$\mathcal{O}$-Convexity: Computing Hulls, Approximations, and Orientation Sets[1]

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Abstract

We continue the investigation of computational aspects of restricted-orientation convexity ($\mathcal{O}$-convexity) in two dimensions. We introduce one notion of an $\mathcal{O}$-halfplane, for a set $\mathcal{O}$ of orientations, and we investigate $\mathcal{O}$-connected convexity. The $\mathcal{O}$-connected convex hull of a finite set X can be computed in time $O(|X| \log |X| + |\mathcal{O}|)$. The $\mathcal{O}$-connected hull is a basis for determining the $\mathcal{O}$-convex hull of a finite set X and a finite set $\mathcal{O}$ of orientations in time $O(|X||\mathcal{O}| \log |X|)$. We also consider two new problems. First, we give an algorithm to determine a minimum-area $\mathcal{O}$-connected convex outer approximation of an $\mathcal{O}$-polygon with n vertices when the number r of $\mathcal{O}$-halfplanes forming the approximation is given. The approximation can be determined in time $O(n^2r+ |\mathcal{O}|)$. Second, we give an algorithm to find the largest orientation set for a simple polygon. This problem can be solved in time $O(n \log n)$, where n is the number of vertices of the polygon. For each of these complexity bounds we assume that $\mathcal{O}$ is sorted.

1 Introduction

Let $\mathcal{O}$ be a set of unit vectors (orientations) in $\mathcal{R}^2$. A line (segment, ray) in the plane is called an **$\mathcal{O}$-line (${\mathcal{O}}$-segment, $\mathcal{O}$-ray)** if its orientation vector is collinear to a vector of $\mathcal{O}$. Güting [3] introduced, essentially, the notion of an **$\mathcal{O}$-oriented polygon**; that is, a polygon whose edges are $\mathcal{O}$-segments.

A planar point set X is **$\mathcal{O}$-convex** if the intersection of X with any $\mathcal{O}$-line is empty or connected. The **$\mathcal{O}$-convex hull** of a planar set is defined to be the smallest $\mathcal{O}$-convex set that contains the set. Rawlins [6] initiated the study of $\mathcal{O}$-convexity, Schuierer [11] examined the relationship of $\mathcal{O}$-convexity and $\mathcal{O}$-visibility. Investigation of these notions in higher dimensions has recently been initiated by Fink and Wood [1, 2] and Metelski [4].

Following Rawlins and Wood [6, 8, 10], we also refer to $\mathcal{O}$-convexity as **restricted-orientation convexity.** This natural generalization (or relaxation) of standard convexity arose in computational geometry in the eighties, initially for two axial orientations in the plane (isothetic convexity or ortho-convexity). It has applications to problems of VLSI layout synthesis, database design, computational morphology, image processing, and stock cutting.

The problem of determining the ortho-convex hull of ortho-polygons was the first computational problem in $\mathcal{O}$-convexity. Consult Wood's review [12] for more details and references. Rawlins and Wood [6, 7] showed that the four kinds of $\mathcal{O}$-convex hull of an $\mathcal{O}$-oriented polygon can be determined in optimal (linear) time.

It should be noted that there are substantial difficulties in developing algorithms for computing the $\mathcal{O}$-convex hulls of disconnected sets. For example, Rawlin's decomposition theorem [6, 9] that

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leads to a natural approach to compute the $\mathcal{O}$-convex hull of a connected planar set, for finite $\mathcal{O}$, is not valid for disconnected sets. The $\mathcal{O}$-convex hull of a disconnected planar ortho-region and a finite point set have been determined for only two orientations [5, 12]. In these cases, **halfplane-convexity,** a restriction of ortho-convexity, plays a key role in the construction of the boundaries of the connected components of the $\mathcal{O}$-convex hull. To compute the $\mathcal{O}$-convex hull of a finite planar point set X, for any finite $\mathcal{O}$, we introduce, in Section 2, an analog of halfplane-convexity (we call it **$\mathcal{O}$-connected convexity**) and we give an algorithm to construct the $\mathcal{O}$-connected convex hull of X in $O(|X| \log |X| + |\mathcal{O}|)$ time and in $O(|X| + |\mathcal{O}|)$ space. An algorithm to compute the **$\mathcal{O}$-convex hull** of a set is given in Section 3. It runs in $O(|X||\mathcal{O}| \log |X|)$ time and uses $O(|X||\mathcal{O}|)$ space.

In Section 4, we consider the construction of an **outer approximation** of a finite planar set X with an $\mathcal{O}$-connected convex polygon $OA(X)$ that is further restricted. It turns out that the approximation problem can be effectively reduced to a polynomial-time solvable network optimization problem.

In all previous work, it is assumed that an orientation set is given and that computational properties of $\mathcal{O}$-convex sets are studied. In Section 5, we consider an "inverse" problem; namely, determine the set $Or(X)$ of all vectors v such that each v-line has a connected intersection with X. The notion of $Or(X)$ leads to measures of the convexity of point sets which may find applications in shape analysis and image processing. An $O(n \log n)$-time algorithm to compute $Or(P)$ of a polygon P with n vertices is also sketched.

2 $\mathcal{O}$-Connected convexity

We begin with the following important notion [7].

Definition 1 *For a planar set X, the convex $\mathcal{O}$-hull of X is the smallest convex $\mathcal{O}$-oriented polygon that contains X.*

If X is an $\mathcal{O}$-oriented polygon with n vertices, then the convex $\mathcal{O}$-hull of (X) can be found in $\Theta(n + |\mathcal{O}|)$ optimal time [7]. Starting from the convex hull of X and exploiting notions of $\mathcal{O}$-oriented supporting lines, antipodal points, and Coxeter's star structure, we obtain the following result; see Fig. 1.

Theorem 1 *Let X be a finite planar point set and $\mathcal{O}$ be a sorted finite-orientation set. Then, the convex $\mathcal{O}$-hull of X can be determined in $O(|X| \log |X| + |\mathcal{O}|)$ time and in $O(|X| + |\mathcal{O}|)$ space,*

Convex cones play an important role in the definition of an analog of halfplane-convexity. Recall that a set C is a **convex cone** if, for all vectors $x, y \in C$ and for all nonnegative real numbers λ and μ, we have $\lambda x + \mu y \in C$. A convex cone C is **pointed** if whenever we have $x \in C$ and $-x \in C$, then $x = 0$. The **conic hull** of a planar set is the smallest convex cone that contains the set.

Since every $\mathcal{O}'$ such that $\mathcal{O} \subseteq \mathcal{O}' \subseteq \mathcal{O} \cup -\mathcal{O}$ defines the same class of restricted-orientation convex sets, we can assume that $\mathcal{O}$ is a symmetric vector set; that is, we have $\mathcal{O} = \mathcal{O} \cup -\mathcal{O}$. Let $cl(X)$ denote the topological closure of a set X.

Definition 2 *A cone is an* **$\mathcal{O}$-cone** *if it is the conic hull of some subset of $\mathcal{O}$. Let a be a point in the plane and C be a maximal pointed $\mathcal{O}$-cone; then, the set $cl(\mathcal{R}^2 \setminus (a + C))$ is an* **$\mathcal{O}$-halfplane** *at the point a; see Fig. 2.*

Definition 3 *The intersection of all $\mathcal{O}$-halfplanes that contain a planar set is called the* **$\mathcal{O}$-connected hull** *of the set. A set is $\mathcal{O}$-connected convex if it equals its $\mathcal{O}$-connected hull.*

Theorem 2 *Let $\mathcal{O}$ be a finite orientation set. Then, all $\mathcal{O}$-connected convex sets are $\mathcal{O}$-convex. Conversely, all closed connected $\mathcal{O}$-convex sets are $\mathcal{O}$-connected convex.*

Theorem 3 *For a sorted finite orientation set $\mathcal{O}$ and a finite set X, the $\mathcal{O}$-connected hull of X can be computed in $O(|X| \log |X| + |\mathcal{O}|)$ time and in $O(|X| + |\mathcal{O}|)$ space.*

3 The $\mathcal{O}$-convex hull of a finite point set

From Theorem 2 it follows that, for closed connected sets, the $\mathcal{O}$-convex hull of X coincides with the $\mathcal{O}$-connected hull of X. As we have remarked, the disconnected case is more complex. Having $\mathcal{O}$-connected convexity as the appropriate generalization of ortho-orientation halfplane-convexity at our disposal, we are able to extend the approach of Ottmann *et al.* [5] to arbitrary finite $\mathcal{O}$ and X. Our algorithm also has the following two steps:

Step 1: Determine maximum subsets of the given set X (clusters) that generate connected components of the $\mathcal{O}$-convex hull of X; see Fig. 3.

Step 2: Construct the boundaries of the components by computing the $\mathcal{O}$-connected hulls of the clusters.

Theorem 4 *For sorted finite $\mathcal{O}$ and X, the $\mathcal{O}$-convex hull of X can be computed in $O(|X||\mathcal{O}| \log |X|)$ time and in $O(|X||\mathcal{O}|)$ space.*

4 $\mathcal{O}$-connected convex approximations

A number of algorithms in the fields of VLSI synthesis and cutting-stock problems are based on outer approximations that simplify the geometric shapes of objects. In the case of finite $\mathcal{O}$, the class of all $\mathcal{O}$-convex sets is too vast and contains rather unusual sets. In this context, $\mathcal{O}$-connected convex sets, which are formed by intersections of $\mathcal{O}$-halfplanes, appear to be more regular and more suitable for approximation.

When the intersection of a finite planar set of $\mathcal{O}$-halfplanes is a polygon, the polygon is clearly an $\mathcal{O}$-connected convex $\mathcal{O}$-oriented polygon. We call it an **fhp-polygon.** The following proposition, which results from Definitions 2 and 3, makes clear the structure of fhp-polygons.

Theorem 5 *Let P be a fhp-polygon that is not classically convex; then, there are maximal pointed $\mathcal{O}$-cones $C_1, \ldots, C_r$ such that P is the set-theoretic difference of the convex $\mathcal{O}$-hull of P and the interior of the cones; see Fig. 4.*

The minimal number of the cones $C_1, \ldots, C_r$, for the fhp-polygon P, is called its **rank.** The rank of a classically convex polygon is defined to be zero. We consider the rank of an fhp-polygon to be a measure of its shape convexity.
Approximation problem: Given an $\mathcal{O}$-oriented polygon P and a positive integer r, determine a minimum-area fhp-polygon $OCA(P, r)$ that contains P and has rank at most r.

Theorem 6 *The approximation problem can be reduced to the problem of searching for a path of maximal total weight, with at most r arcs, in a weighted acyclic digraph. For an $\mathcal{O}$-oriented polygon P with n vertices, $OCA(P, r)$ can be computed in $O(n^2r + |\mathcal{O}|)$ time and in $O(nr + |\mathcal{O}|)$ space.*

5 Computing the cone of convex orientations of a polygon

Definition 4 *Let X be a planar set and $Or(X)$ be the set of all unit planar vectors v that have a connected intersection with X. The set $K(X) = \{\lambda v \mid v \in Or(X) \text{ and } \lambda \geq 0\}$ is the* **cone of convex orientations** *of X.*

Let P be a simple polygon. A chain $(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_{k+1})$, $k \geq 2$ of edges of P is **concave** if the inner angles of the vertices $v_2, v_3, \ldots, v_k$ are of more than π radians.

Theorem 7 *Suppose that $Ch_1, Ch_2, \ldots, Ch_r$ is the set of all maximal concave chains of P; then, $K(P) = \cap_{i=1}^{r} K(Ch_i)$, where the chains are considered to be planar point sets; see Fig. 5.*

This theorem is the basis for our algorithm, which leads to the following result.

Theorem 8 *The cone of convex orientations of a simple polygon with n vertices can be computed in $O(n \log n)$ time and $O(n)$ space.*

References

[1] Eugene Fink and Derick Wood. Fundamentals of restricted-orientation convexity. *Information Sciences*, 1996, to appear.

[2] Eugene Fink and Derick Wood. Generalized halfspaces in restricted-orientation convexity. Unpublished manuscript, 1995.

[3] Ralf Hartmut Güting. Stabbing C-oriented polygons. *Information Processing Letters*, 16:35–40, 1983.

[4] Nikolai N. Metelski. On semispaces of partial convexity. *Doklady of Belarussian Academy of Sciences*, 3:12–16, 1995, (in Russian).

[5] Thomas Ottmann, Eljas Soisalon-Soininen, and Derick Wood. On the definition and computation of rectilinear convex hulls. *Information Sciences*, 33:157–171, 1984.

[6] Gregory J. E. Rawlins. *Explorations in Restricted-Orientation Geometry.* PhD thesis, University of Waterloo, 1987. Technical Report CS-89-48.

[7] Gregory J. E. Rawlins and Derick Wood. Optimal computation of finitely oriented convex hulls. *Information and Computation*, 72:150–166, 1987.

[8] Gregory J. E. Rawlins and Derick Wood. Ortho-convexity and its generalizations. In Godfried T. Toussaint, editor, *Computational Morphology*, pages 137–152. Elsevier Science Publishers B. V., North-Holland, 1988.

[9] Gregory J. E. Rawlins and Derick Wood. A decomposition theorem for convexity spaces. *Journal of Geometry*, 36:143–157, 1989.

[10] Gregory J. E. Rawlins and Derick Wood. Restricted-orientation convex sets. *Information Sciences*, 54:263–281, 1991.

[11] Sven Schuierer. *On Generalized Visibility.* PhD thesis, Universität Freiburg, Germany, 1991.

[12] Derick Wood. An isothetic view of computational geometry. In Godfried T. Toussaint, editor, *Computational Geometry*, pages 429–459. Elsevier Science Publishers B. V., North-Holland, 1985.

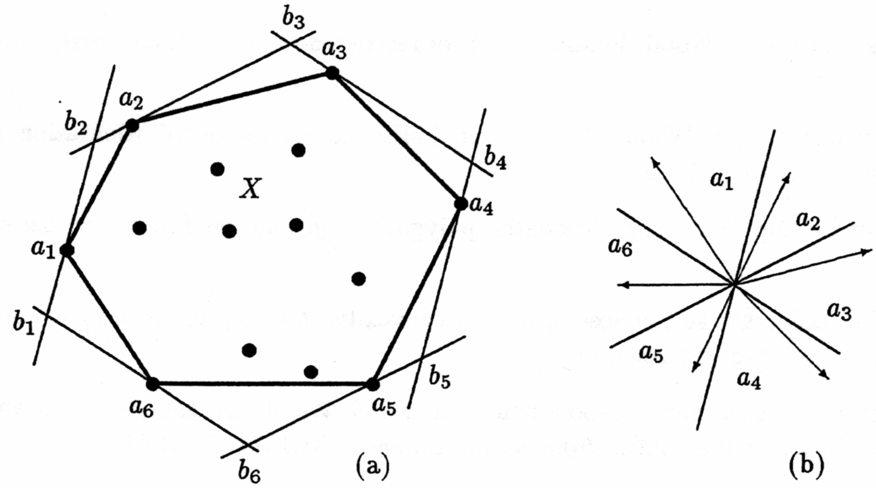

Figure 1: a. The standard convex hull (outlined with heavy lines) and the convex $\mathcal{O}$-hull (vertices: $b_1, b_2, b_3, b_4, b_5, b_6$) of a finite point set X. b. Superposition of Coxeter's star (the arrows) and the given $\mathcal{O}$-lines. The pairs (a_1, a_4), (a_2, a_5), and (a_3, a_6) are antipodal.

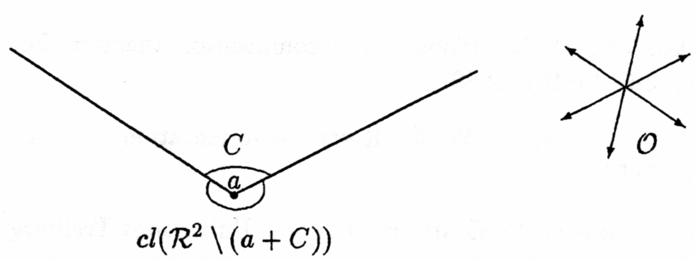

Figure 2: An $\mathcal{O}$-cone C and the $\mathcal{O}$-halfplane $cl(\mathcal{R}^2 \setminus (a + C))$.

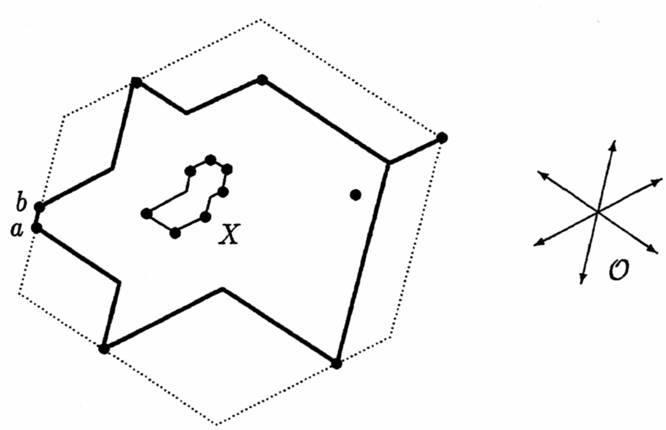

Figure 3: The $\mathcal{O}$-connected hull (outlined with heavy lines) and components of the $\mathcal{O}$-convex hull of a finite set X (the inner polygon, the segment $[a, b]$, and all the other points of X).

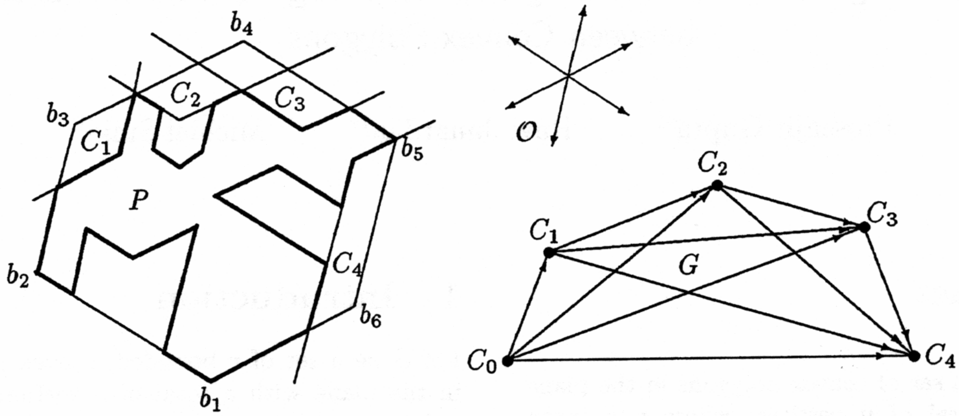

Figure 4: An $\mathcal{O}$-oriented polygon P, its maximal pointed $\mathcal{O}$-cones $C_1, \ldots, C_4$, and its corresponding digraph.

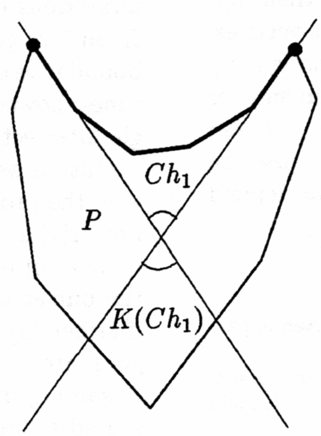

Figure 5: A polygon P with its unique, maximal concave chain Ch_1 such that $K(P) = K(Ch_1)$.