Inner diagonals of convex polytopes

Extended Abstract

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1 Introduction

An *inner diagonal* of a polytope P is a segment that joins two vertices of P and that lies, except for its ends, in P's relative interior. For $1 \leq i \leq d$, an *i-diagonal* of a d-polytope P is a segment $[x, y]$ whose ends are vertices of P and whose *carrier* in P (i.e., the smallest face of P that contains $[x, y]$) is of dimension i. The number of P's i-diagonals is denoted by $\delta_i(P)$. Thus $\delta_1(P)$ is the number of P's edges and $\delta_d(P)$ is the number of P's *inner diagonals*. The focus here is on δ_d.

It seems strange that, except for $d = 2$, so little attention has been paid to inner diagonals of d-polytopes. There is an interesting paper of Mani [10] and an attractive open problem of von Stengel [13] (see Section 4), but we know of little else. Following some preliminaries in Section 2, our main results appear in Section 3. They consist of the determination, for each $f \geq 4$, of both the minimum and the maximum of $\delta_3(P)$ as P ranges over all simple 3-polytopes that have precisely f facets. Section 4 contains some results on inner diagonals of higher-dimensional polytopes, including the determination for all d of the maximum number of inner diagonals of d-polytopes having a given number of vertices.

2 Preliminaries

We use $v(P)$ and $f(P)$ to denote respectively the number of vertices and the number of facets ($(d-1)$-faces) of a d-polytope P. A d-polytope is *simplicial* if each of its facets is a simplex, and it is *simple* if each of its vertices is incident to precisely d edges (equivalently, to precisely d facets).

We say that two vertices of a polytope are *estranged* if they do do not lie together on any facet, and that two facets are *estranged* if they do not share any vertex. Note that two vertices are joined by an inner diagonal if and only if they are estranged. Note also that under the standard polarity operation for polytopes, simple polytopes correspond to simplicial polytopes, the vertices of a polytope correspond to the facets of its polar, and estranged pairs of vertices in a polytope correspond to estranged pairs of facets in its polar. We use the notation $\mathsf{DMAX}_d^V(n)$ (resp. $\mathsf{EMAX}_d^V(n)$) to denote the class of d-polytopes that maximize, among the d-polytopes with n vertices, the number of inner diagonals (resp. estranged pairs of facets). The notations $\mathsf{DMAX}_d^F(n)$ and $\mathsf{EMAX}_d^F(n)$ are defined similarly with respect to fixing the number of facets. The following two remarks are immediate consequences of the basic definitions.

2.1. Remark. *For each d-polytope P, $\sum_i \delta_i(P) = \binom{v(P)}{2}$, with $\delta_1(P) + \delta_d(P) = \binom{v(P)}{2}$ when P is simplicial.*

2.2. Remark. *If a d-polytope P has at least one inner diagonal, then $v(P) \geq d + 2$ and $f(P) \geq 2d$.*

A d-polytope P is called a *pyramid* if there is a facet F of P that contains all but one vertex of P. Each facet that has this property is called a *base* of the pyramid and the remaining vertex is called an *apex*. Note that for each choice of $d \geq 3$, $v > d$, and $f > d$, there is a d-pyramid that has v vertices and there is a d-pyramid that has f facets. Note also that when $d \geq 2$ a d-pyramid has no estranged pair of facets and also no estranged pair of vertices.

When Q is an $(f-1)$-facet $(d-1)$-polytope in $\mathbb{R}^{d-1}$ and F is a facet of Q, a *wedge over Q with foot F* is an f-facet d-polytope W obtained from the product $Q \times [0, 1]$ by (in effect) collapsing $F \times [0, 1]$ to $F \times \{0\}$. We use the following easily verified property of wedges.

2.3. Remark. *If Q is a simple $(d-1)$-polytope and W is a wedge over Q with foot F, then W is simple and $\delta_d(W)$ is twice the number of inner diagonals of Q that miss F.*

For other properties of polytopes, the reader is referred to the books of Grünbaum [5] and Ziegler [15].

3 Inner diagonals of 3-polytopes

This section is concerned with the combinatorial structure of 3-polytopes, or, in view of Steinitz's theorem [12], [5], with 3-connected planar graphs. We shall (mostly) continue to use the geometric language since it is from a geometric viewpoint that the inner diagonals are of special interest.

By adding edges between non-adjacent vertices on a facet (and splitting the facet), we strictly increase the number of inner diagonals.

3.1. Theorem. *For each 3-polytope P with v vertices, $\delta_3 \leq (v^2 - 7v + 12)/2$, with equality if and only if P is simplicial.*

A similar argument, and polarity, yields the following.

3.2. Theorem. *For $f \geq 6$, if $P \in \mathsf{DMAX}_3^F(f)$ then P is simple.*

We now proceed to determine both the minimum and the maximum number of inner diagonals for simple 3-polytopes with a given number $f \geq 6$ of facets. An essential notion is

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that of the *p-vector* $(p_3, p_4, \dots)$ of a 3-polytope P, where p_i is the number of i-gonal facets of P. The p-vector $(p_3, p_4, \dots)$ is also written as $k_1^{p_{k_1}} \dots k_m^{p_{k_m}}$ with $3 \le k_1 < k_2 < \cdots < k_m$, where $k_1, \dots, k_m$ are the values of i for which $p_i > 0$. We occasionally deviate slightly from this practice to permit repetition among the k_i or $p_{k_i} = 0$. For reasons that will immediately become apparent, we are interested in the function

$$\varphi(P) = \sum_i i^2 p_i,$$

where $(p_3, p_4, \dots)$ is the p-vector of P.

3.3. Proposition. *For each simple 3-polytope with f facets, $\delta_1 = 3f - 6$, $\delta_2 = \frac{1}{2}\varphi - 9f + 18$, and $\delta_3 = 2f^2 - 3f - 2 - \frac{1}{2}\varphi$.*

It follows from Proposition 3.3 that if we are restricted to simple 3-polytopes with a given number f of facets, the maximum (resp. minimum) number of inner diagonals is obtained by minimizing (resp. maximizing) the quantity φ over all sequences $(p_3, p_4, ...)$ that have $\sum_i p_i = f$ and are realized as the p-vector of some simple 3-polytope. Easy counting (using Euler's theorem and the fact that $2\delta_1 = 3v$) shows that the equation,

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{i \ge 7} (i - 6) p_i$$

is a necessary condition for such realizability. The quantity p_6 does not appear in this equation, and a theorem of Eberhard [3] asserts that for each sequence satisfying the equation there is at least one value of p_6 for which the sequence with this p_6 in place is indeed the p-vector of a simple 3-polytope. (See Grünbaum [5] for a much simpler proof.) However, for many choices of the reduced sequence $(p_i : 3 \le i \ne 6)$ it is not known which values of p_6 have the stated property.

We turn now to the problem of minimizing δ_3 over all simple 3-polytopes that have a given number f of facets. This brings us back to the *wedges* mentioned in Remark 2.3, for it will turn out that except for additional minimizers when f is 7 or 8, the 3-dimensional wedges are precisely the desired minimizers. We define an *f-wedge* as a 3-polytope that is a wedge over an $(f-1)$-gon K and has as its foot some edge of K. Each f-wedge is a simple 3-polytope with f facets, and for given f all f-wedges are combinatorially equivalent. Figure 1 depicts a 6-wedge and its two inner diagonals.

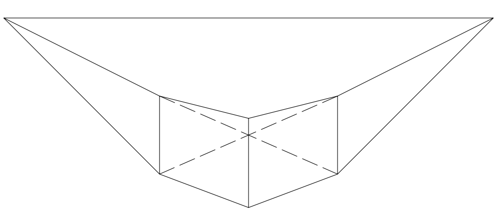

Figure 1: A 6-wedge and its two inner diagonals.

3.4. Lemma. *An f-facet simple 3-polytope is an f-wedge iff its p-vector is $3^2 4^{f-4} (f-1)^2$. Each f-wedge W has $\varphi(W) = 2f^2 + 12f - 44$ and $\delta_3(W) = f^2 - 9f + 20$.*

It follows from an observation of Steinitz [12] (see also [5], p. 243) that for $f \ge 4$, each simple 3-polytope P with $f+1$ facets can (in combinatorial type) be obtained from some simple 3-polytope Q with f facets by (in graph theoretic terms) splitting a facet Y as follows. Two edges of Y are split by adding a vertex in the middle, and the two new vertices are joined by an edge. Let X and Z be the other two facets effected by the split. If the numbers of vertices of X, Y, and Z are respectively α, β, and $\gamma \le \alpha$, and the β-gon Y is split into a ξ-gon and an η-gon with $\eta \le \xi$, then the above sort of transition from the simple 3-polytope Q to the simple 3-polytope P is here called an *(X, Y, Z)-splitting of type* $(\alpha : \xi, \eta : \gamma)$. Note that $\xi + \eta = \beta + 4 \ge 7$. As can be seen in Figure 2, an $(f+1)$-wedge can be obtained from an f-wedge by means of splittings of three different types: $(f-1 : 4, 4 : f-1)$, $(f-1 : 4, 3 : f-1)$, and $(f-1 : f, 3 : 3)$. In each case the value of φ increases by $4f + 14$.

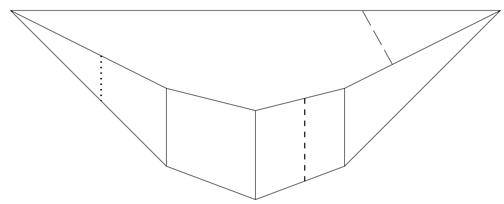

Figure 2: A 7-wedge can be obtained from a 6-wedge by 3 types of split.

3.5. Lemma. *Suppose that Q is a simple 3-polytope with f facets, and P arises from Q by means of a splitting of type $t = (\alpha : \xi, \eta : \gamma)$. Then $\varphi(P) - \varphi(Q) \le 4f + 14$, with equality if and only if*

(a) $X \cap Y \cap Z = \emptyset \ne X \cap Z$, and $t = (f-1 : 4, 4 : f-1)$; or

(b) $X \cap Y \cap Z \ne \emptyset$, $\eta = 3$, and $\alpha + \xi + \gamma = 2f + 2$

In each case, equality implies that all vertices of Q belong to $X \cup Y \cup Z$.

We have seen in Lemma 3.4 that an f-wedge P has $\varphi(P) = 2f^2 + 12f - 44$ and is characterized among simple 3-polytopes by having $3^2 4^{f-4} (f-1)^2$ as its p-vector. We have also seen (in Figure 2) that for $f \ge 6$ an $(f+1)$-wedge can be produced from an f-wedge by means of splittings of three different types: $(f-1 : 4, 4 : f-1)$, $(f-1 : 4, 3 : f-1)$ and $(f-1 : f, 3 : 3)$.

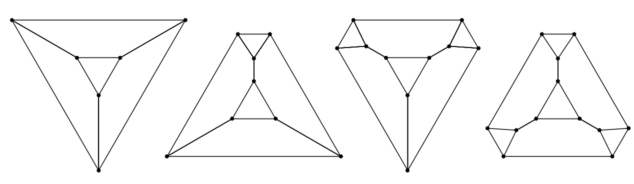

Figure 3: The polytopes T_k obtained by truncating a tetrahedron at $k-4$ of its vertices.

For $5 \le f \le 8$, the T_f of Figure 3 is a simple 3-polytope that has f facets and arises from truncating a tetrahedron at $f-4$ of its vertices. Note that T_5 and T_6 are wedges but T_7 and T_8 are not. The following is proved by inductive application of Lemma 3.5.

3.6. Theorem. *If $f \geq 4$ and P is a simple 3-polytope with f facets, then $\varphi(P) \leq 2f^2+12f-44$, and $\delta_3(P) \geq f^2-9f+20$. Equalities hold if and only if P is an f-wedge or f is 7 or 8 and $P = T_f$.*

We turn now to the problem of maximizing the number of inner diagonals in the class of all simple 3-polytopes having a given number f of facets. The following characterization was suggested by the results of a computer search using an algorithm of Avis [1]. The proof is based on solving a certain class of linear programs over p-vectors, and showing that the solutions are realizable as simple polytopes. For $f \geq 14$ the maximizers turn out to be polytopes whose existence was established by Grünbaum and Motzkin [6]. For $f < 14$, the maximizers are those shown in Figure 4, along with the cube and the dodecahedron.

3.7. Theorem. *For simple 3-polytopes with f facets, the maximum number of inner diagonals and the unique associated p-vector are as follows:*

f	δ_3	*p-vector*
$6 \leq f \leq 10$	$2f^2-20f+52$	$4^{12-f}5^{2f-12}$
11	73	$4^2 5^8 6$
13	128	$4^1 5^{10} 6^2$
$f = 12, 14 \leq f$	$2f^2-21f+64$	$5^{12}6^{f-12}$

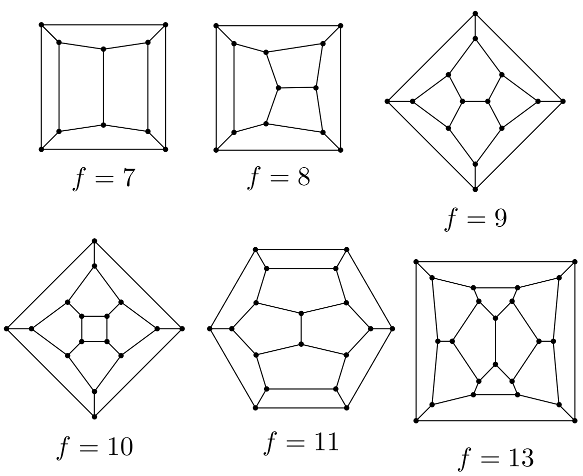

Figure 4: Simple 3-polytopes maximizing the number of inner diagonals for given f.

4 Higher-dimensional polytopes

A polytope is called 2-*neighborly* if each pair of its vertices defines an edge. Consideration of wedges, pyramids and (2-neighborly) cyclic polytopes yields:

4.1. Proposition. *For each $n > d \geq 4$, there exists a simplicial (resp. simple) d-polytope P with $v(P) = n$ (resp. $f(P) = n$) and $\delta_d(P) = 0$.*

Let $P = \operatorname{conv} X$ be a d-polytope. Let $X' = (X \setminus \{x_0\}) \cup \{x_0'\}$, where x_0' is a point of $\mathbb{R}^d$ such that the half-open segment $]x_0, x_0']$ does not intersect any hyperplane determined by points of X. If x_0 is in the interior of P', $P' = \operatorname{conv} X'$ is said to be *obtained from P by pulling x_0*. If x_0' is in the interior of P, P' is said to be *obtained from P by pushing x_0*. Initially we will consider some consequences of pulling; we return to pushing below.

It is shown in [4] and [5] that if $\operatorname{conv} X'$ is obtained from $\operatorname{conv} X$ by pulling, then for $0 \leq r \leq d-1$, the r-faces of $\operatorname{conv} X'$ are precisely the sets of the following two sorts: an r-face of $\operatorname{conv} X$ that misses x_0; a pyramid of the form $\operatorname{conv}(B \cup \{x_0'\})$, where B is an $(r-1)$-face of a facet F of $\operatorname{conv} X$ such that $x_0 \in F \setminus B$. Let $\rho(s,t,P)$ denote the dimension of the carrier of $[s,t]$ in polytope P.

4.2. Lemma. *Suppose that x_0 belongs to the vertex set X of a d-polytope $P \subset \mathbb{R}^d$, and that P' is obtained from P by pulling x_0 to a new position x_0'. Then the following statements are true.*

(a) If $\rho(s,t,P) \in \{1, d\}$ then $\rho(s',t',P') = \rho(s,t,P)$.

(b) If $1 < \rho(s,t,P) < d$, K is the carrier of $[s,t]$, and $x_0 \in K \setminus \{s,t\}$ then $\rho(s,t,P') > \rho(s,t,P)$.

4.3. Proposition. *If $d \geq 2$ and $P \in \mathsf{DMAX}_d^V(v)$ then each facet of P is 2-neighborly.*

We now argue that the v-vertex d-polytopes maximizing the number of inner diagonals are all simplicial. Let $p_k(P)$ denote the number of 2-faces of P with k-vertices. For a v-vertex d-polytope P, define the functions $g_1(P)$ and $g_2(P)$ as follows:

$$g_1(P) = v - (d+1)$$

$$g_2(P) = \delta_1(P) - \left[dv - \binom{d+1}{2}\right] + \sum_{k>3}(k-3)p_k(P)\,.$$

A v-vertex stacked d-polytope is either a d-simplex or is obtained recursively from a $(v-1)$-vertex stacked polytope by erecting a pyramid over one of the facets. Barnette's Lower Bound Theorem [2] says that for each j with $1 \leq j \leq d-1$, each v-vertex simplicial d-polytope has at least as many j-faces as a v-vertex stacked d-polytope, and that within the class of simplicial polytopes, this bound is attained (when $d \geq 4$) only by the stacked polytopes. The Lower Bound Theorem can be recursively reduced to the case $j = 1$ (see [8], Sec. 5 for details); hence for simplicial d-polytopes with $d \geq 4$ the Lower Bound Theorem is equivalent to the statements

$$g_2(P) \geq 0, \qquad \text{and} \qquad \text{(L1)}$$

$$g_2(P) = 0 \qquad \text{iff } P \text{ is stacked}\,. \qquad \text{(L2)}$$

The statement (L1) was proved for rational (not necessarily simplicial) polytopes by Stanley [11]. Kalai [8] generalized this to all polytopes by using notions of rigidity of graphs, in particular a theorem of Whitely [14]. In order to generalize statement (L2) to not necessarily simplicial polytopes, we need the following theorem of Kalai [9]. Let P/F denote the quotient polytope of P with respect to F, i.e., a polytope whose face lattice is isomorphic to the interval $\{G \mid F \subseteq G \subseteq P\}$ of the face lattice of P (see Ziegler [15], p. 57).

4.4. Theorem (Kalai). *For any d-polytope P and any face F of P, $g_2(P) \geq g_2(F) + g_1(F)g_1(P/F) + g_2(P/F)$.*

The following is a consequence of this theorem.

4.5. Proposition. *For $d \geq 4$, if P is a d-polytope with 2-neighborly facets, then $g_2(P) = 0$ if and only if P is stacked.*

Putting together Propositions 4.3 and 4.5, we have

4.6. Theorem. *For a v-vertex d-polytope P,*

$$\delta_d(P) \leq \binom{v}{2} - dv + \binom{d+1}{2}.$$

For $d \geq 4$ this maximum is attained by the stacked polytopes and only by them.

We now consider maximizing the number of inner diagonals among polytopes with a fixed number of facets.

4.7. Theorem. *Within the class of d-polytopes that have a given number $f \geq 2d$ of facets, the maximum possible number of inner diagonals is attained by certain simple d-polytopes.*

For $d \geq 4$ we do not know whether "by certain" in Theorem 4.7 can be replaced by "only by certain." We can however make a few observations about the d-polytopes that maximize the number of inner diagonals among those with a fixed number of facets. The following is proved using a perturbation argument similar to Lemma 4.2.

4.8. Proposition. *If $P \in \mathsf{EMAX}_d^V(n)$ then each facet of P that is not a simplex must intersect every other facet of P.*

Let $\mathrm{dist}(s, t, P)$ (the *distance* from s to t) denote the length (number of edges) of the shortest path from s to t in the graph defined by vertices and edges of polytope P. The *diameter* of a polytope P with vertex set X is defined as the maximum over all $\{x, y\} \subset X$ of $\mathrm{dist}(x, y, P)$. The following can be proved using the notion of pushing a vertex.

4.9. Theorem. *If $P \in \mathsf{EMAX}_d^V(n)$ then P is simplicial or P has diameter at most 4.*

In contrast to the situation (Theorem 4.6) when the number of vertices is fixed, we do not know (for $d \geq 4$) what is the maximum number of inner diagonals when the number of facets is fixed. The present note was motivated by the following conjecture of von Stengel [13], which arose from his study of the Nash equilibrium points of bimatrix games.

4.10. Conjecture (von Stengel). *Within the class of all simple d-polytopes having $2d$ facets, the maximum number of inner diagonals is the 2^{d-1} attained by a d-cube.*

There are two relevant 3-polytopes: the 3-cube with 4 inner diagonals and the 5-wedge with 2. Grünbaum and Sreedharan [7] provided (in dual form) a catalog of the 37 different combinatorial types of simple 4-polytopes with 8 facets. Of these, six have no inner diagonals and only the 4-cube has as many as 8. Hence Conjecture 4.10 holds (even with "attained" replaced by "attained only") for $d \leq 4$. However, the conjecture remains open for all $d \geq 5$.

Acknowledgments

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