Relevant Agents

Marta Bílková

Department of Logic, Charles University in Prague; Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 2,18 2 07 Prague 8

Ondrej Majer

Institute of Philosophy, Academy of Sciences of the Czech Republic Jilská 1, 110 00 Prague 1

Michal Peliš

Institute of Philosophy, Academy of Sciences of the Czech Republic Jilská 1, 110 00 Prague 1

Greg Restall

Department of Philosophy, University of Melbourne Parkville 3010, Australia

Abstract

In [4], Majer and Peliš proposed a relevant logic for epistemic agents, providing a novel extension of the relevant logic R with a distinctive epistemic modality K, which is at the one and the same time factive ($K\varphi \to \varphi$ is a theorem) and an existential normal modal operator ($K(\varphi \vee \psi) \to (K\varphi \vee K\psi)$ is also a theorem). The intended interpretation is that $K\varphi$ holds (relative to a situation s) if there is a resource available at s, confirming φ. In this article we expand the class of models to the broader class of 'general epistemic frames'. With this generalisation we provide a sound and complete axiomatisation for the logic of general relevant epistemic frames. We also show, that each of the modal axioms characterises some natural subclasses of general frames.

Keywords: Modal Logic, Epistemic Logic, Relevant Logic, Substructural Logic, Frame Semantics.

[1] The work of the first and the third author was supported by the grant ICC/08/E018 of the Grant Agency of the Czech Republic (a part of ESF Eurocores-LogICCC project FP006). The work of the second author was supported by grant no. IAA900090703 of the Grant Agency of the Academy of Sciences of the Czech Republic.

1 Introduction

Representation of epistemic states of a rational agent and their changes has been for a long time an important issue in both logic and computer science. The majority solution represents the knowledge operator as a standard necessity-like modal operator; the standard modal axioms (K, T, 4, 5) then reflect epistemic properties (closure, truth, positive and negative introspection). The most popular representation (widely used also in computer science) is based on the epistemic version of S5, in which knowledge turns out to be an indistinguishability between epistemic states. The S5-representation has been extensively criticized (see, e.g., [3,2]) for being unrealistically strong. The agents represented here are 'too perfect'—they are, among other things, logically omniscient (they know all the logical truths) and fully introspective (they are explicitly aware of their both positive and negative knowledge).

Some of the strong properties can be avoided using modal systems weaker than S5. Other ones, like the logical omniscience, require a more essential change of framework—in recent literature we can find solutions based on the framework of dynamic epistemic logic. [2]

Our way to solve the problem of unrealistically strong properties is to employ a system weaker than that of a normal modal logic—the framework of distributive relevant logic. [3] The main reason for choosing relevant logic is that it fits very well our motivations. From a technical point of view we could use even weaker systems, see the section Conclusion. There is a number of ways to introduce modalities in the relevant framework (see [8] for a general overview). However we shall not add epistemic modality quite as an external independent operator; instead we define our knowledge operator using ingredients already contained in the relevant framework. The main reason is that we can provide an intuitively acceptable interpretation of the relational frames for distributive relevant logic and the definition of the epistemic operator naturally follows from this interpretation.

1.1 Rational agent

We assume that formulas of our epistemic language represent some collections of data. Data are typically incomplete and it can well happen they are inconsistent, but we still can work with them - we use them to make decisions and draw conclusions. Our prototypical agent is a scientist dealing with scientific data. Typically, she has to deal with data which are both theoretical and empirical and which are obtained from various sources (results of experiments, articles, books, technical reports etc.) Obviously different experiments might give results which contradict each other (due to an error of either equipment or experimenter) and even theories might explain some phenomena in a ways that are incompatible. Various data might be of a different quality. Obviously

[2] Duc in [2] provides a solution based on modifications of standard Kripke semantics (awareness and impossible worlds) as well as solutions based on a combination of temporal and epistemic logic and complexity approaches (algorithmic knowledge).

[3] The idea of combining epistemic and relevant frameworks has been used in the literature, however with a different aim—see, e.g., [1] and [9].

there is little use of inconsistent parts, but even consistent data are not on the same level - some of them might be results of experiments with a more reliable equipment, some might come from more respectable authorities etc. It is clear that our agent has to discriminate among the available data. Typically she prefers data which are confirmed.

1.2 Information states

Our basic entity will be an information state of an agent. It consists of local information—a collection of data immediately available to the agent (e.g. results of her experiments, observations...) and 'remote information'—collections of local data of another information states (e.g. data obtained by some other scientists or even by herself in past).

Local information consists of two basic kinds: experimental data ('facts')—inputs and outputs of experiments/observations and 'theories/laws'-generalizations extracted from the experimental data. If we consider these two kinds of data from the point of view of a logical framework, we can, with some simplification, see that basic data are typically represented by atoms, their conjunctions and disjunctions, while 'laws' are represented by conditionals (and their combinations).

The agent accepts data as knowledge if they are *confirmed* by some source (we require at least one confirmation, which makes our operator possibility-like). [4] As we shall see in the section 2.4, the relation of confirmation also deals with inconsistency of data as an inconsistent piece of data is never confirmed.

We assume that information states evolve. However, no information is lost—the evolution is in fact an accumulation of information—in this sense it reminds the persistence relation in intuitionistic logic.

2 Frame Semantics

There are more formal systems that can be called relevant logic. Our point of departure will be the distributive relevant logic R of Anderson and Belnap. We base our framework on the relational semantics for this logic, as developed by [5,6,7] and others. Before we give a formal definition, we discuss the elements of the relational semantics from the point of view of our epistemic motivation.

The universe of our semantics consists of *information states* (sometimes also called *situations*)—they play in our framework the same role as possible worlds in Kripke frames. We interpret a current state as data immediately available to our prototypical agent. Unlike possible worlds, states might be incomplete (neither φ nor $\neg\varphi$ is true in s) or inconsistent (both φ and $\neg\varphi$ are true in s).

As we said above, information states evolve. The relation tracing this process (or rather various ways the evolution can proceed) is traditionally called *involvement* and modelled by a partial order.

[4] It might seem that we use the term knowledge in rather specific sense, but let us point out, that there is no universal agreement about the criteria a logical representations of knowledge should obey. Various systems capture some features of knowledge while they leave some other unexplained—modal representation not being an exception.

Formulas are defined in the usual way in the language of relevant logic with a modality K:

$$\varphi ::= p \mid \top \mid t \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid K\varphi$$

Strong logical connectives $\otimes$ (group conjunction, fusion) and $\oplus$ (group disjunction) are definable by implication and negation, as well as the constants f and $\bot$:

$$(\varphi \oplus \psi) \equiv_{def} (\neg\varphi \to \psi) \tag{1}$$

$$(\varphi \otimes \psi) \equiv_{def} \neg(\varphi \to \neg\psi) \tag{2}$$

$$f \equiv_{def} \neg t \tag{3}$$

$$\bot \equiv_{def} \neg\top \tag{4}$$

Classical (weak, lattice) conjunction and disjunction correspond to the situation when the agent combines local data, i.e., data immediately available in the current state.

Implication is a modal connective in the sense that it depends on a neighborhood of a current state, which is given by a ternary *relevance* relation R. In fact it is analogous to the strong (necessary) implication in a standard Kripke frame except the neighborhood of a state s is given by pairs of states (y, z) such that $Rsyz$. The relation R reflects in our interpretation actual experimental setups. Let us call y, z the antecedent and the consequent state, respectively. Antecedent states correspond to some initial data (outcome of measurements or observations) of some experiment, while the related consequent states correspond to the corresponding resulting data of the experiment. Implication then corresponds to a (simple) kind of a rule: if I observe in my current state, that at every experiment (represented by a couple antecedent–consequent state) each observation of φ is followed by an observation of ψ, then I accept 'ψ follows φ' as a rule.

In Kripke models the *negation* of a formula φ is true at a world iff φ is not true there. As states can be incomplete and/or inconsistent, this is not an option any more if one deals with substructural logics. Negation becomes a modal connective and its meaning depends on the states related to the given state by *compatibility* relation C. Informally we can see the compatible states as collections of data our scientist wants to be consistent with (e.g. becuse of their reliabilty, impact etc.) Relevant negation does not correspond straightforwardly to 'necessary false'. We do not require that the negated formula in question is false in the neighborhood of the given state, we just require no state in the neighborhood contains a positive instance of this formula.

From the point of view of our motivation the interpretation of negation is rather straightforward—an agent can explicitly deny some information (a piece of data) only if its negation does not contradict any collection of data she wants to be consistent with. This condition also has a normative side: she has to be skeptical in the sense that she denies everything not positively supported by any of the compatible collection of data. If we want to have negative facts at the same basic level as positive facts, we can read the clause for the definition of compatibility relation in the other direction: the agent can relate her actual state just to the states which do not contradict her negative facts.

Properties of the compatibility relation obviously determine the kind of negation obtained, but as we shall see they moreover influence properties of the epistemic modality defined below. In standard relevant frameworks it is usually assumed that compatibility is symmetric, but it is in general neither reflexive (inconsistent states are not self-compatible) nor transitive.

One of the tricky points in the definition of relational semantics is the definition of truth in a relevant frame (model). If we take a hint from Kripke frames, we should equate truth in a frame with truth in every state. But this would gives us an extremely weak system with some very unpleasant properties [7]. For example the almost uniformly accepted identity axiom ($\alpha \to \alpha$) and the Modus Ponens rule fail to hold in every state. Designers of relevant logics took a different route; instead of requiring truth in all states, they identify truth in a frame with truth in all logically 'well behaved' states. These states are called *logical*. In order to satisfy the 'good behavior' it is enough to require in a state l that all the information in any antecedent situation related to l is contained in the corresponding consequent situation as well. This is ensured by a half of the following condition: x is below y in the involvement relation if and only if there is a logical state l such that $Rlxy$. It is easy to see that logical situations validate both the identity axiom and (implicative) Modus Ponens.

We proceed to the formal definitions now.

Definition 2.1 A *relevant frame* is a tuple $F = (W, L, \leq, C, R)$, where W is a nonempty set of *situations* and

- $\leq$ is an *involvement* relation which is a partial order on W
- R is a ternary *relevance* relation on W satisfying the conditions[5] (where R^2xyzw means $(\exists s)(Rxys$ and $Rszw)$):

$$Rxyz \text{ and } x' \leq x, y' \leq y, z' \geq z \text{ implies } Rx'y'z' \tag{5}$$

$$Rxyz \text{ implies } Ryxz \tag{6}$$

$$Rxxx \tag{7}$$

$$R^2(xy)zw \text{ implies } R^2(xz)yw \tag{8}$$

- L, the set of *logical situations*, is a nonempty upwards closed subset of $(W, \leq)$, satisfying

$$x \leq y \text{ iff there is } z \in L \text{ such that } Rzxy \tag{9}$$

- C is a binary *compatibility* relation on W satisfying the conditions

$$Rxyz \text{ implies } (\forall z')z'Cz(\exists y')y'Cy \text{ where } Rxz'y' \tag{10}$$

$$xCy \text{ and } x' \leq x \text{ and } y' \leq y \text{ implies } x'Cy' \tag{11}$$

$$xCy \text{ implies } yCx \tag{12}$$

$$(\forall x)(\exists y)xCy \tag{13}$$

$$\forall x(\exists u)(xCu \text{ and } \forall z(xCz \text{ implies } z \leq u)) \tag{14}$$

Definition 2.2 A *relevant model* is $M = (F, V)$, where F is a relevant frame and $V : Prop \mapsto \mathcal{P}(W)$, such that each $V(p)$ is an upper subset of $(W, \leq)$, is a persistent

[5] We use the stronger versions of the relation conditions, called 'plump' in [8], because they correspond to structural axioms and we are interested in modal characterizability as well.

valuation of propositional formulas. The valuation generates the following satisfaction relation between states and formulas:

- $x \Vdash p$ iff $x \in V(p)$
- $x \Vdash t$ iff $x \in L$
- $x \Vdash \top$
- $x \Vdash \varphi \wedge \psi$ iff $x \Vdash \varphi$ and $x \Vdash \psi$
- $x \Vdash \varphi \vee \psi$ iff $x \Vdash \varphi$ or $x \Vdash \psi$
- $x \Vdash \neg\varphi$ iff for all y, xCy implies $y \nVdash \varphi$
- $x \Vdash \varphi \rightarrow \psi$ iff for all y, z, $Rxyz$ and $y \Vdash \varphi$ implies $z \Vdash \psi$

The relation between implication and fusion is that they form an adjoint pair (implication is the residuum of the fusion in algebraic terms):

$$\varphi \rightarrow (\psi \rightarrow \chi) \;\equiv\; \varphi \otimes \psi \rightarrow \chi$$

thus, semantically

$$x \Vdash \varphi \otimes \psi \quad \text{iff} \quad \text{there are } y, z \text{ such that } Ryzx \text{ and } y \Vdash \varphi \text{ and } z \Vdash \psi.$$

The validity of formulas in a model, a frame, or in a class of models or frames is defined via validity in all *logical* states.

The results of persistency for all formulas, soundness and completeness of the semantics for the relevant logic R can be extracted as a special case from general definitions and proofs contained in [8].

The original motivation behind an epistemic modality in the framework of experimental data is that data can be accepted as knowledge only if they are *confirmed* by a source. As there are several possibilities what can be counted as a reliable source, we explicitly represent the relation of *being a source* by a new binary relation S on W and use it to define our epistemic modality K:

$$x \Vdash K\varphi \text{ iff for some s where } sSx, s \Vdash \varphi \tag{15}$$

$K\varphi$ thus meaning that φ is supported by at least one source.

Next we explore some possibilities how to define the S-relation and introduce corresponding classes of relevant frames.

2.1 *Classic relevant frames*

The S relation can be seen as determining which states are to be counted as reliable sources. The first attempt uses the ingredients already contained in a frame to define S. Following motivations presented in [4], we start with a requirement that a source state confirming data in a current state shall be compatible with the current state and it should (strictly) precede the current state in the involvement ordering (the second condition was supposed to make the confirmation 'independent'—to exclude the possibility that a state is a source for itself)

$$sSx \text{ iff } s < x \text{ and } sCx \tag{16}$$

Moreover, it is reasonable to require that modal formulas are persistent, this is guaranteed by the following condition:

$$sSx \text{ and } x \leq x' \text{ then } (\exists s')(s \leq s' \wedge s'Sx') \tag{17}$$

The class of frames defined this way however doesn't seem a good candidate to work with since it is an open question how to axiomatize the logic of classical frames in the modal language we have fixed. We conjecture it coincides with the logic of General frames defined below, and that the class of classical frames is not modally definable in the current language due to the presence of an anti-property $\neq$ contained in the definition of S in (16).

We call the class of frames satisfying conditions (16) and (17) the class of *Classic frames* and denote it $\mathcal{F}_c$.

Let us remind that validity in a class of frames is defined as a validity in all logical states in all models based on the frames from the class. Thus:

$$\mathcal{F}_c \Vdash \varphi \text{ iff } (\forall F \in \mathcal{F}_c)(\forall x \in L)(x \Vdash \varphi) \tag{18}$$

We can also weaken our requirements on available sources and admit that in some cases a state can be a source for itself, so we work with $\leq$ instead of $<$. We define a class of *Weak Classic frames* $\mathcal{F}_{wc}$ weakening the property (16) to:

$$sSx \text{ iff } s \leq x \text{ and } sCx \tag{19}$$

This class turns out to be distinguishable from the other two by the validity of the *introspection* axiom $K\varphi \to KK\varphi$. This was one of the axioms we criticised in the introduction and the validity of which we tried to avoid. We shall comment on this later.

2.2 General relevant frames

Providing an axiomatisation for the logic of classic (and weak classic) relevant epistemic frames is an open problem. Upon reflection, however, it is not clear that this class of epistemic frames is the natural target. To count *every* lower compatible state under the current state as a resource for use may be more restrictive than we need. For a more general class of frames, we allow for slightly more variation in the interpretation of K, by generalising the accessibility relation S. It is now required to satisfy the following two conditions: the condition (17) remains unchanged, while we replace 'iff' in the other condition with 'only if':

$$sSx \text{ then } s \leq x \text{ and } sCx \tag{20}$$

We denote the class of general frames $\mathcal{F}_g$ and provide in the next section an axiomatisation of this class. Validity of formulas in the class is defined:

$$\mathcal{F}_g \Vdash \varphi \text{ iff } (\forall F \in \mathcal{F}_g)(\forall x \in L)(x \Vdash \varphi) \tag{21}$$

2.3 Relating the classes of frames

We start with stating some basic properties of all the three classes defined above.

Lemma 2.3 (Persistency for formulas) *$x \Vdash \varphi$ and $x \leq y$ implies $y \Vdash \varphi$.*

Proof. We concentrate on the case of $K\varphi$ only. Suppose that $x \Vdash K\varphi$ and $x \leq y$. We show that $y \Vdash K\varphi$. Since $x \Vdash K\varphi$ there is some sSx satsifying φ. From the condition (17) $(\exists s')(s \leq s' \wedge s'Sy)$. From the induction hypothesis $s' \Vdash \varphi$, but then $y \Vdash K\varphi$ as desired. □

Lemma 2.4 (Selfcompatibility of sources) *sSx implies sCs.*

Proof. Suppose sSx. Then (in $\mathcal{F}_c, \mathcal{F}_{wc}$, and $\mathcal{F}_g$) $s \leq x$ and sCx. From the condition (11) it follows that sCs. □

Since every classic (weak classic) frame is a general frame, we have $\mathcal{F}_c \subseteq \mathcal{F}_g$ and $\mathcal{F}_{wc} \subseteq \mathcal{F}_g$, and thus, for the logics of the classes, the inclusions $L(\mathcal{F}_g) \subseteq L(\mathcal{F}_c)$ and $L(\mathcal{F}_g) \subseteq L(\mathcal{F}_{wc})$ hold. The other inclusions are open question.

We can however distinguish the class $\mathcal{F}_{wc}$ from $\mathcal{F}_g$ (and from $\mathcal{F}_c$ as well) with validity of the axiom of introspection $K\varphi \to KK\varphi$. Thus in particular $L(\mathcal{F}_g) \subset L(\mathcal{F}_{wc})$ is a proper inclusion.

Lemma 2.5 *$K\varphi \to KK\varphi$ fails in $\mathcal{F}_g$ and $\mathcal{F}_c$, while it holds in $\mathcal{F}_{wc}$.*

Proof. To show $K\varphi \to KK\varphi$ holds in $\mathcal{F}_{wc}$ consider $u \in L$, $Ruxy$ where $x \Vdash K\varphi$. We show $y \Vdash KK\varphi$. Since $x \Vdash K\varphi$, then there is s such that sSx and $s \Vdash \varphi$. Because of sCs (Lemma 2.4) and $s \leq s$, we obtain sSs. The situation s is a source for itself. From $s \Vdash K\varphi$ we get $x \Vdash KK\varphi$. By persistence (Lemma 2.3) $y \Vdash KK\varphi$.

To show $K\varphi \to KK\varphi$ fails in the class $\mathcal{F}_c$ and in the class $\mathcal{F}_g$, we consider the following counterexample: $W = L = \{x, s\}$ and sSx (thus in particular $s < x$ and sCx, and sCs by the Lemma 2.4. Moreover we have $Rsxx$ (and many others by the conditions for L and R to be satisfied). But we do not have sSs (as we would be forced in $\mathcal{F}_{wc}$ by $s \leq s$ and sCs). Now we put $V(p) = \{s, x\}$. Then $x \Vdash Kp$ but $x \nVdash KKp$, thus $Kp \to KKp$ fails in a logical state s. □

We are not interested much in the class $\mathcal{F}_{wc}$ itself, but the introspection axiom is interesting from another point of view—we show which class of relevant frames corresponds to the axiom of introspection in Section 4.

One more definition needed to state strong completeness is the *(local) consequence* over the class of frames $\mathcal{F}_g$. It is defined as follows (so that logical validity $\mathcal{F}_g \Vdash \varphi$ corresponds to $t \vDash_{\mathcal{F}_g} \varphi$):

$$\Gamma \vDash_{\mathcal{F}_g} \varphi \text{ iff } (\forall F \in \mathcal{F}_g)(\forall V)(\forall x)((\forall \gamma \in \Gamma)(F, V), x \Vdash \gamma \text{ implies } (F, V), x \Vdash \varphi) \qquad (22)$$

2.4 Basic properties of the modality K

The following examples of valid and invalid schemes hold for the validity in the class of general epistemic frames $\mathcal{F}_g$, as well as in the class of classic epistemic frames $\mathcal{F}_c$. Thus we use the symbol $\models$ and do not mention the class explicitly. We are so far not able to provide any example distinguishing these two classes.

Lemma 2.6 (Monotonicity of K) $\models \varphi \to \psi$ *implies* $\models K\varphi \to K\psi$

Proof. Consider $x \in L$, we show $x \Vdash K\varphi \to K\psi$. Consider any y, z, such that $Rxyz$ and $y \Vdash K\varphi$. We show that $z \Vdash K\psi$.

$y \Vdash K\varphi$ implies there is s such that (sSy and $s \Vdash \varphi$). $s \leq s$ implies there is $t \in L$ such that $Rtss$ and $Rsts$ (by the condition for L). Since in such $t \Vdash \varphi \to \psi$ and $s \Vdash \varphi$, we conclude $s \Vdash \psi$. Thus $y \Vdash K\psi$.

But $y \leq z$ (since $x \in L$ and $Rxyz$), and finally $z \Vdash K\psi$. □

It is to be expected that the proposed semantics based on the class of general frames blocks all the undesirable properties of both material and strict implication. Moreover, we ruled out (at least for some classes of frames) the validity of majority of the properties of standard epistemic logics that we have criticized, in particular, both positive and negative introspection, as well as some closure properties.

The two core modal principles valid in general frames (see Theorem 3.2) are *factivity*

$$K\varphi \to \varphi \tag{23}$$

and *strong factivity*

$$\neg\varphi \wedge K\varphi \to \bot \tag{24}$$

We call the second condition strong factivity, as it says not only that only information warranted here can be known, but that anything 'diswarranted' here is excluded from knowledge.

Factivity vs. strong factivity. Factivity does not follow from strong factivity. For factivity, we would need $\neg\varphi \otimes K\varphi \to \bot$ whence we could residuate to get $K\varphi \to (\neg\varphi \to \bot)$, and then contrapose the consequent $K\varphi \to (\top \to \varphi)$ and use the entailment from $(\top \to \varphi)$ to φ.

Of course, the claim that $\bot$ follows from the *fusion* of $\neg\varphi$ with $K\varphi$ is a stronger claim than it following from their conjunction. In the presence of weakening, however, factivity would follow from strong factivity.

Material factivity. It is worth observing that we also have validity of the weaker claim which we call *material factivity*:

$$\neg(\neg\varphi \wedge K\varphi), \text{ or equivalently, } \neg K\varphi \vee \varphi \tag{25}$$

Material factivity is a weaker condition than factivity since $\varphi \to \psi$ entails $\neg\varphi \vee \psi$ in R. It is also weaker then strong factivity since $(\varphi \to \bot) \to \neg\varphi$ is a theorem of R: $\neg\varphi \leftrightarrow (\varphi \to f)$ is a theorem, and we can prefix $\varphi \to$ on the theorem $\bot \to f$ in the usual manner. But in logics weaker than R, in which the law of excluded middle is rejected, material factivity remains a consequence of strong factivity, but no longer weak factivity.

K-axiom. **K** (in implicational form) would in fact correspond to a 'distribution of confirmation': if an implication is confirmed, then the confirmation of the antecedent implies the confirmation of the consequent. But the S relation is not connected to the R relation in any straightforward way, so existence of sources for $K(\alpha \to \beta)$ and for $K\alpha$ does not force existence of a source for $K\beta$. Hence

$$\not\models K(\alpha \to \beta) \to (K\alpha \to K\beta)$$

Introspection. We defined knowledge as independently confirmed data. In this reading the axioms **4** and **5** rather than to introspection correspond to a 'second order confirmation' (if α is confirmed then the confirmation of α is confirmed as well, similarly for the negative introspection). We showed in Lemma 2.5, that positive introspection fails in $\mathcal{F}_g$ and $\mathcal{F}_c$, while it holds in $\mathcal{F}_{wc}$. It is easy to see that negative introspection fails for all frames:

$$\not\models \neg K\alpha \to K\neg K\alpha$$

Necessity and negation. There is a difference between $s \not\Vdash K\varphi$ and $s \Vdash \neg K\varphi$. The former simply says that φ is not confirmed at the current situation s, while the latter is stronger (at least in the case of selfcompatible situations), it says that φ is not confirmed in any situation compatible with s. From this point of view it is uncontroversial that both $K\varphi$ (confirmation in the current situation) and $\neg K\varphi$ (the lack of confirmation in the compatible situations) might be true in some situation s (in this case s is not compatible with itself).

Closure properties. In the introduction we criticized too strong closure properties of the standard modal representations of knowledge. In fact the question how strong conditions shall be imposed on epistemic states to obtain an adequate representation is one of the crucial choices of the knowledge representation. It is also closely related to the problem of logical omniscience.

We can see the machinery of 'logical expansion' as having two basic ingredients. One is knowledge of all the tautologies of the logical system in question guaranteed by the Necessitation rule. The other is Modal Modus Ponens, which produces all consequences of any new piece of (non-logical) information. Our system turns out to be extremely weak and avoids both of these closure properties and some more. It can be seen as anti-logical and pragmatic—in a sense that our agent believes (accepts) just what is (or was) observed. Even the data corresponding to logical laws have to be confirmed.

Necessitation rule. The rule

$$\frac{\varphi}{K\varphi}$$

common to all normal epistemic logics, guarantees among other things that all the tautologies of the logical system in question are known. In our framework this would mean that all the logical truths are confirmed. This is in general not the case. Let us assume that φ is a valid formula (i.e., $l \Vdash \varphi$, for every logical situation l). The necessity rule would imply the validity of $K\varphi$. However, for $l \Vdash K\varphi$ we need a confirmation from a resource and the conditions for the relation S do not guarantee, that l has a logical situation as a resource (in fact it does not need to have a resource at all), so even if there is a source situation s for l, this situation does not have to be logical and hence the validity of ψ in s is not guaranteed.

Modal Modus Ponens. Closure of knowledge with respect to logical consequence, which is a part of logical omniscience (if an agent knows both φ and $\varphi \to \psi$, then she knows ψ as well) is forced by the validity of the modal Modus Ponens:

$$\frac{K\alpha \quad K(\alpha \to \beta)}{K\beta}$$

It is easy to see that it does not hold in our system. As we noted above, **K** is in fact a 'distribution of confirmation'. If both an implication and its antecedent are confirmed, there is no reason the consequent needs to be confirmed as well.

Let us note, that the weaker version of modal Modus Ponens holds

$$\frac{K\alpha \quad K(\alpha \to \beta)}{\beta}$$

however, it cannot be considered as any kind of omniscience. It just says that if both α and $(\alpha \to \beta)$ are confirmed, then β is a part of currently available data.

This rule holds not only in logical situations, but in all situations. If $K\alpha$ and $K(\alpha \to \beta)$ are true in an $s \in S$, then $s \Vdash \alpha$ and $s \Vdash \alpha \to \beta$ because of **T** axiom (*factivity*). It follows from the assumption $Rsss$ and the definition of implication, that $s \Vdash \beta$ as well.

Contradiction. Contradiction in relevant logic is non-explosive: φ and $\neg\varphi$ might hold in a contradictory situation, but it does not entail an arbitrary formula ψ. (This would require an R-connection to situation where ψ holds.) [6]

$$\not\models (\varphi \wedge \neg\varphi) \to \psi$$

As we noted above, a contradiction cannot be known (it is never confirmed).

$$\not\models K(\varphi \wedge \neg\varphi)$$

This has a trivial consequence, that knowledge of contradiction implies anything ($\models K(\varphi \wedge \neg\varphi) \to \psi$), so, in particular knowledge of contradiction implies knowledge of anything ($\models K(\varphi \wedge \neg\varphi) \to K(\psi)$). Nevertheless this does not lead to any kind of explosion as there is no such situation in which the antecedent is true. In standard models, $K(\varphi \wedge \neg\varphi)$ is never true either, but the reason is that $\varphi \wedge \neg\varphi$ is not true in any state (possible world). In our framework the situation is different: $\varphi \wedge \neg\varphi$ can be true in some situation (the agent obtained contradictory data), but $K(\varphi \wedge \neg\varphi)$ cannot.

Adjunction. Modal adjunction also does not hold—if $K\alpha$ and $K\beta$ are true in s, then obviously $(\alpha \wedge \beta)$ is true there, but $K(\alpha \wedge \beta)$ need not be. [7] Our agent is really careful here. Even if each of α and β are confirmed separately, their conjunction is not accepted as knowledge, unless there is a single resource confirming both of them (which in general does not need to be the case).

[6] The explosion does not occur even in the case of the strong conjunction; $(\varphi \otimes \neg\varphi) \to \psi$ does not hold.

[7] The same negative result holds also for strong conjunction. If $K\alpha$ and $K\beta$ are true in s, then $(\alpha \otimes \beta)$ is true in s (because of factivity and $Rsss$), but $K(\alpha \otimes \beta)$ need not be true in s.

Modal disjunction rule. In our system knowledge distributes with disjunction (see Theorem 3.2). It holds that

$$\frac{K(\alpha \vee \beta)}{K\alpha \vee K\beta}$$

One of the disjuncts has to be confirmed by the same source as the whole disjunction, because a disjunction is true at a source if at least one of the disjuncts is.

3 Axiomatics, Soundness, Completeness

We extend a standard Hilbert style axiomatisation of R in the language $\rightarrow, \wedge, \vee, \neg, \bot$, which can be adopted e.g. from [5], with aditional axioms for the constants t and $\top$ and the modality K. The remaining logical operators are definable as we noted in Section 2.

Definition 3.1 Calculus RK consists of axiom schemes

$\varphi \rightarrow \varphi$	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
$\varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$	$(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$
$(\varphi \wedge \psi) \rightarrow \varphi$	$(\varphi \wedge \psi) \rightarrow \psi$
$\varphi \rightarrow (\varphi \vee \psi)$	$\psi \rightarrow (\varphi \vee \psi)$
$((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \wedge \chi))$	$(\varphi \wedge (\psi \vee \chi)) \rightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
$\neg\neg\varphi \rightarrow \varphi$	$(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$
$(t \rightarrow \varphi) \leftrightarrow \varphi$	$\varphi \rightarrow \top$
$K\varphi \rightarrow \varphi$	$\neg\varphi \wedge K\varphi \rightarrow \bot$
$K(\varphi \vee \psi) \rightarrow K\varphi \vee K\psi$	

and the rules

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \qquad \frac{\varphi \rightarrow \psi}{K\varphi \rightarrow K\psi}$$

Our first result is that this is an axiomatisation of the logic of general frames.

Theorem 3.2 (Soundness) *Any formula provable in* RK *is valid in all general frames.*

Proof. We spell out only soundness of the modal axioms, the soundness of the monotonicity rule has been established in Lemma 2.6.
(factivity) $K\varphi \rightarrow \varphi$ holds, since, whenever sSx and $s \Vdash \varphi$ then by the condition (20) $s \leq x$ and thus $x \Vdash \varphi$.
(strong factivity) $\neg\varphi \wedge K\varphi \rightarrow \bot$ (note that $x \Vdash \varphi \rightarrow \bot$ iff $x \nVdash \varphi$) holds in a logical state u, since if any $x \Vdash \neg\varphi \wedge K\varphi$, then for some s where $sSx, s \Vdash \varphi$, but from the definition of sSx we have sCx: so since $x \Vdash \neg\varphi$, we have $s \nVdash \varphi$, a contradiction. So there is no such x where $\neg\varphi \wedge K\varphi$ holds.
(K commuting with $\vee$) Suppose $x \in L$, $Rxyz$ and $y \Vdash K(\varphi \vee \psi)$ we show that $z \Vdash K\varphi \vee K\psi$. There is sSy satisfying $\varphi \vee \psi$, suppose, e.g., $s \Vdash \varphi$. From $Rxyz$ and $x \in L$ we know $y \leq z$. By the condition (17) there is s' such that $s \leq s'$ and $s'Sz$. Now $s' \Vdash \varphi$ and $z \Vdash K\varphi$. Then $z \Vdash K\varphi \vee K\psi$. □

We prove that the axiomatization RK is strongly complete with respect to the class of general frames, i.e. $\Gamma \nvdash \varphi$ implies $\Gamma \nvDash_{\mathcal{F}_g} \varphi$. We shall adopt the standard technique of *canonical model* construction.

Working with the logic R the natural concept of a theory (or a theory of a state in a model) is given by the notion of *prime theory.* A set of formulas Γ is a theory iff it is closed under provability: if $\varphi \vdash \psi$ and $\varphi \in \Gamma$ then $\psi \in \Gamma$, is closed under conjunction: if $\varphi \wedge \psi \in \Gamma$ then both $\varphi, \psi \in \Gamma$. Warning: in relevant logic, theories do not automatically contain all theorems. (If so, such theory is often called regular.) A theory is *prime* if it is moreover closed under disjunctions: if $\varphi \vee \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

We use the set of all prime theories over RK as the canonical set of points. Note that the canonical model defined this way may contain points in which t is not included, thus it contains nonlogical situations. It is not surprising since e.g. to violate the weakening axiom we need some nonlogical situations included.

To show that any invalid consequence $\Gamma \nvdash \varphi$ can be falsified in a state of the canonical model we need to know that Γ can be expanded to a prime theory in such a way that φ is not included in the expansion. In R this is done extending simultaneously two sets—a pair $\langle \Gamma, \{\varphi\} \rangle$—keeping track on what should and what shouldn't be included simultaneously. (Showing φ is not a theorem corresponds to invalid consequence $t \nvdash \varphi$, thus we want to falsify φ in a logical state.)

Definition 3.3 Let Γ, Δ denote sets of formulas. $P = \langle \Gamma, \Delta \rangle$ is called a *pair* if no conjunction ($\wedge$) of formulas in Γ entails any disjunction ($\vee$) of formulas in Δ.

According to Pair Extension Theorem [8, p. 94] we can extend a pair P to a full pair $P' = \langle \Gamma', \Delta' \rangle$ which is maximal in the sense that $\Gamma' \cup \Delta'$ is the whole language. It follows, that in a full pair Γ' is always a prime theory [8, p. 93].

Theorem 3.4 (Strong Completeness) *The axiomatization* RK *is strongly complete with respect to the class* $\mathcal{F}_g$ *of general frames.*

Proof. We take the usual Henkin-style construction of a canonical model, from [8, Sections 11.3 and 11.4]. To this construction of canonical model for RK, we must pay attention to the behaviour of K, and the accessibility relation S defined on the canonical model, to verify that this model satisfies the frame conditions for S, and hence, is a general frame.

Following the proof in [8], we take the points W_r in the canonical frame to be all the prime theories of the logic RK, and define the canonical frame to be $F_r = (W_r, L_r, \leq_r, C_r, R_r)$, where canonical relations are defined:

- $L_r = \{x \mid \ t \in x\}$
- $R_r xyz$ iff for each $\varphi \to \psi \in x$ where $\varphi \in y; \psi \in z$
- $xC_r y$ iff for each $\neg\varphi \in x, \varphi \notin y$
- $xS_r y$ iff for each $\varphi \in x, K\varphi \in y$

Adding valuation V such that $V(p) = \{x \mid p \in x\}$ we get a canonical model M_r.

It is immediate that membership in the prime theory satisfies the modelling conditions for $\wedge$ and $\vee$, and half of the modelling conditions for the conditional, negation and for K. We have

- if $\varphi \to \psi \in x$, then if $R_r xyz$, and $\varphi \in y$, then $\psi \in z$

- if $\neg\varphi \in x$, then if $xC_r y$, then $\varphi \notin y$
- if $\varphi \in x$, then if $xS_r y$, then $K\varphi \in y$.

To ensure that we have the converse of these, we appeal to Pair Extension Theorem [8]. The proof for R_r and C_r is standard [8, pp. 256, 261], but the proof for S_r we reiterate here:

Lemma 3.5 (Extension of the valuation to formulas) $x \Vdash \varphi$ *iff* $\varphi \in x$

Proof. Everything is clear except of $x \Vdash K\varphi$ iff $K\varphi \in x$:

From left to right: $x \Vdash K\varphi$, thus there is $sS_r x$ satisfying φ. From the definition of S_r, $K\varphi \in x$.

From right to left: Let us assume $K\varphi \in x$, we need an $s \in W_r$ such that $sS_r x$ and $\varphi \in s$.

$P = \langle\{\varphi\}; \{\psi : K\psi \notin x\}\rangle$ is a pair. Because if not, then $\varphi \vdash \psi_1 \vee ... \vee \psi_n$, hence $K\varphi \vdash K(\psi_1 \vee ... \vee \psi_n)$ (monotonicity of K), and $K\varphi \vdash K\psi_1 \vee ... \vee K\psi_n$ (K commuting with $\vee$ axiom). But as $K\varphi \in x$, then $K\psi_1 \vee ... \vee K\psi_n \in x$ (as x is a theory), hence $K\psi_i \in x$ (as x is prime), which is a contradiction with the definition of P.
According to Pair Extension Theorem [8] we can extend P to a full pair $P' = \langle s, r\rangle$. It follows, that s is prime. It remains to show that $sS_r x$. Assume $\alpha \in s$. If $K\alpha \notin x$, then $\alpha \in r$ (definition of P), so $\alpha \notin s$. Contradiction. □

So, the canonical model satisfies the general conditions for a model. It remains to show the canonical frame falls within the class of general frames, checking the canonical relations satisfy the required conditions.

Lemma 3.6 $F_r \in \mathcal{F}_g$

Proof. It is almost immediate that R_r, C_r and S_r, so defined, satisfy the plump versions of conditions which we have used in this paper to define general frames. We skip these cases here and refer to [8] where they can be easily extracted from the general proof.

To make clear the situation of logical states we however show that L_r satisfy the required conditions. First observe that L_r is clearly an upper set w.r.t. inclusion. We show it satisfies the condition (9).

Suppose $R_r xyz$ and $x \in L_r$. To show $y \subseteq z$ suppose $\varphi \in y$. Since $t \in x$ and $t \vdash \varphi \to \varphi$, we have $\varphi \to \varphi \in x$. Since $R_r xyz$ and $\varphi \in y$, we obtain $\varphi \in z$ as desired.

For the converse suppose $y \subseteq z$. We have to find a prime theory x such that $t \in x$ and $R_r xyz$ holds. First observe that $P = \langle\{t\}, \{\varphi \to \psi \mid \varphi \in y, \psi \notin z\}\rangle$ is a pair: if not, $t \vdash \bigvee_{i\in I}(\varphi_i \to \psi_i)$ for some disjunction of implications from the left set in the pair P. Then $\bigwedge_{i\in I}\varphi_i \vdash \bigvee_{i\in I}\psi_i$. But since all $\varphi_i \in y$, we have $\bigvee_{i\in I}\psi_i \in y$ and thus some $\psi_i \in y$ – a contradiction.

According to Pair Extension Theorem [8] we can extend P to a full pair $P' = \langle x, u\rangle$ where x is a prime theory. Moreover, we have $t \in x$ and $x \in L_r$. $R_r xyz$ holds immediatelly from the definition of P: if $\varphi \in y$ and $\psi \notin z$ then, from the definition of P, $\varphi \to \psi \in u$ and thus $\varphi \to \psi \notin x$.

We show that S_r satisfies the modelling condition (20):

Suppose sS_rx. To show that $s \leq x$, reason as follows: If $\varphi \in s$, then $K\varphi \in x$ (by the definition of S_r) and by factivity (any prime theory containing $K\varphi$ is closed under factivity because $K\varphi \vdash \varphi$), $\varphi \in x$ too. Since inclusion on the canonical model is subsethood, we have s $\leq x$.

Suppose sS_rx. We know from the general proof in [8] that C_r is symmetric. It is therefore enough to show that xC_rs. Reason as follows: If $\neg\varphi \in x$, then suppose φ were in s. Then since sS_rx, we have $K\varphi \in x$. But we have $\neg\varphi \in x$. Thus, $\neg\varphi \wedge K\varphi \in x$, which by strong factivity is impossible (x is a prime theory, and no prime theories contain $\perp$). So, if $\neg\varphi \in x$, then $\varphi \notin s$, and hence, xC_rs. From symmetry sS_rx as desired.

Next we make sure that S_r satisfies the other condition (17). Suppose sS_rx and $x \subseteq x'$. We need a prime theory s' such that $s \subseteq s'$ and $s'S_rx'$. We claim we can take $s' = s$. We show sS_rx': suppose $\varphi \in s$, then by sS_rx and definition of S_r, $K\varphi \in x$. Since $x \subseteq x'$, we have $K\varphi \in x'$ as well. □

This finishes the proof of Theorem 3.4. □

4 Correspondence

We address a question of modal correspondence in this section. Not only are factivity and strong factivity consequences of the modelling conditions (20) and (17), we can strengthen this to a correspondence result. The modal axioms of factivity and strong factivity *characterize* the class of general frames $\mathcal{F}_g$ (they hold in a frame F if and only if $F \in \mathcal{F}_g$). We even obtain a more subtle view on the conditions we put on the relation S: factivity corresponds, on frames, to the first half of the condition (20): $sSx \to s \leq x$. Strong factivity corresponds to the other half of condition (20): $sSx \to sCx$. We may split the condition (20) into two conditions:

(S1) sSx implies $s \leq x$

(S2) sSx implies sCx

If we now define two classes of frames $\mathcal{F}_{S1}$ and $\mathcal{F}_{S2}$ (so that their intersection is $\mathcal{F}_g$) as those relevant frames with S added and satisfying (S1) and (17), (or (S2) and (17) respectivelly), we obtain immediatelly sound and complete axiomatizations of the two logics $L(\mathcal{F}_{S1})$ (using factivity) and L($\mathcal{F}_{S2}$) (using strong factivity).

Theorem 4.1 (Correspondence)

- $K\varphi \to \varphi$ *characterizes the class* $\mathcal{F}_{S1}$.
- $\neg\varphi \wedge K\varphi \to \perp$ *characterizes the class* $\mathcal{F}_{S2}$.

Proof. Soundness theorem 3.2 supplies one half of the correspondence result.

For the other half of the correspondence condition for factivity, suppose we have a frame in which the condition $sSx \to s \leq x$ fails. That is, we have a pair of points s, x where sSx and $s \not\leq x$. Let $V(p)$ be the set of all points in which s is included: $V(p) = \{u \mid s \leq u\}$. This is clearly an upper set. Then $s \in V(p)$ but $x \notin V(p)$. It follows that Kp holds at x, since sSx, but p does not. Thus, since by the condition (9) there is some $u \in L$ such that $Ruxx$, we have $u \nVdash Kp \to p$.

For strong factivity, suppose $sSx \to sCx$ fails. That is, we have a pair of points s, x where sSx and not sCx. Let $V(p)$ be the set of all points not compatible with x: $V(p) = \{u \mid \neg uCx\}$. This is upward closed from the conditions posed on C: if not uCx and $u \leq v$, then not vCx by (11). So, p is true at no point compatible with x, hence $\neg p$ holds at x. However, since s is not compatible with x, $s \in V(p)$, and hence Kp holds at x. Thus we have $x \Vdash \neg p \wedge Kp$ and since by the condition (9) there is some $u \in L$ such that $Ruxx$, strong factivity cannot hold in u. □

The following condition (implied by (19) using the fact $sSx \to sSs$) corresponds to the introspection axiom:

$$sSx \to (\exists y \geq s)(\exists t)(yStSx) \tag{26}$$

Theorem 4.2 *The class of relevant frames with S satisfying conditions (17) and (26) corresponds to $K\varphi \to KK\varphi$.*

Proof. Suppose (26) holds in a frame. Suppose $x \Vdash K\varphi$. Then there is sSx and $s \Vdash \varphi$. From (26) there is $y \geq s$, thus $y \Vdash \varphi$ from persistency. There is t such that $yStSx$. Thus $t \Vdash K\varphi$ and $x \Vdash KK\varphi$.

Suppose (26) doesn't hold in a frame. Then there is sSx and for no $y \geq s$ and no t we have $yStSx$. Define $V(p) = \{u \mid u \geq s\}$. Now $x \Vdash Kp$. No t such that tSx sees a p source via S (we have $\neg yStSx$ for each y), thus no such t satisfies Kp and x doesn't satisfy KKp. There must be a logical state u with $Ruxx$ and $u \nVdash Kp \to KKp$. □

5 Conclusion

Our aim was to provide an axiomatisation and completeness for relevant frames with an epistemic modality proposed in [4]. The motivation of the original operator allows for some natural generalisations and we provided an axiomatisation for the generalised operator (the logic of the class $\mathcal{F}_g$), while the axiomatisation for the original one (the logic of the class $\mathcal{F}_c$) is still an open problem. Moreover, we showed that the class of general frames is an intersection of the classes of frames satisfying factivity and the class of frames satisfying strong factivity. We also gave a modal characterisation of the class of epistemic relevant frames in which the introspection axiom holds.

There are several topics related to the subject we did not address here. We shall explore the proof theory of relevant epistemic modalities and define a display calculus for them.

Another thing we did not pay attention to is the modality I adjoint to K (i.e. $\vdash K\varphi \to \psi$ iff $\vdash \varphi \to I\psi$). It has a natural interpretation in our epistemic framework, it corresponds to what we can call *implicit knowledge* (a formula is implicit knowledge in a state iff it is true in all the states, for which the current state is a potential source.) We obtain $\varphi \to I\varphi$ as a theorem (everything explicitly waranted is implicitly known) and $\varphi \to IK\varphi$ (all that holds in a state is at least implicitly known there) as well as

$KI\varphi \to \varphi$ (nothing else can be known to be implicit then facts warranted in the state) are theorems.

Other modalities can of course be considered as e.g. another adjoint pair of natural semantical duals of diamond-like K and box-like I (i.e. a box-like modality acting backwards along S, and a diamond-like modality acting forward along S). To obtain persistent meanings of formulas we would have to add another half of condition (17). Such modalities do not have natural epistemic interpretation in our framework and as such are not a part of the current work, however we conjecture they are not definable from the old two and as such can increase the expressivity of the modal language. We conjecture we would be able e.g. to distinguish between the classes of classical frames, and the class of general frames using them, as well as extend our definability results.

Last we remark we could start with a logic weaker then R without loosing much from our original motivations. Obvious candidates would be logics with a weaker negation obtained, e.g., by weakening conditions for the compatibility relation. This would of course influence the properties the modality K. Another option is to give up the contraction or exchange (commutativity of fusion). A challenging option is to give up distributivity, but this would mean to use more complicated frame semantics.

References

[1] Cheng, J., *A Strong Relevant Logic Model of Epistemic Processes in Scientific Discovery*, in: E. Kawaguchi, H. Kangassalo, H. Jaakkola and I. Hamid, editors, *Information Modelling and Knowledge Bases XI*, IOS Press, 2000 pp. 136–159.

[2] Duc, H. N., "Resource-Bounded Reasoning about Knowledge," Ph.D. thesis, Faculty of Mathematics and Informatics, University of Leipzig (2001).

[3] Fagin, R., J. Halpern, Y. Moses and M. Vardi, "Reasoning about Knowledge," The MIT Press, 2003.

[4] Majer, O. and M. Peliš, *Epistemic Logic with Relevant Agents*, in: M. Peliš, editor, *The Logica Yearbook 2008*, College Publications, 2009 pp. 123–135.

[5] Mares, E. D., "Relevant Logic," Cambridge University Press, 2004.

[6] Paoli, F., "Substructural Logics: A Primer," Kluwer, 2002.

[7] Restall, G., *Negation in Relevant Logics: How I stopped worrying and learned to love the Routley Star*, in: D. Gabbay and H. Wansing, editors, *What is Negation?*, The Applied Logic Series **13**, Kluwer, 1999 pp. 53–76.

[8] Restall, G., "An Introduction to Substructural Logics," Routledge, 2000.

[9] Wansing, H., *Diamonds Are a Philosopher's Best Friends*, Journal of Philosophical Logic **31** (2002), pp. 591–612.