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Research article

Nonlinear differential equations of fourth-order: Qualitative properties of the solutions

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Abstract: In this paper, we study the oscillation of solutions for a fourth-order neutral nonlinear differential equation, driven by a p-Laplace differential operator of the form

$$\begin{cases} \left(r(t)\,\Phi_{p_1}[w'''(t)]\right)' + q(t)\,\Phi_{p_2}(u(\vartheta(t))) = 0, \\ r(t) > 0,\ r'(t) \geq 0,\ t \geq t_0 > 0, \end{cases}$$

The oscillation criteria for these equations have been obtained. Furthermore, some examples are given to illustrate the criteria.

Keywords: fourth-order differential equations; neutral delay; oscillation
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1. Introduction

In this article, we study the oscillatory behavior of the fourth-order neutral nonlinear differential equation of the form

$$\begin{cases} \left(r(t)\,\Phi_{p_1}[w'''(t)]\right)' + q(t)\,\Phi_{p_2}(u(\vartheta(t))) = 0, \\ r(t) > 0,\ r'(t) \geq 0,\ t \geq t_0 > 0, \end{cases} \tag{1.1}$$

where $w(t) := u(t) + a(t)\,u(\tau(t))$ and the first term means the p-Laplace type operator ($1 < p < \infty$). The main results are obtained under the following conditions:

L1: $\Phi_{p_i}[s] = |s|^{p_i-2}s,\ i = 1, 2,$

L2: $r \in C[t_0, \infty)$ and under the condition

$$\int_{t_0}^{\infty} \frac{1}{r^{1/(p_1-1)}(s)} \mathrm{d}s = \infty. \tag{1.2}$$

L3: $a, q \in C[t_0, \infty)$, $q(t) > 0$, $0 \leq a(t) < a_0 < \infty$, $\tau, \vartheta \in C[t_0, \infty)$, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \vartheta(t) = \infty$

By a solution of (1.1) we mean a function $u \in C^3[t_u, \infty)$, $t_u \geq t_0$, which has the property $r(t)(w'''(t))^{p_1-1} \in C^1[t_u, \infty)$, and satisfies (1.1) on $[t_u, \infty)$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_u, \infty)$, and otherwise it is called to be nonoscillatory. (1.1) is said to be oscillatory if all its solutions are oscillatory.

We point out that delay differential equations have applications in dynamical systems, optimization, and in the mathematical modeling of engineering problems, such as electrical power systems, control systems, networks, materials, see [1]. The p-Laplace equations have some significant applications in elasticity theory and continuum mechanics.

During the past few years, there has been constant interest to study the asymptotic properties for oscillation of differential equations with p-Laplacian like operator in the canonical case and the noncanonical case, see [2–4, 11] and the numerical solution of the neutral delay differential equations, see [5–7]. The oscillatory properties of differential equations are fairly well studied by authors in [16–27]. We collect some relevant facts and auxiliary results from the existing literature.

Liu et al. [4] studied the oscillation of even-order half-linear functional differential equations with damping of the form

$$\begin{cases} \left(r(t)\,\Phi\left(y^{(n-1)}(t)\right)\right)' + a(t)\,\Phi\left(y^{(n-1)}(t)\right) + q(t)\,\Phi(y(g(t))) = 0, \\ \Phi = |s|^{p-2}\,s,\ t \geq t_0 > 0, \end{cases}$$

where n is even. This time, the authors used comparison method with second order equations.

The authors in [9, 10] have established sufficient conditions for the oscillation of the solutions of

$$\begin{cases} \left(r(t)\left|y^{(n-1)}(t)\right|^{p-2} y^{(n-1)}(t)\right)' + \sum_{i=1}^{j} q_i(t)\,g(y(\vartheta_i(t))) = 0, \\ j \geq 1,\ t \geq t_0 > 0, \end{cases}$$

where n is even and $p > 1$ is a real number, in the case where $\vartheta_i(t) \geq \upsilon$ (with $r \in C^1((0, \infty), \mathbb{R})$, $q_i \in C([0, \infty), \mathbb{R})$, $i = 1, 2, .., j$).

We point out that Li et al. [3] using the Riccati transformation together with integral averaging technique, focuses on the oscillation of equation

$$\begin{cases} \left(r(t)\,|w'''(t)|^{p-2}\,w'''(t)\right)' + \sum_{i=1}^{j} q_i(t)\,|y(\delta_i(t))|^{p-2}\,y(\delta_i(t)) = 0, \\ 1 < p < \infty,\ ,\ t \geq t_0 > 0. \end{cases}$$

Park et al. [8] have obtained sufficient conditions for oscillation of solutions of

$$\begin{cases} \left(r(t)\left|y^{(n-1)}(t)\right|^{p-2} y^{(n-1)}(t)\right)' + q(t)\,g(y(\delta(t))) = 0, \\ 1 < p < \infty,\ ,\ t \geq t_0 > 0. \end{cases}$$

As we already mentioned in the Introduction, our aim here is complement results in [8–10]. For this purpose we discussed briefly these results.

In this paper, we obtain some new oscillation criteria for (1.1). The paper is organized as follows. In the next sections, we will mention some auxiliary lemmas, also, we will use the generalized Riccati transformation technique to give some sufficient conditions for the oscillation of (1.1), and we will give some examples to illustrate the main results.

2. Main results

For convenience, we denote

$$\begin{aligned}
A(t) &= q(t)(1-a_0)^{p_2-1} M^{p_1-p_2}(\vartheta(t)),\\
B(t) &= (p_1-1)\varepsilon\frac{\vartheta^2(t)\zeta\vartheta'(t)}{r^{1/(p_1-1)}(t)},\\
\phi_1(t) &= \int_t^\infty A(s)\,\mathrm{d}s,\\
R_1(t) &:= (p_1-1)\mu\frac{t^2}{2r^{1/(p_1-1)}(t)},\\
\xi(t) &:= q(t)(1-a_0)^{p_2-1} M_1^{p_2-p_1}\varepsilon_1\left(\frac{\vartheta(t)}{t}\right)^{3(p_2-1)},\\
\eta(t) &:= (1-a_0)^{p_2/p_1} M_2^{p_2/(p_1-2)}\int_t^\infty\left(\frac{1}{r(\delta)}\int_\delta^\infty q(s)\frac{\vartheta^{p_2-1}(s)}{s^{p_2-1}}\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}\delta,\\
\xi_*(t) &= \int_t^\infty \xi(s)\,\mathrm{d}s,\ \eta_*(t) = \int_t^\infty \eta(s)\,\mathrm{d}s,
\end{aligned}$$

for some $\mu\in(0,1)$ and every M_1, M_2 are positive constants.

Definition 1. A sequence of functions $\{\delta_n(t)\}_{n=0}^\infty$ and $\{\sigma_n(t)\}_{n=0}^\infty$ as

$$\begin{aligned}
&\delta_0(t) = \xi_*(t),\ \text{and } \sigma_0(t) = \eta_*(t),\\
&\delta_n(t) = \delta_0(t) + \textstyle\int_t^\infty R_1(t)\,\delta_{n-1}^{p_1/(p_1-1)}(s)\,\mathrm{d}s,\ n>1\\
&\sigma_n(t) = \sigma_0(t) + \textstyle\int_t^\infty \sigma_{n-1}^{p_1/(p_1-1)}(s)\,\mathrm{d}s,\ n>1.
\end{aligned}\tag{2.1}$$

We see by induction that $\delta_n(t)\le\delta_{n+1}(t)$ and $\sigma_n(t)\le\sigma_{n+1}(t)$ for $t\ge t_0,\ n>1$.

In order to discuss our main results, we need the following lemmas:

Lemma 2.1. [12] If the function w satisfies $w^{(i)}(\nu)>0$, $i=0,1,...,n$, and $w^{(n+1)}(\nu)<0$ eventually. Then, for every $\varepsilon_1\in(0,1)$, $w(\nu)/w'(\nu)\ge\varepsilon_1\nu/n$ eventually.

Lemma 2.2. [13] Let $u(t)$ be a positive and n-times differentiable function on an interval $[T,\infty)$ with its nth derivative $u^{(n)}(t)$ non-positive on $[T,\infty)$ and not identically zero on any interval of the form $[T',\infty)$, $T'\ge T$ and $u^{(n-1)}(t)u^{(n)}(t)\le 0,\ t\ge t_u$ then there exist constants $\theta,\ 0<\theta<1$ and $\varepsilon>0$ such that

$$u'(\theta t)\ge\varepsilon t^{n-2}u^{(n-1)}(t),$$

for all sufficient large t.

Lemma 2.3 [14] Let $u \in C^n([t_0,\infty),(0,\infty))$. Assume that $u^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$ and that there exists a $t_1 \geq t_0$ such that $u^{(n-1)}(t)\,u^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} u(t) \neq 0$, then for every $\mu \in (0,1)$ there exists $t_\mu \geq t_1$ such that

$$u(t) \geq \frac{\mu}{(n-1)!} t^{n-1} \left|u^{(n-1)}(t)\right| \text{ for } t \geq t_\mu.$$

Lemma 2.4. [15] Assume that (1.2) holds and u is an eventually positive solution of (1.1). Then, $\left(r(t)(w'''(t))^{p_1-1}\right)' < 0$ and there are the following two possible cases eventually:

$$\begin{aligned} (\mathbf{G}_1)\ w^{(k)}(t) &> 0,\ k=1,2,3,\\ (\mathbf{G}_2)\ w^{(k)}(t) &> 0,\ k=1,3, \text{ and } w''(t)<0. \end{aligned}$$

Theorem 2.1. Assume that

$$\liminf_{t\to\infty} \frac{1}{\phi_1(t)} \int_t^\infty B(s)\, \phi_1^{\frac{p_1}{(p_1-1)}}(s)\,\mathrm{d}s > \frac{p_1-1}{p_1^{\frac{p_1}{(p_1-1)}}}. \tag{2.2}$$

Then (1.1) is oscillatory.

proof. Assume that u be an eventually positive solution of (1.1). Then, there exists a $t_1 \geq t_0$ such that $u(t) > 0$, $u(\tau(t)) > 0$ and $u(\vartheta(t)) > 0$ for $t \geq t_1$. Since $r'(t) > 0$, we have

$$w(t) > 0,\ w'(t) > 0,\ w'''(t) > 0,\ w^{(4)}(t) < 0 \text{ and } \left(r(t)(w'''(t))^{p_1-1}\right)' \leq 0, \tag{2.3}$$

for $t \geq t_1$. From definition of w, we get

$$\begin{aligned} u(t) &\geq w(t) - a_0 u(\tau(t)) \geq w(t) - a_0 w(\tau(t)) \\ &\geq (1-a_0)\, w(t), \end{aligned}$$

which with (1.1) gives

$$\left(r(t)(w'''(t))^{p_1-1}\right)' \leq -q(t)(1-a_0)^{p_2-1} w^{p_2-1}(\vartheta(t)). \tag{2.4}$$

Define

$$\varpi(t) := \frac{r(t)(w'''(t))^{p_1-1}}{w^{p_1-1}(\zeta\vartheta(t))}. \tag{2.5}$$

for some a constant $\zeta \in (0,1)$. By differentiating and using (2.4), we obtain

$$\begin{aligned} \varpi'(t) \leq\ & \frac{-q(t)(1-a_0)^{p_2-1} w^{p_2-1}(\vartheta(t)).}{w^{p_1-1}(\zeta\vartheta(t))} \\ & -(p_1-1)\frac{r(t)(w'''(t))^{p_1-1} w'(\zeta\vartheta(t))\,\zeta\vartheta'(t)}{w^{p_1}(\zeta\vartheta(t))}. \end{aligned}$$

From Lemma 2.2, there exist constant $\varepsilon > 0$, we have

$$\varpi'(t) \leq -q(t)(1-a_0)^{p_2-1} w^{p_2-p_1}(\vartheta(t))$$

$$-\left(p_1-1\right)\frac{r\left(t\right)\left(w'''\left(t\right)\right)^{p_1-1}\varepsilon\vartheta^2\left(t\right)w'''\left(\vartheta\left(t\right)\right)\zeta\vartheta'\left(t\right)}{w^{p_1}\left(\zeta\vartheta\left(t\right)\right)}.$$

Which is

$$\begin{aligned}\varpi'\left(t\right) &\le -q\left(t\right)\left(1-a_0\right)^{p_2-1}w^{p_2-p_1}\left(\vartheta\left(t\right)\right)\\ &\quad -\left(p_1-1\right)\varepsilon\frac{r\left(t\right)\vartheta^2\left(t\right)\zeta\vartheta'\left(t\right)\left(w'''\left(t\right)\right)^{p_1}}{w^{p_1}\left(\zeta\vartheta\left(t\right)\right)},\end{aligned}$$

by using (2.5) we have

$$\varpi'\left(t\right)\le -q\left(t\right)\left(1-a_0\right)^{p_2-1}w^{p_2-p_1}\left(\vartheta\left(t\right)\right)-\left(p_1-1\right)\varepsilon\frac{\vartheta^2\left(t\right)\zeta\vartheta'\left(t\right)}{r^{1/(p_1-1)}\left(t\right)}\varpi^{p_1/(p_1-1)}\left(t\right). \tag{2.6}$$

Since $w'\left(t\right)>0$, there exist a $t_2\ge t_1$ and a constant $M>0$ such that

$$w\left(t\right)>M.$$

Then, (2.6), turns to

$$\begin{aligned}\varpi'\left(t\right) &\le -q\left(t\right)\left(1-a_0\right)^{p_2-1}M^{p_2-p_1}\left(\vartheta\left(t\right)\right)\\ &\quad -\left(p_1-1\right)\varepsilon\frac{\vartheta^2\left(t\right)\zeta\vartheta'\left(t\right)}{r^{1/(p_1-1)}\left(t\right)}\varpi^{p_1/(p_1-1)}\left(t\right),\end{aligned}$$

that is

$$\varpi'\left(t\right)+A\left(t\right)+B\left(t\right)\varpi^{p_1/(p_1-1)}\left(t\right)\le 0.$$

Integrating the above inequality from t to l , we get

$$\varpi\left(l\right)-\varpi\left(t\right)+\int_t^l A\left(s\right)\mathrm{d}s+\int_t^l B\left(s\right)\varpi^{p_1/(p_1-1)}\left(s\right)\mathrm{d}s\le 0.$$

Letting $l\to\infty$ and using $\varpi>0$ and $\varpi'<0$, we have

$$\varpi\left(t\right)\ge\phi_1\left(t\right)+\int_t^\infty B\left(s\right)\varpi^{p_1/(p_1-1)}\left(s\right)\mathrm{d}s.$$

This implies

$$\frac{\varpi\left(t\right)}{\phi_1\left(t\right)}\ge 1+\frac{1}{\phi_1\left(t\right)}\int_t^\infty B\left(s\right)\phi_1^{p_1/(p_1-1)}\left(s\right)\left(\frac{\varpi\left(s\right)}{\phi_1\left(s\right)}\right)^{p_1/(p_1-1)}\mathrm{d}s. \tag{2.7}$$

Let $\lambda=\inf_{t\ge T}\varpi\left(t\right)/\phi_1\left(t\right)$ then obviously $\lambda\ge 1$. Thus, from (2.2) and (2.7) we see that

$$\lambda\ge 1+\left(p_1-1\right)\left(\frac{\lambda}{p_1}\right)^{p_1/(p_1-1)}$$

or

$$\frac{\lambda}{p_1}\ge\frac{1}{p_1}+\frac{\left(p_1-1\right)}{p_1}\left(\frac{\lambda}{p_1}\right)^{p_1/(p_1-1)},$$

which contradicts the admissible value of $\lambda\ge 1$ and $\left(p_1-1\right)>0$.

Therefore, the proof is complete.

Theorem 2.2. Assume that

$$\liminf_{t\to\infty}\frac{1}{\xi_*(t)}\int_t^\infty R_1(s)\,\xi_*^{p_1/(p_1-1)}(s)\,\mathrm{d}s > \frac{(p_1-1)}{p_1^{p_1/(p_1-1)}} \tag{2.8}$$

and

$$\liminf_{t\to\infty}\frac{1}{\eta_*(t)}\int_{t_0}^\infty \eta_*^2(s)\,\mathrm{d}s > \frac{1}{4}. \tag{2.9}$$

Then (1.1) is oscillatory.

proof. Assume to the contrary that (1.1) has a nonoscillatory solution in $[t_0,\infty)$. Without loss of generality, we let u be an eventually positive solution of (1.1). Then, there exists a $t_1 \geq t_0$ such that $u(t) > 0$, $u(\tau(t)) > 0$ and $u(\vartheta(t)) > 0$ for $t \geq t_1$. From Lemma 2.4 there is two cases $(\mathbf{G}_1)$ and $(\mathbf{G}_2)$.

For case $(\mathbf{G}_1)$. Define

$$\omega(t) := \frac{r(t)\,(w'''(t))^{p_1-1}}{w^{p_1-1}(t)}.$$

By differentiating ω and using (2.4), we obtain

$$\omega'(t) \leq -q(t)(1-a_0)^{p_2-1}\frac{w^{p_2-1}(\vartheta(t))}{w^{p_1-1}(t)} - (p_1-1)\frac{r(t)\,(w'''(t))^{p_1-1}}{w^{p_1}(t)}w'(t). \tag{2.10}$$

From Lemma 2.1, we get

$$\frac{w'(t)}{w(t)} \leq \frac{3}{\varepsilon_1 t}.$$

Integrating again from t to $\vartheta(t)$, we find

$$\frac{w(\vartheta(t))}{w(t)} \geq \varepsilon_1\frac{\vartheta^3(t)}{t^3}. \tag{2.11}$$

It follows from Lemma 2.3 that

$$w'(t) \geq \frac{\mu_1}{2}t^2 w'''(t), \tag{2.12}$$

for all $\mu_1 \in (0,1)$ and every sufficiently large t. Since $w'(t) > 0$, there exist a $t_2 \geq t_1$ and a constant $M > 0$ such that

$$w(t) > M, \tag{2.13}$$

for $t \geq t_2$. Thus, by (2.10), (2.11), (2.12) and (2.13), we get

$$\omega'(t) + q(t)(1-a_0)^{p_2-1}M_1^{p_2-p_1}\varepsilon_1\left(\frac{\vartheta(t)}{t}\right)^{3(p_2-1)} + \frac{(p_1-1)\,\mu t^2}{2r^{1/(p_1-1)}(t)}\omega^{p_1/(p_1-1)}(t) \leq 0,$$

that is

$$\omega'(t) + \xi(t) + R_1(t)\,\omega^{p_1/(p_1-1)}(t) \leq 0. \tag{2.14}$$

Integrating (2.14) from t to l , we get

$$\omega(l) - \omega(t) + \int_t^l \xi(s)\,\mathrm{d}s + \int_t^l R_1(s)\,\omega^{p_1/(p_1-1)}(s)\,\mathrm{d}s \leq 0.$$

Letting $l \to \infty$ and using $\omega > 0$ and $\omega' < 0$, we have

$$\omega(t) \geq \xi_*(t) + \int_t^\infty R_1(s)\,\omega^{p_1/(p_1-1)}(s)\,\mathrm{d}s. \tag{2.15}$$

This implies

$$\frac{\omega(t)}{\xi^*(t)} \geq 1 + \frac{1}{\xi_*(t)}\int_t^\infty R_1(s)\,\xi_*^{p_1/(p_1-1)}(s)\left(\frac{\omega(s)}{\xi_*(s)}\right)^{p_1/(p_1-1)}\mathrm{d}s. \tag{2.16}$$

Let $\lambda = \inf_{t\geq T}\omega(t)/\xi_*(t)$ then obviously $\lambda \geq 1$. Thus, from (2.8) and (2.16) we see that

$$\lambda \geq 1 + (p_1 - 1)\left(\frac{\lambda}{p_1}\right)^{p_1/(p_1-1)}$$

or

$$\frac{\lambda}{p_1} \geq \frac{1}{p_1} + \frac{(p_1-1)}{p_1}\left(\frac{\lambda}{p_1}\right)^{p_1/(p_1-1)},$$

which contradicts the admissible value of $\lambda \geq 1$ and $(p_1 - 1) > 0$.

For case $(\mathbf{G}_2)$. Integrating (2.4) from t to m, we obtain

$$r(m)\left(w'''(m)\right)^{p_1-1} - r(t)\left(w'''(t)\right)^{p_1-1} \leq -\int_t^m q(s)\,(1-a_0)^{p_2-1}\,w^{p_2-1}(\vartheta(s))\,\mathrm{d}s. \tag{2.17}$$

From Lemma 2.1, we get that

$$w(t) \geq \varepsilon_1 t w'(t) \text{ and hence } w(\vartheta(t)) \geq \varepsilon_1 \frac{\vartheta(t)}{t} w(t). \tag{2.18}$$

For (2.17), letting $m \to \infty$ and using (2.18), we see that

$$r(t)\left(w'''(t)\right)^{p_1-1} \geq \varepsilon_1 (1-a_0)^{p_2-1}\,w^{p_2-1}(t)\int_t^\infty q(s)\,\frac{\vartheta^{p_2-1}(s)}{s^{p_2-1}}\mathrm{d}s.$$

Integrating this inequality again from t to ∞, we get

$$w''(t) \leq -\varepsilon_1 (1-a_0)^{p_2/p_1}\,w^{p_2/p_1}(t)\int_t^\infty \left(\frac{1}{r(\delta)}\int_\delta^\infty q(s)\,\frac{\vartheta^{p_2-1}(s)}{s^{p_2-1}}\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}\delta, \tag{2.19}$$

for all $\varepsilon_1 \in (0,1)$. Define

$$y(t) = \frac{w'(t)}{w(t)}.$$

By differentiating y and using (2.13) and (2.19), we find

$$\begin{aligned} y'(t) &= \frac{w''(t)}{w(t)} - \left(\frac{w'(t)}{w(t)}\right)^2 \\ &\leq -y^2(t) - (1-a_0)^{p_2/p_1}\,M^{(p_2/p_1)-1}\int_t^\infty \left(\frac{1}{r(\delta)}\int_\delta^\infty q(s)\,\frac{\vartheta^{p_2-1}(s)}{s^{p_2-1}}\mathrm{d}s\right)^{1/(p_1-1)}\mathrm{d}\delta, \end{aligned} \tag{2.20}$$

hence

$$y'(t)+\eta(t)+y^2(t)\le 0. \tag{2.21}$$

The proof of the case where $(\mathbf{G}_2)$ holds is the same as that of case $(\mathbf{G}_1)$. Therefore, the proof is complete.

Theorem 2.3. Let $\delta_n(t)$ and $\sigma_n(t)$ be defined as in (2.1). If

$$\limsup_{t\to\infty}\left(\frac{\mu_1 t^3}{6r^{1/(p_1-1)}(t)}\right)^{p_1-1}\delta_n(t)>1 \tag{2.22}$$

and

$$\limsup_{t\to\infty}\lambda t\sigma_n(t)>1, \tag{2.23}$$

for some n, then (1.1)is oscillatory.

proof. Assume to the contrary that (1.1) has a nonoscillatory solution in $[t_0,\infty)$. Without loss of generality, we let u be an eventually positive solution of (1.1). Then, there exists a $t_1\ge t_0$ such that $u(t)>0$, $u(\tau(t))>0$ and $u(\vartheta(t))>0$ for $t\ge t_1$. From Lemma 2.4 there is two cases.

In the case $(\mathbf{G}_1)$, proceeding as in the proof of Theorem 2.2, we get that (2.12) holds. It follows from Lemma 2.3 that

$$w(t)\ge\frac{\mu_1}{6}t^3w'''(t). \tag{2.24}$$

From definition of $\omega(t)$ and (2.24), we have

$$\frac{1}{\omega(t)}=\frac{1}{r(t)}\left(\frac{w(t)}{w'''(t)}\right)^{p_1-1}\ge\frac{1}{r(t)}\left(\frac{\mu_1}{6}t^3\right)^{p_1-1}.$$

Thus,

$$\omega(t)\left(\frac{\mu_1 t^3}{6r^{1/(p_1-1)}(t)}\right)^{p_1-1}\le 1.$$

Therefore,

$$\limsup_{t\to\infty}\omega(t)\left(\frac{\mu_1 t^3}{6r^{1/(p_1-1)}(t)}\right)^{p_1-1}\le 1,$$

which contradicts (2.22).

The proof of the case where $(\mathbf{G}_2)$ holds is the same as that of case $(\mathbf{G}_1)$. Therefore, the proof is complete.

Corollary 2.1. Let $\delta_n(t)$ and $\sigma_n(t)$ be defined as in (2.1). If

$$\int_{t_0}^{\infty}\xi(t)\exp\left(\int_{t_0}^{t}R_1(s)\,\delta_n^{1/(p_1-1)}(s)\,\mathrm{d}s\right)\mathrm{d}t=\infty \tag{2.25}$$

and

$$\int_{t_0}^{\infty}\eta(t)\exp\left(\int_{t_0}^{t}\sigma_n^{1/(p_1-1)}(s)\,\mathrm{d}s\right)\mathrm{d}t=\infty, \tag{2.26}$$

for some n, then (1.1) is oscillatory.

proof. Assume to the contrary that (1.1) has a nonoscillatory solution in $[t_0,\infty)$. Without loss of generality, we let u be an eventually positive solution of (1.1). Then, there exists a $t_1\ge t_0$ such that $u(t)>0$, $u(\tau(t))>0$ and $u(\vartheta(t))>0$ for $t\ge t_1$. From Lemma 2.4 there is two cases $(\mathbf{G}_1)$ and $(\mathbf{G}_2)$.

In the case ($\mathbf{G}_1$), proceeding as in the proof of Theorem 2, we get that (2.15) holds. It follows from (2.15) that $\omega(t) \geq \delta_0(t)$. Moreover, by induction we can also see that $\omega(t) \geq \delta_n(t)$ for $t \geq t_0,\ n > 1$. Since the sequence $\{\delta_n(t)\}_{n=0}^{\infty}$ monotone increasing and bounded above, it converges to $\delta(t)$. Thus, by using Lebesgue's monotone convergence theorem, we see that

$$\delta(t) = \lim_{n\to\infty} \delta_n(t) = \int_t^{\infty} R_1(t)\,\delta^{p_1/(p_1-1)}(s)\,\mathrm{d}s + \delta_0(t)$$

and

$$\delta'(t) = -R_1(t)\,\delta^{p_1/(p_1-1)}(t) - \xi(t). \tag{2.27}$$

Since $\delta_n(t) \leq \delta(t)$, it follows from (2.27) that

$$\delta'(t) \leq -R_1(t)\,\delta_n^{1/(p_1-1)}(t)\,\delta(t) - \xi(t).$$

Hence, we get

$$\delta(t) \leq \exp\left(-\int_T^t R_1(s)\,\delta_n^{1/(p_1-1)}(s)\,\mathrm{d}s\right)\left(\delta(T) - \int_T^t \xi(s)\exp\left(\int_T^s R_1(\delta)\,\delta_n^{1/(p_1-1)}(\delta)\,\mathrm{d}\delta\right)\mathrm{d}s\right).$$

This implies

$$\int_T^t \xi(s)\exp\left(\int_T^s R_1(\delta)\,\delta_n^{1/(p_1-1)}(\delta)\,\mathrm{d}\delta\right)\mathrm{d}s \leq \delta(T) < \infty,$$

which contradicts (2.25). The proof of the case where ($\mathbf{G}_2$) holds is the same as that of case ($\mathbf{G}_1$). Therefore, the proof is complete.

Example 2.1. Consider the differential equation

$$\left(u(t) + \frac{1}{2}u\left(\frac{t}{2}\right)\right)^{(4)} + \frac{q_0}{t^4}u\left(\frac{t}{3}\right) = 0, \tag{2.28}$$

where $q_0 > 0$ is a constant. Let $p_1 = p_2 = 2$, $r(t) = 1$, $a(t) = 1/2$, $\tau(t) = t/2$, $\vartheta(t) = t/3$ and $q(t) = q_0/t^4$. Hence, it is easy to see that

$$\begin{aligned} A(t) &= q(t)(1-a_0)^{(p_2-1)} M^{p_2-p_1}(\vartheta(t)) = \frac{q_0}{2t^4}, \\ B(t) &= (p_1-1)\,\varepsilon\frac{\vartheta^2(t)\,\zeta\vartheta'(t)}{r^{1/(p_1-1)}(t)} = \frac{\varepsilon t^2}{27} \end{aligned}$$

and

$$\phi_1(t) = \frac{q_0}{6t^3},$$

also, for some $\varepsilon > 0$, we find

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{\phi_1(t)} \int_t^{\infty} B(s)\,\phi_1^{p_1/(p_1-1)}(s)\,\mathrm{d}s &> \frac{(p_1-1)}{p_1^{p_1/(p_1-1)}}. \\ \liminf_{t\to\infty} \frac{6\varepsilon q_0 t^3}{972} \int_t^{\infty} \frac{\mathrm{d}s}{s^4} &> \frac{1}{4} \\ q_0 &> 121.5\varepsilon. \end{aligned}$$

Hence, by Theorem 2.1, every solution of Eq (2.28) is oscillatory if $q_0 > 121.5\varepsilon$.
Example 2.2. Consider a differential equation

$$(u(t) + a_0 u(\tau_0 t))^{(n)} + \frac{q_0}{t^n} u(\vartheta_0 t) = 0, \tag{2.29}$$

where $q_0 > 0$ is a constant. Note that $p = 2,\ t_0 = 1,\ r(t) = 1,\ a(t) = a_0,\ \tau(t) = \tau_0 t,\ \vartheta(t) = \vartheta_0 t$ and $q(t) = q_0/t^n$.
Easily, we see that condition (2.8) holds and condition (2.9) satisfied.

Hence, by Theorem 2.2, every solution of Eq (2.29) is oscillatory.
Remark 2.1. Finally, we point out that continuing this line of work, we can have oscillatory results for a fourth order equation of the type:

$$\begin{cases} \left(r(t)\,|y'''(t)|^{p_1-2}\,y'''(t)\right)' + a(t)\,f(y'''(t)) + \sum_{i=1}^{j} q_i(t)\,|y(\sigma_i(t))|^{p_2-2}\,y(\sigma_i(t)) = 0, \\ t \geq t_0,\ \sigma_i(t) \leq t,\ j \geq 1,,\ 1 < p_2 \leq p_1 < \infty. \end{cases}$$

3. Conclusion

The paper is devoted to the study of oscillation of fourth-order differential equations with p-Laplacian like operators. New oscillation criteria are established by using a Riccati transformations, and they essentially improves the related contributions to the subject.
Further, in the future work we get some Hille and Nehari type and Philos type oscillation criteria of (1.1) under the condition $\int_{\upsilon_0}^{\infty} \frac{1}{r^{1/(p_1-1)}(s)}\mathrm{d}s < \infty$.

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Conflict of interest

The author declares that there is no competing interest.

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