Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(12): 4341–4365
DOI: 10.3934/era.2022220
Received: 27 May 2022
Revised: 19 September 2022
Accepted: 20 September 2022
Published: 28 September 2022

Research article

Semi-Jordan curve theorem on the Marcus-Wyse topological plane

Sang-Eon Han*

Department of Mathematics Education, Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju-City Jeonbuk 54896, Republic of Korea

* **Correspondence:** Email: sehan@jbnu.ac.kr; Tel: +82632704449.

Abstract: The paper initially develops the semi-Jordan curve theorem on the digital plane with the Marcus-Wyse topology, i.e., MW-topological plane or $(\mathbb{Z}^2, \gamma)$ for brevity. We first prove that while every simple closed MW-curve is semi-open in $(\mathbb{Z}^2, \gamma)$, it may not be semi-closed. Given a simple closed MW-curve with l elements, denoted by SC_γ^l, after establishing a continuous analog of SC_γ^l denoted by $\mathcal{A}(SC_\gamma^l)$, we initially show that $\mathcal{A}(SC_\gamma^l)$ is both semi-open and semi-closed in $(\mathbb{R}^2, \mathcal{U})$, where $(\mathbb{R}^2, \mathcal{U})$ is the 2-dimensional real plane $\mathbb{R}^2$ with the usual topology $\mathcal{U}$. Furthermore, we find a condition for $\mathcal{A}(SC_\gamma^l)$ to separate $(\mathbb{R}^2, \mathcal{U})$ into exactly two non-empty components, compared to a typical Jordan curve theorem on $(\mathbb{R}^2, \mathcal{U})$. Since not every SC_γ^l always separates $(\mathbb{Z}^2, \gamma)$ into two nonempty components, we find a condition for $SC_\gamma^l, l \neq 4$, to separate $(\mathbb{Z}^2, \gamma)$ into exactly two components. The semi-Jordan curve theorem on the MW-topological plane plays an important role in applied topology such as digital topology, mathematical morphology as well as computer science.

Keywords: semi-Jordan curve theorem; semi-open; semi-closed; Alexandroff space; Marcus-Wyse topology; Marcus-Wyse (MW-, for brevity) topological plane; semi-homeomorphism; continuous analog of a digital object; digital-topological group; digital topology

1. Introduction

In this paper, we use the notation $\mathbb{Z}$ (resp. $\mathbb{N}$ and $\mathbb{R}$) to indicate the set of integers (resp. natural numbers and real numbers). Besides, since we will often use the name "Marcus-Wyse" in this paper, we will take the term "MW-" instead of "Marcus-Wyse", if there is no danger of ambiguity. Besides, $\mathbb{Z}_o$ (resp. $\mathbb{N}_e$) means the set of odd integers (resp. even natural numbers) and further, we will use the notation "$\subset$" (resp. $X^\sharp$) to denote a 'proper subset or equal' (resp. the cardinality of the given set X). The notation " $:=$" will be used to introduce a new notion or a terminology. In addition, let us denote a simple closed MW-curve with l elements by SC_γ^l, $4 \leq l \in \mathbb{N}_e \setminus \{6\}$ (see Definition 2.1(3) in detail).

Indeed, the well-known Jordan curve theorem on the 2-dimensional real space [1] has some limitations of dealing with digital objects on $\mathbb{Z}^2$ from the viewpoints of applied sciences such as digital topology and digital geometry. Thus, in relation to the establishment of several types of Jordan curve theorems in digital topological settings, there are many works including the papers [2–12]. In the literature, to do this work, digital graph theory [7–9] and several types of topologies have been used such as Khalimsky, Marcus-Wyse, Alexandroff topology, pretopology, and so on. However, given a certain topological space $(\mathbb{Z}^2, T)$, the earlier works did not examine topological features of J and $\mathbb{Z}^2 \setminus J$, where J is a simple closed digital curve in $(\mathbb{Z}^2, T)$. Since both J and $\mathbb{Z}^2 \setminus J$ may not be either a closed or an open set in $(\mathbb{Z}^2, T)$, we need to intensively study some topological features of both J and $\mathbb{Z}^2 \setminus J$. Furthermore, with a certain topological space $(\mathbb{Z}^2, T)$, since the number of the components of $\mathbb{Z}^2 \setminus J$ can be very important from the viewpoint of mathematics, we need to intensively investigate this topic. For instance, on the MW-topological plane, i.e., $(\mathbb{Z}^2, \gamma)$, the present paper clearly shows that the number of the components of the complement of SC_γ^l in $(\mathbb{Z}^2, \gamma)$ depends on the situation. Besides, we also find that topological features of the sets SC_γ^l and $\mathbb{Z}^2 \setminus SC_\gamma^l$ are so related to the semi-closedness and semi-openness in $(\mathbb{Z}^2, \gamma)$. In detail, see [13, 14] or Section 3 in the present paper. Indeed, there are lots of works studying various properties of semi-closed and semi-open subsets of a topological space [13–19]. Based on this approach, the present paper will partially use these works.

The aim of the present paper is initially to propose the semi-Jordan curve theorem on the digital plane with the MW-topology (or $(\mathbb{Z}^2, \gamma)$) because it has something quite independent from the earlier results in the literature including the papers [3, 4, 6, 8, 9, 11, 12]. To propose this theorem and support some utilities, we will mainly deal with the following topics.

- Examination of many types of SC_γ^l with respect to the semi-closedness and semi-openness in $(\mathbb{Z}^2, \gamma)$.
- Establishment of a method for making a continuous analog of SC_γ^l and an investigation of some topological features of $\mathcal{A}(SC_\gamma^l)$ with respect to the semi-openness and semi-closedness in $(\mathbb{R}^2, \mathcal{U})$, where $(\mathbb{R}^2, \mathcal{U})$ is the 2-dimensional real space with the usual topology.
- Given an SC_γ^l, $l \neq 4$, how to separate $(\mathbb{R}^2, \mathcal{U})$ in terms of $\mathcal{A}(SC_\gamma^l)$?
- Assume the two subspaces (X, γ_X) and (Y, γ_Y) that are MW-homeomorphic to SC_γ^l. Then we will examine if the number of the components of $\mathbb{R}^2 \setminus \mathcal{A}(X)$ is equal to that of $\mathbb{R}^2 \setminus \mathcal{A}(Y)$. Besides, we strongly need to further compare the number of the components of $\mathbb{Z}^2 \setminus X$ and that of $\mathbb{Z}^2 \setminus Y$.
- Given an SC_γ^l, we need to examine if the number of the components of $\mathbb{Z}^2 \setminus SC_\gamma^l$ is a topological invariant.
- Under what condition, does SC_γ^l separate $(\mathbb{Z}^2, \gamma)$ into exactly two components?
- Development of the semi-Jordan curve theorem on the MW-topological plane. Besides, given an SC_γ^l, we investigate how to separate $(\mathbb{Z}^2, \gamma)$ with respect to the semi-Jordan curve theorem.
- Investigation of some properties of many kinds of SC_γ^l relating to the semi-Jordan curve theorem.

After addressing these topics with a success, we can confirm that the semi-Jordan curve theorem has strong advantages and some utilities compared with the earlier works in the literature because it does not have any paradox raised in the Rosenfeld's approach and further, it proceeds with the topological structures, which makes a distinction from the Rosenfeld's approach.

This paper is organized as follows: Section 2 provides some basic notions related to the digital k-connectivity on $\mathbb{Z}^2$ and the MW-topology. Section 3 studies some tools discriminating between semi-open and semi-closed sets in $(\mathbb{Z}^2, \gamma)$ and further, investigates various properties of semi-closed or semi-open subsets in $(\mathbb{Z}^2, \gamma)$. In Section 4, after examining if a simple closed MW-curve is semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$, we prove the semi-openness of each SC_γ^l and further, the semi-closedness of it is related to the number of l. Section 5 suggests a method for establishing a continuous analog of an SC_γ^l denoted by $\mathcal{A}(SC_\gamma^l)$ by using the local rule in [20, 21] considered on $\mathbb{R}^2$. Besides, we find some conditions for $\mathcal{A}(SC_\gamma^l)$ to separate $(\mathbb{R}^2, \mathcal{U})$ into exactly two components, compared to the typical Jordan curve theorem in $(\mathbb{R}^2, \mathcal{U})$ that is the 2-dimensional real plane. Furthermore, we prove that $\mathcal{A}(SC_\gamma^l)$ is both a semi-open and a semi-closed subset of $(\mathbb{R}^2, \mathcal{U})$. Meanwhile, every semi-closed $SC_\gamma^l, l \neq 4$, is proved to separate $(\mathbb{Z}^2, \gamma)$ into many semi-open components whose number depends on the number l of SC_γ^l. Section 6 proposes the semi-Jordan curve theorem on the MW-topological plane. Besides, a semi-open SC_γ^l is also proved to separate $(\mathbb{Z}^2, \gamma)$ into semi-closed or semi-open components whose number depends on the situation. More precisely, after proving that SC_γ^l separates $(\mathbb{Z}^2, \gamma)$ into many semi-closed or semi-open components depending on the situation, we find a condition for SC_γ^l to separate $(\mathbb{Z}^2, \gamma)$ into exactly two components. Besides, given two simple closed MW-curve with l elements X and Y, we first prove that the number of components of X^c need not be equal to that of Y^c. Section 7 refers to some advantages and utilities of MW-topological structure and the semi-Jordan curve theorem on $(\mathbb{Z}^2, \gamma)$. Section 8 concludes the paper with summary and a further work.

2. Preliminaries

To study digital objects in $\mathbb{Z}^2$, many basic notations will be used such as a digital 4- and 8-neighborhood of a point $p \in \mathbb{Z}^2$ [7, 8], as follows:
Based on the digital 4- and 8-connectivity in [7, 8, 22], for a point $p = (x, y) \in \mathbb{Z}^2$, the following notations will be often used later [7, 8].

$$\begin{Bmatrix} N_4(p) = \{(x \pm 1, y), p, (x, y \pm 1)\} \\ N_8(p) = \{(x \pm 1, y), p, (x, y \pm 1), (x \pm 1, y \pm 1)\} \end{Bmatrix}$$

which is respectively called the 4-neighborhood and 8-neighborhood of a point p.
Then we recall that distinct points $p, q \in \mathbb{Z}^2$ are 4-(resp. 8-)adjacent if and only if $p \in N_4(q) \setminus \{q\}$ (resp. $p \in N_8(q) \setminus \{q\}$) or $q \in N_4(p) \setminus \{p\}$ (resp. $q \in N_8(p) \setminus \{p\}$) [7, 8].

We now recall an Alexandroff topological structure using the study of some properties of MW-topological spaces. More precisely, a topological space (X, T) is called an Alexandroff space if every point $x \in X$ has the smallest open neighborhood in (X, T) [24]. As an Alexandroff topological space [24, 25], the Marcus-Wyse topological space, denoted by $(\mathbb{Z}^2, \gamma)$, was established and there are many studies including the papers [5,6,26]. Indeed, the MW-topology, denoted by $(\mathbb{Z}^2, \gamma)$, is generated by the set of all $U(p)$ in (2.2) below, i.e., $\{U(p) \mid p \in \mathbb{Z}^2\}$, as a base [27], where for each point $p = (x, y) \in \mathbb{Z}^2$

$$U(p) := \begin{Bmatrix} N_4(p) \text{ if } x + y \text{ is even, and} \\ \{p\} : \text{ else.} \end{Bmatrix} \tag{2.2}$$

In the paper we call a point $p = (x_1, x_2)$ *doubly even* if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; *even* if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and *odd* if $x_1 + x_2$ is an odd number [12].

In all subspaces of $(\mathbb{Z}^2, \gamma)$ of Figures 1–7 the symbols $\diamond$ and $\bullet$ mean a *doubly even point or even point* and an *odd point*, respectively. In view of (2.2), we can obviously obtain the following: Under

$(\mathbb{Z}^2, \gamma)$, the singleton $\{\diamond\}$ is a closed set and $\{\bullet\}$ is an open set. Besides, for a subset $X \subset \mathbb{Z}^2$, the subspace induced by $(\mathbb{Z}^2, \gamma)$ is obtained, denoted by (X, γ_X) and called an MW-topological space. Hereinafter, for our purpose, we will use the notations

$$\begin{cases} (\mathbb{Z}^2)_e := \{p \in \mathbb{Z}^2 \mid p \text{ is a doubly even or even point in } \mathbb{Z}^2\}, \text{ and} \\ (\mathbb{Z}^2)_o := \{p \in \mathbb{Z}^2 \mid p \text{ is an odd point in } \mathbb{Z}^2\}. \end{cases} \tag{2.3}$$

In terms of this perspective, it turns out that the *minimal (open) neighborhood* of the point $p := (p_1, p_2)$ of $\mathbb{Z}^2$, denoted by $SN_\gamma(p) \subset \mathbb{Z}^2$, is obtained, as follows [26, 28]:

$$SN_\gamma(p) = \begin{cases} \{p\} \text{ if } p \in (\mathbb{Z}^2)_o, \text{ and} \\ N_4(p) \text{ if } p \in (\mathbb{Z}^2)_e. \end{cases} \tag{2.4}$$

Hereinafter, in (X, γ_X), for $p \in X$ we use the notation $SN_\gamma(p) := SN_\gamma(p) \cap X$ for short if there is no danger of ambiguity. Using the smallest open set of (2.4), the notion of an MW-adjacency in $(\mathbb{Z}^2, \gamma)$ is defined, as follows: For distinct points p, q in $(\mathbb{Z}^2, \gamma)$, we say that p is MW-adjacent to q [26] if

$$p \in SN_\gamma(p) \text{ or } q \in SN_\gamma(q)$$

In view of the properties of (2.2) and (2.4), we obviously obtain the following:

Based on the structure of (2.4), for a point $p := (p_1, p_2)$ of $\mathbb{Z}^2$, the closure of the singleton $\{p\}$ is denoted by $Cl_\gamma\{p\}) \subset \mathbb{Z}^2$ as follows [26]:

$$Cl_\gamma(\{p\}) = \begin{cases} \{p\} \text{ if } p \in (\mathbb{Z}^2)_e, \text{ and} \\ N_4(p) \text{ if } p \in (\mathbb{Z}^2)_o. \end{cases} \tag{2.5}$$

Hereinafter, in relation to the study of MW-topological spaces, we will use the term Cl for brevity instead of Cl_γ if there is no danger of confusion.

Definition 2.1. *[26] Let $X := (X, \gamma_X)$ be an MW-topological space. Then we define the following:*
(1) An MW-path from x to y in X is defined as a sequence $(p_i)_{i\in[0,l]_\mathbb{Z}} \subset X, l \in \mathbb{N}$, in X such that $p_0 = x$, $p_l = y$ and each point p_i is MW-adjacent to p_{i+1} and $i \in [0, l-1]_\mathbb{Z}$. The number l is the length of this path. In particular, a singleton in $(\mathbb{Z}^2, \gamma)$ is assumed to be an MW-path.
(2) Distinct points $x, y \in X$ are called MW-path connected if there is a finite MW-path $(p_i)_{i\in[0,m]_\mathbb{Z}}$ on X with $p_0 = x$ and $p_m = y$. For arbitrary points $x, y \in X$, if there is an MW-path $(p_i)_{i\in[0,m]_\mathbb{Z}} \subset X$ such that $p_0 = x$ and $p_m = y$, then we say that X is MW-path connected (or MW-connected).
(3) A simple closed MW-curve (resp. simple MW-path) with l elements in X means a finite MW-path $(p_i)_{i\in[0,l-1]_\mathbb{Z}}$, $4 \le l \in \mathbb{N}_e \setminus \{6\}$ in $\mathbb{Z}^2$ such that the points p_i and p_j are MW-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$ (resp. $|i - j| = 1$). Then we use the notation SC^l_γ to denote a simple closed MW-curve with l elements.

As for some properties of SC^l_γ, it is clear that $SC^{l_1}_\gamma$ is MW-homeomorphic to $SC^{l_2}_\gamma$ if and only if $l_1 = l_2$ [26].

3. Some properties of semi-open and semi-closed sets

This section first recalls the concepts of a semi-open and a semi-closed set. Namely, a subset A of a topological space (X, T) is said to be *semi-open* if there is an open set O in (X, T) such that $O \subset A \subset Cl(O)$. Besides, we say that a subset B of a topological space (X, T) is *semi-closed* if the complement of B in X, i.e., B^c, is semi-open in (X, T). Then it turns out that a subset A of (X, T) is semi-open if and only if $A \subset Cl(Int(A))$ [13] and a subset B of (X, T) is semi-closed if and only if $Int(Cl(B)) \subset B$ [29]. Hence, in (X, T), it is clear that each of the empty set and the total set is both semi-open and semi-closed. Besides, "open" (resp. "closed") is stronger than "semi-open" (resp. "semi-closed"). The notions of semi-openness and semi-closedness enable us to get the following [13,19,31]:
($\star$1) The intersection of two semi-open sets need not be semi-open.
($\star$2) The union of two semi-closed sets need not be semi-closed.
($\star$3) The union of two semi-open sets is semi-open.
($\star$4) The intersection of two semi-closed sets is semi-closed.

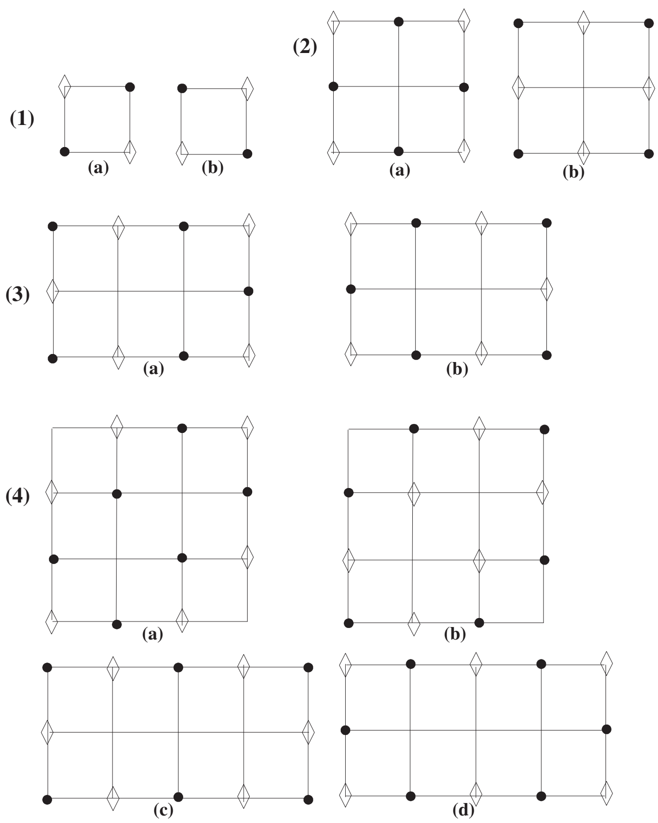

Figure 1. Examples of several types of SC_γ^l, where $l \in \{4, 8, 10, 12\}$. Namely, (1) SC_γ^4 is both semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$. (2) As for SC_γ^8, while the object of (a) is not semi-closed but semi-open, the object of (b) is both semi-closed and semi-open in $(\mathbb{Z}^2, \gamma)$. (3) As for SC_γ^{10}, the objects of (a) and (b) are both semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$. (4) As for SC_γ^{12}, while the object of (a) is not semi-closed but semi-open, each of (b)–(d) is both semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$.

Remark 3.1. *[30, 31] In $(\mathbb{Z}^2, \gamma)$, the following are obtained:*
(1) The singleton $\{p\}$ is both semi-closed and semi-open, where $p \in (\mathbb{Z}^2)_o$. Namely, $\mathbb{Z}^2 \setminus \{p\}$ is both semi-closed and semi-open, where $p \in (\mathbb{Z}^2)_o$.
(2) The singleton $\{q\}$ is not semi-open but semi-closed, where $q \in (\mathbb{Z}^2)_e$. Namely, $\mathbb{Z}^2 \setminus \{q\}$ is semi-open, where $q \in (\mathbb{Z}^2)_e$.

Let us further establish some techniques to examine if a set in $(\mathbb{Z}^2, \gamma)$ is semi-open or semi-closed. In $(\mathbb{Z}^2, \gamma)$, for a set $X \subset \mathbb{Z}^2$, we will take the following notation [31].

$$X_{op} := \{x \,|\, x \text{ is an odd point in } X\}. \tag{3.1}$$

Besides, the topological structure of $(\mathbb{Z}^2, \gamma)$ enables us to get the following [31]:

Remark 3.2. *[31] In $(\mathbb{Z}^2, \gamma)$, we have the following:*
(1) For $x, y \in \mathbb{Z}^2$, $x \in SN_\gamma(y)$ if and only if $y \in Cl(x)$, i.e., $y \in Cl_\gamma(x)$(see the properties of (2.4) and (2.5) in the present paper).
(2) If X is an open set in $(\mathbb{Z}^2, \gamma)$, then there is at least an odd point x in X (see the property of (2.3)).
(3) The set X_{op} of (3.1) is an open set in $(\mathbb{Z}^2, \gamma)$.

Given a set X in $(\mathbb{Z}^2, \gamma)$, to further examine if the set X is semi-open or semi-closed in $(\mathbb{Z}^2, \gamma)$, we now introduce the following two theorems that will be strongly used in discriminating against subsets based on the semi-openness and semi-closedness of the MW-topological space.

Theorem 3.3. *[31] In $(\mathbb{Z}^2, \gamma)$, a (non-empty) set $X(\subset \mathbb{Z}^2)$ is semi-open if and only if each $x \in X$, $SN_\gamma(x) \cap X_{op} \neq \emptyset$, where $SN_\gamma(x)$ is assumed in $(\mathbb{Z}^2, \gamma)$.*

Since this theorem strongly plays an important role in studying many results in the present paper, to make Theorem 3.3 self-contained, we suggest a proof briefly, as follows: In case $X = \emptyset$, the proof is straightforward. Let us assume that X is not an empty set.
($\Rightarrow$) According to the choice of a point $x \in X$, we can consider the following two cases.
(Case 1) Assume that $x(\in X)$ is an odd point. From the hypothesis, we have $x \in X \subset Cl(Int(X))$ so that we obtain

$$SN_\gamma(x) \cap Int(X) \neq \emptyset. \tag{3.2}$$

Since $SN_\gamma(x) = \{x\}$, we obtain $x \in Int(X)$ and further, $x \in X_{op}$. Hence, owing to (3.2), we have $SN_\gamma(x) \cap X_{op} \neq \emptyset$.

(Case 2) Assume that $x \in X$ is a doubly even or even point. Owing to the hypothesis, we obtain $x \in Cl(Int(X))$ that leads to the following property as mentioned in (3.2).

$$SN_\gamma(x) \cap Int(X) \neq \emptyset.$$

Since $SN_\gamma(x) \cap Int(X)$ is a non-empty open set in $(\mathbb{Z}^2, \gamma)$, by Remark 3.2(2), we now take an odd point z in $(\mathbb{Z}^2, \gamma)$ such that

$$z \in SN_\gamma(x) \cap Int(X). \tag{3.3}$$

By the property of (3.3), since $z \in Int(X) \subset X$, we have $z \in X_{op}$ (see Remark 3.2(2)) so that $z \in SN_\gamma(x) \cap X_{op} \neq \emptyset$. In addition, we see that the point z is indeed MW-adjacent to x.

(⇐) According to the choice of a point $x \in X$, we can consider the following two cases.
(Case 1) For an arbitrary point $x \in X$, assume that x is an odd point in $(\mathbb{Z}^2, \gamma)$. Since $\{x\} = SN_\gamma(x)$, owing to the hypothesis of $SN_\gamma(x) \cap X_{op} \neq \emptyset$, we have $x \in X_{op}$, i.e., $\{x\} \cap X_{op} \neq \emptyset$. Furthermore, owing to the identity $SN_\gamma(x) = \{x\}$, by Remark 3.2(3), it is clear that

$$x \in X_{op} \Rightarrow \{x\} \subset Int(X) \Rightarrow x \in Cl(Int(X)). \tag{3.4}$$

(Case 2) For an arbitrary point $x \in X$, assume that x is a doubly even or even point in $(\mathbb{Z}^2, \gamma)$. Owing to the hypothesis, since $SN_\gamma(x) \cap X_{op} \neq \emptyset$, by Remark 3.2(2) and (3), there is an odd point z in $(\mathbb{Z}^2, \gamma)$ such that $z \in SN_\gamma(x) \cap X_{op}$ because $SN_\gamma(x) \cap X_{op}$ is an open set in $(\mathbb{Z}^2, \gamma)$. Hence we get $z \in SN_\gamma(x)$, by Remark 3.2(1), we have

$$x \in Cl(\{z\}) \subset Cl(Int(X)) \Rightarrow x \in Cl(Int(X)). \tag{3.5}$$

Owing to both (3.4) and (3.5), we obtain $X \subset Cl(Int(X))$ which prove the assertion.

Owing to the notion of semi-closedness, using Theorem 3.4, we obtain the following:

Theorem 3.4. *[31] In $(\mathbb{Z}^2, \gamma)$, $B(\subset \mathbb{Z}^2)$ is semi-closed if and only if each $x \in \mathbb{Z}^2 \backslash B$, $SN_\gamma(x) \cap (\mathbb{Z}^2 \backslash B)_{op} \neq \emptyset$, where $SN_\gamma(x)$ is assumed in $(\mathbb{Z}^2, \gamma)$.*

As examples for Theorems 3.3 and 3.4, see the cases referred to in Remark 3.1(1)–(3).
In view of Theorems 3.3 and 3.4, we have the following:

Remark 3.5. *In $(\mathbb{Z}^2, \gamma)$, assume a connected subset X with $X^\sharp \geq 2$. Then it is semi-open and it may not be semi-closed.*

4. Classification of simple closed MW-curves with respect to the semi-closedness

To classify all types of SC_γ^l with respect to the semi-openness and semi-closedness, based on the topological features of SC_γ^l, it suffices to consider the only case of $l \in \{2m \mid m \in \mathbb{N} \setminus \{1, 3\}\}$ because no SC_γ^6 exists. Hereinafter, when studying semi-topological features of a set $X \subset \mathbb{Z}^2$, we assume that the set X is considered in $(\mathbb{Z}^2, \gamma)$.

Theorem 4.1. *Given an SC_γ^l, $4 \leq l \in \mathbb{N}_e \setminus \{6\}$, the semi-topological features of SC_γ^l in $(\mathbb{Z}^2, \gamma)$ are determined according to the number l, as follows:*
(1) SC_γ^l is always semi-open for any l, where $l \in \{2m \mid m \in \mathbb{N} \setminus \{1, 3\}\}$.
(2) SC_γ^l is always both semi-open and semi-closed whenever $l \in \{4, 10\}$.

Proof: (1) Given an $SC_\gamma^l := (x_i)_{i \in [0, l-1]_\mathbb{Z}}$, take any element $x_i \in SC_\gamma^l$. Then $SN_\gamma(x_i)$ in $(\mathbb{Z}^2, \gamma)$ has the following property,

$$SN_\gamma(x_i) \cap (SC_\gamma^l)_{op} \neq \emptyset.$$

By Theorem 3.3, we conclude that SC_γ^l is semi-open in $(\mathbb{Z}^2, \gamma)$.

(2) (2-1) In the case of SC_γ^4, let $Y := \mathbb{Z}^2 \setminus SC_\gamma^4$. Then, for any $p \in Y$ we have

$$SN_\gamma(p) \cap (Y)_{op} \neq \emptyset,$$

which implies that Y is semi-open in $(\mathbb{Z}^2, \gamma)$ (see Theorem 3.3). Hence SC_γ^4 is semi-closed in $(\mathbb{Z}^2, \gamma)$. Also, using a method similar to the proof of (1), it is clear that SC_γ^4 is semi-open in $(\mathbb{Z}^2, \gamma)$.

(2-2) In the case of SC_γ^{10} (see the objects in Figure 1(3)(a),(b)), let $W := \mathbb{Z}^2 \setminus SC_\gamma^{10}$. Using a method similar to the proof of (2-1) above, by Theorems 3.3 and 3.4, we prove that SC_γ^{10} is both semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$.

Remark 4.2. *In the case of $l \notin \{4, 10\}$, SC_γ^l may not be semi-closed. The semi-closedness of SC_γ^l depends on the situation.*

Proof: In the case of SC_γ^l, where $l \notin \{4, 10\}$, the semi-closedness of SC_γ^l depends on the situation. More precisely, given an $SC_\gamma^l := (x_i)_{i \in [0,l-1]_\mathbb{Z}}$, $l \notin \{4, 10\}$, assume that SC_γ^l does not have the subsequence

$$X_1 := (x_{t-3(mod\, l)}, x_{t-1(mod\, l)}, x_{t+1(mod\, l)}, x_{t+3(mod\, l)})$$

whose each element is an odd point (i.e., $X_1 \subset SC_\gamma^l \cap (\mathbb{Z}^2)_o$) and $X_1 \subset N_4(x), x \in \mathbb{Z}^2 \setminus SC_\gamma^l$. Then, by Theorem 3.4, SC_γ^l is semi-closed. For instance, since no SC_γ^6 exists, it suffices to mention that SC_γ^l, $l \notin \{4, 10\}$, is semi-closed depending on the situation. As suggested in Figure 1(2), while the object SC_γ^8 of (a) is not semi-closed (see Theorem 3.4) but semi-open, the object of (b) is both semi-closed and semi-open.

In view of Theorem 4.1, we obtain the following:

Proposition 4.3. *There are two types of $SC_\gamma^l, l \notin \{4, 10\}$, with respect to the semi-closedness.*

Proof: Using Theorems 3.3 and 3.4, we prove the assertion. As mentioned in the proof of Theorem 4.1, we need to consider the following two cases:
(Case 1) Assume an $SC_\gamma^l := (x_i)_{i \in [0,l-1]_\mathbb{Z}}, l \notin \{4, 10\}$, such that SC_γ^l does not have the subsequence

$$X_1 := (x_{t-3(mod\, l)}, x_{t-1(mod\, l)}, x_{t+1(mod\, l)}, x_{t+3(mod\, l)}) \tag{4.1}$$

whose each element is an odd point and $X_1 \subset N_4(x), x \in \mathbb{Z}^2 \setminus SC_\gamma^l$. Then, by Theorem 3.4, SC_γ^l is semi-closed.
(Case 2) Assume an $SC_\gamma^l := (x_i)_{i \in [0,l-1]_\mathbb{Z}}, l \notin \{4, 10\}$, in which there is the subsequence X_1 of (4.1) whose each element is an odd point and $X_1 \subset N_4(x), x \in \mathbb{Z}^2 \setminus SC_\gamma^l$. Then, by Theorem 3.4, SC_γ^l is not semi-closed but only semi-open owing to Theorem 3.3. For instance, since SC_γ^{12} in Figure 2(a) does not satisfy the condition of Theorem 3.4, it is not semi-closed in $(\mathbb{Z}^2, \gamma)$ (see the points $p_1 := (0, 0), p_2 := (1, 1)$ in $\mathbb{Z}^2 \setminus SC_\gamma^{12}$ as in Figure 2(b)).

Example 4.1. *In Figure 1, some examples for several types of SC_γ^l are shown for $l \in \{4, 8, 10, 12\}$. In view of Theorems 3.3, 3.4 and 4.1, and Proposition 4.3, we obtain the following:*
(1) SC_γ^4 in Figure 1 is both semi-open and semi-closed.
(2) As for the SC_γ^8 in Figure 1(2), the object of (a) is semi-open instead of semi-closed because it satisfies only the condition of Theorem 3.3 instead of that of Theorem 3.4. However, the object of (b) is both semi-open and semi-closed (see also Theorems 3.3 and 3.4).
(3) As for the SC_γ^{10} in Figure 1(3), each of (a) and (b) is both semi-open and semi-closed (see Theorems 3.3 and 3.4).
(4) As for the SC_γ^{12} in Figure 1(4), the object of (a) is semi-open instead of semi-closed because it

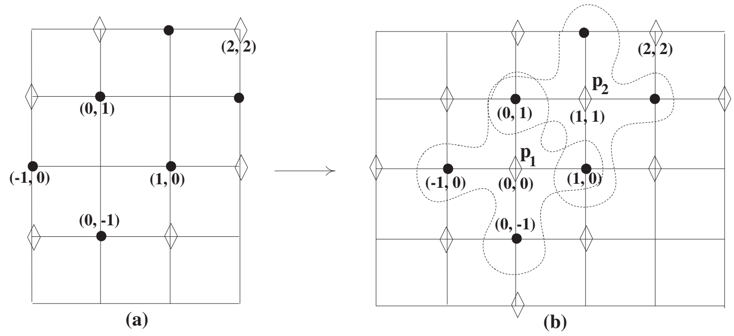

Figure 2. In $(\mathbb{Z}^2, \gamma)$, consider the SC_γ^{12} in Figure 2(a). As shown in Figure 2(b), owing to the two points p_1, p_2 in $\mathbb{Z}^2 \setminus SC_\gamma^{12}$ in Figure 2(a), we conclude that SC_γ^{12} is not semi-closed in $(\mathbb{Z}^2, \gamma)$ because $Int(Cl(SC_\gamma^{12})) \not\subseteq SC_\gamma^{12}$ (see the points p_1 and p_2 in (b)). However, we obtain $SC_\gamma^{12} \subset Cl(Int(SC_\gamma^{12}))$ that implies the semi-openness of SC_γ^{12}.

satisfies only the condition of Theorem 3.3 instead of that of Theorem 3.4. To be specific, based on an SC_γ^{12} in Figure 2(a), consider the object in Figure 2(b). Since the set $Cl(SC_\gamma^{12})$ contains the open sets $N_4(p_i) = SN_\gamma(p_i), i \in \{1,2\}$, $p_1 = (0,0)$ and $p_2 = (1,1)$, so that $Int(Cl(SC_\gamma^{12})) \not\subseteq SC_\gamma^{12}$, which implies the non-semi-closedness of SC_γ^{12} (see the proof of Proposition 4.3). However, the object of Figure 1(4)(b) is both semi-open and semi-closed.

5. Establishment of a continuous analog of SC_γ^l, $\mathcal{A}(SC_\gamma^l) \subset \mathbb{R}^2$, with respect to the MW-topology

This section introduces a method for establishing a continuous analog of an object on $\mathbb{Z}^2$ with respect to the MW-topology. The local rule introduced in Definition 5.1 below will be used in this work and has been widely used in digitization and digital-based rough set theory [20].

Definition 5.1. *[20] For each point $p := (p_1, p_2) \in \mathbb{Z}^2$, the continuous analog of the given point $p \in \mathbb{Z}^2$ with respect to the MW-topology, denoted by A_p, is defined by:*

$$A_p = \left\{ \begin{array}{l} [p_1 - 0.5, p_1 + 0.5] \times [p_2 - 0.5, p_2 + 0.5], \\ \textit{if } p \textit{ is a doubly even point } (\textit{see Figure 3(1)}); \\ [p_1 - 0.5, p_1 + 0.5] \times [p_2 - 0.5, p_2 + 0.5] \setminus \{q_1, q_2, q_3, q_4\}, \\ \textit{where } q_i, i \in [1,4]_{\mathbb{Z}} \\ \textit{such that } q_1 = (p_1 - 0.5, p_2 - 0.5), q_2 = (p_1 - 0.5, p_2 + 0.5), \\ q_3 = (p_1 + 0.5, p_2 + 0.5), q_4 = (p_1 + 0.5, p_2 - 0.5), \\ \textit{if } p \textit{ is an even point } (\textit{see Figure 3(2)}); \textit{ and} \\ (p_1 - 0.5, p_1 + 0.5) \times (p_2 - 0.5, p_2 + 0.5), \\ \textit{if } p \textit{ is an odd point } (\textit{see Figure 3(3)-(4)}). \end{array} \right\} \tag{5.1}$$

Hereinafter, we assume the set A_p to be a subspace of $(\mathbb{R}^2, \mathcal{U})$. Using the local rule around a point $p \in \mathbb{Z}^2$ as in Definition 5.1, we define the following:

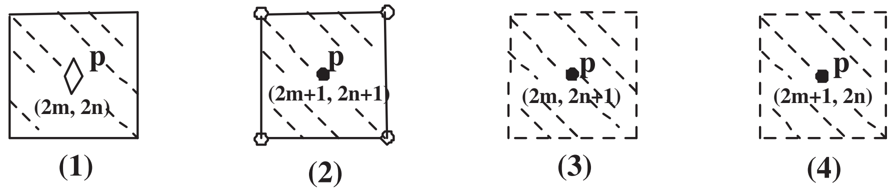

Figure 3. Configurations of $A_p(\subset \mathbb{R}^2)$, $p \in \mathbb{Z}^2$ in Definition 5.1, according to the point $p \in \mathbb{Z}^2$ as stated in (5.1), where the point p of (1) is a doubly even point, the point p of (2) is an even point, and each of the points p of (3) and (4) is an odd point.

Definition 5.2. *A continuous analog of* $X(\subset \mathbb{Z}^2)$ *is defined as* $\mathcal{A}(X) = \bigcup_{p\in X} A_p$ *by taking the following way.*

$$\left\{\begin{array}{l}\mathcal{A} : P(\mathbb{Z}^2) \to P(\mathbb{R}^2) \text{ for } X(\subset \mathbb{Z}^2) \text{ defined by} \\ \mathcal{A}(X) := \bigcup_{p\in X} A_p.\end{array}\right\}$$

Then we assume $\mathcal{A}(X)$ *to be* $(\mathcal{A}(X), \mathcal{U}_{\mathcal{A}(X)})$ *as a subspace of* $(\mathbb{R}^2, \mathcal{U})$.
In particular, the set $\mathcal{A}(\mathbb{Z}^2)$ *is defined as* $\mathcal{A}(\mathbb{Z}^2) = \bigcup_{p\in\mathbb{Z}^2} A_p = \mathbb{R}^2$ *by taking the following way.*

$$\mathbb{Z}^2 \to \mathcal{A}(\mathbb{Z}^2) := \bigcup_{p\in\mathbb{Z}^2} A_p = \mathbb{R}^2.$$

Then we assume $\mathcal{A}(\mathbb{Z}^2)$ *to be* $(\mathcal{A}(\mathbb{Z}^2), \mathcal{U})$*, i.e.,* $(\mathcal{A}(\mathbb{Z}^2), \mathcal{U}) = (\mathbb{R}^2, \mathcal{U})$.

Remark 5.3. *The operator* $\mathcal{A}$ *need not preserve an M-homeomorphism into a homeomorphism in* $(\mathbb{R}^2, \mathcal{U})$.

Definitions 5.1 and 5.2 enable us to get the following:

Lemma 5.4. *(1) In case X is a connected subset of* $(\mathbb{Z}^2, \gamma)$*,* $\mathcal{A}(X)$ *is also a connected subset of* $(\mathbb{R}^2, \mathcal{U})$.
(2) In case Y is a disconnected subset of $(\mathbb{Z}^2, \gamma)$*,* $\mathcal{A}(Y)$ *may not be a disconnected subset of* $(\mathbb{R}^2, \mathcal{U})$. *Namely, the connectedness of* $\mathcal{A}(Y)$ *in* $(\mathbb{R}^2, \mathcal{U})$ *depends on the situation.*

Proof: (1) Given a connected subset X of $(\mathbb{Z}^2, \gamma)$, we obtain $\mathcal{A}(X) = \bigcup_{p\in X} A_p$ that is a connected subset of $(\mathbb{R}^2, \mathcal{U})$ (see Definition 5.1).
(2) As an example, consider the set $\{p, q\}$, where $p, q \in (\mathbb{Z}^2)_e$, $p \neq q$, and $q \in N_8(p)$. While the set $\{p, q\}$ is a disconnected subset of $(\mathbb{Z}^2, \gamma)$, $\mathcal{A}(\{p, q\}) = A_p \cup A_q$ is a connected subset of $(\mathbb{R}^2, \mathcal{U})$ (see Definition 5.1). For instance, in Figure 4(1), consider the two points $p := (0, 0)$ and $q := (1, -1)$ in SC^4_γ. Then the set $\{p, q\}$ supports the assertion.

Theorem 5.5. *Given an* SC^l_γ*,* $\mathcal{A}(SC^l_\gamma)$ *is both semi-open and semi-closed in* $(\mathbb{R}^2, \mathcal{U})$.

Proof: Since $\mathcal{A}(SC^l_\gamma) \subset Cl(Int(\mathcal{A}(SC^l_\gamma)))$, the proof of the semi-openness of $\mathcal{A}(SC^l_\gamma)$ is completed. Besides, since we obtain $Int(Cl(\mathcal{A}(SC^l_\gamma)) \subset \mathcal{A}(SC^l_\gamma)$, the proof of the semi-closedness of $\mathcal{A}(SC^l_\gamma)$ is also completed.

To support Theorem 5.5, we can suggest the following examples. The set $\mathcal{A}(SC_\gamma^4)$ is both semi-open and semi-closed in $(\mathbb{R}^2, \mathcal{U})$. Similarly, each of the sets $\mathcal{A}(SC_\gamma^8)$ and $\mathcal{A}(SC_\gamma^{10})$ is also both semi-open and semi-closed in $(\mathbb{R}^2, \mathcal{U})$.

Proposition 5.6. *Given an SC_γ^l, we obtain the following:*
(1) In the case of SC_γ^4, $\mathcal{A}(SC_\gamma^4)$ does not separates $(\mathbb{R}^2, \mathcal{U})$ into two non-empty components. Namely, $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^4)$ has the only one non-empty component.
(2) In the case of $SC_\gamma^l, l \in \{8, 10\}$, $\mathcal{A}(SC_\gamma^l)$ separates $(\mathbb{R}^2, \mathcal{U})$ into exactly two non-empty components.
(3) For any l of $SC_\gamma^l, l \notin \{4, 8, 10\}$, $\mathcal{A}(SC_\gamma^l)$ separates $(\mathbb{R}^2, \mathcal{U})$ into more than or equal to two non-empty components.

Proof: (1) Given an SC_γ^4, we obtain $\mathcal{A}(SC_\gamma^4) = \bigcup_{p \in SC_\gamma^4} A_p$ as shown in Figure 4(1)(b) that is a subspace of $(\mathbb{R}^2, \mathcal{U})$, where $\mathcal{A}(SC_\gamma^4)$ is both semi-open and semi-closed in $(\mathbb{R}^2, \mathcal{U})$ (see Figure 4(1)(a),(b)). Then it is clear that $\mathcal{A}(SC_\gamma^4)$ does not separate $(\mathbb{R}^2, \mathcal{U})$ into two nonempty components.
(2) In the case of $l \in \{8, 10\}$, we obtain $\mathcal{A}(SC_\gamma^l)$ as a subspace of $(\mathbb{R}^2, \mathcal{U})$ whose complement of $\mathcal{A}(SC_\gamma^l)$ in $\mathbb{R}^2$ consists of only two non-empty components. See the process of obtaining $\mathcal{A}(SC_\gamma^8)$ in Figure 4(2) from (a) to (b) and (c) to (d).
(3) For any l of $SC_\gamma^l, l \notin \{4, 8, 10\}$, $\mathcal{A}(SC_\gamma^l)$ separates $(\mathbb{R}^2, \mathcal{U})$ to obtain that $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ has more than or equal to two non-empty components. For instance, as shown in each of the objects SC_γ^{12} in Figure 4(4)(a),(b), we see that $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{12})$ has only two non-empty components.
Meanwhile, as for the SC_γ^{12} as in Figure 4(c) and SC_γ^{18} in Figure 5(a), each of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{12})$ and $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{18})$ has more than two components. To be specific, we need to strongly recognize that the set $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{12})$ of Figure 4(4)(b) has only two components, i.e., we have $C(p_1) = C(p_2)$, where $C(p_i)$ means the component containing the given point p_i in $(\mathbb{R}^2, \mathcal{U})$, $i \in \{1, 2\}$. Similarly, $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{18})$ of Figure 5(d) has only two components. However, as shown in Figure 4(4) from (c) to (d), we see that $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{12})$ has three disjoint non-empty components such as

$$C(q_1), C(q_2), \text{ and } \mathbb{R}^2 \setminus (C(q_1) \cup C(q_2) \cup \mathcal{A}(SC_\gamma^{12})).$$

Example 5.1. *(1) Given the SC_γ^4 in Figure 4(1)(a), $\mathcal{A}(SC_\gamma^4)$ is obtained as in Figure 4(1)(b).*
(2) Given the SC_γ^8 in Figure 4(2)(a), $\mathcal{A}(SC_\gamma^8)$ is obtained as in Figure 4(2)(b) which leads that $\mathcal{A}(SC_\gamma^8)$ separates $(\mathbb{R}^2, \mathcal{U})$ into two components.
(3) Given the SC_γ^{10} in Figure 4(3)(a), $\mathcal{A}(SC_\gamma^{10})$ is obtained as in Figure 4(3)(b).
(4) Consider the SC_γ^{12} in Figure 4(4)(a), $\mathcal{A}(SC_\gamma^{12})$ is obtained as in Figure 4(4)(b). Note that in Figure 4(4)(b) the set $A_{p_1} \cup A_{p_2}$ is connected in $(\mathbb{R}^2, \mathcal{U})$.
However, as for the SC_γ^{12} in Figure 4(4)(c), we see $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{12})$ has three non-empty components because $C_{q_1} \neq C_{q_2}$ as in Figure 4(4)(c),(d), where $q_1 = (1, 0)$ and $q_2 = (2, 1)$ (see Definition 5.1).
(5) Given the SC_γ^{12} in Figure 4(5)(a), $\mathcal{A}(SC_\gamma^{12})$ is obtained, as in Figure 4(5)(b) to show that $\mathcal{A}(SC_\gamma^{12})$ separate $(\mathbb{R}^2, \mathcal{U})$ into exactly two disjoint components.

In view of Proposition 5.6, we have the following query.
Under what condition, does $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ have exactly two non-empty components?

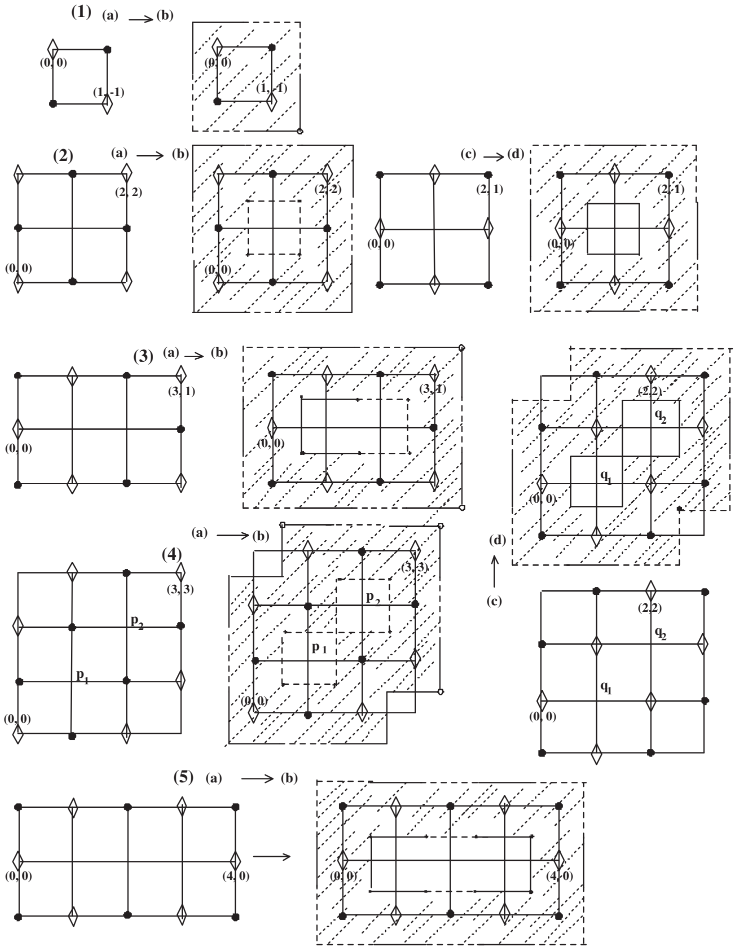

Figure 4. Given several types of SC^l_γ, configuration of $\mathcal{A}(SC^l_\gamma)$ according to the given $SC^l_\gamma, l \in \{4, 8, 10, 12\}$ mentioned in Proposition 5.6. More precisely, (1) $SC^4_\gamma \to \mathcal{A}(SC^4_\gamma)$ (2) $SC^8_\gamma \to \mathcal{A}(SC^8_\gamma)$ (3) $SC^{10}_\gamma \to \mathcal{A}(SC^{10}_\gamma)$ (4) Two types of processes for obtaining $\mathcal{A}(SC^{12}_\gamma)$ from the non-semi-closed SC^{12}_γ of (a) and the semi-closed SC^{12}_γ of (c) (5) $SC^{12}_\gamma \to \mathcal{A}(SC^{12}_\gamma)$.

Theorem 5.7. *Assume an $SC^l_\gamma := (d_i)_{i\in[0,l-1]_\mathbb{Z}}, l \neq 4$, satisfying the following property:*

$$\left\{\begin{array}{l} \textit{There are no distinct elements } d_{t_1}, d_{t_2} \textit{ in } SC^l_\gamma \cap (\mathbb{Z}^2)_e \\ \textit{such that } d_{t_2} \in N_8(d_{t_1}) \textit{ and } Con(d_{t_1}) \cap \{d_{t_2}\} = \emptyset, \\ \textit{where } Con(d_{t_1}) \textit{ is the connected maximal subset of } N_8(d_{t_1}) \cap SC^l_\gamma \\ \textit{containing the point } d_{t_1}. \end{array}\right\} \tag{5.2}$$

(1) Then $\mathcal{A}(SC^l_\gamma)$ separates $(\mathbb{R}^2, \mathcal{U})$ into exactly two components in $(\mathbb{R}^2, \mathcal{U})$ that are both semi-open and semi-closed.
(2) One of the components of $\mathbb{R}^2 \setminus \mathcal{A}(SC^l_\gamma)$ is bounded and the other is unbounded in $\mathbb{R}^2$.

Before proving the assertion, we need to recognize that the hypothesis requires that SC^l_γ always satisfies the property of (5.2). In particular, we need to focus on the part "$SC^l_\gamma \cap (\mathbb{Z}^2)_e$" of (5.2).

Proof: Owing to the hypothesis, assume an $SC_\gamma^l := (d_i)_{i\in[0,l-1]_\mathbb{Z}}, l \neq 4$, that does not contain the subset

$$X_1 := \{d_{t_1}, d_{t_2}\}(\subset SC_\gamma^l \cap (\mathbb{Z}^2)_e) \tag{5.3}$$

having the following property:

$$d_{t_2} \in N_8(d_{t_1}) \text{ and } Con(d_{t_1}) \cap \{d_{t_2}\} = \emptyset,$$

where $Con(d_{t_1})$ is the connected maximal subset of $N_8(d_{t_1}) \cap SC_\gamma^l$ containing the point d_{t_1}. Then, owing to the notion of (5.1) and the features of SC_γ^l, $\mathcal{A}(SC_\gamma^l)$ separates $(\mathbb{R}^2, \mathcal{U})$ into exactly two both semi-open and semi-closed components in $(\mathbb{R}^2, \mathcal{U})$.

For instance, consider the case of SC_γ^{12} given in Figure 4(4)(a). Then we obtain $\mathcal{A}(SC_\gamma^{12})$ to separate $(\mathbb{R}^2, \mathcal{U})$ into exactly two both semi-open and semi-closed components in $(\mathbb{R}^2, \mathcal{U})$. More precisely, as in Figure 4(4)(a),(b), owing to the property of (5.1), the set $A_{p_1} \cup A_{p_2}$ is a connected subset of $(\mathbb{R}^2, \mathcal{U})$.

Meanwhile, without the hypothesis, we can consider the following case. Assume an $SC_\gamma^l := (d_i)_{i\in[0,l-1]_\mathbb{Z}}, l \neq 4$, in which there is a subset $X_1(\subset SC_\gamma^l \cap (\mathbb{Z}^2)_e)$ of (5.3) such that $d_{t_2} \in N_8(d_{t_1})$ and $Con(d_{t_1}) \cap \{d_{t_2}\} = \emptyset$. To be specific, see the two points r_1 and r_2 in Figure 5(a). Then, $\mathcal{A}(SC_\gamma^l)$ does not separate $(\mathbb{R}^2, \mathcal{U})$ into exactly two both semi-open and semi-closed components in $(\mathbb{R}^2, \mathcal{U})$. As another example, see the objects in Figure 4(4)(c),(d). To be specific, in Figure 4(4)(c),(d), the set A_{q_1} is not connected with A_{q_2} (see the property of (5.1)).
(2) With the hypothesis, it is clear that one of the components of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ is bounded and the other is unbounded in $\mathbb{R}^2$.

Example 5.2. *Consider the object SC_γ^{18} in Figure 5(c). Then we observe that the SC_γ^{18} of Figure 5(c) satisfies the property of (5.2) and $\mathcal{A}(SC_\gamma^{18})$ separates $(\mathbb{R}^2, \mathcal{U})$ into exactly two components such as*

$$C(q_1) \textit{ and } \mathbb{R}^2 \setminus (C(q_1) \cup \mathcal{A}(SC_\gamma^{18}))$$

because $C(q_1) = C(q_2) = C(q_3) = C(q_4)$ in $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{18})$ of Figure 5(d).

Theorem 5.8. *Assume the subspaces (X, γ_X) and (Y, γ_Y) that are MW-homeomorphic to SC_γ^l. Then the number of the components of $\mathbb{R}^2 \setminus \mathcal{A}(X)$ need not be equal to that of $\mathbb{R}^2 \setminus \mathcal{A}(Y)$.*

Proof: It suffices to prove it by suggesting a counterexample. Given the two types of SC_γ^{18} as in Figure 5(a),(c), we obtain the corresponding two types of $\mathcal{A}(SC_\gamma^{18})$ as in Figure 5(b),(d).
For our purpose, let A_1 (resp. A_2) be the set $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{18})$, where SC_γ^{18} is the object in Figure 5(a) (resp. Figure 5(c)). Then it is clear that A_1 has four components and A_2 has the only two components as in Figure 5(b),(d).

Remark 5.9. *The number the components of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ need not be equal to that of $\mathbb{Z}^2 \setminus SC_\gamma^l$.*

Proof: As a counterexample, see the objects in Figure 5(c),(d). As shown in Figure 5(c), consider the given SC_γ^{18} in Figure 5(c). While $\mathbb{Z}^2 \setminus SC_\gamma^{18}$ has four components as in Figure 5(c) in $(\mathbb{Z}^2, \gamma)$ such as

$$C(q_i) = \{q_i\}, i \in \{1,2\}, C(q_3) = \{q_3, q_4\}, \text{ and } \mathbb{Z}^2 \setminus (\cup_{i\in[1,3]_\mathbb{Z}} C(q_i) \cup SC_\gamma^{18}),$$

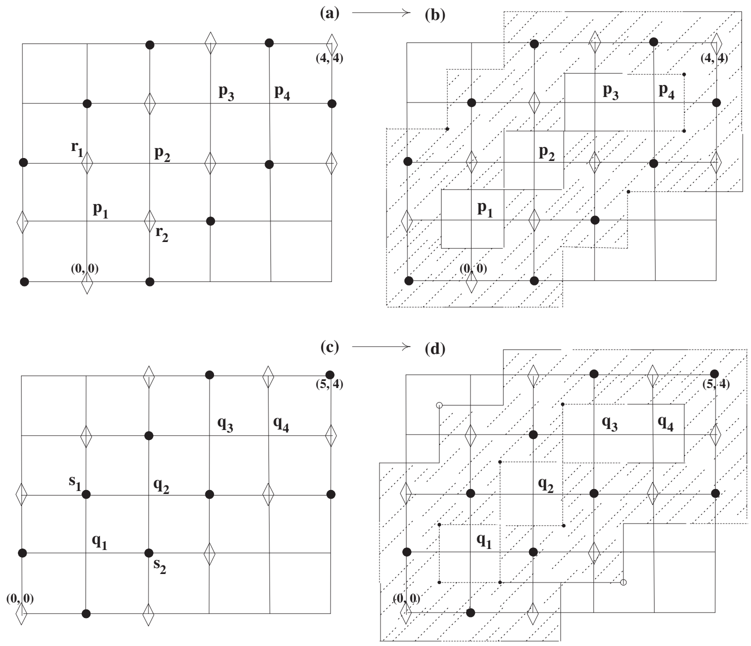

Figure 5. Assume the two types of SC_γ^{18} in (a) and (c) in Example 5.2. The set in (b) is the set $\mathcal{A}(SC_\gamma^{18})$ obtained from the object of (a) and the set $\mathcal{A}(SC_\gamma^{18})$ in (d) is derived from the object of (c).

the set $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{18})$ has exactly two components as in Figure 5(d) because $C(q_1) = C(q_2) = C(q_3)$ as the subspaces of $(\mathbb{R}^2, \mathcal{U})$, i.e., $C(q_1)$ is connected with $C(q_2)$ and $C(q_2)$ is also connected with $C(q_3)$. To be specific, see the cases of SC_γ^{18} in Figure 5(c) and $\mathcal{A}(SC_\gamma^{18})$ in Figure 5(d) stated in Example 5.2, which completes the proof.

Lemma 5.10. *Assume an $SC_\gamma^l := (d_i)_{i\in[0,l-1]_\mathbb{Z}}, l \in \{4, 8, 10\}$. Then the number of the components of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ in $(\mathbb{R}^2, \mathcal{U})$ is equal to that of $\mathbb{Z}^2 \setminus SC_\gamma^l$ in $(\mathbb{Z}^2, \gamma)$.*

Proof: (Case 1) Consider an $SC_\gamma^4 := (d_i)_{i\in[0,3]_\mathbb{Z}}$. As proved in Proposition 5.6, $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^4)$ in $(\mathbb{R}^2, \mathcal{U})$ has one component. Besides, $\mathbb{Z}^2 \setminus SC_\gamma^4$ also has one component in $(\mathbb{Z}^2, \gamma)$.
(Case 2) Consider an $SC_\gamma^8 := (d_i)_{i\in[0,7]_\mathbb{Z}}$. As proved in Proposition 5.6, $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^8)$ in $(\mathbb{R}^2, \mathcal{U})$ has exactly two components. Besides, $\mathbb{Z}^2 \setminus SC_\gamma^8$ also has exactly two components in $(\mathbb{Z}^2, \gamma)$.
(Case 3) Consider an $SC_\gamma^{10} := (d_i)_{i\in[0,9]_\mathbb{Z}}$. As proved in Proposition 5.6, $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{10})$ in $(\mathbb{R}^2, \mathcal{U})$ has exactly two components. Besides, $\mathbb{Z}^2 \setminus SC_\gamma^{10}$ also has exactly two components in $(\mathbb{Z}^2, \gamma)$.

Unlike Lemma 5.10, owing to Remark 5.9, let us now find a condition for comparing between the number of the components of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ in $(\mathbb{R}^2, \mathcal{U})$ and that of $\mathbb{Z}^2 \setminus SC_\gamma^l$ in $(\mathbb{Z}^2, \gamma)$, as follows:

Proposition 5.11. *Assume an $SC_\gamma^l := (c_i)_{i\in[0,l-1]_\mathbb{Z}}, l \notin \{4, 8, 10\}$, having the following property:*

$$\left\{\begin{array}{l} \textit{There are no distinct elements } c_{t_1}, c_{t_2} \textit{ in } SC_\gamma^l \cap (\mathbb{Z}^2)_o \\ \textit{such that } c_{t_2} \in N_8(c_{t_1}) \textit{ and } Con(c_{t_1}) \cap \{c_{t_2}\} = \emptyset, \\ \textit{where } Con(c_{t_1}) \textit{ is the connected maximal subset of } N_8(c_{t_1}) \cap SC_\gamma^l \\ \textit{containing the point } c_{t_1}. \end{array}\right\} \tag{5.4}$$

Then the number of the components of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ *in* $(\mathbb{R}^2, \mathcal{U})$ *is equal to that of* $\mathbb{Z}^2 \setminus SC_\gamma^l$ *in* $(\mathbb{Z}^2, \gamma)$.

Proof: First of all, without the hypothesis, we need to show that the assertion does not hold. Suppose an $SC_\gamma^l := (c_i)_{i\in[0,l-1]_\mathbb{Z}}, l \notin \{4, 8, 10\}$, that does not satisfy the property of (5.4). For instance, as shown in Figure 5(c),(d), the given SC_γ^{18} in Figure 5(c) does not satisfy the property of (5.4), i.e., see the two points s_1 and s_2 in Figure 5(c). Based on the SC_γ^{18}, we find that $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{18})$ has only two components in $(\mathbb{R}^2, \mathcal{U})$ (see Example 5.2). Meanwhile, $\mathbb{Z}^2 \setminus SC_\gamma^{18}$ has four components in $(\mathbb{Z}^2, \gamma)$.
Next, owing to the property of (5.1), the topological feature of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ in $(\mathbb{R}^2, \mathcal{U})$, and that of $\mathbb{Z}^2 \setminus SC_\gamma^l$ in $(\mathbb{Z}^2, \gamma)$, the proof is completed.

Example 5.3. *Let us consider* SC_γ^{18} *in Figure 5(a). Then it is clear that it satisfies the property of (5.4) so that we obtain* $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{18})$ *consisting of four components in* $(\mathbb{R}^2, \mathcal{U})$ *(see Figure 5(b)) and* $\mathbb{Z}^2 \setminus SC_\gamma^{18}$ *is composed of four components in* $(\mathbb{Z}^2, \gamma)$ *(see Figure 5(a)).*

6. Semi-Jordan curve theorem on the MW-topological plane

In 1970, Rosenfeld [9–11] initially considered the digital topological version of the typical Jordan curve theorem (see also [22]). Consider $SC_k^{2,l}$ in a binary digital picture $D := (\mathbb{Z}^2, k, \bar{k}, SC_k^{2,l})$. Then the $\bar{k}$-components are called white components of D and $SC_k^{2,l}$ is said to be a black component (or equivalently, k-component) of the digital picture [22], where we say that a k-component of a non-empty digital image (X, k) is a maximal k-connected subset of (X, k) [22]. To be precise, given an $SC_8^{2,4}$ on $\mathbb{Z}^2$, to evade from the so-called "digital connectivity paradox" [10, 11], the papers [7–9] considered it in a binary digital picture $D := (\mathbb{Z}^2, 8, 4, SC_8^{2,4})$ instead of $(\mathbb{Z}^2, 8, 8, SC_8^{2,4})$. In the digital picture $(\mathbb{Z}^2, 8, 4, SC_8^{2,4})$, to avoid the digital connectivity paradox, it turns out that $SC_8^{2,4}$ should be considered to be an $(8, 4)$-binary digital image in the given digital picture D above. Namely, the part $\mathbb{Z}^2 \setminus SC_8^{2,4}$ should be considered with 4-connectivity. Then the given set $SC_8^{2,4}$ separates $\mathbb{Z}^2$ into the two 4-components A and B [22] such that

$$\begin{Bmatrix} \mathbb{Z}^2 \setminus SC_8^{2,4} = A \cup B \text{ such that } A \cap B = \emptyset \text{ and further,} \\ \text{each element of } A \text{ is not 4-adjacent to that of } B. \end{Bmatrix} \tag{6.1}$$

Based on this approach, it is clear that one of the sets A and B is finite and the other is infinite. Then we call a finite set A the interior of the given set $B := SC_8^{2,4} \cup A$ in the digital picture $(\mathbb{Z}^2, 8, 4, B)$. Similarly, as a general case of $SC_8^{2,4}$, we can consider $SC_k^{2,l} (\neq SC_4^{2,4})$ in the digital picture $D := (\mathbb{Z}^2, k, \bar{k}, SC_k^{2,l})$ as mentioned above, where $(k, \bar{k}) \in \{(4, 8), (8, 4)\}$. Then, it also separates $\mathbb{Z}^2$ into two non-empty $\bar{k}$-components A and B such that

$$\begin{Bmatrix} \mathbb{Z}^2 \setminus SC_k^{2,l} = A \cup B \text{ such that } A \cap B = \emptyset \text{ and further,} \\ \text{each element of } A \text{ is not } \bar{k}\text{-adjacent to that of } B. \end{Bmatrix} \tag{6.2}$$

Meanwhile, unlike this approach followed from Rosenfeld's work, in the category of MW-topological spaces, given an SC_γ^l in $(\mathbb{Z}^2, \gamma)$, we now raise the following queries.
(Q1) How can we propose an MW-topological version of the typical Jordan curve theorem?
(Q2) What differences are there between an MW-topological version of the well-known Jordan curve theorem and the typical Jordan curve theorem on $(\mathbb{R}^2, \mathcal{U})$?

(Q3) What differences are there between the Jordan curve theorem in an MW-topological setting and the digital Jordan curve theorem established by Rosenfeld?

The paper [12] also studied several types of digital Jordan curve theorems with nine pretopologies on $\mathbb{Z}^2$. Besides, the paper [3] also proposed a computational topological version of the curve and surface theorem. In addition, there are some studies on the digital versions of the Jordan curve theorem in digital spaces including the papers [6, 12, 22]. However, to study this topic more intensively from the viewpoint of the MW-topology, we strongly need to have an approach using semi-topological structures. To study some properties of the semi-closedness or semi-openness of $\mathbb{Z}^2 \setminus SC^l_\gamma$, we first recall that SC^4_γ does not separate $(\mathbb{Z}^2, \gamma)$ into exactly two non-empty components (see Figure 1(1)(a),(b)). Furthermore, we have the following (see Figure 4(1)(a), (2)(a), (3)(a)).

Remark 6.1. *$\mathbb{Z}^2 \setminus SC^4_\gamma$ has the only one non-empty component that is both semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$.*

Lemma 6.2. *(1) SC^8_γ separates $(\mathbb{Z}^2, \gamma)$ into exactly two semi-closed components. One of these components need not be semi-open in $(\mathbb{Z}^2, \gamma)$.*
(2) SC^{10}_γ separates $(\mathbb{Z}^2, \gamma)$ into exactly two components that are both semi-open and semi-closed.

Proof: Based on the SC^8_γ in Figure 1(2)(a),(b), and SC^{10}_γ in Figure 1(3)(a),(b), by Theorems 3.3 and 3.4, the proof is clearly completed. In particular, note that the components of $\mathbb{Z}^2 \setminus SC^8_\gamma$ are obviously semi-closed in $(\mathbb{Z}^2, \gamma)$. Indeed, in the case of SC^8_γ in Figure 1(2)(a), one of them is not semi-open. To be specific, consider SC^8_γ in Figure 1(2)(a), the finite component of $\mathbb{Z}^2 \setminus SC^8_\gamma$ is not semi-open. Meanwhile, the components of $\mathbb{Z}^2 \setminus SC^{10}_\gamma$ are both semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$ (see Theorems 3.3 and 3.4).

Unlike the case of $SC^l_\gamma := (c_i)_{i \in [0, l-1]_\mathbb{Z}}, l \in \{4, 8, 10\}$, motivated by Theorem 5.7 and Proposition 5.11, we obtain the following:

Proposition 6.3. *Assume $SC^l_\gamma := (c_i)_{i \in [0, l-1]_\mathbb{Z}}, l \neq 4$, satisfying the following property:*

$$\left\{\begin{array}{l} \textit{There are no distinct elements } c_{t_1}, c_{t_2} \textit{ in } SC^l_\gamma \\ \textit{such that } c_{t_2} \in N_8(c_{t_1}) \textit{ and } Con(c_{t_1}) \cap \{c_{t_2}\} = \emptyset, \\ \textit{where } Con(c_{t_1}) \textit{ is the connected maximal subset of } N_8(c_{t_1}) \cap SC^l_\gamma \\ \textit{containing the point } c_{t_1}. \end{array}\right\} \tag{6.3}$$

Then SC^l_γ separates $(\mathbb{Z}^2, \gamma)$ into exactly two semi-closed components, e.g., A and B. Namely, a partition $\{A, B, SC^l_\gamma\}$ of $(\mathbb{Z}^2, \gamma)$ exists.

Proof: First of all, we need to strongly point out an importance of the given hypothesis. Without the hypothesis, as shown in Figure 4(4)(a),(c), since the given SC^{12}_γ in Figure 4(4)(a),(c) do not satisfy the hypothesis of (6.3), they do not separate $(\mathbb{Z}^2, \gamma)$ into exactly two components. Besides, some components of $\mathbb{Z}^2 \setminus SC^{12}_\gamma$ cannot be semi-open. To be specific, in the SC^{12}_γ of Figure 4(4)(a), each of $C(p_i) = \{p_i\}, i \in \{1, 2\}$, is not semi-open in $(\mathbb{Z}^2, \gamma)$.

As another case, since the given SC^{12}_γ in Figure 4(4)(c) does not satisfy the property of (6.3) either, it does not separate $(\mathbb{Z}^2, \gamma)$ into exactly two components. Besides, note that none of the objects in Figure 5(a),(c), Figure 6, and Figure 7(1),(2) satisfies the property of (6.3) either.

In addition, the condition of (6.3) does not support the semi-openness of the component of $\mathbb{Z}^2 \setminus SC^l_\gamma$.

For instance, consider SC_γ^8 in Figure 1(2)(a), one component of $\mathbb{Z}^2 \setminus SC_\gamma^8$ is not semi-open in $(\mathbb{Z}^2, \gamma)$ as mentioned in the proof of Lemma 6.2.

Meanwhile, with the hypothesis of (6.3), owing to the features of SC_γ^l, it is clear that SC_γ^l separates $(\mathbb{Z}^2, \gamma)$ into exactly two components (see Figure 1(4)(c),(d)) which are semi-closed in $(\mathbb{Z}^2, \gamma)$.

Owing to Remark 6.1, based on Theorem 5.7, we can define the notions of Definition 6.4 below because given an SC_γ^l satisfying the hypothesis of (5.2), $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ has exactly two components in $(\mathbb{R}^2, \mathcal{U})$ and further, one of them is bounded and the other is unbounded (see the cases of SC_γ^{38} in Figure 7(1) and SC_γ^{28} in Figure 7(3)).

Definition 6.4. *Assume an SC_γ^l satisfying the property of (5.2), $l \neq 4$. Then we define the following two notions.*
(1) $I(SC_\gamma^l) := B(\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)) \cap \mathbb{Z}^2$, where $B(\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l))$ means the bounded component of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$.
(2) $O(SC_\gamma^l) := Ub(\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)) \cap \mathbb{Z}^2$, where $Ub(\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l))$ stands for the unbounded component of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$.

Comparing the condition of Proposition 6.3 and that of Definition 6.4, we can note that the former is stronger than the latter. Hereinafter, the two notions $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$ in Definition 6.4 are called "inside" and "outside" of SC_γ^l in $(\mathbb{Z}^2, \gamma)$, respectively. In particular, note that these notions are not related to the notions of interior and exterior of a set of $(\mathbb{Z}^2, \gamma)$.

Remark 6.5. *(1) In Definition 6.4, the hypothesis of (5.2) is strongly required to establish both $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$ because it supports the assertion of Theorem 5.7 so that the set $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^l)$ has exactly two components of which one of them is bounded and the other is unbounded.*
(2) Without the hypothesis of (5.2), we have some difficulties in establishing the notions of $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$. For instance, consider the case of SC_γ^{42} in Figure 6. In particular, see the points c_2 and c_{38}, and c_4 and c_{36}. Then, owing to these points, it is clear that this SC_γ^{42} does not satisfy the property of (5.2). Hence we have some difficulties in establishing $I(SC_\gamma^{42})$ because $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{42})$ does not have exactly two components. As another case, consider the case of SC_γ^{28} in Figure 7(2). In particular, consider the two points c_{19} and c_{25}. Then they clearly does not satisfy the property of (5.2) so that $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{28})$ has three components, e.g., two bounded components and one unbounded component. More precisely, since the set A_{q_2} is a bounded component of $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{28})$, A_{q_2} is not related to the set $Ub(\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{28})) \cap \mathbb{Z}^2$ which comes across some difficulties in establishing $O(SC_\gamma^{28})$.
(3) As a good example for Definition 6.4, consider the SC_γ^{38} in Figure 7(1). First, see the two points d_{29} and d_{35} of the SC_γ^{38} in Figure 7(1). Then they can be admissible to establish $O(SC_\gamma^{38})$. Besides, see the two points d_{11} and d_{17} of the SC_γ^{38} in Figure 7(1). Then they can be admissible to establish the notion of $I(SC_\gamma^{38})$ because A_{q_1} is connected with A_{q_2}.
Similarly, the SC_γ^{28} in Figure 7(3) also a good example for Definition 6.4.

Example 6.1. *(1) As for the SC_γ^{42} in Figure 6, it does not satisfy the property of (5.2) (see the points c_2 and c_{38}, and c_4 and c_{36}). Hence, as mentioned in Theorem 5.7, $\mathcal{A}(SC_\gamma^{42})$ does not separate $(\mathbb{R}^2, \mathcal{U})$ into two components. Indeed, $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{42})$ consists of four components as follows: $C(p_i) = A_{p_i}, i \in [1,2]_\mathbb{Z}$, $C(p_3) = C(p_4)$ and $C(q_1) = C(q_2)$.*
Furthermore, the set $\mathbb{Z}^2 \setminus SC_\gamma^{42}$ has six components such as $C(p_i), i \in [1,4]_\mathbb{Z}$ and $C(q_j), j \in [1,2]_\mathbb{Z}$.
(2) Given the SC_γ^{38} in Figure 7(1), it satisfies the property of (5.2). Hence the set $\mathbb{R}^2 \setminus \mathcal{A}(SC_\gamma^{38})$ consists

of the exactly two components. Hence we obtain

$$I(SC_\gamma^{38}) = C(q_1) \cup C(q_2) \text{ and } O(SC_\gamma^{38}) = C(q_3) \cup C(q_4), \text{ where}$$

$C(q_i)$ is the component of q_i in $(\mathbb{Z}^2, \gamma)$ containing the given point q_i, $i \in [1,4]_\mathbb{Z}$.
Meanwhile, the set $\mathbb{Z}^2 \setminus SC_\gamma^{38}$ consists of four components in $(\mathbb{Z}^2, \gamma)$, e.g., $C(q_i), i \in [1,4]_\mathbb{Z}$.
(3) Given the SC_γ^{28} in Figure 7(2), it does not satisfy the property of (5.2). Indeed, the set $\mathbb{Z}^2 \setminus SC_\gamma^{28}$ consists of three components as follows: $C(q_i), i \in [1,3]_\mathbb{Z}$ and $C(q_1)$ consists of eleven elements as in Figure 7(2). In particular, $C(q_2) = \{q_2\}$ and $(C(q_3))^\sharp = \aleph_0$ that is the cardinal number of the set of natural numbers.
(4) Given the SC_γ^{28} in Figure 7(3), it is clear that this object satisfies the hypothesis of Definition 6.4. Hence we obtain $I(SC_\gamma^{28}) = C(q_1)$ and $O(SC_\gamma^{28}) = C(q_2) \cup C(q_3)$.

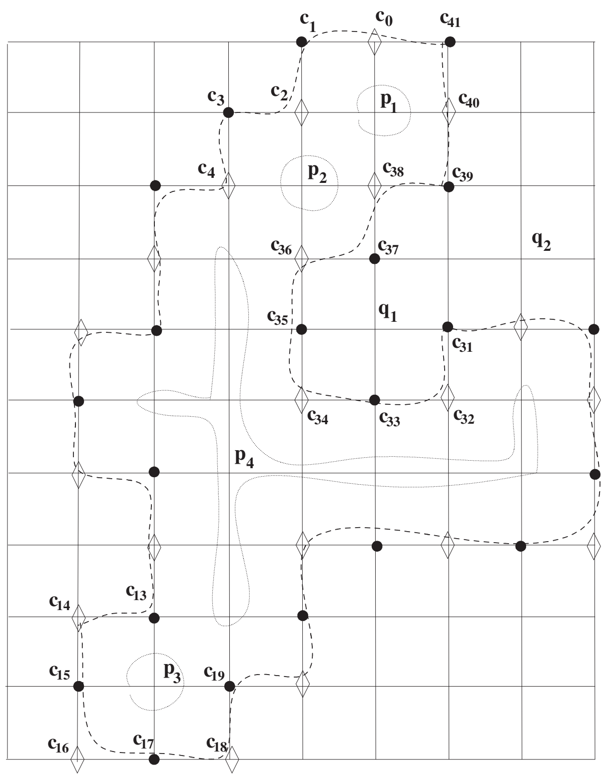

Figure 6. In $(\mathbb{Z}^2, \gamma)$, based on the non-satisfaction of the property of (5.2) of SC_γ^{42}, there are some difficulties in establishing $I(SC_\gamma^{42})$.

Owing to Definition 6.4, we have the following:

Theorem 6.6. *Assume an SC_γ^l, $l \neq 4$, satisfying the property of (5.2).*
Then, a partition $\{I(SC_\gamma^l), O(SC_\gamma^l), SC_\gamma^l\}$ of $\mathbb{Z}^2$ exists.

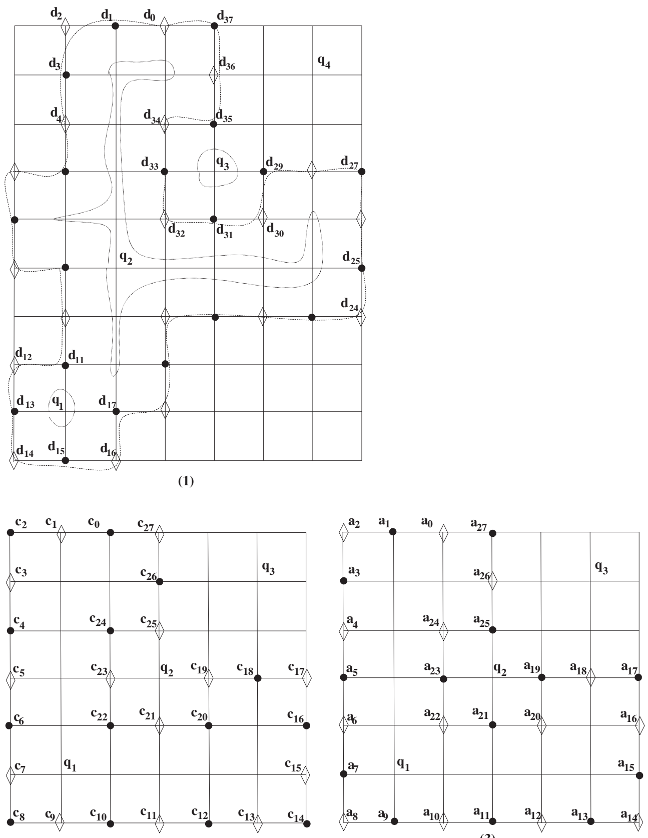

Figure 7. In $(\mathbb{Z}^2, \gamma)$, (1) based on the $SC_\gamma^{38} := (d_i)_{i\in[0,37]_\mathbb{Z}}$ satisfying the property of (5.2), $\mathbb{Z}^2 \setminus SC_\gamma^{38}$ has four non-empty components which implies that $I(SC_\gamma^{38}) = C(q_1) \cup C(q_2)$ and $O(SC_\gamma^{38}) = C(q_3) \cup C(q_4)$ such that $C(q_1) \cap C(q_2) = \emptyset$ and $C(q_3) \cap C(q_4) = \emptyset$, where $C(q_4) = \mathbb{Z}^2 \setminus (I(SC_\gamma^{38}) \cup C(q_3) \cup SC_\gamma^{38})$ and each of $C(q_i), i \in [1, 4]_\mathbb{Z}$, is not an empty set.
(2) Based on the $SC_\gamma^{28} := (c_i)_{i\in[0,27]_\mathbb{Z}}$ in Figure 7(2), $\mathbb{Z}^2 \setminus SC_\gamma^{28}$ indeed has three components. However, since it does not satisfy the property of (5.2), both $I(SC_\gamma^{28})$ and $O(SC_\gamma^{28})$ are not considered.

Proof: Owing to Definition 6.4, the proof is completed.

Remark 6.7. *(1) Each of $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$ need not be connected.*
(2) The number of the components of $I(SC_\gamma^l)$ depends on the situation.
(3) Each of the sets $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$ need not be an open set or a closed set in $(\mathbb{Z}^2, \gamma)$.

Proof: (1) As an example, consider the SC_γ^{38} in Figure 7(1). It is clear that the given SC_γ^{38} satisfies the hypothesis of Definition 6.4. While each of $I(SC_\gamma^{38})$ and $O(SC_\gamma^{38})$ exists, they are not connected in $(\mathbb{Z}^2, \gamma)$. To be specific, it turns out that

$$I(SC_\gamma^{38}) = C(q_1) \cup C(q_2) \text{ and } O(SC_\gamma^{38}) = C(q_3) \cup C(q_4)$$

such that $C(q_1) \cap C(q_2) = \emptyset$, $C(q_3) \cap C(q_4) = \emptyset$, and each of $C(q_i), i \in [1, 4]_\mathbb{Z}$ is not an empty set.
(2) From the above (3) of Example 6.1, it is clear that the number of the components of $I(SC_\gamma^l)$ depends

on the number l.
(3) As an example, consider the SC_γ^8 in Figure 4(2)(a). Then $I(SC_\gamma^8)$ is not an open set. Besides, consider the SC_γ^{38} in Figure 7(1). Then $O(SC_\gamma^{38})$ is not an open set.

Let us now investigate the semi-openness or semi-closedness of $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$.

Example 6.2. *Given any SC_γ^8, there is a partition of $(\mathbb{Z}^2, \gamma)$, i.e., $\{I(SC_\gamma^8), SC_\gamma^8, O(SC_\gamma^8)\}$ such that each of $I(SC_\gamma^8)$ and $O(SC_\gamma^8)$ is semi-closed and both of them are connected. Namely, SC_γ^8 separates $(\mathbb{Z}^2, \gamma)$ with exactly two components.*

Lemma 6.8. *$I(SC_\gamma^{10})$ is both semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$.*

Proof: As shown in Figure 1(4)(c),(d), owing to Theorems 3.3 and 3.4, $I(SC_\gamma^8)$ is clearly both semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$.

Remark 6.9. *The semi-openness or semi-closedness of $I(SC_\gamma^l)$ depends on the situation, $l \notin \{4, 10\}$.*

Proof: The semi-topological features of $I(SC_\gamma^8)$ is determined according to the two types of SC_γ^8 in Figure 1(2)(a),(b). To be specific, based on the SC_γ^8 in Figure 1(2)(a), we have $I(SC_\gamma^8)$ that is only semi-closed instead of semi-open in $(\mathbb{Z}^2, \gamma)$. Meanwhile, for the case of SC_γ^8 in Figure 1(2)(b), $I(SC_\gamma^8)$ is proved to be both semi-open and semi-closed in $(\mathbb{Z}^2, \gamma)$.
Next, let us consider the case SC_γ^l, $12 \leq l \in \mathbb{N}_e$. Then the semi-openness or semi-closedness of $I(SC_\gamma^l)$ depends on the situation (see Figure 5(c) and 7(1),(3)). For instance, the SC_γ^{18} in Figure 5(c) has $I(SC_\gamma^{18})$ that is not semi-open but semi-closed.

Corollary 6.10. *Let X and Y be simple closed MW-curves with l elements in $(\mathbb{Z}^2, \gamma)$ and each of them satisfies the property of (6.3). Then the number of the components of X^c is equal to that of Y^c.*

Proof: By Proposition 6.3, the proof is completed.

Let E be the SC_γ^{28} in Figure 7(3) and F be the SC_γ^{28} satisfying the property of (6.3). While $O(E)$ is not connected and $O(F)$ is connected. Thus we obtain the following.

Remark 6.11. *Without the condition relating to the property (6.3), each of $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$ may not be connected.*

Theorem 6.12. *Assume that the subspaces (X, γ_X) and (Y, γ_Y) are MW-homeomorphic to SC_γ^l.*
(1) $(I(X), \gamma_{I(X)})$ need not MW-homeomorphic to $(I(Y), \gamma_{I(X)})$.
(2) $(O(X), \gamma_{O(X)})$ need not MW-homeomorphic to $(O(Y), \gamma_{O(X)})$.

Proof: To disprove these assertions (1) and (2), we will use some examples.
(1) Consider the two SC_γ^{12} in Figures 4(4)(a) and 5(a). For our purpose, let A be the SC_γ^{12} in Figure 4(4)(a) and B be the SC_γ^{12} in Figure 4(5)(a). While $I(A)$ is not connected and $I(B)$ is connected, which completes the proof.
(2) For our purpose, let C be the SC_γ^{28} in Figure 7(3) and D be the SC_γ^{28} satisfying the property of (6.3). While $O(C)$ is not connected and $O(D)$ is connected, which completes the proof.

Corollary 6.13. *(1) $O(SC_\gamma^l)$ need not be semi-open.*
(2) The number of the components of $O(SC_\gamma^l)$ need not depend on the number l.

Proof: To disprove these assertions (1) and (2), we will use some examples.
(1) Let us consider the SC_γ^{28} in Figure 7(3). Then it is clear that $O(SC_\gamma^{28})$ is not semi-open.
(2) In view of the SC_γ^{28} in Figure 7(3), the proof is completed. To be specific, let A be the SC_γ^{28} in Figure 7(3) and B be the SC_γ^{28} satisfying the property of (6.3). Then $O(A)$ has two components and $O(B)$ has the only one component.

Unlike Theorem 6.6, let us find a condition to separate $(\mathbb{Z}^2, \gamma)$ into exactly two components, as follows:

Theorem 6.14. *Let SC_γ^l satisfy the property of (6.3), $l \neq 4$. Under $(\mathbb{Z}^2, \gamma)$, we obtain the following:*
(1) There is a partition $\{I(SC_\gamma^l), O(SC_\gamma^l), SC_\gamma^l\}$ such that each of $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$ is connected.
(2) Each of $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$ is semi-closed. While $O(SC_\gamma^l)$ is semi-open, $I(SC_\gamma^l)$ need not be semi-open.

Proof: (1) It suffices to prove the connectedness of both $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$. Owing to the hypothesis of the property of (6.3), each of $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$ is proved to be connected because there are not two points

$$\begin{Bmatrix} c_{t_1}, c_{t_2} \in SC_\gamma^l := (c_i)_{i\in[0,l-1]_\mathbb{Z}}, l \neq 4 \text{ such that} \\ c_{t_2} \in N_8(c_{t_1}) \text{ and } Con(c_{t_1}) \cap \{c_{t_2}\} = \emptyset. \end{Bmatrix} \tag{6.4}$$

Indeed, this property of (6.4) supports the connectedness of both $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$.
(2) With the hypothesis, using Theorems 3.3 and 3.4, we prove that both $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$ are semi-closed. To be specific, for any point $x \in \mathbb{Z}^2 \setminus I(SC_\gamma^l)$, owing to the hypothesis, we always obtain

$$SN_\gamma(x) \cap (\mathbb{Z}^2 \setminus I(SC_\gamma^l))_{op} \neq \emptyset,$$

which implies the semi-closedness of $I(SC_\gamma^l)$.
Similarly, for any point $x \in \mathbb{Z}^2 \setminus O(SC_\gamma^l)$, owing to the hypothesis, we also obtain

$$SN_\gamma(x) \cap (\mathbb{Z}^2 \setminus O(SC_\gamma^l))_{op} \neq \emptyset,$$

which implies the semi-closedness of $O(SC_\gamma^l)$.
Besides, by Theorem 3.3, owing to the hypothesis, $O(SC_\gamma^l)$ is proved to be semi-open. However, $I(SC_\gamma^l)$ need not be semi-open. For instance, for the SC_γ^8 in Figure 4(2)(a). Then $I(SC_\gamma^8)$ is not a semi-open in $(\mathbb{Z}^2, \gamma)$.

7. Advantages and utilities of MW-topological structure and the semi-Jordan curve theorem on $(\mathbb{Z}^2, \gamma)$

When studying digital objects X in $\mathbb{Z}^2$, the properties of (2.4) and (2.5) enable us to get the following utilities of the MW-topological structure of X.

Remark 7.1. *(Utilities of the MW-topological structure)*
(1) When studying a self-homeomorphism of $(\mathbb{Z}^2, \gamma)$, we should consider the following map [31]

$$\begin{cases} h : (\mathbb{Z}^2, \gamma) \to (\mathbb{Z}^2, \gamma) \text{ defined by:} \\ \text{for each point } x := (x_1, x_2) \in \mathbb{Z}^2, \\ h(x) = (x_1 + t_1, x_2 + t_2), \\ \text{where } t_i \in \mathbb{Z}_o \text{ for each } i \in [1,2]_{\mathbb{Z}}, \text{ or} \\ h(x) = (x_1 + 2m_1, x_2 + 2m_2), \\ \text{for some } m_i \in \mathbb{Z}, i \in M \subset [1,2]_{\mathbb{Z}}. \end{cases} \tag{7.1}$$

Since the modern electronic devices are usually operated on the finite digital planes with more than ten million pixels to support the high-level display resolution, the mapping of (7.1) can be very admissible. At the moment, note that the following map g cannot be a homeomorphism, where

$$\begin{cases} g : (\mathbb{Z}^2, \gamma) \to (\mathbb{Z}^2, \gamma) \text{ defined by:} \\ \text{for each point } x := (x_1, x_2) \in \mathbb{Z}^2, \\ g(x) = (x_1 + t_1, x_2 + t_2) \\ \text{such that there is at least } t_i \in \mathbb{Z}_o, i \in M \subsetneq [1,2]_{\mathbb{Z}}. \end{cases} \tag{7.2}$$

(2) Since the MW-topological structure is one of the fundamental frames, motivated by this structure, some more generalized topological structures on $\mathbb{Z}^n$ *can be established.*
(3) Based on the MW-topological structure of $\mathbb{Z}^2$*, we can obtain the* 4*-digital adjacency induced by the given topological structure [26]. In detail, for distinct elements* $x, y \in (\mathbb{Z}^2, \gamma)$*, they are MW-adjacent if* $x \in SN_\gamma(y)$ *or* $y \in SN_\gamma(x)$ *[26]. Namely, the MW-adjacency is equivalent to the* 4*-adjacency of* $\mathbb{Z}^2$ *as in (2.2).*

Example 7.1. *The map* $g : (\mathbb{Z}^2, \gamma) \to (\mathbb{Z}^2, \gamma)$ *defined by* $g(x_1, x_2) = (x_1 + 2m_1 + 1, x_2 + 2m_2), m_1, m_2 \in \mathbb{Z}$ *cannot be a homeomorphism. Meanwhile, it is clear that the map* $h : (\mathbb{Z}^2, \gamma) \to (\mathbb{Z}^2, \gamma)$ *defined by* $h(x_1, x_2) = (x_1 + 2m_1 + 1, x_2 + 2m_2 + 1)$ *or* $(x_1 + 2m_1, x_2 + 2m_2)$*,* $m_1, m_2 \in \mathbb{Z}$*, is a homeomorphism.*

Remark 7.2. *(Advantages of the semi-Jordan curve theorem)*
(1) Unlike the typical Jordan curve theorem in a digital topological setting estabsihed by Rosenfeld [22], no paradox exists in the semi-Jordan curve theorem in the MW-topological structure.
(2) Based on the semi-Jordan curve theorem in the MW-topological structure, we can consider a digital topological version of the typical Jordan curve theorem in terms of a simple closed 4*-curve in the digital plane* $(\mathbb{Z}^2, 4, 8)$*.*
(3) When digitizing a set X in the 2*-dimensional real space with respect to the MW-topological structure, we can use a local rule in [20] to obtain a digitized set* $D_\gamma(X) \subset \mathbb{Z}^2$ *from X which is used in the fields of mathematical morphology, rough set theory, digital geometry, and so on [20, 21].*

8. Concluding remark and further work

After developing the semi-Jordan curve theorem in the MW-topological setting, we have studied various properties of it. In particular, we have found a condition for SC_γ^l to separate $(\mathbb{Z}^2, \gamma)$ with exactly two components, Furthermore, we studied many semi-topological properties of both $I(SC_\gamma^l)$ and $O(SC_\gamma^l)$. As a further work, we can compare among several kinds of digital versions of the typical

Jordan curve theorem and the combinatorial version of the Jordan curve theorem in [3]. Besides, based on the digital-topological group structure in [32], we can further examine a topological group structure of SC^l_γ. In addition, based on the established structure in [33], we can study covering spaces in the category of MW-topological spaces.

Acknowledgments

The author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2019R1I1A3A03059103). Besides, this work was supported under the framework of international cooperation program managed by the National Research Foundation of Korea (2021K2A9A2A06039864).

Conflict of interest

The author declares no conflict of interest.

References

1. J. R. Munkres, *Topology A First Course*, Prentice-Hall, Inc., 1975.
2. D. Fajardo-Rojas, N. Jonard-Pérez, A Jordan curve theorem for 2-dimensional tillings, *Topol. Appl.*, **300** (2021), 107773. https://doi.org/10.1016/j.topol.2021.107773
3. S. E. Han, Jordan surface theorem for simple closed SST-surfaces, *Topol. Appl.*, **272** (2020), 106953. https://doi.org/10.1016/j.topol.2019.106953
4. G. T. Herman, Oriented surfaces in digital spaces, *CVGIP: Graphical Models Image Proc.*, **55** (1993), 381–396. https://doi.org/10.1006/cgip.1993.1029
5. E. Khalimsky, R. Kopperman, P. R. Meyer, Computer graphics and connected topologies on finite ordered sets, *Topol. Appl.*, **36** (1990), 1–17. https://doi.org/10.1016/0166-8641(90)90031-V
6. C. O. Kiselman, Digital Jordan curve theorem, in *International Conference on Discrete Geometry for Computer Imagery*, (2000), 46–56. https://doi.org/10.1007/3-540-44438-6_5
7. A. Rosenfeld, Connectivity in digital pictures, *J. Assoc. Comput. Math.*, **17** (1970), 146–160. https://doi.org/10.1145/321556.321570
8. A. Rosenfeld, Arcs and curves in digital pictures, *J. Assoc. Comput. Math.*, **20** (1973), 81–87. https://doi.org/10.1145/321738.321745
9. A. Rosenfeld, Adjacency in digital pictures, *Inf. Control*, **26** (1974), 24–33. https://doi.org/10.1016/S0019-9958(74)90696-2
10. A. Rosenfeld, A converse to the Jordan curve theorem for digital curves, *Inf. Control*, **29** (1975), 292–293. https://doi.org/10.1016/S0019-9958(75)90459-3
11. A. Rosenfeld, Digital topology, *Am. Math. Monthly*, **86** (1979), 76–87. https://doi.org/10.1080/00029890.1979.11994873

12. J. Šlapal, A quotient-universal digital topology, *Theor. Comput. Sci.*, **405** (2008), 164–175. https://doi.org/10.1016/j.tcs.2008.06.035
13. N. Levine, Semi-open sets and semi-continuity in topological spaces, *Am. Math. Monthly*, **70** (1963), 36–41. https://doi.org/10.2307/2312781
14. N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo*, **19** (1970), 89–96. https://doi.org/10.1007/BF02843888
15. S. G. Crosseley, A note on semitopological properties, *Proc. Am. Math. Soc.*, **72** (1978), 409–412. https://doi.org/10.1090/S0002-9939-1978-0507348-9
16. T. Hamlett, A correction to the paper "Semi-open sets and semi-continuity in topological spaces" by Norman Levine, *Proc. Am. Math. Soc.*, **49** (1975), 458–460. https://doi.org/10.2307/2040665
17. H. Maki, K. Balachandran, R. Devi, Remarks on semi-generalized closed sets and generalized semi-closed sets, *Kyungpook Math. J.*, **36** (1996), 155–163.
18. A. S. Mashhour, M. E. Abd El-Monsef, S.N. El-Deeb, On precontinuous an weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, **53** (1982), 47–53.
19. T. Noiri, A note on semi-homeomorphism, *Bull. Calcutta Math. Soc.*, **76** (1984), 1–3.
20. S. E. Han, Roughness measures of locally finite covering rough sets, *Int. J. Approx. Reason.*, **105** (2019), 368–385. https://doi.org/10.1016/j.ijar.2018.12.003
21. S. E. Han, W. Yao, Homotopy based on Marcus-Wyse topology and its applications, *Topol. Appl.*, **201** (2016), 358–371. https://doi.org/10.1016/j.topol.2015.12.047
22. T. Y. Kong, A. Rosenfeld, *Topological Algorithms for the Digital Image Processing*, Elsevier Science, Amsterdam, 1996.
23. S. E. Han, Non-product property of the digital fundamental group, *Inf. Sci.*, **171** (2005), 73–91. https://doi.org/10.1016/j.ins.2004.03.018
24. P. Alexandorff, Uber die Metrisation der im Kleinen kompakten topologischen Räume, *Math. Ann.*, **92** (1924), 294–301. https://doi.org/10.1007/BF01448011
25. P. Alexandorff, Diskrete Räume, *Mat. Sb.*, **2** (1937), 501–518.
26. S. E. Han, Generalizations of continuity of maps and homeomorphisms for studying 2D digital topological spaces and their applications, *Topol. Appl.*, **196** (2015), 468–482. https://doi.org/10.1016/j.topol.2015.05.024
27. D. Marcus, F. Wyse, Solution to problem 5712, *Am. Math. Monthly*, **77** (1970), 1119. https://doi.org/10.2307/2316121
28. S. E. Han, Continuities and homeomorphisms in computer topology and their applications, *J. Korean Math. Soc.*, **45** (2008), 923–952. https://doi.org/10.4134/JKMS.2008.45.4.923
29. S. G. Crosseley, S. K. Hildebrand, Semi-closure, *Texas. J. Sci.*, **22** (1971), 99–112.
30. S. E. Han, Low-level separation axioms from the viewpoint of computational topology, *Filomat*, **33** (2019), 1889–1901. https://doi.org/10.2298/FIL1907889H
31. S. E. Han, Semi-topological properties of the Marcus-Wyse topological spaces, *AIMS Math.*, **7** (2022), 12742–12759. https://doi.org/10.3934/math.2022705

32. S. E. Han, Digitally topoloigical groups, *Electron. Res. Arch.*, **30** (2022), 2356–2384. https://doi.org/10.3934/era.2022120

33. S. E. Han, The most refined axiom for a digital covering space, *Mathematics*, **8** (2020), 1868. https://doi.org/10.3390/math8111868

AIMS Press