# On Stable Line Segments in Triangulations

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## 1 Overview

Let $S$ be a set of $n$ points in the plane and $E$ denote the set of all the line segments with endpoints in $S$. A line segment $\overline{pq}$ with $p, q \in S$ is called a **stable line segment** of all triangulations of $S$, if no line segment in $E$ properly intersects $\overline{pq}$. The intersection of all possible triangulations of $S$ then is the set of all stable line segments in $S$, denoted by $SL(S)$.

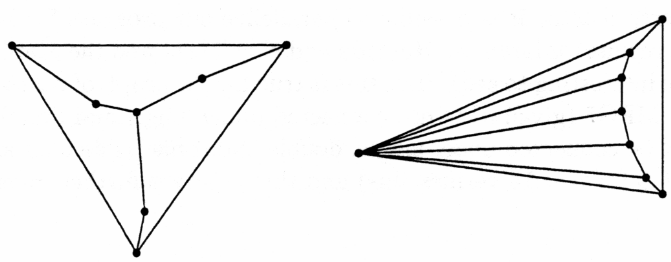
As a combinatorial problem, various properties of stable line segments of a set of planar points have been investigated in [Xu92]. It is shown that the maximum number of stable line segments in $S$ is $2(n-1)$. There is an interesting relationship between stable line segments and so-called extreme line segments $EL(S)$ [Ed86]. A line segment $\overline{pq}$ with $p, q \epsilon S$ is called an extreme line segment if $\{\overline{pq}\} = E \cap H$ for some open half-plane $H$ [Ed86]. Then, we have that

$$CH(S) \subseteq EL(S) \subseteq SL(S).$$

A more important property is the relationship between $SL(S)$ and so-called $k$-optimal triangulations. Let $T(S)$ denote a triangulation of $S$. $T(S)$ is called a **$k$-optimal triangulation** for $4 \leq k < n$, denoted by $LOT_k(S)$, if every $k$-sided simple polygon drawn from $T(S)$ is optimally triangulated by some edges of $T(S)$.

Let $SL_k(S)$ denote the intersection of all possible $LOT_k(S)$'s (i.e., the set of edges that are in every $LOT_k(S)$). Let $MWT(S)$ denote a minimum weight triangulation of $S$. Then, we have that

$$SL(S) \subseteq SL_4(S) \subseteq \cdots \subseteq SL_k(S) \cdots \subseteq SL_{n-1}(S) \subseteq MWT(S).$$



(a) (b)

Figure 1:

In some special cases of $S$, $SL(S)$ forms a connected graph as shown in Figure 1. Thus, an $MWT(S)$ can be constructed in polynomial time using the dynamic programming algorithm proposed in [Gi79, Kl80].

So far the structure properties of $SL(S)$ have been thoroughly studied, but not its algorithmic issue.

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A recent result on finding a subgraph $LOT(S)$ of $SL_4(S)$ [DM96] implies an $O(n^4)$ time and $O(n^3)$ space algorithm for finding $SL(S)$ since it is not difficult to show that

$$SL(S) \subseteq LOT(S) \subseteq SL_4(S).$$

In this paper, we shall propose two algorithms for computing $SL(S)$. One is an $O(n^2 \log n)$ time and $O(n)$ space algorithm and the other is an $O(n^2)$ time and $O(n^2)$ space algorithm.

## 2 Introduction

A triangulation of a planar point set $S$ is defined as a maximal set of non-crossing line segments which have both endpoints in $S$. A minimum weight triangulation of $S$ (denoted $MWT(S)$) is a triangulation among all possible triangulations over $S$ such that the sum of its total edge lengths is minimal. To compute an $MWT$ of a point set is an outstanding open problem, whose complexity status is unknown since 1975 [SH75, GJ79]. An $O(n^3)$ time dynamic programming algorithm for constructing an $MWT$ of a simply polygon was given independently in [Gi79, Kl80]. Based on the above mentioned dynamic programming algorithm, Anagnostou and Corneil [AC93] designed an $O(n^{3k+1})$ time algorithm for computing an $MWT$ of a point set with k nested convex polygons, and later Meijer and Rappaport [MR92] improved the time complexity to $O(n^k)$ when each of the $k$ nested polygons degenerated into a straight line segment. Xu and others [Xu92, CGJ95] showed that if a subgraph of an $MWT$ with k connected components is given, then an $MWT$ can be found in $O(n^{k+2})$ time. Up to now, none of the existing algorithms for finding an $MWT$ of a general point set achieves polynomial time bound. An alternative direction is to identify a subset of line segments in $E$ belonging to an $MWT$. The advantage of this direction is two-fold. The more such line segments are identified, the more likely the resulting subgraph will connect all the points in $S$. Then, the ultimate solution can be found in $O(n^{k+2})$ time by using dynamic programming. On the other hand, it was shown in [XZ96] that finding more line segments within an $MWT$ can improve the performance of some heuristics.

Several investigations have reported on the subgraphs of an $MWT$, [BDE96, CX96, DM96, Ke94, Xu92, Xu96, YXY94]. A trivial case is the set of line segments in all triangulations of a given point set S (i.e., a set of stable line segments $SL(S)$). No detailed work was done on the algorithms for computing $SL(S)$. In the following section, we shall propose two algorithms for computing $SL(S)$.

## 3 Algorithmic Issues

Let $J$ denote the set of all triangulations of a point set $S$, then we have the following obvious facts:

**Fact 1.** $SL(S) = \cap_{T(S) \in J} T(S)$, and

**Fact 2.** $\overline{pq} \in SL(S)$ **iff** no line segment with endpoints in $S$ properly intersects $\overline{pq}$.

Note that the Delaunay triangulation of $S$, $DT(S)$, belongs to $J$. By Fact 1, we first construct the Delaunay triangulation $DT(S)$ and then test whether the line segments in $DT(S)$ are also in $SL(S)$. Note that the number of line segments in $DT(S)$ is linearly proportional to $n$, it is easy to design an $O(n^3)$ time algorithm by testing all possible intersections of the line segments with Delaunay edges.

With a more detailed geometric analysis, we can improve the time complexity from $O(n^3)$ to $O(n^2 \log n)$ and space complexity from $O(n^2)$ to $O(n)$ or time complexity to $O(n^2)$ and space complexity remains as $O(n^2)$.

## 3.1 Algorithm 1

**Lemma 1** *Let $\overline{pq}$ be a line segment, $\{p,q\} \cup S$ be a simple point set, $|S| = n$. To determine whether there is a line segment with two endpoints in $S$ that properly intersects $\overline{pq}$ can be answered in $O(n \log n)$ time and $O(n)$ space.*

**Proof** First, by a rigid motion we can transform point $p$ to the origin and point $q$ on the $x$-axis and denote its coordinates $(x^*, 0), x^* > 0$. This can be done in $O(n)$ time. In the new coordinate system, $S$ becomes $S'$, $p \to p'$ and $q \to q'$, $p' = (0,0)$ and $q' = (x^*, 0)$, and $r = (x(r), y(r))$ in $S'$. If no points in $S'$ are below (or above) $x$-axis, then no line segment with two endpoints in $S'$ intersects the line segment $L(p', q')$. If there are points with $y(p_i) > 0$ and $y(p_j) < 0$ for $p_i, p_j \in S'$, we divide $S'$ into two subsets

$$
\begin{aligned}
S'_+ &= \{p \mid y(p) > 0, p \in S'\} \\
S'_- &= \{p \mid y(p) < 0, p \in S'\}
\end{aligned}
$$

This step can be done in $O(n)$ time. For convenience of discussion, we assume there is no point $r$ in $S'$ with $x(r) = 0$ or $x(r) = x^*$, so we can divide $S'_+$ and $S'_-$ into $S'_+(-1), S'_+(0)$, $S'_+(+1)$ and $S'_-(-1), S'_-(0)$, $S'_-(+1)$ as follows:

$$
\begin{aligned}
S'_+(-1) &= \{p \mid x(p) < 0, p \in S'_+\} \\
S'_+(+0) &= \{p \mid 0 < x(p) < x^*, p \in S'_+\} \\
S'_+(+1) &= \{p \mid x(p) > x^*, p \in S'_+\} \\
S'_-(-1) &= \{p \mid x(p) < 0, p \in S'_-\} \\
S'_-(-0) &= \{p \mid 0 < x(p) < x^*, p \in S'_-\} \\
S'_-(+1) &= \{p \mid x(p) > x^*, p \in S'_-\}
\end{aligned}
$$

and clearly we have

$$
\begin{aligned}
S'_+ &= S'_+(-1) \cup S'_+(0) \cup S'_+(+1) \\
S'_- &= S'_-(-1) \cup S'_-(0) \cup S'_-(+1).
\end{aligned}
$$

This step can be completed in $O(n)$ time. As shown in Figure 2(A), if $S'_+(0) \neq \emptyset$ and $S'_-(0) \neq \emptyset$, then there is a line segment with one endpoint in $S'_+(0)$ and the other in $S'_-(0)$ which intersects $L(p', q')$ (Figure 2(B)). To determine whether or not $S'_-(0)$ and $S'_+(0)$ is empty needs only $O(n)$ time. So without lose of generality, we can assume that $S'_+(0) = \emptyset$.

Second, we sort points in $S'_+(-1)$ and $S'_+(+1)$ lexicographically by polar angle at $p'$ and $q'$ respectively. In the new polar coordinate system, $S'_+(+1)$ becomes $S'_+(+1)_{p'}$ and $S'_+(-1)$ becomes $S'_+(-1)_{p'}$, similarly $S'_+(+1)_{q'}$ and $S'_+(-1)_{q'}$. Let $|\ S'_+(+1)\ | = m_1, |\ S'_+(-1)\ | = m_2, \alpha_{p'}(r)$ denote the polar angle of $r$ from origin $p'$ and $\alpha_{q'}(r)$ denote the polar angle of $r$ from $q'$. We have

$$
\begin{aligned}
S'_+(+1)_{p'} &= \{p_i^+ \mid \alpha_{p'}(p_{i+1}^+) > \alpha_{p'}(p_i^+), i = 1, 2, \cdots, m_1 - 1\} \\
S'_+(+1)_{q'} &= \{q_i^+ \mid \alpha_{q'}(q_{i+1}^+) > \alpha_{q'}(q_i^+), i = 1, 2, \cdots, m_1 - 1\} \\
S'_+(-1)_{p'} &= \{p_i^- \mid \alpha_{p'}(p_{i+1}^-) > \alpha_{p'}(p_i^-), i = 1, 2, \cdots, m_2 - 1\} \\
S'_+(-1)_{q'} &= \{q_i^- \mid \alpha_{q'}(q_{i+1}^-) > \alpha_{q'}(q_i^-), i = 1, 2, \cdots, m_2 - 1\}.
\end{aligned}
$$

The above sorting step can be done in $O(n \log n)$ time [PS85]. Let $u \in S'_-$. We discuss whether there is a line segment with one endpoint u and another endpoint in $S'_+$ that crosses $\overline{pq}$ in following three cases.

Case 1: $u \in S'_-(0)$.

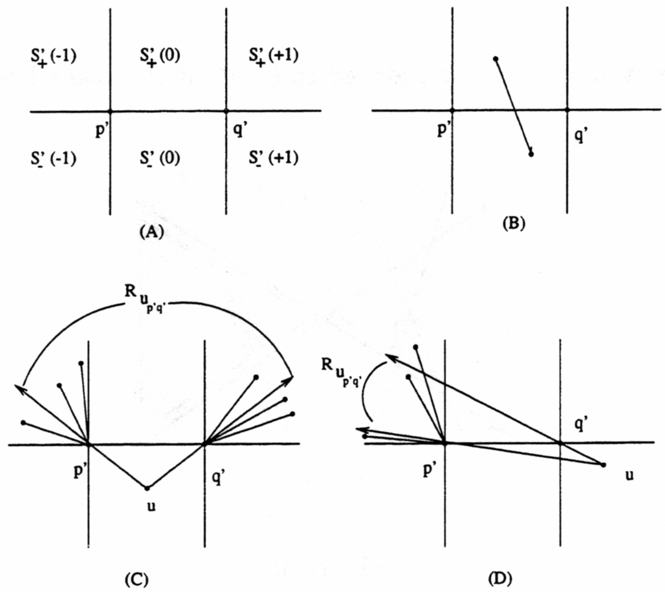


Figure 2:

Construct two rays $up'$ and $uq'$, let $\alpha_{up'}$ and $\alpha_{uq'}$ be the polar angles of $up'$ and $uq'$ in polar coordinate system with anchor points $p'$ and $q'$ respectively. Testing the rank of $\alpha_{up'}$ in $S'_+(+1)_{p'}$ and $\alpha_{uq'}$ in $S'_+(-1)q'$, can be done in $O(\log n)$ time binary search. This way we can find out whether there exists a point in $S'_+(+1) \cup S'_+(-1)$ lying in the angle region $R_{up'q'}$ between the two rays $up'$ and $uq'$. (Figure 2(C)).

Case 2: $u \in S'_-(+1)$.

Construct two rays $up'$ and $uq'$, let $\alpha_{up'}$ and $\alpha_{uq'}$ be the same as in Case 1. Test the ranks of $\alpha_{up'}$ and $\alpha_{uq'}$ in $S'_+(-1)_{p'}$ and $S'_+(-1)_{q'}$. (Note only points in $S'_+(-1)$ needs to be tested, since $u \in S'_-(+1)$). Thus, in $O(\log n)$ time we can find out whether there exists a point in $S'_+(-1)$ which lies in the angular region $R_{u_{p'q'}}$. (Figure 2(D)).

Case 3: $u \in S'_-(-1)$.

A similarly analysis as in Case 2.

It has shown in the above discussion that the total computations to determine whether a line segment with two endpoints in $S$ intersects $\overline{pq}$ take at most $O(n \log n)$ time. □

In the following, $LI(S, \overline{pq})$ denotes the above algorithm that answers whether or not there exists a line segment in $E$ that crosses $\overline{pq}$. By the above lemma, algorithm $LI(S, \overline{pq})$ takes $O(n \log n)$ time and $O(n)$ space.

**Lemma 2** *Let $\overline{pq}$ be a line segment, and let $\{p, q\} \cup S$ be a point set in general positions, $\mid S \mid = n$. Let $E$ denote all the line segments in $S$. Then, whether or not $\overline{pq}$ crosses an element of $E$ can be answered in $O(nlogn)$ time and $O(n)$ storage.*

**Theorem 1** *$SL(S)$ can be found in $O(n^2 \log n)$ time and $O(n)$ space, where $\mid S \mid = n$.*

**Proof** It is clear that $SL(S)$ must be contained in the Delaunay triangulation $DT(S)$. Thus, we start with $DT(S)$, which can be constructed in $O(n \log n)$ time and $O(n)$ space. Using algorithm $LI(S, \overline{pq})$ we test if an edge $\overline{pq}$ of $DT(S)$ belongs to $SL(S)$ in $O(n \log n)$ time and $O(n)$ space. The theorem follows since the number of edges in $DT(S)$ is $O(n)$.

□

### 3.2 Algorithm 2

The above time complexity bound can be reduced to $O(n^2)$ if the space bound increases to $O(n^2)$.

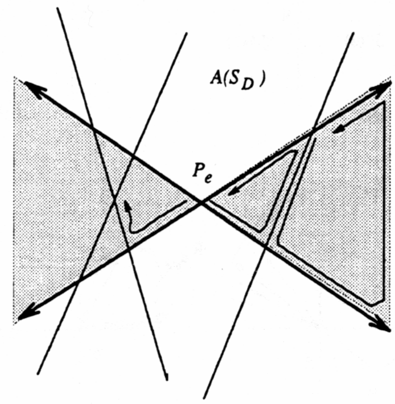


Figure 3:

**Algorithm 2**

- Find the arrangement for $n$ lines, where each line is the dual of a point of $S$ in the dual plane. Denote this arrangement as $A(S_D)$.
- Find $DT(S)$; For each Delaunay edge $e$ of $DT(S)$ DO.
  - Let $p_e$ be the intersection point of the dual lines of the endpoints of $e$. Let $W(p_e)$ be the double wedge determined by these two dual lines. Traverse the portion of $A(S_D)$ inside $W(p_e)$, starting at $p_e$. (Refer to Figure 3.)
  - If a vertex of $A(S_D)$ is found properly inside $W(p_e)$, then report '$e$ is not in $SL$';
  - Otherwise, report '$e$ is in $SL$'
- EndDo.

**Theorem 2** *$SL(S)$ can be found in $O(n^2)$ time and $O(n^2)$ space, where $|S| = n$.*

## 4 Concluding Remarks

We proposed an $O(n^2 \log n)$ time and $O(n)$ space algorithm and an $O(n^2)$ time and space algorithm for finding $SL(S)$. It is interesting to see whether $SL_4(S)$ can be computed in polynomial time.

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