Observations and Computations in Sylvester-Gallai Theory

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Abstract

We bring together several new results related to the classical Sylvester-Gallai Theorem and its recently noted sharp dual. In 1951 Dirac and Motzkin conjectured that a configuration of n not all collinear points must admit at least $n/2$ ordinary connecting lines. There are two known counterexamples, when $n = 7$ and $n = 13$. We provide a construction that yields both counterexamples and show that the common construction cannot be extended to provide additional counterexamples.

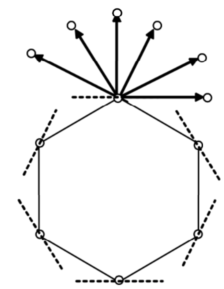

Figure 1: Böröczky even n example with $n = 12$ points and 6 ordinary lines.

1 Introduction

In 1893 J. J. Sylvester posed the following celebrated problem [13]: Given a collection of points in the plane, not all lying on a line, prove that there exists a line which passes through precisely two of the points. Sylvester's problem was reposed by Erdős in 1944 [5] and then solved the same year by T. Gallai [7]. Today, the positive result to Sylvester's problem is usually referred to as Sylvester's Theorem or the Theorem of Sylvester and Gallai. Sylvester's Theorem holds equally in the Euclidean and Real Projective Planes. We write $\mathbb{RP}^2$ to denote the real projective plane. See [1] and [6] for excellent treatments of Sylvester-Gallai theory.

A line which passes through precisely two points in a configuration of points is referred to as an **ordinary line**. Analogously, given an arrangement of lines, a point lying at the intersection of precisely two lines is referred to as an **ordinary point**.

Much work has gone into obtaining lower bounds on the number of ordinary lines in a collection of points satisfying the hypothesis of Sylvester's Theorem. Dirac [4] and Motzkin [12] separately conjectured that, given n points as in the statement of Sylvester's Theorem, there must be at least $n/2$ ordinary lines. For even n there is a family of examples due to Böröczky (as cited in [2]) with n points and precisely $n/2$ ordinary lines. These examples are most easily verified in the real projective plane. One starts with the vertices of a regular $n/2$-gon and then adds the $n/2$ points at infinity determined by the line through any pair of vertices. Each vertex of the original regular $n/2$-gon then determines an ordinary line (a dashed line in the figure) with precisely one of the $n/2$ points at infinity. Note that there is nothing special about this example residing in $\mathbb{RP}^2$; the points and connecting lines can easily enough be rotated into $\mathbb{R}^2$.

In 1958 Kelly and Moser [8] found an example of 7 points with just 3 ordinary lines. They also showed that a set of n not all collinear points must admit at least $3n/7$ ordinary lines. Then in 1968 McKee [2] found an example of 13 points with just 6 ordinary lines. Finally in 1993 Csima and Sawyer [3] showed that except for the case of $n = 7$ there must be at least $6n/13$ ordinary lines in a configuration of n not all collinear points. The Dirac-Motzkin conjecture thus stands as follows: For n not all collinear points, $n \neq 7, 13$, there must be at least $n/2$ ordinary lines.

By projective duality, Sylvester's Theorem is equivalent to the statement that in an arrangement of lines in $\mathbb{RP}^2$, not all of which pass through a single point, there must be an ordinary point. For given n, a lower bound on the number of ordinary lines amongst not all collinear point configurations (in $\mathbb{R}^2$ or $\mathbb{RP}^2$) of size n, corresponds to the same lower bound on the number of ordinary points amongst not all coincident line arrangements in $\mathbb{RP}^2$ of the same size n. In [10, 11] Lenchner showed that a sharper dual version of Sylvester's Theorem actually holds, namely that given an arrangement of lines in $\mathbb{R}^2$, not all of which are parallel and not all of which are coincident, then there must be an ordinary point - indeed, that given n such lines, there must be at least $(5n + 6)/39$ ordinary points.

In this paper we look at the sharp dual results in [10, 11] from a different angle and prove a couple of related theorems which give insight into the $n/2$ conjecture. We also provide a common construction for the $n = 7$ and

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$n = 13$ counterexamples and show that the common construction cannot be extended to produce additional counterexamples.

2 Lines or Pseudolines and the Ordinary Points they may contain

To obtain our results, we use the following key lemmas and definitions. In what follows we shall consider Sylvester's problem in its dual formulation in the real projective plane.

Definition Say that an ordinary point p is **attached** to a line l, not containing p, if l together with two lines crossing at p form a triangular face of the arrangement. See Figure 2.

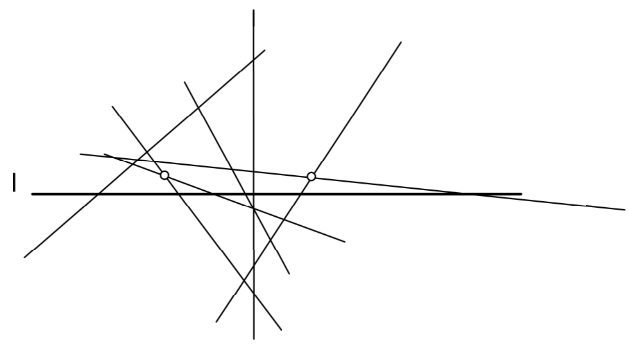

Figure 2: An example of a line l with two ordinary points attached.

The following two lemmas are due to Kelly and Moser [8].

Lemma 1 *If a line l of an arrangement $\mathcal{A}$ contains no ordinary points, then there are at least 3 ordinary points attached to l.*

Lemma 2 *If a line l of an arrangement $\mathcal{A}$ contains a single ordinary point, then the line l has at least 2 ordinary points attached to it.*

Definition A line is said to be of **type** (n, m) if it contains n ordinary points and has m ordinary points attached to it.

The following lemma is due to Csima and Sawyer [3]:

Lemma 3 *Suppose $\mathcal{A}$ is a finite arrangement of lines in $\mathbb{RP}^2$ having two lines of type $(2, 0)$ that intersect in an ordinary point. Then $\mathcal{A}$ is graph theoretically isomorphic to the Kelly-Moser arrangement (Figure 3).*

The following simple observation is what is used to drive home the Kelly-Moser and Csima-Sawyer results.

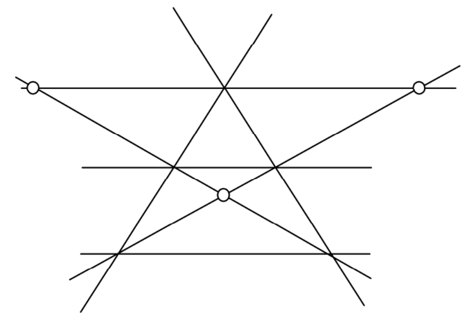

Figure 3: The Kelly-Moser arrangement of 7 lines with just 3 ordinary points.

Lemma 4 *In an arrangement of lines in $\mathbb{RP}^2$, an ordinary point can have at most 4 lines counting that point as an attachment.*

Proof. An ordinary point is contained in 2 crossing lines, and hence a vertex of 4 faces; it can therefore be attached to at most 4 lines. □

Definition An arrangement of **pseudolines** in $\mathbb{RP}^2$ is a family of simple closed curves each pair of which has exactly one point in common, and at this common point the curves cross.

We note that both the Csima-Sawyer and Kelly-Moser Theorems, in their dual formulations, are true if we replace lines in $\mathbb{RP}^2$ by pseudolines in $\mathbb{RP}^2$. The pseudoline version of the Kelly-Moser Theorem was proved in 1972 by Kelly and Rottenberg [9] and the pseudoline version of the Csima-Sawyer Theorem was remarked to hold without changing any of the basic lemmas in the original Csima-Sawyer article [3]. Lemmas 1 - 4 are valid both for lines and pseudolines. Lenchner's Theorem 4 from [11] which says that an arrangement of n, not all coincident, not all parallel lines in $\mathbb{R}^2$ must contain at least $\lceil (5n + 6)/39 \rceil$ ordinary points, relies solely on the Csima-Sawyer Theorem, and so can be reinterpreted in the context of pseudolines as follows:

Theorem 5 *Given an arrangement of n pseudolines in $\mathbb{RP}^2$, one can find at least $\lceil (5n+6)/39 \rceil$ ordinary points off any pseudoline not already part of the arrangement.*

We can also say something about ordinary points off pseudolines that *are* part of the arrangement.

Theorem 6 *Given an arrangement of n pseudolines in $\mathbb{RP}^2$, no $n - 1$ of which pass through a common point, one can find at least $\lceil (5n - 6)/39 \rceil$ ordinary points off any pseudoline in the arrangement.*

Proof. Given an arrangement $\mathcal{A}$ as in the statement of the Theorem, Csima-Sawyer again guarantees at least $\lceil 6n/13 \rceil$ ordinary points as long as $n \neq 7$. We consider the case $n = 7$ separately.

If our result were false, then more than $\frac{6n}{13} - \frac{5n-6}{39} = \frac{n}{3} + \frac{2}{13}$ of those ordinary points would have to lie on a fixed pseudoline $l \in \mathcal{A}$. Now consider the arrangement $\mathcal{A}'$ consisting of the $n-1$ elements of $\mathcal{A}$ other than l. The removal of l "kills off" at least $\frac{n}{3} + \frac{2}{13}$ ordinary points and creates at most $\frac{n-1-(\frac{n}{3}+\frac{2}{13})}{2} = \frac{n}{3} - \frac{15}{26}$ new ones since an ordinary point can only be created where previously two pseudolines from $\mathcal{A}$ intersected in l.

Now, by assumption the pseudolines of $\mathcal{A}'$ do not all pass through a common point, so as long as $n \neq 8$ Csima-Sawyer guarantees there are at least $6(n-1)/13$ ordinary points. Thus, there must have been at least $\frac{6(n-1)}{13} - (\frac{n}{3} - \frac{15}{26}) > \frac{5n-6}{39}$ ordinary points off of l, contradicting our choice of l.

Finally, if $n = 7$ or 8 then $\lceil (5n-6)/39 \rceil = 1$ so if the Theorem were false all ordinary points of $\mathcal{A}$ would have to lie on a single pseudoline l. Removing l kills all the at least 3 ordinary points and so creates at most 2 new ones, leaving at least one which all along must have been off of l. □

One is led to ask whether, in "Sylvester-critical" arrangements, i.e. arrangements with $\leq n/2$ ordinary points, if it is possible that a small number of pseudolines can actually contain all the ordinary points. Looking at Theorem 6 we might think it possible for just two, or some other small number of pseudolines to contain all the ordinary points in such arrangements. In the case of the "near-pencil" (not a Sylvester-critical arrangement) with all lines but one going through a common point, all the ordinary points lie on a single line. If $\mathcal{A}$ is an arrangement, we use the notation $|\mathcal{A}|$ to denote the number of lines or pseudolines in $\mathcal{A}$. A sub-collection $\mathcal{B} \subset \mathcal{A}$ **spans the ordinary points**, if all ordinary points of $\mathcal{A}$ are contained in $\mathcal{B}$.

Theorem 7 *Let $\{\mathcal{A}_i\}_{i=1}^{\infty}$ be a family of arrangements in $\mathbb{RP}^2$ with $|\mathcal{A}_i| \nearrow \infty$ and such that if $|\mathcal{A}_i| = n_i$ then the number of ordinary points in $\mathcal{A}_i$ is $\leq n_i/2$. The size of sub-collections $\mathcal{B}_i \subset \mathcal{A}_i$ spanning the ordinary points is unbounded.*

Proof. Suppose we could always find a spanning sub-collection of pseudolines of size $\leq k$. Consider $N = |\mathcal{A}|$ with $N \gg k^2$. Then almost all ordinary points are contained in the intersection of one of the k lines and a line of type $(1, 2+), (2, 0+), (3, 0+)$ etc. where the notation $x+$ means that the given line has at *least* x ordinary points attached.

We show that in all such cases we would end up with too many attachments. First, suppose all $m \leq n/2$ ordinary points result from intersections of the k lines with $(1, 2)$ lines. We use the notation $\sim$ to denote "on the order of." There would have to be m such lines, leaving $\sim n-m$ lines of the form $(0, 3+)$. But this would give rise to at least, on the order of $2m + 3(n-m) = 3n - m$ total attachments. But $3n - m > 4m$ since $m \leq n/2$, which contradicts the fact that we can have at most 4 attachments per ordinary point (Lemma 4).

Next suppose that all $m \leq n/2$ ordinary points result from intersections of the k lines with $(2, 0+)$ lines. There must be $\sim m/2$ of these, leaving on the order of $n-m/2$ of type $(0, 3+)$, yielding at least $\sim 9n/4$ attachments, which again is too many.

Intersections of the k lines with $(3, 0+)$ lines require even more attachments. It is easy to see that intersection of the k lines with lines of type $(2, 0)$ requires the fewest attachments, but as noted even these are not adequate. The theorem is thus established. □

In light of the argument in the preceding proof, the following lemma is not too surprising. It is asserted to be true without proof in Borwein and Moser's 1990 survey article [1].

Lemma 8 *An arrangement of n not all coincident lines in $\mathbb{RP}^2$ with fewer than $n/2$ ordinary points must have at least one line of type* $(2, 0)$.

Proof. The idea is that without lines of type $(2, 0)$, if one first writes down the lines contributing to k ordinary points through intersection, then because of the constraint of at most 4 attachments per ordinary point (Lemma 4), one is forced to have at least $2k$ lines.

Suppose first that we had just lines of type $(1, 2+)$. It is plain that $2k$ of these form exactly k ordinary points, with no room for additional lines because of the at least $4k$ attachments. How about lines of type $(2, 1+)$? k of these lines contribute k ordinary points and any more than k additional lines of type $(0, 3+)$ would yield more than $4k$ total attachments. Finally, k lines of type $(3, 0+)$ yield $3k/2$ ordinary points, and $2k$ additional lines of type $(0, 3+)$ would already yield at least $6k$ attachments, which is capacity. It is easy to verify that lines of type $(4, 0+)$ and higher do worse. We thus conclude that an arrangement of lines as in the hypothesis of the lemma must contain lines of type $(2, 0)$. □

In fact, the preceding argument shows that it is only lines of type $(2, 0)$ that really contribute to an arrangement of n lines having fewer than the *critical* number $(n/2)$ of ordinary points.

3 A Unified View of the $n = 7$ and $n = 13$ Examples

There are various ways to view the Kelly-Moser and McKee counterexamples to the Dirac-Motzkin $n/2$ conjecture. The literature of this subject considers these examples to be "sporadic," or not related. However, they can be seen to come from a common construction.

We work in $\mathbb{RP}^2$. For the Kelly-Moser example, start with the vertices of two equal sized equilateral triangles, glue the triangles together along one edge, and add a center point. This gives a configuration with 5 points and 4 ordinary lines. Now greedily add points at infinity with the objective of reducing the ratio of ordinary lines to points. Since the 4 original ordinary lines formed 2 pairs of parallel lines, we greedily add two points, one to kill each pair of parallel lines. See Figure 4(a).

For the McKee example, start with the vertices of two equal sized regular pentagons, glue the pentagons together along an edge, add a center point, and again greedily add points at infinity with the objective of reducing the ratio of ordinary lines to points. We start with 9 points and 12 ordinary lines. There are two sets of 4 ordinary lines all sharing a common direction (slope), so we greedily add points at infinity to remove these. In the process we create two new finite ordinary lines, and one additional ordinary line at infinity. Adding vertical and horizontal points at infinity saturates this example, yielding McKee's configuration of 13 points and 6 ordinary lines as shown in Figure 4(b).

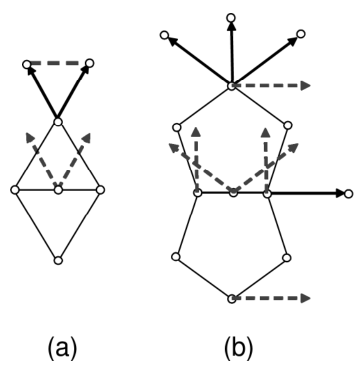

Figure 4: A common view of the (a) Kelly-Moser and (b) McKee examples.

We may try the same procedure with equal sized regular 7-gons. In this case we start with 13 points and 36 ordinary lines. There are two sets of 6 ordinary lines sharing a common direction, so adding corresponding points at infinity yields a configuration with 15 points and 27 ordinary lines. Adding vertical and horizontal points at infinity is the next most productive thing to do, leaving a configuration of 17 points and 22 ordinary lines. See Figure 5.

Unfortunately there is nothing very productive to do at this stage and the example becomes saturated with 21 points and 26 ordinary lines. Pairs of regular 9 and 11-gons only fair worse.

4 Conclusion

The Dirac-Motzkin $n/2$ conjecture remains. We know that one way of trying to extend the $n = 7$ and $n = 13$ examples is not fruitful, but perhaps there is another. In addition there is the question of whether the asymptotic $5n/39$ bound is tight for the number of ordinary points avoiding any particular pseudoline. We conjecture that the $5n/39$ bound is not tight, and in fact should be the same asymptotic bound as for the number of total ordinary points.

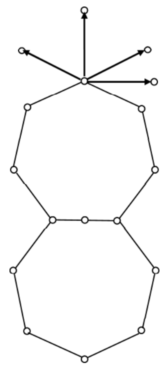

Figure 5: Glued 7-gons have 17 points with 22 ordinary lines after adding the most productive points at infinity. The ordinary lines are too numerous to draw.

References

[1] P. Borwein and W. O. J. Moser. A survey of Sylvester's problem and its generalizations. *Aequationes Math.*, 40:111–135, 1990.

[2] D. W. Crowe and T. A. McKee. Sylvester's problem on collineary points. *Math. Mag.*, 41:30–34, 1968.

[3] J. Csima and E. Sawyer. There exist $6n/13$ ordinary points. *Disc. and Comput. Geom.*, 9:187–202, 1993.

[4] G. Dirac. Collinearity properties of sets of points. *Quart. J. Math.*, 2:221–227, 1951.

[5] P. Erdös. Problem number 4065. *Amer. Math. Monthly*, 51:169, 1944.

[6] S. Felsner. *Geometric Graphs and Arrangements.* Vieweg and Sohn-Verlag, Wiesbaden, Germany, 2004.

[7] T. Gallai. Solution to problem number 4065. *Amer. Math. Monthly*, 51:169–171, 1944.

[8] L. Kelly and W. Moser. On the number of ordinary lines determined by n points. *Canad. J. of Math.*, 10:210–219, 1958.

[9] L. M. Kelly and R. Rottenberg. Simple points on pseudoline arrangements. *Pacific J. Math.*, 40:617–622, 1972.

[10] J. Lenchner. On the dual and sharpened dual of Sylvester's theorem in the plane. *IBM Research Report, RC23411*, 2004.

[11] J. Lenchner. Wedges in Euclidean arrangements. *Proc. of the Japan Conf on Disc. and Comp. Geom.*, 2004.

[12] T. Motzkin. The lines and planes connecting the points of a finite set. *Trans. of the Amer. Math. Soc.*, 70:451–464, 1951.

[13] J. J. Sylvester. Mathematical question 11851. *Educational Times*, 69:98, 1893.