Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 16291–16307.
DOI: 10.3934/math.2023834
Received: 14 December 2022
Revised: 12 March 2023
Accepted: 20 March 2023
Published: 08 May 2023

Research article

Oscillation theorems for fourth-order quasi-linear delay differential equations

Fahd Masood[1,2], **Osama Moaaz**[1,3], **Shyam Sundar Santra**[4,5,*], **U. Fernandez-Gamiz**[6] **and Hamdy A. El-Metwally**[1]

[1] Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
[2] Department of mathematics, Faculty of Education and science, University of Saba Region, Marib,Yemen
[3] Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia
[4] Department of Mathematics, JIS College of Engineering, Kalyani, Nadia, West Bengal 741235, India
[5] Department of Mathematics, Applied Science Cluster, University of Petroleum and Energy Studies (UPES), Dehradun, Uttarakhand 248007, India
[6] Nuclear Engineering and Fluid Mechanics Department, University of the Basque Country UPV/EHU, Vitoria-Gasteiz, Spain

* **Correspondence:** Email: shyam 01.math@gmail.com.

Abstract: In this paper, we deal with the asymptotic and oscillatory behavior of quasi-linear delay differential equations of fourth order. We first find new properties for a class of positive solutions of the studied equation, $\mathcal{N}_a$. As an extension of the approach taken in [1], we establish a new criterion that guarantees that $\mathcal{N}_a = \emptyset$. Then, we create a new oscillation criterion.

Keywords: oscillatory behavior; nonoscillatory behavior; delay differential equation; fourth order
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

It is ideal to represent phenomena and real-world problems in numerous applied sciences using delay differential equations (DDEs), a type of functional differential equation, that is characterized by taking into account the temporal memory of events. In both pure and applied mathematics, physics, meteorology, engineering, and population dynamics, there are many applications for the study of functional differential equations. The properties of these equations of different orders are a topic that is addressed by all of these sciences. For global existence and uniqueness theorems for differential equations, pure mathematics focuses on the existence and uniqueness of solutions. Applied mathematics, however, places a greater emphasis on the careful justification of the qualitative behavior of solutions (oscillation, periodicity, stability, global attractivity, Hopf bifurcation, control, synchronization, etc.) see [2–5].

Finding sufficient conditions to assure that all solutions of DDE oscillate is one of the main aims of oscillation theory. Ladde et al. [6] were among the first to outline oscillation theory, covered the work up until 1984. The focus of this book is on how divergent arguments affect the oscillation of solutions. The book by Gyori and Ladas [7], which made significant contributions to the development of linearized oscillation theory and the relationship between the distribution of the roots of characteristic equations and the oscillation of all solutions, is one of the key works in the field of oscillation theory.

The deflection of buckling beams with constant or changing cross-sections, electromagnetic waves, three-layer beams, gravity-driven flows, etc., are only a few examples of the many disciplines of applied mathematics and physics from which the fourth-order differential equations are formed. Due to its widespread use in the study of physical sciences, mechanics, radio technology, lossless high-speed computer networks, control systems, life sciences, and population growth, the oscillation theory of fourth-order differential equations has recently attracted a lot of attention, see [8–10].

In recent years, oscillation theory has received significant attention from researchers who have conducted various studies to understand the oscillation behavior of functional differential equations of different orders. This area of research continues to be active, with new findings emerging frequently. Specifically, when investigating the oscillatory behavior of functional differential equations, the second-order equations received the most attention from researchers [11–19], followed by the third-order equations [20, 21], whereas the fourth-order and higher-order differential equations received comparatively less attention [22, 23]. Investigation of the oscillatory behavior of solutions of the fourth-order quasi-linear DDE

$$\left(a(t)\left(u'''(t)\right)^{\alpha}\right)' + q(t)\,u^{\alpha}(\sigma(t)) = 0,\ t \geq t_0, \tag{1.1}$$

is the main topic of this paper, where we assume the following constraints during the study:

(H$_1$) $\alpha > 0$ is a ratio of two odd integers, $a \in \mathbf{C}^1([t_0,\infty),(0,\infty))$, $q \in \mathbf{C}([t_0,\infty),[0,\infty))$, and $a'(t) \geq 0$.
(H$_2$) $\sigma \in \mathbf{C}^1([t_0,\infty),\mathbb{R})$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, and $\lim_{t\to\infty}\sigma(t) = \infty$.

A function $u \in \mathbf{C}^3([t_*,\infty),\mathbb{R})$, $t_* \geqslant t_0$, is said to be a solution of (1.1) if it has the property $a(u''')^{\alpha} \in \mathbf{C}^1([t_*,\infty),\mathbb{R})$, and satisfies equation (1.1) for $t \geq t_*$. We consider only those solutions u of (1.1) which satisfy $\sup\{|u(t)| : t \geqslant t_1\} > 0$, for all $t_1 \geq t_*$. A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

In the next part of the introduction, we review some important results that dealt with the oscillation of DDEs of even orders.

Agarwal et al. [24] established criteria for oscillation of the nth-order DDE

$$\left(\left|u^{(n-1)}(t)\right|^{\alpha-1} u^{(n-1)}(t)\right)' + F(t,u(\sigma(t))) = 0, \tag{1.2}$$

where $t \geq t_0$, n is even, $F \in \mathbf{C}([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and $\operatorname{sgn} F(t,u) = \operatorname{sgn} u$.

Theorem 1.1. *[24, Corollary 2.1] If there exist $\rho, \mu \in \mathbf{C}^1\left([t_0, \infty), \mathbb{R}^+\right)$ such that*

$$\mu(t) \leq \inf\{t, \sigma(t)\}, \ \lim_{t\to\infty} \mu(t) = \infty, \ \mu'(t) > 0$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s) q(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(s))^{\alpha+1}}{(\rho(s)\mu'(s))^{\alpha}}\right) \mathrm{d}s = \infty,$$

then the DDE

$$\left(|u'(t)|^{\alpha-1} u'(t)\right)' + q(t) |u(\sigma(t))|^{\alpha-1} u(\sigma(t)) = 0,$$

is oscillatory.

Theorem 1.2. *[24, Theoerem 2.3] If $F(t,u)\ \operatorname{sgn} u \geq q(t)|u|^{\alpha}$ for $u \neq 0$ and $\alpha > 0$, and*

$$\limsup_{t\to\infty} t^{\alpha(n-1)} \int_{\gamma(t)}^{\infty} q(s)\,\mathrm{d}s > ((n-1)!)^{\alpha},$$

then (1.2) is oscillatory, where $\gamma(t) := \sup\{s \geq t_0 : \sigma(s) \leq t\}$.

In both canonical

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\,\mathrm{d}s = \infty, \tag{1.3}$$

and non-canonical cases, Baculıkova et al. [25] studied the asymptotic and oscillatory properties of the nth-order DDE

$$\left(a(t)\left(u^{(n-1)}(t)\right)^{\alpha}\right)' + q(t) f(u(\sigma(t))) = 0, \tag{1.4}$$

where $uf(u) > 0$ for $u \neq 0$, $f(u)$ is nondecreasing, and

$$-f(-xy) \geq f(xy) \geq f(x) f(y), \text{ for } xy > 0.$$

Theorem 1.3. *[25, Corollary 1] Assume that (1.3) holds, $f\left(u^{1/\alpha}\right)/u \geq 1$ for $0 < |u| \leq 1$, and for some $\delta \in (0,1)$,*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s) f\left(\frac{\delta}{(n-1)!} \frac{\sigma^{n-1}(s)}{r^{1/\alpha}(\sigma(s))}\right) \mathrm{d}s > \frac{1}{\mathrm{e}}.$$

Then, (1.4) is oscillatory.

Koplatadze et al. [1] established sufficient conditions for the DDE

$$u^{(n)}(t) + q(t) u(\sigma(t)) = 0, \ n \geq 2,$$

to have Properties A and B, and considered the odd and even cases for the order.

For neutral equations, Li and Rogovchenko [26] investigated the oscillatory behavior of the neutral DDE

$$(u(t) + p(t) u(\tau(t)))^{(n)} + q(t) u(\sigma(t)) = 0, \ n \geq 4. \tag{1.5}$$

They derived two oscillation results which complement and improve the results in [27–29]. Baculikova and Dzurina [30] introduced comparison theorems for the oscillation of (1.5).

For second-order, recently, Baculikova [31] and Baculikova and Dzurina [32] extended the results in [1] to the non-canonical case of the DDE

$$(a(t)u'(t))' + q(t)f(u(\sigma(t))) = 0,$$

and the canonical case of the DDE

$$(a(t)u'(t))^{\alpha} + q(t)u^{\alpha}(\sigma(t)) = 0.$$

In this paper, in the canonical case, we begin by finding some monotonic and asymptotic properties of a class of positive solutions to the DDE (1.1). Then, as an extension of the results in [1], we deduce a new condition that excludes positive solutions in the class under study. Moreover, we introduce a criterion that guarantees the oscillation of all solutions of the studied equation.

2. Preliminary results

We begin with some useful lemmas concerning the monotonic properties of the nonoscillatory solutions of the studied equations. To simplify the presentation of the main results, we define the following functions: $\rho'_{+}(t) := \max\{0, \rho'(t)\}$,

$$\eta_0(t) := \int_{t_0}^{t} \frac{1}{a^{1/\alpha}(s)} \mathrm{d}s, \ \eta_i(t) := \int_{t_0}^{t} \eta_{i-1}(s)\,\mathrm{d}s, \ i = 1, 2,$$

and

$$\widehat{q}(t) := \begin{cases} \eta_2^{\alpha}(\sigma(t))\eta_0^{-1}(\sigma(t))q(t), & \text{for } \alpha \geq 1; \\ \eta_2^{\alpha}(\sigma(t))\eta_0^{-1}(t)\,q(t), & \text{for } \alpha < 1. \end{cases}$$

Lemma 2.1. *[33, Lemma 2.2.3] Let $w \in \mathbf{C}^n([t_0, \infty), (0, \infty))$, $w^{(n)}$ be of fixed sign and not identically zero on $[t_0, \infty)$ and assume that there exists $t_1 \geq t_0$ such that $w^{(n-1)}(t)w^{(n)}(t) \leq 0$ for all $t_1 \geq t_0$. If $\lim_{t\to\infty} w(t) \neq 0$, then there exists $t_\mu \in [t_1, \infty)$ such that*

$$w(t) \geq \frac{\mu}{(m-1)!} t^{n-1} \left| w^{(n-1)}(t) \right|,$$

for every $\mu \in (0, 1)$ and $t \geq t_\mu$.

Lemma 2.2. *[34] Let $w \in \mathbf{C}^m([t_0, \infty), (0, \infty))$, $w^{(i)}(t) > 0$ for $i = 1, 2, ..., m$, and $w^{(m+1)}(t) \leq 0$, eventually. Then, eventually, $w(t)/w'(t) \geq \epsilon t/m$ for every $\epsilon \in (0, 1)$.*

Lemma 2.3. *[35] Let $A > 0$ and B be real numbers. Then*

$$B\phi - A\phi^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}. \tag{2.1}$$

Lemma 2.4. *Assume that u is an eventually positive solution of (1.1). Then u satisfies one of the following cases, eventually:*

$$\begin{aligned} (\mathrm{P}_1) \ &: \ u > 0, \ u' > 0, \ u'' > 0, \ u''' > 0, \ (a(u''')^{\alpha})' < 0, \\ (\mathrm{P}_2) \ &: \ u > 0, \ u' > 0, \ u'' < 0, \ u''' > 0, \ (a(u''')^{\alpha})' < 0. \end{aligned}$$

Notation 1. *The class of all eventually positive solutions satisfying case* (P_1) *or* (P_2)*, in Lemma 2.4, is denoted by* $\mathcal{N}_a$ *or* $\mathcal{N}_b$*, respectively.*

Lemma 2.5. *Assume that* $u \in \mathcal{N}_a$*. If*

$$\int_{t_0}^{\infty} \eta_2^{\alpha}(\sigma(s))\, q(s)\, \mathrm{d}s = \infty, \tag{2.2}$$

then
$(\mathrm{B}_{1,1})$ $u \geqslant a^{1/\alpha} u''' \eta_2$;
$(\mathrm{B}_{1,2})$ u''/η_0 *and* u/η_2 *are decreasing*;
$(\mathrm{B}_{1,3})$ $u \geqslant u'' \eta_2/\eta_0$;
$(\mathrm{B}_{1,4})$ $\lim_{t\to\infty} u(t)/\eta_2(t) = 0$;
$(\mathrm{B}_{1,5})$ $\lim_{t\to\infty} u''(t)/\eta_0(t) = 0$.

Proof. $(\mathrm{B}_{1,1})$ The monotonicity of $a^{1/\alpha} u'''$ implies that

$$\begin{aligned} u''(t) &\geq \int_{t_1}^{t} a^{1/\alpha}(s) u'''(s) \frac{1}{a^{1/\alpha}(s)} \mathrm{d}s \geq a^{1/\alpha}(t) u'''(t) \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)} \mathrm{d}s \\ &\geq a^{1/\alpha}(t) u'''(t)\, \eta_0(t). \end{aligned} \tag{2.3}$$

Integrating twice more from t_1 to t, we obtain

$$u' \geq a^{1/\alpha} u''' \eta_1,$$

and

$$u \geq a^{1/\alpha} u''' \eta_2.$$

$(\mathrm{B}_{1,2})$ From (2.3), we obtain

$$\left(\frac{u''}{\eta_0}\right)' = \frac{a^{1/\alpha} u''' \eta_0 - u''}{a^{1/\alpha} \eta_0^2} \leq 0.$$

Since $u''/\eta_0$is decreasing, then

$$u'(t) \geq \int_{t_1}^{t} \frac{u''(s)}{\eta_0(s)} \eta_0(s)\, \mathrm{d}s \geq \frac{u''(t)}{\eta_0(t)} \eta_1(t). \tag{2.4}$$

From this we deduce that

$$\left(\frac{u'}{\eta_1}\right)' = \frac{u'' \eta_1 - \eta_0 u'}{\eta_1^2} \leq 0.$$

Since u'/η_1 is decreasing, then

$$u(t) \geq \int_{t_1}^{t} \frac{u'(s)}{\eta_1(s)} \eta_1(s)\, \mathrm{d}s \geq \frac{u'(t)}{\eta_1(t)} \eta_2(t). \tag{2.5}$$

Consequently

$$\left(\frac{u}{\eta_2}\right)' = \frac{u' \eta_2 - \eta_1 u}{\eta_2^2} \leq 0.$$

($B_{1,3}$) From (2.4) and (2.5), we find

$$u \geq \frac{\eta_2}{\eta_0} u''.$$

($B_{1,4}$) Since u/η_2 is positive and decreasing, $\lim_{t\to\infty} u(t)/\eta_2(t) = l_1 \geqslant 0$. We claim that $l_1 = 0$. If not, then $u(t)/\eta_2(t) \geqslant l_1 > 0$ eventually. Integrating (1.1) from t_1 to t, we have

$$\begin{aligned} a(t_1)\left(u'''(t_1)\right)^{\alpha} &\geq \int_{t_1}^{t} q(s) u^{\alpha}(\sigma(s)) \mathrm{d}s \\ &\geq \int_{t_1}^{t} q(s) \eta_2^{\alpha}(\sigma(s)) \frac{u^{\alpha}(\sigma(s))}{\eta_2^{\alpha}(\sigma(s))} \mathrm{d}s \\ &\geq l_1^{\alpha} \int_{t_1}^{t} q(s) \eta_2^{\alpha}(\sigma(s)) \mathrm{d}s \to \infty \text{ as } t \to \infty, \end{aligned}$$

which contradicts (2.2). So that, $l_1 = 0$.

($B_{1,5}$) Since u''/η_0 is positive and decreasing, $\lim_{t\to\infty} u''(t)/\eta_0(t) = l_2 \geqslant 0$. We claim that $l_2 = 0$. If not, then $u''(t)/\eta_0(t) \geqslant l_2 > 0$ eventually. Integrating (1.1) from t_1 to t, we have

$$a(t_1)\left(u'''(t_1)\right)^{\alpha} \geq \int_{t_1}^{t} q(s) u^{\alpha}(\sigma(s)) \mathrm{d}s.$$

From (2.4) and (2.5), we get

$$u \geq \frac{u''}{\eta_0} \eta_2.$$

Therefore,

$$\begin{aligned} a(t_1)\left(u'''(t_1)\right)^{\alpha} &\geq \int_{t_1}^{t} q(s) \eta_2^{\alpha}(\sigma(s)) \frac{(u''(\sigma(s)))^{\alpha}}{\eta_0^{\alpha}(\sigma(s))} \mathrm{d}s \\ &\geq l_2^{\alpha} \int_{t_1}^{t} q(s) \eta_2^{\alpha}(\sigma(s)) \mathrm{d}s \to \infty \text{ as } t \to \infty, \end{aligned}$$

which contradicts (2.2). So that $l_2 = 0$. Hence, the proof of the lemma is complete. □

Since η_0 is increasing, there exists $\lambda \geq 1$ such that

$$\frac{\eta_0(t)}{\eta_0(\sigma(t))} \geq \lambda. \tag{2.6}$$

Lemma 2.6. *Assume that $u \in \mathcal{N}_a$, and there exists a $\delta > 0$ such that*

$$\frac{1}{\alpha} a^{1/\alpha}(t)\, \eta_2^{\alpha}(\sigma(t)) \eta_0(t) q(t) \geq \delta. \tag{2.7}$$

Then

($B_{2,1}$) *$u''/\eta_0^{1-\delta}$ is decreasing;*

($B_{2,2}$) *$u''/\eta_0^{\delta_0}$ is increasing, where $\delta_0 = \delta^{1/\alpha}\lambda^{\delta}$.*

Proof. Assume that $u \in \mathcal{N}_a$. It follows from (2.7) that

$$\begin{aligned}\int_{t_0}^{t} \eta_2^{\alpha}(\sigma(s))\, q(s)\,\mathrm{d}s &\geq \alpha\delta \int_{t_0}^{t} \frac{1}{a^{1/\alpha}(s)\,\eta_0(s)}\mathrm{d}s \\ &= \alpha\delta \ln \frac{\eta_0(t)}{\eta_0(t_0)} \to \infty \text{ as } t \to \infty.\end{aligned}$$

So (2.7) guarantees condition (2.2).
$(B_{2,1})$ Note that $(B_{1,5})$ in Lemma 2.5 implies

$$\lim_{t\to\infty} a^{1/\alpha}(t)\, u'''(t) = 0. \tag{2.8}$$

By integrating (1.1) from t to ∞, we conclude that

$$a^{1/\alpha}(t)\, u'''(t) = \left(\int_{t}^{\infty} q(s) u^{\alpha}(\sigma(s))\,\mathrm{d}s\right)^{1/\alpha}. \tag{2.9}$$

We have

$$\left(a\left(u'''\right)^{\alpha}\right)' = \alpha\left(a^{1/\alpha}u'''\right)'\left(a^{1/\alpha}u'''\right)^{\alpha-1}.$$

Putting into (1.1), we obtain

$$\left(a^{1/\alpha}(t)\, u'''(t)\right)' + \frac{1}{\alpha}\left(a^{1/\alpha}(t)u'''(t)\right)^{1-\alpha} q(t)u^{\alpha}(\sigma(t)) = 0. \tag{2.10}$$

Then, $\phi = a^{1/\alpha}u'''$ is a positive decreasing function and satisfies

$$\phi'(t) + \frac{1}{\alpha} q(t)\phi^{1-\alpha}(t)\, u^{\alpha}(\sigma(t)) = 0. \tag{2.11}$$

On the other hand, $(B_{1,1})$ in Lemma 2.5 implies

$$u \geq a^{1/\alpha}u'''\eta_2 = \phi\eta_2,$$

and so

$$u^{\alpha}(\sigma(t)) \geq \phi^{\alpha}(\sigma(t))\,\eta_2^{\alpha}(\sigma(t)) \geq \phi^{\alpha}(t)\,\eta_2^{\alpha}(\sigma(t)).$$

Substituting the previous inequality into (2.11), we have

$$\phi'(t) + \frac{1}{\alpha} q(t)\eta_2^{\alpha}(\sigma(t))\,\phi(t) \leq 0. \tag{2.12}$$

By using (2.7), we obtain

$$\phi' + \frac{\delta}{a^{1/\alpha}\eta_0}\phi \leq 0,$$

which implies

$$-\phi'\eta_0 \geq \delta\frac{\phi}{a^{1/\alpha}} = \delta u'''.$$

We present the auxiliary function

$$y = (1-\delta)\, u'' - a^{1/\alpha}\eta_0 u'''. \tag{2.13}$$

Differentiating y, we get

$$y' = -\delta u''' - \phi' \eta_0 \geq -\delta u''' + \delta u''' = 0.$$

Therefore, the function y is increasing and has constant sign, eventually. If $y(t) \leq 0$ for $t \geq t_1$, then this implies that $u''/\eta_0^{1-\delta}$ is increasing. Using this fact together with (2.7) and (2.9), we have

$$\begin{aligned} a^{1/\alpha}(t)\, u'''(t) &\geq \left(\int_t^\infty q(s) u^\alpha(\sigma(s)) \mathrm{d}s \right)^{1/\alpha} \\ &\geq \left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)\, \eta_0(s)} \frac{u^\alpha(\sigma(s))}{\eta_2^\alpha(\sigma(s))} \mathrm{d}s \right)^{1/\alpha}. \end{aligned}$$

Since u/η_2 is decreasing, then

$$a^{1/\alpha}(t)\, u'''(t) \geq \left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)\, \eta_0(s)} \frac{u^\alpha(s)}{\eta_2^\alpha(s)} \mathrm{d}s \right)^{1/\alpha}. \tag{2.14}$$

From $(B_{1,3})$ in Lemma 2.5, we find

$$a^{1/\alpha}(t)\, u'''(t) \geq \left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)\, \eta_0^{\alpha+1}(s)} \left(u''(s) \right)^\alpha \mathrm{d}s \right)^{1/\alpha}.$$

Since $u''/\eta_0^{1-\delta}$ is increasing, then

$$\begin{aligned} a^{1/\alpha}(t)\, u'''(t) &\geq \left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)} \frac{1}{\eta_0^{\alpha\delta+1}(s)} \left(\frac{u''(s)}{\eta_0^{1-\delta}(s)} \right)^\alpha \mathrm{d}s \right)^{1/\alpha} \\ &\geq \frac{u''(t)}{\eta_0^{1-\delta}(t)} \left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)} \frac{1}{\eta_0^{\alpha\delta+1}(s)} \mathrm{d}s \right)^{1/\alpha} \\ &\geq \frac{u''(t)}{\eta_0(t)}. \end{aligned}$$

It follows from the last inequality that $(u''/\eta_0)' \geq 0$. This is a contradiction and we deduce that

$$y = (1-\delta)\, u'' - a^{1/\alpha} \eta_0(t)\, u''' \geq 0,$$

which implies that

$$\left(\frac{u''}{\eta_0^{1-\delta}} \right)' = \frac{a^{1/\alpha} \eta_0 u''' - (1-\delta)\, u''}{a^{1/\alpha} \eta_0^{2-\delta}} \leq 0.$$

$(B_{2,2})$ From (2.7) and (2.9), we have

$$\begin{aligned} a^{1/\alpha}(t)\, u'''(t) &\geq \left(\int_t^\infty q(s) u^\alpha(\sigma(s)) \mathrm{d}s \right)^{1/\alpha} \\ &\geq \left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)\, \eta_0(s)} \frac{u^\alpha(\sigma(s))}{\eta_2^\alpha(\sigma(s))} \mathrm{d}s \right)^{1/\alpha}. \end{aligned}$$

From ($B_{1,3}$) in Lemma 2.5, we get

$$\begin{aligned} a^{1/\alpha}(t)\,u'''(t) &\geq \left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)\,\eta_0(s)} \frac{(u''(\sigma(s)))^\alpha}{\eta_0^\alpha(\sigma(s))}\mathrm{d}s\right)^{1/\alpha} \\ &\geq \left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)\,\eta_0(s)} \frac{1}{\eta_0^{\alpha\delta}(\sigma(s))}\left(\frac{u''(\sigma(s))}{\eta_0^{1-\delta}(\sigma(s))}\right)^\alpha \mathrm{d}s\right)^{1/\alpha}. \end{aligned}$$

Since $u''/\eta_0^{1-\delta}$ is decreasing, then

$$a^{1/\alpha}(t)\,u'''(t) \geq \left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)\,\eta_0(s)} \frac{1}{\eta_0^{\alpha\delta}(\sigma(s))} \frac{(u''(s))^\alpha}{\eta_0^{\alpha(1-\delta)}(s)}\mathrm{d}s\right)^{1/\alpha}.$$

Since u'' is increasing, then

$$a^{1/\alpha}(t)\,u'''(t) \geq u''(t)\left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)\,\eta_0^{1+\alpha}(s)} \frac{\eta_0^{\alpha\delta}(s)}{\eta_0^{\alpha\delta}(\sigma(s))}\mathrm{d}s\right)^{1/\alpha}.$$

Using (2.6), we obtain

$$\begin{aligned} a^{1/\alpha}(t)\,u'''(t) &\geq u''(t)\left(\int_t^\infty \frac{\alpha\delta}{a^{1/\alpha}(s)\,\eta_0^{1+\alpha}(s)}\lambda^{\alpha\delta}\mathrm{d}s\right)^{1/\alpha} \\ &\geq \delta^{1/\alpha}\lambda^\delta u''(t)\left(\int_t^\infty \frac{\alpha}{a^{1/\alpha}(s)\,\eta_0^{1+\alpha}(s)}\lambda^{\alpha\delta}\mathrm{d}s\right)^{1/\alpha} \\ &\geq \delta^{1/\alpha}\lambda^\delta \frac{u''(t)}{\eta_0(t)}. \end{aligned}$$

Then

$$a^{1/\alpha}u''' \geq \delta_0 \frac{u''}{\eta_0},$$

or equivalently

$$a^{1/\alpha}u'''\eta_0 - \delta_0 u'' \geq 0. \tag{2.15}$$

From the last inequality, we deduce that

$$\left(\frac{u''}{\eta_0^{\delta_0}}\right)' = \frac{a^{1/\alpha}u'''\eta_0 - \delta_0 u''}{a^{1/\alpha}\eta_0^{1+\delta_0}} \geq 0,$$

which means that $u''/\eta_0^{\delta_0}$ is increasing. Thus, the proof is complete. □

Lemma 2.7. *Assume that $u \in \mathcal{N}_a$, and (2.6) and (2.7) hold for some $\lambda \geq 1$ and $\delta \in (0,1)$. Then, the DDE*

$$\left(a^{1/\alpha}(t)\,z'(t)\right)' + \kappa\widehat{q}(t)z(\sigma(t)) = 0, \tag{2.16}$$

has a positive solution, where

$$\kappa := \begin{cases} \frac{1}{\alpha}(1-\delta)^{1-\alpha}\lambda^{\delta(\alpha-1)} & \text{for } \alpha \geq 1; \\ \frac{1}{\alpha}\delta^{\frac{1-\alpha}{\alpha}}\lambda(1-\delta_0)^{\frac{\alpha-1}{\alpha}} & \text{for } \alpha < 1. \end{cases}$$

Proof. Assume that $u \in \mathcal{N}_a$. We have

$$\left(a\left(u'''\right)^{\alpha}\right)' = \alpha\left(a^{1/\alpha}u'''\right)'\left(a^{1/\alpha}u'''\right)^{\alpha-1}.$$

Using this relation in (1.1), we get

$$\left(a^{1/\alpha}(t)\,u'''(t)\right)' + \frac{1}{\alpha}\left(a^{1/\alpha}(t)\,u'''(t)\right)^{1-\alpha} q(t)u^{\alpha}(\sigma(s)) = 0.$$

From $(B_{1,3})$ in Lemma 2.5, we have

$$\left(a^{1/\alpha}(t)\,u'''(t)\right)' + \frac{1}{\alpha}\left(a^{1/\alpha}(t)u'''(t)\right)^{1-\alpha} q(t)\,\frac{\eta_2^{\alpha}(\sigma(t))}{\eta_0^{\alpha}(\sigma(t))}\left(u''(\sigma(t))\right)^{\alpha} \leq 0. \tag{2.17}$$

Since $u''/\eta_0^{1-\delta}$ is decreasing, then

$$u'' \geq \frac{a^{1/\alpha}u'''}{1-\delta}\eta_0. \tag{2.18}$$

For $\alpha \geq 1$, we get

$$\left(a^{1/\alpha}u'''\right)^{1-\alpha} \geq \frac{\left(u''\right)^{1-\alpha}}{\eta_0^{1-\alpha}}(1-\delta)^{1-\alpha}. \tag{2.19}$$

Since $u''/\eta_0^{1-\delta}$ is decreasing, we find

$$u''(t) \leq \frac{u''(\sigma(t))}{\eta_0^{1-\delta}(\sigma(t))}\eta_0^{1-\delta}(t).$$

Hence

$$\left(u''(t)\right)^{1-\alpha} \geq \frac{\left(u''(\sigma(t))\right)^{1-\alpha}}{\left(\eta_0^{1-\delta}(\sigma(t))\right)^{1-\alpha}}\left(\eta_0^{1-\delta}(t)\right)^{1-\alpha}. \tag{2.20}$$

Substituting (2.20) into (2.19), we arrive at

$$\left(a^{1/\alpha}(t)\,u'''(t)\right)^{1-\alpha} \geq \frac{(1-\delta)^{1-\alpha}\,\eta_0^{\delta(\alpha-1)}(t)}{\left(\eta_0^{1-\delta}(\sigma(t))\right)^{1-\alpha}}\left(u''(\sigma(t))\right)^{1-\alpha}.$$

From (2.6), we obtain

$$\left(a^{1/\alpha}(t)\,u'''(t)\right)^{1-\alpha} \geq \frac{(1-\delta)^{1-\alpha}\,\lambda^{\delta(\alpha-1)}}{\eta_0^{1-\alpha}(\sigma(t))}\left(u''(\sigma(t))\right)^{1-\alpha}. \tag{2.21}$$

Combining (2.17) and (2.21), we have

$$\left(a^{1/\alpha}(t)\,u'''(t)\right)' + \frac{(1-\delta)^{1-\alpha}\,\lambda^{\delta(\alpha-1)}}{\alpha}\frac{\eta_2^{\alpha}(\sigma(t))}{\eta_0(\sigma(t))}q(t)\,u''(\sigma(t)) \leq 0,$$

or equivalently

$$\left(a^{1/\alpha}(t)\,u'''(t)\right)' + \kappa_1\widehat{q}(t)u''(\sigma(t)) \leq 0.$$

Letting $z := u''$, we get that z satisfies the linear differential inequality

$$\left(a^{1/\alpha}(t)\, z'(t)\right)' + \kappa_1 \widehat{q}(t) z(\sigma(t)) \le 0.$$

Corollary 1 in [36] ensures that the corresponding DDE (2.16) has a positive solution.

For $\alpha < 1$, from (2.9), we get

$$\left(a^{1/\alpha}(t)\, u'''(t)\right)' + \frac{1}{\alpha}\left(\int_t^{\infty} q(s) u^{\alpha}(\sigma(s))\, \mathrm{d}s\right)^{\frac{1-\alpha}{\alpha}} q(t) u^{\alpha}(\sigma(t)) = 0.$$

From $(B_{1,3})$ in Lemma 2.5, we obtain

$$\left(a^{1/\alpha}(t)\, u'''(t)\right)' + \frac{1}{\alpha}\left(\int_t^{\infty} q(s)\frac{\eta_2^{\alpha}(\sigma(s))}{\eta_0^{\alpha}(\sigma(s))}\left(u''(\sigma(s))\right)^{\alpha} \mathrm{d}s\right)^{\frac{1-\alpha}{\alpha}} q(t)\frac{\eta_2^{\alpha}(\sigma(t))}{\eta_0^{\alpha}(\sigma(t))}\left(u''(\sigma(t))\right)^{\alpha} \le 0.$$

Since $u''/\eta_0^{\delta_0}$ is increasing, we arrive at

$$\begin{aligned}&\left(a^{1/\alpha}(t)\, u'''(t)\right)' + \frac{1}{\alpha}\left(\frac{u''(\sigma(t))}{\eta_0^{\delta_0}(\sigma(t))}\right)^{1-\alpha} \times \\ &\left(\int_t^{\infty} q(s)\frac{\eta_2^{\alpha}(\sigma(s))}{\eta_0^{\alpha}(\sigma(s))}\eta_0^{\alpha\delta_0}(\sigma(s))\, \mathrm{d}s\right)^{\frac{1-\alpha}{\alpha}} q(t)\frac{\eta_2^{\alpha}(\sigma(t))}{\eta_0^{\alpha}(\sigma(t))}\left(u''(\sigma(t))\right)^{\alpha} \le 0\,.\end{aligned}$$

Therefore,

$$\left(a^{1/\alpha}(t)\, u'''(t)\right)' + \frac{1}{\alpha}\frac{q(t)}{\eta_0^{\delta_0(1-\alpha)}(\sigma(t))}\frac{\eta_2^{\alpha}(\sigma(t))}{\eta_0^{\alpha}(\sigma(t))}\left(\int_t^{\infty} q(s)\frac{\eta_2^{\alpha}(\sigma(s))}{\eta_0^{\alpha(1-\delta_0)}(\sigma(s))}\mathrm{d}s\right)^{\frac{1-\alpha}{\alpha}} u''(\sigma(t)) \le 0. \tag{2.22}$$

Using (2.6) and (2.7), we have

$$\begin{aligned}\int_t^{\infty} q(s)\frac{\eta_2^{\alpha}(\sigma(s))}{\eta_0^{\alpha(1-\delta_0)}(\sigma(s))}\mathrm{d}s &\ge \int_t^{\infty} \alpha\delta\frac{1}{a^{1/\alpha}(s)\,\eta_0(s)\,\eta_0^{\alpha(1-\delta_0)}(\sigma(s))}\mathrm{d}s \\ &\ge \int_t^{\infty} \alpha\delta\frac{\eta_0^{\alpha(1-\delta_0)}(s)}{\eta_0^{\alpha(1-\delta_0)}(\sigma(s))}\frac{1}{\eta_0^{\alpha(1-\delta_0)}(s)}\frac{1}{a^{1/\alpha}(s)\,\eta_0(s)}\mathrm{d}s \\ &\ge \alpha\delta\lambda^{\alpha(1-\delta_0)}\int_t^{\infty}\frac{\eta_0^{\alpha(\delta_0-1)-1}(s)}{a^{1/\alpha}(s)}\mathrm{d}s \\ &= \frac{\delta\lambda^{\alpha(1-\delta_0)}}{1-\delta_0}\eta_0^{\alpha(\delta_0-1)}(t)\,.\end{aligned}$$

From (2.22), we obtain

$$\left(a^{1/\alpha}(t)\, u'''(t)\right)' + \frac{1}{\alpha}\frac{\delta^{\frac{1-\alpha}{\alpha}}\lambda^{(1-\alpha)(1-\delta_0)}}{(1-\delta_0)^{\frac{1-\alpha}{\alpha}}} q(t)\frac{\eta_2^{\alpha}(\sigma(t))}{\eta_0^{\alpha}(\sigma(t))}\frac{\eta_0^{(\delta_0-1)(1-\alpha)}(t)}{\eta_0^{\delta_0(1-\alpha)}(\sigma(t))} u''(\sigma(t)) \le 0,$$

which in view of (2.6) yields

$$\left(a^{1/\alpha}(t)\, u'''(t)\right)' + \frac{1}{\alpha}\frac{\delta^{\frac{1-\alpha}{\alpha}}\lambda}{(1-\delta_0)^{\frac{1-\alpha}{\alpha}}}\frac{\eta_2^{\alpha}(\sigma(t))}{\eta_0(t)} q(t) u''(\sigma(t)) \le 0,$$

or equivalently

$$\left(a^{1/\alpha}(t)\, u'''(t)\right)' + \kappa_2 \widehat{q}(t) u''(\sigma(t)) \le 0.$$

As in the case of $\alpha \ge 1$, we can complete the proof of this case. The proof of the lemma is complete. □

3. Oscillatory criteria

Theorem 3.1. *Assume that (2.6) and (2.7) hold for some* $\lambda \geq 1$ *and* $\delta \in (0,1)$*. If*

$$\limsup_{t\to\infty}\left\{\eta_0^{\delta-1}(\sigma(t))\int_{t_1}^{\sigma(t)}\frac{\eta_0(s)}{\eta_0^{\delta-1}(\sigma(s))}\widehat{q}(s)\,\mathrm{d}s + \eta_0^{\delta}(\sigma(t))\int_{\sigma(t)}^{t}\frac{\widehat{q}(s)}{\eta_0^{\delta-1}(\sigma(s))}\mathrm{d}s+\eta_0^{1-\delta_0}(\sigma(t))\int_t^{\infty}\eta_0^{\delta_0}(\sigma(s))\widehat{q}(s)\,\mathrm{d}s\right\}>\frac{1}{\kappa}, \tag{3.1}$$

and there is a $\rho \in C([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\rho(s)\int_s^{\infty}\left(\frac{1}{a(\varrho)}\int_{\varrho}^{\infty}q(v)\left(\frac{\sigma(v)}{v}\right)^{\alpha/\epsilon}\mathrm{d}v\right)^{1/\alpha}\mathrm{d}\varrho-\frac{(\rho'_+(s))^2}{4\rho(s)}\right)\mathrm{d}s=\infty, \tag{3.2}$$

for some $\epsilon \in (0,1)$*, then (1.1) is oscillatory.*

Proof. Assume the contrary that u is an eventually positive solution of (1.1). From Lemma 2.4, $u \in \mathcal{N}_a$ or $u \in \mathcal{N}_b$.

Assume first that $u \in \mathcal{N}_a$. It follows from Lemma 2.7, Eq (2.16) has a positive solution. An integration of (2.16) from t to ∞ yields

$$z'(t) \geq \frac{\kappa_1}{a^{1/\alpha}(t)}\int_t^{\infty}\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s.$$

Integrating once more from t_1 to t, we obtain

$$\begin{aligned} z(t) &\geq \kappa\int_{t_1}^{t}\frac{1}{a^{1/\alpha}(\varrho)}\int_{\varrho}^{\infty}\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s\mathrm{d}\varrho \\ &= \kappa\int_{t_1}^{t}\frac{1}{a^{1/\alpha}(\varrho)}\left(\int_{\varrho}^{t}\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s\mathrm{d}\varrho+\int_t^{\infty}\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s\mathrm{d}\varrho\right). \end{aligned}$$

Thus, we get

$$z(t) \geq \kappa\int_{t_1}^{t}\eta_0(s)\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s+\kappa\eta_0(t)\int_t^{\infty}\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s.$$

Hence,

$$\begin{aligned} z(\sigma(t)) &\geq \kappa\int_{t_1}^{\sigma(t)}\eta_0(s)\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s+\kappa\eta_0(\sigma(t))\int_{\sigma(t)}^{\infty}\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s \\ &\geq \kappa\int_{t_1}^{\sigma(t)}\eta_0(s)\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s+\kappa\eta_0(\sigma(t))\int_{\sigma(t)}^{t}\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s \\ &\quad +\kappa\eta_0(\sigma(t))\int_t^{\infty}\widehat{q}(s)\,z(\sigma(s))\,\mathrm{d}s. \end{aligned}$$

Using the facts that $z/\eta_0^{1-\delta}$ is decreasing and $z/\eta_0^{\delta_0}$ is increasing, we arrive at

$$\frac{1}{\kappa} \geq \frac{1}{\eta_0^{1-\delta}(\sigma(t))}\int_{t_1}^{\sigma(t)}\frac{\eta_0(s)}{\eta_0^{\delta-1}(\sigma(s))}\widehat{q}(s)\,\mathrm{d}s+\eta_0^{\delta}(\sigma(t))\int_{\sigma(t)}^{t}\eta_0^{1-\delta}(\sigma(s))\widehat{q}(s)\,\mathrm{d}s$$

$$+\frac{1}{\eta_0^{\delta_0-1}(\sigma(t))}\int_t^\infty \eta_0^{\delta_0}(\sigma(s))\widehat{q}(s)\,z\mathrm{d}s.$$

This is a contradiction.

Assume now that $u \in \mathcal{N}_b$. Integrating (1.1) from t to ∞ and using the fact that $(a(u''')^\alpha)' \leq 0$, we obtain

$$-a(t)(u'''(t))^\alpha = -\int_t^\infty q(s)\,u^\alpha(\sigma(s))\,\mathrm{d}s. \tag{3.3}$$

Using Lemma 2.2, we find $u \geq \epsilon t u'$ for all $\epsilon \in (0,1)$. Integrating this inequality from σ to t, we get

$$\frac{u(\sigma(t))}{u(t)} \geq \left(\frac{\sigma(t)}{t}\right)^{1/\epsilon}.$$

Therefore, (3.3) becomes

$$a(t)(u'''(t))^\alpha \geq \int_t^\infty q(s)\left(\frac{\sigma(s)}{s}\right)^{\alpha/\epsilon} u(s)\,\mathrm{d}s.$$

Since $u'(t) > 0$, then

$$a(t)(u'''(t))^\alpha \geq u^\alpha(t)\int_t^\infty q(s)\left(\frac{\sigma(s)}{s}\right)^{\alpha/\epsilon} u(s)\,\mathrm{d}s,$$

or equivalently

$$u'''(t) \geq u(t)\left(\frac{1}{a(t)}\int_t^\infty q(s)\left(\frac{\sigma(s)}{s}\right)^{\alpha/\epsilon} u(s)\,\mathrm{d}s\right)^{1/\alpha}.$$

Integrating this inequality from t to ∞, we have

$$u''(t) \leq -u(t)\int_t^\infty \left(\frac{1}{a(\varrho)}\int_\varrho^\infty q(s)\left(\frac{\sigma(s)}{s}\right)^{\alpha/\epsilon} u(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}\varrho. \tag{3.4}$$

Now, define

$$w := \rho\frac{u'}{u}.$$

Then, $w(t) \geq 0$ for $t \geq t_1 \geq t_0$ and

$$\begin{aligned} w' &= \rho'\frac{u'}{u} + \rho\frac{u''}{u} - \rho\frac{(u')^2}{u^2} \\ &= \rho\frac{u''}{u} + \frac{\rho'}{\rho}w - \frac{1}{\rho}w^2. \end{aligned}$$

Hence, by (3.4), we get

$$w'(t) \leq -\rho(t)\int_t^\infty \left(\frac{1}{a(\varrho)}\int_\varrho^\infty q(s)\left(\frac{\sigma(s)}{s}\right)^{\alpha/\epsilon}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}\varrho + \frac{\rho'_+(t)}{\rho(t)}w(t) - \frac{1}{\rho(t)}w^2(t). \tag{3.5}$$

Using Lemma 2.3 with $B = \rho'_+/\rho$, and $A = 1/\rho$, we obtain

$$\frac{\rho'_+}{\rho}w - \frac{1}{\rho}w^2 \leq \frac{(\rho'_+)^2}{4\rho}.$$

Consequently, (3.5) leads to

$$w'(t) \leq -\rho(t) \int_t^\infty \left(\frac{1}{a(\varrho)} \int_\varrho^\infty q(s) \left(\frac{\sigma(s)}{s} \right)^{\alpha/\epsilon} \mathrm{d}s \right)^{1/\alpha} \mathrm{d}\varrho + \frac{(\rho'_+(t))^2}{4\rho(t)}.$$

Integrating this inequality from t_1 to t, we have

$$\int_{t_1}^t \left(\rho(s) \int_s^\infty \left(\frac{1}{a(\varrho)} \int_\varrho^\infty q(v) \left(\frac{\sigma(v)}{v} \right)^{\alpha/\epsilon} \mathrm{d}v \right)^{1/\alpha} \mathrm{d}\varrho - \frac{(\rho'_+(s))^2}{4\rho(s)} \right) \mathrm{d}s \leq w(t_1),$$

which contradicts (3.2). Hence, the proof of this theorem is complete. □

Example 3.1. *Consider the DDE*

$$\left(t^\gamma (u'''(t))^\alpha\right)' + \frac{q_0}{t^{3\alpha-\gamma+1}} u^\alpha(\sigma_0 t) = 0, \tag{3.6}$$

with $\gamma < \alpha$, $\sigma_0 \in (0,1)$ *and* $q_0 > 0$. *By comparing (1.1) and (3.6) we see that* $a(t) = t^\gamma$, $\sigma(t) = \sigma_0 t$. *Then*
$\eta_0(t) = \frac{t^{1-\gamma/\alpha}}{1-\gamma/\alpha}$, $\eta_1(t) = \frac{t^{2-\gamma/\alpha}}{(1-\gamma/\alpha)(2-\gamma/\alpha)}$, $\eta_2(t) = \frac{t^{3-\gamma/\alpha}}{(1-\gamma/\alpha)(2-\gamma/\alpha)(3-\gamma/\alpha)}$, $q(t) = \frac{q_0}{t^{3\alpha-\gamma+1}}$, $\lambda = \frac{\eta_0(t)}{\eta_0(\sigma(t))} = \sigma_0^{\gamma/\alpha-1}$, $\delta = \frac{\sigma_0^{3\alpha-\gamma} q_0}{\alpha(1-\gamma/\alpha)^{\alpha+1}(2-\gamma/\alpha)^\alpha(3-\gamma/\alpha)^\alpha}$, $\delta_0 = \delta^{1/\alpha}\left(\frac{1}{\sigma_0}\right)^{\delta(1-\gamma/\alpha)}$, *and condition (3.1) in Theorem 3.1 leads to*

$$\frac{q_0}{1-\delta}\sigma_0^{3\alpha-\gamma-\delta(1-\gamma/\alpha)} + \frac{q_0}{\delta}\sigma_0^{3\alpha-\gamma-\delta(1-\gamma/\alpha)}\left(1-\sigma_0^{\delta(1-\gamma/\alpha)}\right) + \frac{q_0}{1-\delta_0}\sigma_0^{3\alpha-\gamma} > \frac{\theta}{\kappa_1}, \tag{3.7}$$

where

$$\theta = (1-\gamma/\alpha)^{\alpha+1}(2-\gamma/\alpha)^\alpha(3-\gamma/\alpha)^\alpha.$$

Also, condition (3.2) in Theorem 3.1 is met where $\rho(t) = t^\alpha$ *and*

$$q_0 > \frac{\alpha^{2\alpha}(3\alpha-\gamma)}{(2\sigma_0)^\alpha}. \tag{3.8}$$

Now, by using Theorem 3.1, Eq (3.6) with $\alpha > 1$ *is oscillatory provided that (3.7) and (3.8) are satisfied. Setting values for* γ *and* α*, the above criteria generated the oscillatory results of Eq (3.6).*

4. Conclusions

In this paper, we investigated the asymptotic properties of positive solutions for fourth-order quasi-linear DDEs in the canonical case. There are new conditions that ensure that Eq (1.1) has no positive solutions. In addition, we prove an important theorem that ensures all solutions of Eq (1.1) are oscillatory if certain criteria are met. Finally, we provided an example that supports our research and illustrates the significance of the results. In our future study, we will try to generalize these criteria to include the n-th order DDE.

Acknowledgements

We would like to thank to the referees for the useful comments that helped us to improve the original manuscript.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. R. Koplatadze, G. Kvinikadze, I. P. Stavroulakis, Properties A and B of nth order linear differential equations with deviating argument, *Georgian Math. J.*, **6** (1999), 553–566. https://doi.org/10.1515/GMJ.1999.553
2. W. Wang, Further results on mean-square exponential Input-to-State stability of stochastic delayed Cohen-Grossberg neural networks, *Neural Process. Lett.*, 2022. https://doi.org/10.1007/s11063-022-10974-8
3. C. Huang, B. Liu, H. Yang, J. Cao, Positive almost periodicity on SICNNs incorporating mixed delays and D operator, *Nonlinear Anal. Model. Control*, **27** (2022), 719–739. https://doi.org/10.15388/namc.2022.27.27417
4. X. Zhang, H. Hu, Convergence in a system of critical neutral functional differential equations, *Appl. Math. Lett.*, **107** (2020), 106385. https://doi.org/10.1016/j.aml.2020.106385
5. K. S. Chiu, T. Li, Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments, *Math. Nachr.*, **292** (2019), 2153–2164. https://doi.org/10.1002/mana.201800053
6. G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, New York: Marcel Dekker, 1987.
7. I. Gyori, G. Ladas, *Oscillation theory of delay differential equations with applications*, Oxford: Clarendon Press, 1991.
8. M. N. Oguztoreli, R. B. Stein, An analysis of oscillations in neuro-muscular systems, *J. Math. Biol.*, **2** (1975), 87–105. https://doi.org/10.1007/BF00275922
9. J. Džurina, S. R. Grace, I. Jadlovská, T. Li, On the oscillation of fourth-order delay differential equations, *Adv. Differ. Equ.*, **2019** (2019), 118. https://doi.org/10.1186/s13662-019-2060-1
10. T. Li, Y. V. Rogovchenko, On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations, *Appl. Math. Lett.*, **67** (2017), 53–59. https://doi.org/10.1016/j.aml.2016.11.007
11. G. E. Chatzarakis, J. Dzurina, I. Jadlovska, New oscillation criteria for second-order half-linear advanced differential equations, *Appl. Math. Comput.*, **347** (2019), 404–416. https://doi.org/10.1016/j.amc.2018.10.091
12. O. Bazighifan, C. Cesarano, Some new oscillation criteria for second order neutral differential equations with delayed arguments, *Mathematics*, **7** (2019), 619. https://doi.org/10.3390/math7070619
13. J. Džurina, S. R. Grace, I. Jadlovská, T. Li, Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term, *Math. Nachr.*, **293** (2020), 910–922. https://doi.org/10.1002/mana.201800196

14. C. Jayakumar, S. S. Santra, D. Baleanu, R. Edwan, V. Govindan, A. Murugesan, et al., Oscillation result for half-linear delay difference equations of second-order, *Math. Biosci. Eng.*, **19** (2022), 3879–3891. http://dx.doi.org/10.3934/mbe.2022178

15. S. S. Santra, A. Scapellato, Some conditions for the oscillation of second-order differential equations with several mixed delays, *J. Fixed Point Theory Appl.*, **24** (2022), 18. https://doi.org/10.1007/s11784-021-00925-6

16. O. Bazighifan, S. S. Santra, Second-order differential equations: Asymptotic behavior of the solutions, *Miskolc Math. Notes*, **23** (2022), 105–115. http://dx.doi.org/10.18514/MMN.2022.3369

17. S. S. Santra, A. Scapellato, O. Moaaz, Second-order impulsive differential systems of mixed type: oscillation theorems, *Bound. Value Probl.*, **2022** (2022), 67. https://doi.org/10.1186/s13661-022-01648-4

18. S. S. Santra, D. Baleanu, K. M. Khedher, O. Moaaz, First-order impulsive differential systems: sufficient and necessary conditions for oscillatory or asymptotic behavior, *Adv. Differ. Equ.*, **2021** (2021), 283. https://doi.org/10.1186/s13662-021-03446-1

19. A. K. Tripathy, S. S. Santra, Necessary and sufficient conditions for oscillation of second-order differential equations with nonpositive neutral coefficient, *Math. Bohem.*, **146** (2021), 185–197.

20. J. Alzabut, S. R. Grace, S. S. Santra, G. N. Chhatria, Asymptotic and oscillatory behaviour of third order non-linear differential equations with canonical operator and mixed neutral terms, *Qual. Theory Dyn. Syst.*, **22** (2023), 15. https://doi.org/10.1007/s12346-022-00715-6

21. S. R. Grace, G. N. Chhatria, On oscillatory behaviour of third-order half-linear dynamic equations on time scales, *Opus. Math.*, **42** (2022), 849–865. http://dx.doi.org/10.7494/OpMath.2022.42.6.849

22. O. Bazighifan, Nonlinear differential equations of fourth-order: qualitative properties of the solutions, *AIMS Math.*, **5** (2020), 6436–6447. http://dx.doi.org/10.3934/math.2020414

23. A. Almutairi, O. Bazighifan, B. Almarri, M. A. Aiyashi, K. Nonlaopon, Oscillation criteria of solutions of fourth-order neutral differential equations, *Fractal Fract.*, **5** (2021), 155. https://doi.org/10.3390/fractalfract5040155

24. R. P. Agarwal, S. R. Grace, D. O'Regan, Oscillation criteria for certain nth order differential equations with deviating arguments, *J. Math. Analy. Appl.*, **262** (2001), 601–622. https://doi.org/10.1006/jmaa.2001.7571

25. B. Baculikova, J. Dzurina, J. R. Graef, On the oscillation of higher order delay differential equations, *J. Math. Sci.*, **184** (2012), 398–400.

26. T. Li, Y. V. Rogovchenko, Oscillation criteria for even-order neutral differential equations, *Appl. Math. Lett.*, **61** (2016), 35–41. https://doi.org/10.1016/j.aml.2016.04.012

27. R. P. Agarwal, M. Bohner, T. Li, C. Zhang, A new approach in the study of oscillatory behavior of even-order neutral delay differential equations, *Appl. Math. Comput.*, **225** (2013), 787–794. https://doi.org/10.1016/j.amc.2013.09.037

28. B. Baculıkova, J. Dzurina, T. Li, Oscillation results for even-order quasilinear neutral functional differential equations, *Electron. J. Diffe. Eq.*, **2011** (2011), 1–9.

29. T. Li, Z. Han, P. Zhao, S. Sun, Oscillation of even-order neutral delay differential equations, *Adv. Diff. Equ.*, **2010** (2010), 184180. https://doi.org/10.1155/2010/184180

30. B. Baculıkova, J. Dzurina, Oscillation theorems for higher order neutral differential equations, *Appl. Math. Comput.*, **219** (2012), 3769–3778. https://doi.org/10.1016/j.amc.2012.10.006

31. B. Baculikova, Oscillation of second-order nonlinear noncanonical differential equations with deviating argument, *Appl. Math. Lett.*, **91** (2019), 68–75. https://doi.org/10.1016/j.aml.2018.11.021

32. B. Baculíková, J. Dzurina, Oscillatory criteria via linearization of half-linear second order delay differential equations, *Opusc. Math.*, **40** (2020), 523–536. https://doi.org/10.7494/OpMath.2020.40.5.523

33. R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for difference and functional differential equations*, Dordrecht: Kluwer Academic, 2000.

34. I. T. Kiguradze, T. A. Chanturiya, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Dordrecht: Kluwer Academic, 1993.

35. C. Zhang, R. P. Agarwal, M. Bohner, T. Li, New results for oscillatory behavior of even-order half-linear delay differential equations, *Appl. Math. Lett.*, **26** (2013), 179–183. https://doi.org/10.1016/j.aml.2012.08.004

36. T. Kusano, M. Naito, Comparison theorems for functional differential equations with deviating arguments, *J. Math. Soc. Jpn.*, **3** (1981), 509–532.

AIMS Press