseteq R_v$; and
4. for each w, ν_w is a function that to each ordinary propositional symbol p assigns a subset of D such that $w \leq v$ implies $\nu_w(p) \subseteq \nu_v(p)$.

The elements of the set W are states of knowledge and for any such state w, the relation R_w is the set of known relationships between possible worlds and the set $\nu_w(p)$ is the set of possible worlds at which p is known to be true. Note that the definition requires that the epistemic partial order $\leq$ preserves these kinds of knowledge, that is, if an advance to a greater state of knowledge is made, then what is known is preserved.

Given a restricted model $\mathfrak{M} = (W, \leq, D, \{R_w\}_{w \in W}, \{\nu_w\}_{w \in W})$, an *assignment* is a function $\mathbf{n}$ that to each nominal assigns an element of D. The relation $\mathfrak{M}, \mathbf{n}, w, d \models \phi$ is defined by induction, where w is an element of W, $\mathbf{n}$ is an assignment, d is an element of D, and ϕ is a formula.

$$\begin{array}{rll}
\mathfrak{M}, \mathbf{n}, w, d \models p & \text{iff} & d \in \nu_w(p) \\
\mathfrak{M}, \mathbf{n}, w, d \models i & \text{iff} & d = \mathbf{n}(i) \\
\mathfrak{M}, \mathbf{n}, w, d \models \phi \wedge \psi & \text{iff} & \mathfrak{M}, \mathbf{n}, w, d \models \phi \text{ and } \mathfrak{M}, \mathbf{n}, w, d \models \psi \\
\mathfrak{M}, \mathbf{n}, w, d \models \phi \vee \psi & \text{iff} & \mathfrak{M}, \mathbf{n}, w, d \models \phi \text{ or } \mathfrak{M}, \mathbf{n}, w, d \models \psi \\
\mathfrak{M}, \mathbf{n}, w, d \models \phi \rightarrow \psi & \text{iff} & \text{for all } v \geq w, \\
 & & \mathfrak{M}, \mathbf{n}, v, d \models \phi \text{ implies } \mathfrak{M}, \mathbf{n}, v, d \models \psi \\
\mathfrak{M}, \mathbf{n}, w, d \models \bot & \text{iff} & \text{falsum} \\
\mathfrak{M}, \mathbf{n}, w, d \models \Box\phi & \text{iff} & \text{for all } v \geq w, \text{ for all } e \in D, \\
 & & dR_v e \text{ implies } \mathfrak{M}, \mathbf{n}, v, e \models \phi \\
\mathfrak{M}, \mathbf{n}, w, d \models \Diamond\phi & \text{iff} & \text{for some } e \in D, dR_w e \text{ and } \mathfrak{M}, \mathbf{n}, w, e \models \phi \\
\mathfrak{M}, \mathbf{n}, w, d \models @_i\phi & \text{iff} & \mathfrak{M}, \mathbf{n}, w, \mathbf{n}(i) \models \phi \\
\mathfrak{M}, \mathbf{n}, w, d \models A\phi & \text{iff} & \text{for all } v \geq w, \text{ for all } e \in D, \mathfrak{M}, \mathbf{n}, v, e \models \phi \\
\mathfrak{M}, \mathbf{n}, w, d \models E\phi & \text{iff} & \text{for some } e \in D, \mathfrak{M}, \mathbf{n}, w, e \models \phi
\end{array}$$

[2]Compare to Definition 2, p. 237, of the paper [9]. The differences are the following: i) In [9] the set W need not be finite. ii) Instead of D there is a family $\{D_w\}_{w \in W}$ of non-empty sets such that $w \leq v$ implies $D_w \subseteq D_v$, R_w is a binary relation on D_w, and $\nu_w(p)$ is a subset of D_w. iii) There is a family $\{\sim_w\}_{w \in W}$ where $\sim_w$ is an equivalence relation on D_w such that $w \leq v$ implies $\sim_w \subseteq \sim_v$ and such that if $d \sim_w d'$, $e \sim_w e'$, and $dR_w e$, then $d' R_w e'$, and similarly, if $d \sim_w d'$ and $d \in \nu_w(p)$, then $d' \in \nu_w(p)$. The equivalence relations are used for the interpretation of nominals. Such a model for intuitionistic hybrid logic corresponds to a standard model for intuitionistic first-order logic with equality where equality is interpreted using the equivalence relations, cf. [16].

This semantics can be looked upon in two different ways: As indicated above, it can be seen as a restricted variant of the semantics given in [9], but it can also be seen as a hybridized version of a semantics given in the paper [12]. In the latter paper, the epistemic worlds of the semantics are thought of as experts and the epistemic partial order is thought of as a relation of dominance between experts: One expert dominates another one if whatever the first expert says is true is also said to be true by the second expert.

As pointed out in [12], the intuitionistic semantics for modal logic is in a certain sense equivalent to the many-valued semantics. This also holds in the hybrid-logical case. In what follows, we outline this equivalence. It can be shown that given a restricted model $\mathfrak{M} = (W, \leq, D, \{R_w\}_{w\in W}, \{\nu_w\}_{w\in W})$, cf. Definition 2, and an assignment $\mathbf{n}$, the $\leq$-closed subsets of W ordered by $\subseteq$ constitute a finite Heyting algebra, and moreover, a many-valued model $(D, R^*, \mathbf{n}, \nu^*)$ can be defined by letting

- $R^*(d, e) = \{w \in W \mid dR_w e\}$ and
- $\nu^*(d, p) = \{w \in W \mid d \in \nu_w(p)\}$.

By a straightforward extension of the corresponding proof in [12], it can be proved that for any formula ϕ, it is the case that $\nu^*(d, \phi) = \{w \in W \mid \mathfrak{M}, \mathbf{n}, w, d \models \phi\}$. Conversely, given a finite Heyting algebra $\mathcal{T}$ and a many-valued model $(D, R, \mathbf{n}, \nu)$, a restricted model $\mathfrak{M} = (W, \subseteq, D, \{R^*_w\}_{w\in W}, \{\nu^*_w\}_{w\in W})$ can be defined by letting

- $W = \{w \mid w$ is a proper prime filter in $\mathcal{T}\}$,
- $dR^*_w e$ if and only if $R(d, e) \in w$, and
- $d \in \nu^*_w(p)$ if and only if $\nu(d, p) \in w$.

Details can be found in the paper [12]. Again, by a straightforward extension of the corresponding proof in that paper, it can be proved that for any formula ϕ, it is the case that $\mathfrak{M}, \mathbf{n}, w, d \models \phi$ if and only if $\nu(d, \phi) \in w$.

Thus, in the above sense the intuitionistic semantics for hybrid logic is equivalent to the many-valued semantics for hybrid logic. It is an interesting question whether there is such an equivalence if instead of the restricted models of Definition 2 one considers the more general models for intuitionistic hybrid logic given in the paper [9][3]. We shall leave this to further work.

3 A tableau calculus for MVHL

In the following we will present a tableau calculus for MVHL. The basic notions for tableaux are defined as usual (see e.g. [14]). The formulas

[3] As indicated in the previous footnote, in the intuitionistic semantics of [9], nominals are interpreted using a family $\{\sim_w\}_{w\in W}$ of equivalence relations, not identity. This seems to imply that in an equivalent many-valued semantics, nominals should be allowed to take on arbitrary truth-values, not just top and bottom.

occurring in our tableaux will all be of the form $@_i(a \to \varphi)$ or $@_i(\varphi \to a)$ prefixed either a T or an F, where $i \in \mathsf{NOM}$ and $a \in \mathcal{T}$. That is, the formulas occurring in our tableaux will be *signed formulas* of hybrid logic. A signed formula of the form $T@_i(a \to \varphi)$ is used to express that the formula $a \to \varphi$ is true at i, that is, receives the value $\top$ at i. If $\nu(\mathbf{n}(i), a \to \varphi) = \top$ then, by definition of ν, $a \Rightarrow \nu(\mathbf{n}(i), \varphi) = \top$. By definition of relative pseudo-complement we then get that $\top$ is the greatest element of $\mathcal{T}$ satisfying $a \wedge \top \leq \nu(\mathbf{n}(i), \varphi)$. In other words, we simply have $a \leq \nu(\mathbf{n}(i), \varphi)$. Thus what is expressed by a formula $T@_i(a \to \varphi)$ is that the truth value of φ at i is greater than or equal to a. Symmetrically, a signed formula of the formula $T@_i(\varphi \to a)$ expresses that the truth value of φ at i is less than or equal to a. Dually, a signed formula of the form $F@_i(a \to \varphi)$ ($F@_i(\varphi \to a)$) expresses that the truth value of φ at i is *not* greater than or equal to (less than or equal to) a.

The tableau rules are divided into four classes; Branch Closing Rules, Non-modal Rules, Modal Rules and Hybrid Rules. The Branch Closing Rules and Propositional Rules are direct translations of Fitting's corresponding rules for the pure modal case [13].

Branch closing rules:

A tableau branch Θ is said to be *closed* if one of the following holds:

1. $T@_i(a \to b) \in \Theta$, for some a, b with $a \not\leq b$.
2. $F@_i(a \to b) \in \Theta$, for some a, b with $a \leq b$, $a \neq \bot$, and $b \neq \top$.
3. $F@_i(\bot \to \varphi) \in \Theta$, for some formula φ.
4. $F@_i(\varphi \to \top) \in \Theta$, for some formula φ.
5. $T@_i(b \to \varphi), F@_i(a \to \varphi) \in \Theta$, for some a, b with $a \leq b$.
6. $T@_j(a \to i), F@_i(b \to j) \in \Theta$, for some $a, b \neq \bot$.
7. $T@_i(i \to a) \in \Theta$, for some nominal i and truth value a with $a \neq \top$.

The two last conditions, 6 and 7, have no counterpart in Fitting's system, but are required in ours to deal with the semantics chosen for nominals. Note that if a formula $F@_i(a \to i)$ with $a \neq \top$ occurs on a branch then the branch can also be closed: In case $a = \bot$, condition 3 immediately implies closure. If $a \neq \bot$ then using the reversal rule ($\mathbf{F} \geq$) (see below), we can add a formula $T@_i(i \to b)$ to the branch, where b is one of the maximal members of $\mathcal{T}$ not above a. Because b is not above a, b cannot be $\top$. Thus condition 7 implies closure.

Non-modal rules:

The tableau rules for the propositional connectives and the rules capturing the properties of the Heyting algebra are given in Figure 1 and Figure 2, respectively. The rules of Figure 2 are called *reversal rules*, as in [13]. The

$$\frac{T@_i(a \to (\varphi \wedge \psi))}{\begin{array}{c} T@_i(a \to \varphi) \\ T@_i(a \to \psi) \end{array}}\ (\mathbf{T}\wedge)^1 \qquad \frac{F@_i(a \to (\varphi \wedge \psi))}{F@_i(a \to \varphi) \mid F@_i(a \to \psi)}\ (\mathbf{F}\wedge)^1$$

$$\frac{T@_i((\varphi \vee \psi) \to a)}{\begin{array}{c} T@_i(\varphi \to a) \\ T@_i(\psi \to a) \end{array}}\ (\mathbf{T}\vee)^2 \qquad \frac{F@_i((\varphi \vee \psi) \to a)}{F@_i(\varphi \to a) \mid F@_i(\psi \to a)}\ (\mathbf{F}\vee)^2$$

$$\frac{F@_i(a \to (\varphi \to \psi))}{\begin{array}{c|c|c} T@_i(b_1 \to \varphi) & \cdots & T@_i(b_n \to \varphi) \\ F@_i(b_1 \to \psi) & \cdots & F@_i(b_n \to \psi) \end{array}}\ (\mathbf{F}\to)^3 \qquad \frac{T@_i(a \to (\varphi \to \psi))}{F@_i(b \to \varphi) \mid T@_i(b \to \psi)}\ (\mathbf{T}\to)^4$$

[1] Where $a \neq \bot$.
[2] Where $a \neq \top$.
[3] Where $a \neq \bot$ and $b_1, ..., b_n$ are all the members of $\mathcal{T}$ with $b_i \leq a$ except $\bot$.
[4] Where $a \neq \bot$ and b is any member of $\mathcal{T}$ with $b \leq a$ except $\bot$.

Figure 1. Propositional Rules for **MVHL**.

reversal rules together with the closure rules ensure that no formula can be assigned more than one truth value (relative to a given world and a given branch).

Modal rules:

These modal rules, presented in Figure 3, differ from the ones of Fitting and heavily employs the hybrid logic machinery. Note that the tableau rules contain formulas of the form $T@_i(a \leftrightarrow \Diamond j)$. Such formulas are simply used as shorthand notation for the occurrence of both the formulas $T@_i(a \to \Diamond j)$ and $T@_i(\Diamond j \to a)$. In each of the rules of our calculus, the leftmost premise is called the *principal premise*. If α is a signed formula on one of the forms $T@_i(a \to \varphi)$, $T@_i(\varphi \to a)$, $F@_i(a \to \varphi)$ or $F@_i(\varphi \to a)$, we call φ the *body* of α and i its *prefix*. If α and β are two signed formulas such that the body of α is a subformula of the body of β, then α is said to be a *quasi-subformula* of β.

Hybrid rules:

These hybrid rules, presented in Figure 4, are inspired by the standard rules from classical hybrid logic (see [1, 6, 4]). Note that for the **(NOM)** rule, two versions are needed. Furthermore a new rule is needed due to the fact that we are in a many-valued setting, this is the rule **(NOM EQ)**, which ensures our semantic definition of nominals as being $\top$ in exactly one world.

A *tableau proof* of a formula ϕ is a closed tableau with root $F@_i(\top \to \phi)$, where i is an arbitrary nominal not occurring in ϕ. The intuition here is

$$\frac{F@_i(a \to \varphi)}{T@_i(\varphi \to b_1) \;\Big|\; \cdots \;\Big|\; T@_i(\varphi \to b_n)}\;(\mathbf{F}\geq)^{1,2} \qquad \frac{T@_i(a \to \varphi)}{F@_i(\varphi \to b)}\;(\mathbf{T}\geq)^{1,3}$$

$$\frac{F@_i(\varphi \to a)}{T@_i(b_1 \to \varphi) \;\Big|\; \cdots \;\Big|\; T@_i(b_n \to \varphi)}\;(\mathbf{F}\leq)^{1,4} \qquad \frac{T@_i(\varphi \to a)}{F@_i(b \to \varphi)}\;(\mathbf{T}\leq)^{1,5}$$

[1] φ is a formula other than a propositional constant from $\mathcal{T}$.
[2] Where $b_1, ..., b_n$ are all maximal members of $\mathcal{T}$ with $a \nleq b_i$ and $a \neq \bot$.
[3] Where b is any maximal member of $\mathcal{T}$ with $a \nleq b$ and $a \neq \bot$.
[4] Where $b_1, ..., b_n$ are all minimal members of $\mathcal{T}$ with $b_i \nleq a$ and $a \neq \top$.
[5] Where b is any minimal member of $\mathcal{T}$ with $b \nleq a$ and $a \neq \top$.

Figure 2. Reversal Rules for **MVHL**.

$$\frac{F@_i(a \to \Box\varphi)}{\begin{array}{c|c|c} T@_i(b_1 \leftrightarrow \Diamond j) & \cdots & T@_i(b_n \leftrightarrow \Diamond j) \\ F@_j((a \sqcap b_1) \to \varphi) & \cdots & F@_j((a \sqcap b_n) \to \varphi) \end{array}}\;(\mathbf{F}\Box)^{1}$$

$$\frac{T@_i(a \to \Box\varphi) \qquad T@_i(b \to \Diamond j)}{T@_j((a \sqcap b) \to \varphi)}\;(\mathbf{T}\Box)$$

$$\frac{F@_i(\Diamond\varphi \to a)}{\begin{array}{c|c|c} T@_i(b_1 \leftrightarrow \Diamond j) & \cdots & T@_i(b_n \leftrightarrow \Diamond j) \\ F@_j(\varphi \to (b_1 \Rightarrow a)) & \cdots & F@_j(\varphi \to (b_n \Rightarrow a)) \end{array}}\;(\mathbf{F}\Diamond)^{1,2}$$

$$\frac{T@_i(\Diamond\varphi \to a) \qquad T@_i(b \to \Diamond j)}{T@_j(\varphi \to (b \Rightarrow a))}\;(\mathbf{T}\Diamond)^{2}$$

$$\frac{F@_i(E\varphi \to a)}{F@_j(\varphi \to a)}\;(\mathbf{FE})^{3} \qquad \frac{T@_i(E\varphi \to a)}{T@_j(\varphi \to a)}\;(\mathbf{TE})^{4}$$

$$\frac{T@_i(a \to A\varphi)}{T@_j(a \to \varphi)}\;(\mathbf{TA})^{4} \qquad \frac{F@_i(a \to A\varphi)}{F@_j(a \to \varphi)}\;(\mathbf{FA})^{3}$$

[1] Where $\mathcal{T} = \{b_1, ..., b_n\}$ and j is a nominal new to the branch.
[2] Where the principal premise is a quasi-subformula of the root formula.
[3] Where j is a nominal new to the branch.
[4] Where j is a nominal already occurring on the branch.

Figure 3. Modal Rules for **MVHL**.

$$\frac{T@_i(@_j\varphi \to a)}{T@_j(\varphi \to a)}\ (@_L) \qquad \frac{T@_i(a \to @_j\varphi)}{T@_j(a \to \varphi)}\ (@_R)$$

$$\frac{F@_i\varphi \qquad T@_i(a \to j)}{F@_j\varphi}\ (\textbf{F-NOM})^{1,2} \qquad \frac{T@_i\varphi \qquad T@_i(a \to j)}{T@_j\varphi}\ (\textbf{T-NOM})^{1,2}$$

$$\frac{T@_k(\Diamond i \to b) \qquad T@_i(a \to j)}{T@_k(\Diamond j \to b)}\ (\textbf{BRIDGE}_L)^1$$

$$\frac{T@_k(b \to \Diamond i) \qquad T@_i(a \to j)}{T@_k(b \to \Diamond j)}\ (\textbf{BRIDGE}_R)^1$$

$$\frac{T@_i(\top \to j) \qquad T@_j(\top \to k)}{T@_i(\top \to k)}\ (\textbf{TRANS})$$

$$\frac{T@_i(a \to j)}{T@_i(\top \to j)}\ (\textbf{NOM EQ})^1$$

[1] Where $a \neq \bot$.
[2] Where the principal premise is a quasi-subformula of the root formula.

Figure 4. Hybrid Rules for **MVHL**.

that the root formula $F@_i(\top \to \phi)$ asserts that ϕ does *not* have the value $\top$, and if the tableau closes, this assertion is refuted. If i is a nominal occurring in the root formula of a tableau then i is called a *root nominal* of the tableau. Other nominals occurring on the tableau are called *non-root nominals*.

4 Termination

The tableau calculus presented above is not terminating. This is due to the rules (**TA**) and (**FA**) for the global modality A. If the rules for the global modalities—(**FE**), (**TE**), (**TA**) and (**FA**)—are all removed, we obtain a tableau calculus for the many-valued hybrid logic with these modalities removed. We will refer to this calculus as the *basic calculus*, and refer to its tableaux as *basic tableaux*. In the following we will prove that the basic calculus terminates. The proof closely follows the method introduced in [4].

If α and β are signed formulas on a tableau branch, then β is said to be *produced* by α if β is one of the conclusions of a rule application with principal premise α. The signed formula β is said to be *indirectly produced* by α if there exists a sequence of signed formulas $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ in which each formula is produced by its predecessor. We now have the following result.

LEMMA 3 (**Quasi-subformula Property).** *Let $\mathcal{T}$ be a basic tableau. For any signed formula α occurring on $\mathcal{T}$, one of the following holds:*

1. *α is a quasi-subformula of the root formula of $\mathcal{T}$.*
2. *α is a formula of one of the forms $T@_i(a \to \Diamond j)$, $T@_i(\Diamond j \to a)$, $F@_i(a \to \Diamond j)$ or $F@_i(\Diamond j \to a)$, for which one of the following holds:*
   - *(a) j is a root nominal.*
   - *(b) α is indirectly produced by (**F**$\Box$) or (**F**$\Diamond$) by a number of applications of the reversal rules.*

Proof. The proof goes by induction on the construction of $\mathcal{T}$. In the basic case α is just the root formula, which of course is of type 1. Now assume that α have been introduced by one of the propositional rules. These rules does not take premises of type 2 and thus by induction they must be of type 1. But then the conclusions produced by these rules must also be of type 1, thus α must be of type 1. If α have been produced by once of the reversal rules by a formula of type 1, then α will also by of type 1 and if α is produced by a formula of type 2, α is also of type 2. Now the modal rules. If α have been produced by the rule (**T**$\Box$) then the principal premise can not be a formula of type 2 and thus by induction it must be of type 1. But then so is α. Similar for the rule (**T**$\Diamond$) where the side condition insures that the principal premise is of type 1. If α is introduced by on of the rules (**F**$\Box$) or (**F**$\Diamond$) again the premise must be of type 1. These rules produce two formulas, the first one is by definition of type 2b and the second must be of type 1 since the premise is. Thus in this case α is either of type 1 or type 2b. Finally for the hybrid rules. In the rules (**TRANS**), (**NOM EQ**), ($@_L$) or ($@_R$) the premises can not be of type 2 and thus by induction they must be of type 1. But then the conclusions will also be of type 1. Now if the rule used is (**T-NOM**) or (**F-NOM**) then the side condition insures that the principal premise are of type 1. But then the conclusion will also be of type 1. Now assume that one of the rules (**BRIDGE**$_L$) or (**BRIDGE**$_R$) have been applied to produce α. Then the non-principal premise can not be of type 1 and thus must be of type 2, which implies that j is a root nominal. Thus the conclusion α must be of type 2a. This completes the proof. ■

Note that in the basic calculus the only rules that can introduce new nominals to a tableau are (**F**$\Box$) and (**F**$\Diamond$).

DEFINITION 4. Let Θ be a branch of a basic tableau. If a nominal j has been introduced to the branch by applying either (**F**$\Box$) or (**F**$\Diamond$) to a premise with prefix i then we say that j is *generated* by i on Θ, and we write $i \prec_\Theta j$.

LEMMA 5. *Let Θ be a branch of a basic tableau. The graph $G = (N^\Theta, \prec_\Theta)$, where N^Θ is the set of nominals occurring on Θ, is a finite set of well-founded, finitely branching trees.*

Proof. That G is wellfounded follows from the observation that if $i \prec_\Theta j$, then the first occurrence of i on Θ is before the first occurrence of j. That G is finitely branching is shown as follows. For any given nominal i the number of nominals j satisfying $i \prec_\Theta j$ is bounded by the number of applications of ($\mathbf{F}\Box$) and ($\mathbf{F}\Diamond$) to premises of the form $F@_i(a \rightarrow \Box\varphi)$ and $F@_i(\Diamond\varphi \rightarrow a)$. So to prove that G is finitely branching, we only need to prove that for any given i the number of such premises is finite. However, this follows immediately from the fact that all such premises must be quasi-subformulas of the root formula (cf. Lemma 3 and the condition on applications of ($\mathbf{F}\Diamond$)). What is left is to prove that G is a finite set of trees. This follows from the fact that each nominal in N^Θ can be generated by at most one other nominal, and the fact that each nominal in N^Θ must have one of the finitely many root nominals of Θ as an ancestor. ∎

LEMMA 6. *Let Θ be a branch of a basic tableau. Then Θ is infinite if and only if there exists an infinite chain of nominals*

$$i_1 \prec_\Theta i_2 \prec_\Theta i_3 \prec_\Theta \cdots .$$

Proof. The 'if' direction is trivial. To prove the 'only if' direction, let Θ be any infinite tableau branch. Θ must contain infinitely many distinct nominals, since it follows immediately from Lemma 3 that a tableau with finitely many nominals can only contain finitely many distinct formulas. This implies that the graph $G = (N^\Theta, \prec_\Theta)$ defined as in Lemma 5 must be infinite. Since by Lemma 5, G is a finite set of wellfounded, finitely branching trees, G must then contain an infinite path $(i_1, i_2, i_3, \dots)$. Thus we get an infinite chain $i_1 \prec_\Theta i_2 \prec_\Theta i_3 \prec_\Theta \cdots$. ∎

DEFINITION 7. Let Θ be a branch of a basic tableau, and let i be a nominal occurring on Θ. We define $m_\Theta(i)$ to be the maximal length of any formula with prefix i occurring on Θ.

LEMMA 8 (**Decreasing length).** *Let Θ be a branch of a basic tableau. If $i \prec_\Theta j$ then $m_\Theta(i) > m_\Theta(j)$.*

Proof. For any signed formula α, we will use $|\alpha|$ to denote the length of α. Assume $i \prec_\Theta j$. Let α be a signed formula satisfying: 1) α has maximal length among the formulas on Θ with prefix j; 2) α is the earliest occurring formula on Θ with this property. We need to prove $m_\Theta(i) > |\alpha|$. The formula α can not have been introduced on Θ by applying any of the propositional rules (Figure 1), since this would contradict maximality of α. It can not have been directly produced by any of the reversal rules (Figure 2) either, since this would contradict the choice of α as the earliest possible on Θ of maximal length with prefix j. By the same argument, α can not have been directly produced by any of the rules ($\mathbf{BRIDGE}_L$), ($\mathbf{BRIDGE}_R$), (**TRANS**) or (**NOM EQ**). Assume now α has been introduced by applying ($@_L$) or ($@_R$) to a premise of the form $T@_k(@_j\varphi \rightarrow a)$

or $T@_k(a \rightarrow @_j\varphi)$. By Lemma 3, the premise must be a quasi-subformula of the root formula. Thus j must be a root nominal. However, this is a contradiction, since by assumption j is generated by i, and can thus not be a root nominal. Thus neither ($@_L$) nor ($@_R$) can have been the rule producing α. Now assume that α has been produced by an application of either (**F-NOM**) or (**T-NOM**). Since α has index j, the non-principal premise used in this rule application must have the form $T@_i(a \rightarrow j)$. By Lemma 3, this premise must be a quasi-subformula of the root formula, and thus j is again a root nominal, which is a contradiction. Thus α can not have been produced by (**F-NOM**) or (**T-NOM**) either. Thus α must have been introduced by one of the rules ($\mathbf{F}\Box$), ($\mathbf{T}\Box$), ($\mathbf{F}\Diamond$) or ($\mathbf{T}\Diamond$). Consider first the case of the ($\mathbf{F}\Box$) and ($\mathbf{F}\Diamond$) rules. If an instance of one of these produced α, then this instance must have been applied to a premise β with prefix i, since we have assumed $i \prec_\Theta j$ and by Lemma 5 there cannot be an $i' \neq i$ satisfying $i' \prec_\Theta j$. (Note that if α is of the form $T@_j(b \rightarrow \Diamond k)$ or $T@_j(\Diamond k \rightarrow b)$ produced by a formula $F@_j(a \rightarrow \Box\varphi)$ or $F@_j(\Diamond\varphi \rightarrow a)$, this would lead to a contradiction with the assumption that α has maximal length with prefix j and is the earliest occurring formula with this property.) Since the rules in question always produce conclusions that are shorter than their premises, β must be longer than α. Since β is a formula with prefix i we then get:

(1) $\quad m_\Theta(i) \geq |\beta| > |\alpha|\,,$

as required. Now consider finally the case where α has been produced by either ($\mathbf{T}\Box$) or ($\mathbf{T}\Diamond$). Then α has been produced by a rule instance with non-principal premise of the form $T@_k(b \rightarrow \Diamond j)$. Since j is not a root nominal, this premise can not be a quasi-subformula of the root formula. Neither can it be of the tybe ($2a$) mentioned in lemma 3. It must thus be of type ($2b$), that is, it must be indirectly produced by formulas of the form $T@_k(b_m \rightarrow \Diamond j')$ or $T@_k(\Diamond j' \rightarrow b_m)$ obtained as conclusion by applications of **($\mathbf{F}\Box$)** or **($\mathbf{F}\Diamond$)**. Since only reversal rules have been applied in the indirect production from these conclusions, we must have $j = j'$ and thus $k \prec_\Theta j$. Since we already have $i \prec_\Theta j$ we get $k = i$, using Lemma 5. We can conclude that the non-principal premise of the rule instance producing α must have the form $T@_i(b \rightarrow \Diamond j)$, and thus the principal premise must be a formula β with index i. Since the rules in question always produce conclusions that are shorter than their premises, β must be longer than α. Since β is a formula with prefix i we then again get the sequence of inequalities (1), as required. ■

We can now finally prove termination of the basic calculus.

THEOREM 9 (**Termination of the basic calculus).** *Any tableau in the basic calculus is finite.*

Proof. Assume there exists an infinite basic tableau. Then it must have an infinite branch Θ. By Lemma 6, there exists an infinite chain

$$i_1 \prec_\Theta i_2 \prec_\Theta i_3 \prec_\Theta \cdots .$$

Now by Lemma 8 we have

$$m_\Theta(i_1) > m_\Theta(i_2) > m_\Theta(i_3) > \cdots$$

which is a contradiction, since $m_\Theta(i)$ is a non-negative number for any nominal i. ■

5 Completeness of the basic calculus

In this section we prove completeness of the basic calculus, that is, the calculus without the global modalities. In this connection we remark that we have proved completeness for a calculus including the global modalities similar to the calculus of the present paper. Let Θ be an open saturated branch in the tableau calculus. We will use this branch to construct a model $\mathcal{M}_\Theta = \langle W_\Theta, R_\Theta, \mathbf{n}_\Theta, \nu_\Theta \rangle$. The set of worlds, W_Θ is simply defined to be the set of nominals occurring on Θ. The definition of the other elements of the model requires a bit more work. First we define the mapping $\mathbf{n}_\Theta$.

Fix a choice function σ that for any given set of nominals on Θ returns one of these nominals. We now define the mapping $\mathbf{n}_\Theta$ in the following way:

$$\mathbf{n}_\Theta(i) = \begin{cases} \sigma\{j \mid T@_i(\top \to j) \in \Theta\} & \text{if } \{j \mid T@_i(\top \to j) \in \Theta\} \neq \emptyset \\ i & \text{otherwise.} \end{cases}$$

A nominal i is called an *urfather* on Θ if $i = \mathbf{n}_\Theta(j)$ for some nominal j.

LEMMA 10. *Let Θ be a saturated tableau branch. Then we have the following properties:*

1. *If $T@_i\varphi \in \Theta$ is a quasi-subformula of the root formula then $T@_{\mathbf{n}_\Theta(i)}\varphi \in \Theta$. Similarly, if $F@_i\varphi \in \Theta$ is a quasi-subformula of the root formula then $F@_{\mathbf{n}_\Theta(i)}\varphi \in \Theta$.*
2. *If $T@_i(\top \to j) \in \Theta$ then $\mathbf{n}_\Theta(i) = \mathbf{n}_\Theta(j)$.*
3. *If i is an urfather on Θ then $\mathbf{n}_\Theta(i) = i$.*

Proof. First we prove (i). Assume $T@_i\varphi \in \Theta$ is a quasi-subformula of the root formula. If $\mathbf{n}_\Theta(i) = i$ then there is nothing to prove. So assume $\mathbf{n}_\Theta(i) = \sigma\{j \mid T@_i(\top \to j) \in \Theta\}$. Then $T@_i(\top \to \mathbf{n}_\Theta(i)) \in \Theta$, and by applying (**T-NOM**) to premises $T@_i\varphi$ and $T@_i(\top \to \mathbf{n}_\Theta(i))$ we get $T@_{\mathbf{n}_\Theta(i)}\varphi$, as needed. The case of $F@_i\varphi \in \Theta$ is proved similarly, using (**F-NOM**) instead of (**T-NOM**). We now prove (ii). Assume $T@_i(\top \to j) \in \Theta$. To prove $\mathbf{n}_\Theta(i) = \mathbf{n}_\Theta(j)$ it suffices to prove that for all nominals k, $T@_i(\top \to k) \in \Theta \Leftrightarrow T@_j(\top \to k) \in \Theta$. So let k be an arbitrary nominal. If $T@_i(\top \to k) \in \Theta$ then we can apply (**T-NOM**) (since $T@_i(\top \to k)$ is a quasi-subformula of the root formula by Lemma 3) to premises $T@_i(\top \to k)$ and $T@_i(\top \to j)$ to obtain the conclusion $T@_j(\top \to k)$, as required. If conversely $T@_j(\top \to k) \in \Theta$ then we can apply (**TRANS**) to premises $T@_i(\top \to j)$ and $T@_j(\top \to k)$ to obtain the conclusion $T@_i(\top \to k)$, as required. We finally prove (iii). Assume i is an urfather. Then $i = \mathbf{n}_\Theta(j)$ for some j. If $j = i$ we are done. Otherwise we have $i = \mathbf{n}_\Theta(j) = \sigma\{k \mid T@_j(\top \to k) \in \Theta\}$ and thus $T@_j(\top \to i) \in \Theta$. This implies $i = \mathbf{n}_\Theta(j) = \mathbf{n}_\Theta(i)$, using item (ii). ■

We now turn to the definition of ν_Θ. As in [13] we will not define a particular valuation ν of the propositional variables occuring on the branch, but only show that any valuation assigning values between a certain lower and upper bound (both given by the branch Θ) will do. Let us first define these bounds.

DEFINITION 11. For a formula φ in the language of MVHL and a nominal i, define:

$$\mathit{bound}^{\Theta,i}(\varphi) = \bigsqcap\{a \mid T@_i(\varphi \to a) \in \Theta\}$$
$$\mathit{bound}_{\Theta,i}(\varphi) = \bigsqcup\{a \mid T@_i(a \to \varphi) \in \Theta\}$$

The intuition is that $\mathit{bound}^{\Theta,i}(\varphi)$ is an upper bound for the truth value of φ at the world i decided by the branch Θ and $\mathit{bound}_{\Theta,i}(\varphi)$ is a lower bound for this truth value.

The following lemma corresponds to Lemma 6.4 of [13] and can be proved in the same way. It ensures that we can actually always chose a value between the lower and the upper bounds.

LEMMA 12. *For all i on Θ and all formulas φ of MVHL*

$$\mathit{bound}_{\Theta,i}(\varphi) \leq \mathit{bound}^{\Theta,i}(\varphi).$$

Later we will show that any valuation assigning a value to p between $\mathit{bound}_{\Theta,i}(p)$ and $\mathit{bound}^{\Theta,i}(p)$ at the world $\mathbf{n}_\Theta(i)$ will do for the truth value of p at this world.

The following lemma corresponds to Proposition 6.5 in [13] and is proven in the same way.

LEMMA 13. *Let φ be any formula in the MVHL language other than a propositional constant from $\mathcal{T}$, and let $a \in \mathcal{T}$, then:*

- *(i) If $T@_i(a \to \varphi) \in \Theta$, then $a \leq \mathit{bound}_{\Theta,i}(\varphi)$.*
- *(ii) If $T@_i(\varphi \to a) \in \Theta$, then $\mathit{bound}^{\Theta,i}(\varphi) \leq a$.*
- *(iii) If $F@_i(a \to \varphi) \in \Theta$, then $a \not\leq \mathit{bound}^{\Theta,i}(\varphi)$.*
- *(iv) If $F@_i(\varphi \to a) \in \Theta$, then $\mathit{bound}_{\Theta,i}(\varphi) \not\leq a$.*

The accessibility relation R_Θ is defined as follows:

$$R_\Theta(i,j) = \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

We have the following result, which we are going to use in proving completeness.

LEMMA 14. *If* $T@_i(c \leftrightarrow \Diamond j) \in \Theta$ *then* $R_\Theta(i, \mathbf{n}_\Theta(j)) = c$.

Proof. We will prove $R_\Theta(i, \mathbf{n}_\Theta(j)) \geq c$ and $R_\Theta(i, \mathbf{n}_\Theta(j)) \leq c$. First we prove $R_\Theta(i, \mathbf{n}_\Theta(j)) \geq c$. Since $T@_i(c \leftrightarrow \Diamond j) \in \Theta$ we have $T@_i(c \rightarrow \Diamond j) \in \Theta$, and thus

$$\begin{array}{rcl} R_\Theta(i, \mathbf{n}_\Theta(j)) & = & \bigsqcup\{b \mid T@_i(b \rightarrow \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = \mathbf{n}_\Theta(j)\} \\ & \geq & \bigsqcup\{b \mid T@_i(b \rightarrow \Diamond j) \in \Theta\} \\ & \geq & c. \end{array}$$

We now prove $R_\Theta(i, \mathbf{n}_\Theta(j)) \leq c$. By definition of $\mathbf{n}_\Theta$ we have either $\mathbf{n}_\Theta(j) = j$ or $T@_j(\top \rightarrow \mathbf{n}_\Theta(j)) \in \Theta$. If $T@_j(\top \rightarrow \mathbf{n}_\Theta(j)) \in \Theta$ then since $T@_i(\Diamond j \rightarrow c) \in \Theta$ we get $T@_i(\Diamond \mathbf{n}_\Theta(j) \rightarrow c) \in \Theta$, using ($\mathbf{BRIDGE}_L$). If $\mathbf{n}_\Theta(j) = j$ we obviously also have $T@_i(\Diamond \mathbf{n}_\Theta(j) \rightarrow c) \in \Theta$. Applying Lemma 13 (ii) we then get $bound^{\Theta,i}(\Diamond \mathbf{n}_\Theta(j)) \leq c$. Thus

$$\begin{array}{rcll} R_\Theta(i, \mathbf{n}_\Theta(j)) & = & \bigsqcup\{b \mid T@_i(b \rightarrow \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = \mathbf{n}_\Theta(j)\} & \\ & \leq & \bigsqcup\{b \mid T@_i(b \rightarrow \Diamond \mathbf{n}_\Theta(j)) \in \Theta\} & (\text{using } (\mathbf{BRIDGE}_R)) \\ & = & bound_{\Theta,i}(\Diamond \mathbf{n}_\Theta(j)) & \\ & \leq & bound^{\Theta,i}(\Diamond \mathbf{n}_\Theta(j)) & (\text{using Lemma 12}) \\ & \leq & c, & \end{array}$$

as required. ■

The theorem we need for completeness now may be stated in the following way:

THEOREM 15. *Let* ν *be a valuation such that for all propositional variables* p *and all urfather nominals* i

$$bound_{\Theta,i}(p) \leq \nu(i, p) \leq bound^{\Theta,i}(p).$$

Then for all subformulas φ *of the body of root formula of* Θ

$$bound_{\Theta,i}(\varphi) \leq \nu(i, \varphi) \leq bound^{\Theta,i}(\varphi).$$

Proof. By induction on φ. The base cases are where φ is a propositional variable p, a value $c \in \mathcal{T}$ or a nominal j. The case where φ is p follows directly by the assumption. The case where φ is c is easy: First note that for any truth values a, b, if $T@_i(a \rightarrow b) \in \Theta$ then $a \leq b$. This follows from closure rule 1 presented in Section 3. Thus we get:

$$\begin{array}{rcl} bound_{\Theta,i}(c) & = & \bigsqcup\{a \mid T@_i(a \rightarrow c) \in \Theta\} \\ & \leq & c \\ & \leq & \bigsqcap\{a \mid T@_i(c \rightarrow a) \in \Theta\} \\ & = & bound^{\Theta,i}(c). \end{array}$$

Now assume φ is a nominal j. By definition of ν, $\nu(i,j)$ is $\top$ if $\mathbf{n}_\Theta(j) = i$ and $\bot$ otherwise. Assume first $\mathbf{n}_\Theta(j) = i$. Then $\nu(i,j)$ is $\top$, so trivially we have $\mathit{bound}_{\Theta,i}(j) \leq \nu(i,j)$. We thus only need to prove $\nu(i,j) \leq \mathit{bound}^{\Theta,i}(j)$, that is, we need to prove $\top = \mathit{bound}^{\Theta,i}(j) = \bigsqcap\{a \mid T@_i(j \to a) \in \Theta\}$. This amounts to showing that, for all $a \in \mathcal{T}$, $T@_i(j \to a) \in \Theta$ implies $a = \top$. Assume towards a contradiction that, for some a, $T@_i(j \to a) \in \Theta$ and $a \neq \top$. Since we have assumed $\mathbf{n}_\Theta(j) = i$, by definition of $\mathbf{n}_\Theta$ we get that either $j = i$ or $T@_j(\top \to i) \in \Theta$. If $j = i$ then we have that Θ contains a formula of the form $T@_i(i \to a)$ where $a \neq \top$. This immediately contradicts closure rule 7. Assume instead $T@_j(\top \to i) \in \Theta$. Since we also have $T@_i(j \to a) \in \Theta$ where $a \neq \top$, we can apply $(\mathbf{T} \leq)$ to conclude that that Θ must contain a formula of the form $F@_i(t \to j)$ where t is some truth value different from $\bot$. Since Θ then contains both $T@_j(\top \to i)$ and $F@_i(t \to j)$ where $t \neq \bot$, we get a contradiction by closure rule 6. Assume now $\mathbf{n}_\Theta(j) \neq i$. Then $\nu(i,j) = \bot$, and the inequality $\nu(i,j) \leq \mathit{bound}^{\Theta,i}(j)$ thus holds trivially. To prove the other inequality, $\mathit{bound}_{\Theta,i}(j) \leq \nu(i,j)$, we need to show that if $T@_i(a \to j) \in \Theta$ then $a = \bot$. Thus assume toward a contradiction that $T@_i(a \to j) \in \Theta$ and $a \neq \bot$. Then rule (**NOM EQ**) implies $T@_i(\top \to j) \in \Theta$. Thus, by item 2 of Lemma 10, we get $\mathbf{n}_\Theta(i) = \mathbf{n}_\Theta(j)$. Since i is assumed to be an urfather, item 3 of Lemma 10 implies $\mathbf{n}_\Theta(i) = i$. Thus we get $\mathbf{n}_\Theta(j) = \mathbf{n}_\Theta(i) = i$, contradiction the assumption.

Now for the induction step. First the case where φ is $@_j\psi$: Note that $\nu(i, @_j\psi) = \nu(\mathbf{n}_\Theta(j), \psi)$ and by induction hypothesis, since $\mathbf{n}_\Theta(j)$ is an urfather,

$$\mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \leq \nu(\mathbf{n}_\Theta(j), \psi) \leq \mathit{bound}^{\Theta,\mathbf{n}_\Theta(j)}(\psi).$$

Now by the rule $(\mathbf{@}_R)$, if $T@_i(a \to @_j\psi) \in \Theta$ then $T@_j(a \to \psi) \in \Theta$, for all $a \in \mathcal{T}$. Thus we get that

$$\begin{array}{rcll}
\mathit{bound}_{\Theta,i}(@_j\psi) & = & \bigsqcup\{a \mid T@_i(a \to @_j\psi) \in \Theta\} & \\
& \leq & \bigsqcup\{a \mid T@_j(a \to \psi) \in \Theta\} & \\
& \leq & \bigsqcup\{a \mid T@_{\mathbf{n}_\Theta(j)}(a \to \psi) \in \Theta\} & \text{(using 1 of Lemma 10)} \\
& = & \mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) & \\
& \leq & \nu(\mathbf{n}_\Theta(j), \psi) & \\
& = & \nu(i, @_j\psi). &
\end{array}$$

Similar by the $(\mathbf{@}_L)$ rule, $T@_i(@_j\psi \to a) \in \Theta$ implies that $T@_j(\psi \to a) \in$

Θ, for all $a \in \mathcal{T}$. Hence

$$\begin{aligned}
\nu(i, @_j\psi) &= \nu(\mathbf{n}_\Theta(j), \psi) \\
&\leq \mathit{bound}^{\Theta, \mathbf{n}_\Theta(j)}(\psi) \\
&= \bigsqcap\{a \mid T@_{\mathbf{n}_\Theta(j)}(\psi \to a) \in \Theta\} \\
&\leq \bigsqcap\{a \mid T@_j(\psi \to a) \in \Theta\} \quad \text{(using 1 of Lemma 10)} \\
&\leq \bigsqcap\{a \mid T@_i(@_j\psi \to a) \in \Theta\} \\
&= \mathit{bound}^{\Theta, i}(@_j\psi),
\end{aligned}$$

and the @-case is done.

In case φ is $\Diamond\psi$, we need to prove that

$$\mathit{bound}_{\Theta,i}(\Diamond\psi) \leq \nu(i, \Diamond\psi) \leq \mathit{bound}^{\Theta,i}(\Diamond\psi),$$

which by definition amounts to showing that

$$\begin{aligned}
\bigsqcup\{a \mid T@_i(a \to \Diamond\psi) \in \Theta\} &\leq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\} \\
&\leq \bigsqcap\{a \mid T@_i(\Diamond\psi \to a) \in \Theta\}.
\end{aligned}$$

Proving the first inequality amounts to showing that if $T@_i(a \to \Diamond\psi) \in \Theta$ then

$$a \leq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\}.$$

To prove this assume toward a contradiction that

$$T@_i(a \to \Diamond\psi) \in \Theta \text{ and } a \not\leq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\},$$

for an $a \in \mathcal{T}$. Then choose a $b \in \mathcal{T}$ such that $b \geq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\}$ and b is a maximal member of $\mathcal{T}$ with $a \not\leq b$. Then by the reversal rule **(T$\geq$)**, $F@_i(\Diamond\psi \to b) \in \Theta$. Then using the **(F$\Diamond$)** rule there is a $c \in \mathcal{T}$ and a $j \in \Theta$ such that $T@_i(c \leftrightarrow \Diamond j) \in \Theta$ and $F@_j(\varphi \to (c \Rightarrow b)) \in \Theta$. Since $T@_i(c \leftrightarrow \Diamond j) \in \Theta$, Lemma 14 implies $R_\Theta(i, \mathbf{n}_\Theta(j)) = c$. Applying 1 of Lemma 10 to the formula $F@_j(\varphi \to (c \Rightarrow b)) \in \Theta$ we get $F@_{\mathbf{n}_\Theta(j)}(\varphi \to (c \Rightarrow b)) \in \Theta$. Now (iv) of Lemma 13 implies $\mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \not\leq c \Rightarrow b$. This further implies that $(\mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \sqcap c) \not\leq b$. But by the induction hypothesis $\mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \leq \nu(\mathbf{n}_\Theta(j), \psi)$ and thus

$$\begin{aligned}
\mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \sqcap c &= \mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \sqcap R_\Theta(i, \mathbf{n}_\Theta(j)) \\
&\leq \nu(\mathbf{n}_\Theta(j), \psi) \sqcap R_\Theta(i, \mathbf{n}_\Theta(j)) \\
&\leq \bigsqcup\{R_\Theta(i, \mathbf{n}_\Theta(j)) \sqcap \nu(\mathbf{n}_\Theta(j), \psi) \mid j \in \Theta\} \\
&\leq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\} \leq b,
\end{aligned}$$

which of course is a contradiction.

In order to prove that

$$\bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\} \leq \bigsqcap\{a \mid T@_i(\Diamond\psi \to a) \in \Theta\},$$

we must show that if $T@_i(\Diamond\psi \to a) \in \Theta$, then $R_\Theta(i,j) \sqcap \nu(j,\psi) \leq a$ for all $j \in \Theta$. Thus assume that $T@_i(\Diamond\psi \to a) \in \Theta$ and that $R_\Theta(i,j) \neq \bot$ (or else it's trivial) for an arbitrary $j \in \Theta$. Since $R_\Theta(i,j) \neq \bot$, the definition of R implies that j must be an urfather. Furthermore,

$$R_\Theta(i,j) = \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

Let b and k be chosen arbitrarily such that $T@_i(b \to \Diamond k) \in \Theta$ and $\mathbf{n}_\Theta(k) = j$. Then by the $(\mathbf{T}\Diamond)$ rule, $T@_k(\psi \to (b \Rightarrow a)) \in \Theta$. Using 1 of Lemma 10 we get $T@_{\mathbf{n}_\Theta(k)}(\psi \to (b \Rightarrow a)) \in \Theta$, that is, $T@_j(\psi \to (b \Rightarrow a)) \in \Theta$. Now, by induction hypothesis, since j is an urfather,

$$\nu(j,\psi) \leq \mathit{bound}^{\Theta,j}(\psi) \leq b \Rightarrow a.$$

Since k and b were chosen arbitrarily with $T@_i(b \to \Diamond k) \in \Theta$ and $\mathbf{n}_\Theta(k) = j$, we get

$$\nu(j,\psi) \leq \bigsqcap\{b \Rightarrow a \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

We now get

$$\begin{aligned}
R_\Theta(i,j) \sqcap \nu(j,\psi) \quad &\leq \quad \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\
&\qquad \sqcap \bigsqcap\{b \Rightarrow a \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\
&\leq \quad \bigsqcup\{b \sqcap (b \Rightarrow a) \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\
&\leq \quad \bigsqcup\{a \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\
&\leq \quad a.
\end{aligned}$$

Because $j \in \Theta$ was arbitrary it follows that it holds for all $j \in \Theta$ and the proof of this case is completed.

In case φ is $\Box\psi$, we need to prove that

$$\begin{aligned}
\bigsqcup\{a \mid T@_i(a \to \Box\psi) \in \Theta\} \quad &\leq \quad \bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\} \\
&\leq \quad \bigsqcap\{a \mid T@_i(\Box\psi \to a) \in \Theta\}.
\end{aligned}$$

To prove the first inequality we need to prove that if $j \in \Theta$, then

$$a \leq R_\Theta(i,j) \Rightarrow \nu(j,\psi), \tag{2}$$

for all $a \in \mathcal{T}$ with $T@_i(a \to \Box\psi) \in \Theta$. So let $a \in \mathcal{T}$ be given arbitrarily such that $T@_i(a \to \Box\psi) \in \Theta$. Note that (2) is equivalent to

$$a \sqcap R_\Theta(i,j) \leq \nu(j,\psi).$$

By definition of R_Θ we have

$$R_\Theta(i,j) = \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

Let b and k be chosen arbitrarily such that $T@_i(b \to \Diamond k) \in \Theta$ and $\mathbf{n}_\Theta(k) = j$. Then by the **(T□)**-rule it follows that $T@_k((a \sqcap b) \to \psi) \in \Theta$. By 1 of Lemma 10 this implies $T@_j((a \sqcap b) \to \psi) \in \Theta$. Thus we get $bound_{\Theta,j}(\psi) \geq (a \sqcap b)$. Since b and k were chosen arbitrarily with the properties $T@_i(b \to \Diamond k) \in \Theta$ and $\mathbf{n}_\Theta(k) = j$ we then get

$$bound_{\Theta,j}(\psi) \geq \bigsqcup\{a \sqcap b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

Using this inequality and the induction hypothesis we now get

$$\begin{aligned} a \sqcap R_\Theta(i,j) &= a \sqcap \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\ &= \{a \sqcap b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\ &\leq bound_{\Theta,j}(\psi) \leq \nu(j,\psi). \end{aligned}$$

Since a was arbitrary this holds for all $a \in \mathcal{T}$ and the inequality have been proven.

To show the other inequality we need to show that

$$\text{if } T@_i(\Box\psi \to a) \in \Theta \text{ then } \bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\} \leq a.$$

If $a = \top$ then this is trivial. Thus assume towards a contradiction that there is an $a \neq \top$ with $T@_i(\Box\psi \to a) \in \Theta$ and $\bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\} \not\leq a$. Now let $b \leq \bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\}$ be a minimal member of $\mathcal{T}$ such that $b \not\leq a$. Then by the reversal rule **(T≤)**, $F@_i(b \to \Box\psi) \in \Theta$. Hence by the **(F□)**-rule there is a nominal $k \in \Theta$ and a $c \in \mathcal{T}$ such that $T@_i(c \leftrightarrow \Diamond k) \in \Theta$ and $F@_k((b \sqcap c) \to \psi) \in \Theta$. From the first it follows that $R_\Theta(i, \mathbf{n}_\Theta(k)) = c$, using Lemma 14. From the second it follows that $F@_{\mathbf{n}_\Theta(k)}((b \sqcap c) \to \psi) \in \Theta$, using 1 of Lemma 10, and thus, by *(iii)* of Lemma 13, $b \sqcap c \not\leq bound^{\Theta,\mathbf{n}_\Theta(k)}(\psi)$. But then from the induction hypothesis it follows that

$$b \sqcap c \not\leq \nu(\mathbf{n}_\Theta(k),\psi) \leq bound^{\Theta,\mathbf{n}_\Theta(k)}(\psi).$$

Hence

$$b \not\leq c \Rightarrow \nu(\mathbf{n}_\Theta(k),\psi) = R_\Theta(i,\mathbf{n}_\Theta(k)) \Rightarrow \nu(\mathbf{n}_\Theta(k),\psi).$$

But by the assumption on b we also have that

$$b \leq \bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\} \leq R_\Theta(i,\mathbf{n}_\Theta(k)) \Rightarrow \nu(\mathbf{n}_\Theta(k),\psi),$$

and a contradiction have been reached. This concludes the □ case and thus the entire proof of the theorem. ■

Now completeness can easily be proven, in the following sense.

THEOREM 16. *If there is no tableau proof of the formula φ, then there is a model $\mathcal{M} = \langle W, R, \mathbf{n}, \nu \rangle$ and a $w \in W$ such that $\nu(w, \varphi) \neq \top$.*

Proof. Assume that there is no tableau proof of the formula φ. Then there is an saturated tableau with a open branch Θ starting with the formula $F@_i(\top \to \varphi)$ for a nominal i not in φ. By item 1 of Lemma 10 it follows that also $F@_{\mathbf{n}_\Theta(i)}(\top \to \varphi) \in \Theta$.

The model $\mathcal{M}_\Theta = \langle W_\Theta, R_\Theta, \mathbf{n}_\Theta, \nu_\Theta \rangle$ can now be constructed such that ν_Θ satisfies the assumption of Theorem 15. Since $F@_{\mathbf{n}_\Theta(i)}(\top \to \varphi) \in \Theta$ it follows by Lemma 13 that $\top \not\leq \mathit{bound}^{\Theta,\mathbf{n}_\Theta(i)}(\varphi)$. But by Theorem 15, since φ is a subformula of the root formula and $\mathbf{n}_\Theta(i)$ is an urfather, we know that $\nu_\Theta(\mathbf{n}_\Theta(i), \varphi) \leq \mathit{bound}^{\Theta,\mathbf{n}_\Theta(i)}(\varphi)$ and it thus follows that $\top \not\leq \nu_\Theta(\mathbf{n}_\Theta(i), \varphi)$ and the proof is completed. ■

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