On the size of the Euclidean sphere of influence graph

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Abstract

Let V be a set of distinct points in the Euclidean plane. For each point $x \in V$, let s_x be the ball centered at x with radius equal to the distance from x to its nearest neighbour. We refer to these balls as the *spheres of influence* of the set V. The *sphere of influence graph* on V is defined as the graph where (x, y) is an edge if and only if s_x and s_y intersect. In this extended abstract, we demonstrate that no Euclidean planar sphere of influence graph (E-SIG) contains more than $15n$ edges.

1 Introduction

In 1980, Godfried Toussaint proposed the sphere of influence graph as a geometric tool for capturing the underlying structures of dot patterns [1, 2, 3]. As is often the case, along with a new graph comes a host of open problems. Toussaint posed the question, "Does there exist a constant c such that a sphere of influence graph in the Euclidean plane (E-SIG) has at most cn edges?"

The question remained unanswered for five years, until it was solved by David Avis and Joe Horton [4] in the affirmative, providing the constant $c = 29$. They proved that given a sphere of influence graph $G(V)$ on a point set V, the vertex x_1 that has the smallest sphere of influence has at most 29 incoming edges. Any edge of $G(V)$ not touching x_1 is an edge of $G(V \setminus \{x_1\})$ since removing x_1 can only increase the radii of the spheres. That $G(V)$ contains at most $29n$ edges now follows by induction on the cardinality of V.

It was later realized that the theorem by Avis and Horton had been proven in a different form forty years earlier. In 1945, Abram Besicovitch required (and proved) the following lemma [5]:

Lemma 1 (Besicovitch, 1945) *Given a set , of coplanar circles, the center of no one of them being in the interior of another, and U the circle (or a circle) of , whose radius does not exceed the radius of any other circle of , , then the number of circles meeting U does not exceed 21.*

Although the upper bound of 21 stated by Besicovitch is not a tight bound, it served his purpose for the problem at hand. The number was improved to its lowest possible at 18 by E. R. Reifenberg in 1948 [6] and independently by Paul Bateman and Paul Erdős in 1951 [7]. Since planar Euclidean spheres of influence are a collection of circles such that no interior of any circle contains the centre of any other, Lemma 1 can be reworded to apply directly to sphere of influence graphs. Thus by induction we can show that no sphere of influence graph of n vertices contains more than $18n$ edges. We can also make a statement concerning a similar graph, the *closed sphere of influence graph*, in which the spheres of influence are closed balls rather than open. Therefore we draw an

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edge between two vertices if their spheres intersect, whether or not the intersection is proper (has nonzero area). The difference is relevant in Figure 1, where the centre vertex has 12 neighbours in the open SIG, but 18 neighbours in the closed SIG. Here too the upper bound on the maximum size is $18n$.

We can reduce this bound to $17.5n$ with a simple realization. Let x_1 be the vertex with the smallest sphere of influence, of radius r_1. This radius is determined by the distance between x_1 and its nearest neighbour, say x_2. Since r_1 is the smallest distance between any two vertices, then x_1 is also the nearest neighbour of x_2. Thus $r_1 = r_2$. Therefore r_1 and r_2 (the radius of the sphere of influence of x_2) are both the smallest radii over all spheres, so x_1 and x_2 each have at most 18 neighbours. One edge is shared by x_1 and x_2, so the two vertices are adjacent to at most 35 edges. Performing the induction on two vertices at a time instead of one yields a bound of $35n/2$ edges, or $17.5n$. This bound is attributed to Katchalski.

More recently, T. S. Michael and Thomas Quint have also produced proofs that the E-SIG contains no more than $17.5n$ edges [8, 9]. Their methods are more graph-theoretic than the previous methods discussed and provide insight into the workings of the sphere of influence graph in general, not just in the Euclidean plane. However, due to the brevity of this abstract we will not delve into them here.

Where is this upper bound headed? The aim, of course, is to find the optimal constant, joining the upper and lower bounds. For an idea of where the tight bound may lie, we consider the closed sphere of influence graph. We see that the hexagonal lattice has 18 neighbours per vertex, for $9n$ edges in total. In Figure 1, the centre vertex is a closed SIG neighbour of all the other drawn vertices. David Avis conjectures that the hexagonal lattice is optimal in that $9n$ is the most number of edges possible for a closed E-SIG. Since the open E-SIG is a subset of the closed E-SIG, the conjecture implies that the tight bound for the open E-SIG is no more than $9n$.

In this extended abstract we improve the upper bound on the size of the E-SIG to $15n$.

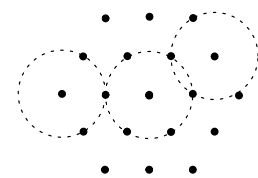

Figure 1: Subset of the hexagonal lattice.

2 An upper bound of $15n$

In this section, we sketch the proof of the following theorem.

Theorem 2 *No open or closed sphere of influence graph of n vertices in the Euclidean plane contains more than $15n$ edges.*

To facilitate our proof, we assign *weights*, or numerical values, to the edges of the E-SIG as follows. First, we replace each undirected edge $\{a, b\}$ with two directed edges, (a, b) and (b, a). Let the radii of the spheres of influence of a and b be r_a and r_b. Then (a, b) is given a weight of 1 if $r_a \leq 2r_b/3$, a weight of 1/2 if $2r_b/3 < r_a < 3r_b/2$, and a weight of 0 otherwise. We refer to this graph as the *weighted sphere of influence graph*, or WSIG.

Our goal is to utilize the WSIG in determining a new upper bound for the E-SIG.

Lemma 3 *On any point set V, the total weight of all edges in the WSIG of V is equal to the number of edges in the SIG of V.*

Lemma 3 implies that if we can prove that no WSIG of n vertices has edges whose total weight is greater than $15n$, then we have also proven Theorem 2. This is exactly the method behind our proof, which we state with the following theorem.

Theorem 4 *There exists no node in the WSIG for which the weights of outgoing edges sum to greater than 15.*

We prove this theorem by demonstrating that it follows from the next theorem, Theorem 5, and then by proving the latter. The next two theorems, greatly inspired by the work of Bateman and Erdős [7], discuss fitting configurations of points into annuli.

Theorem 5 *Let the term* admissible point of weight $1/2$ *refer to a point* p *in the annulus* $1 \leq \rho \leq 5/3$ *such that no other admissible point is within distance* $2/3$ *of* p*. Let the term* admissible point of weight 1 *refer to a point* q *in the annulus* $1.5 \leq \rho \leq 2.5$ *such that*

- *no other admissible point is within distance 1.5 of* q*, except*
- *for each admissible point of weight 1/2 which has polar co-ordinates* $(5/3, \theta)$*, there exists a point in space* (r, θ) *where* $5/3 \leq r \leq 2.5$ *such that* (r, θ) *is at least distance 1.5 from* q*.*

Then it is impossible to fit any combination of admissible points in the annulus $1 \leq \rho \leq 2.5$ *such that their total weights sum to a value greater than 15.*

We delay the proofs of the last two theorems for now and instead show that Theorem 5 implies Theorem 4. We first require the following lemma.

Lemma 6 *In polar co-ordinates, let* $X = (x, \theta_x)$ *and* $Y = (y, \theta_y)$ *be the centres of two circles that do not contain each other's centres but that both intersect* $\rho = 1$*. Furthermore, we impose the condition that* X *and* Y *lie outside the disk* $\rho \leq R$*, for some* $R > 1$*. Then the points* $A = (R, \theta_x)$ *and* $B = (R, \theta_y)$ *are at least distance* $R - 1$ *apart.*

We are now ready to prove the following theorem.

Theorem 7 *Theorem 5 implies Theorem 4.*

Proof (sketch). Let O be some node in the WSIG with at least one outgoing edge of non-zero weight. Without loss of generality, assume that O is at the origin and that the sphere of influence of O has radius 1. Thus we have a set Δ of circles of radius at least $2/3$ which intersect the circle $\rho = 1$ such that the centre of no circle is contained in any other. Also, since the sphere of influence of O has radius 1, no circle in Δ is centered inside $\rho < 1$.

It suffices to show that we can construct a set Δ^* of admissible points where each circle in Δ of radius

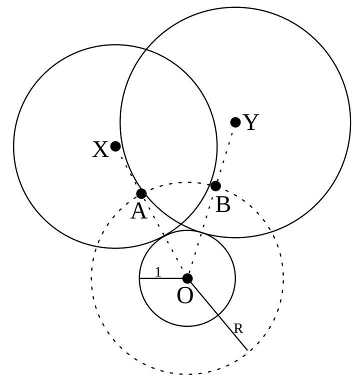

Figure 2: Lemma 6.

more than $2/3$ but less than 1.5 corresponds to a point in Δ^* with weight $1/2$, and where each circle in Δ of radius 1.5 or greater corresponds to a point in Δ^* with weight 1. Due to Lemma 6, no matter where the circles in the set Δ are centered, we can find corresponding points for Δ^* in the annulus $1 \leq \rho \leq 2.5$.

Furthermore, we demand that both correspondences be *bijective*, meaning that every circle in Δ corresponds uniquely to a point in Δ^* and vice versa.

□

To prove Theorem 5, we require the following lemma, a generalization from a similar lemma by Bateman and Erdős [7].

Lemma 8 *Label the origin as* O*. Let* r*,* R*, and* τ *be such that* $0 < R - \tau \leq r \leq R$*. Suppose that we have two points* P *and* Q *which lie in the annulus* $r \leq \rho \leq R$ *and which have mutual distance* τ*. Then the minimum value* $\Phi_\tau(r, R)$ *of* $m\angle POQ$ *has the smaller of the two values*

$$\Phi_\tau(r, R) = \arccos \frac{(R/\tau)^2 + (r/\tau)^2 - 1}{2Rr/\tau}, \textit{ and}$$

$$\Phi_\tau(r, R) = \arccos(1 - \frac{1}{2(R/\tau)^2}) = 2 \arcsin \frac{\tau}{2R}.$$

Proof. It suffices to consider the case where $OQ = R$ and $PQ = \tau$. Let $OP = \rho$. Our problem can be reduced to finding the ρ which yields the minimum value of $m\angle POQ$. Let $f(\rho) = m\angle POQ = \arccos[(R)^2 + (\rho)^2 - \tau^2)/(2R\rho)]$ for ρ in the interval $r \leq \rho \leq R$. If we differentiate, we see that $f(\rho)$ cannot have an interior minimum in this interval. Thus the minimum is the smaller of $f(r)$ and $f(R)$, which are the two values described in the lemma.

□

We will sketch the proof of Theorem 5 here. The method of proof is simple. Lemma 8 provides us with the minimum angle subtended by any two admissible points. Therefore, if we place a configuration of admissible points inside the annulus $1 \leq \rho \leq 2.5$, we can compute a lower bound on the sum of the angles between radially consecutive points. If this sum is greater than $360°$, then this configuration is impossible as it cannot fit in the annulus. For example, we can prove the following lemma.

Lemma 9 *It is impossible to have 11 admissible points of weight 1.*

Proof. Admissible points of weight 1 are distance 1.5 apart in the annulus $1.5 \leq \rho \leq 2.5$. Therefore, by Lemma 8, the angle between any two radially consecutive such points is at least $\Phi_{1.5}(1.5, 2.5)$. If we compute $\Phi_{1.5}(1.5, 2.5)$, we find that it is slightly greater than $33.5°$. Therefore, our lemma follows from the fact that $11\Phi_{1.5}(1.5, 2.5) > 368°.5$.

□

We can use this method to show that no configuration exists such that some vertex has outgoing edges whose weights total more than 15. The last lemma implies that proving the impossibility of the following six combinations are sufficient to prove the theorem.

- 10 points of weight 1 and 11 of weight 1/2.
- 9 points of weight 1 and 13 of weight 1/2.
- 8 points of weight 1 and 15 of weight 1/2.
- 7 points of weight 1 and 17 of weight 1/2.
- 6 points of weight 1 and 19 of weight 1/2.
- 26 points of non-zero weight.

These six cases complete our proof of Theorem 5. We have shown that this implies Theorem 4, which states that no node in the WSIG has outgoing edges whose weights sum to greater than 15. This, in turn, implies our main result, Theorem 2, which states that no sphere of influence graph in the Euclidean plane contains more than $15n$ edges.

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