Probabilistic Bounds on the Length of a Longest Edge in Delaunay Graphs of Random Points in d-Dimensions *

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Abstract

Motivated by low energy consumption in geographic routing in wireless networks, there has been recent interest in determining bounds on the length of edges in the Delaunay graph of randomly distributed points. Asymptotic results are known for random networks in planar domains. In this paper, we obtain upper and lower bounds that hold with parametric probability in any dimension, for points distributed uniformly at random in domains with and without boundary. The results obtained are asymptotically tight for all relevant values of such probability and constant number of dimensions, and show that the overhead produced by boundary nodes in the plane holds also for higher dimensions. To our knowledge, this is the first comprehensive study on the lengths of long edges in Delaunay graphs.

1 Introduction

We study the length of a longest Delaunay edge for points randomly distributed in multidimensional Euclidean spaces. In particular, we consider the Delaunay graph for a set of n points distributed uniformly at random in a d-dimensional body of unit volume. It is known that the probability that uniformly distributed random points are not in general position [1] is negligible and therefore it is safe to focus on generic sets of points [8], which we do throughout the paper.

The motivation to study such settings comes from the Random Geometric Graph (RGG) model in which n nodes are distributed uniformly at random in a disk or, more generally, according to some specified density function on d-dimensional Euclidean space [16]. The problem has attracted recent interest because of its applications in energy-efficient geometric routing and flooding in wireless sensor networks (see, e.g., [7, 11, 12, 13]).

Related Work. Kozma, Lotker, Sharir, and Stupp [11] show that the asymptotic length of a longest Delaunay edge depends on the sum, σ, of the distances to the boundary of its endpoints. More specifically, their bounds are $O(\sqrt[3]{(\log n)/n})$ if $\sigma \leq ((\log n)/n)^{2/3}$, $O(\sqrt{(\log n)/n})$ if $\sigma \geq \sqrt{(\log n)/n}$, and $O((\log n)/(n\sigma))$ otherwise. Kozma et al. also show, in the same setting, that the expected sum of the squares of all Delaunay edge lengths is $O(1)$. In [5] the authors consider the Delaunay triangulation of an infinite random (Poisson) point set in d dimensional space. In particular, they study different properties of the subset of those Delaunay edges completely included in a cube $[0, n^{1/d}] \times \cdots \times [0, n^{1/d}]$. For the maximum length of a Delaunay edge in this setting, they observe that in expectation is in $\Theta(\log^{1/d} n)$.

The lengths of longest edges in geometric graphs induced by random point sets has also been studied for graphs related to the Delaunay, including Gabriel graphs [18] and relative neighborhood (RNG) graphs [17, 19]. In particular, Wan and Yi [18] show that for n points uniformly distributed in a unit-area disk, the ratio of the length of a longest Gabriel edge to $\sqrt{(\ln n)/(\pi n)}$ is asymptotically almost surely equal to 2, and the expected number of "long" Gabriel edges, of length at least $2\sqrt{(\ln n + \xi)/(\pi n)}$, is asymptotically almost surely equal to $2e^{-\xi}$, for any fixed ξ. In [9], while studying the maximum degree of Gabriel and Yao graphs, the authors observe that the probability that the maximum edge length is greater than $3\sqrt{(\log n)/n}$ tends to zero, bound that they claim becomes $O(((\log n)/n)^{1/d})$ for d dimensions. An overview of related problems can be found in [1].

Interest in bounding the length of a longest Delaunay edge in two-dimensional spaces has grown out of extensive algorithmic work [6, 4, 10] aimed at reducing the energy consumption of geographically routing messages in Radio Networks. Multidimensional Delaunay graphs

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[1] A set of $d+1$ points in d-dimensional Euclidean space is said to be *in general position* if no hyperplane contains all of them. We say that such a set is *generic*, or *degenerate* otherwise.

are well studied in computational geometry from the point of view of efficient algorithms to construct them (see [8] and references therein), but only limited results are known regarding probabilistic analysis of Delaunay graphs in higher dimensions [14].

Overview of Our Results. We study the probabilistic length of longest Delaunay edges for points distributed in geometric domains in two and more dimensions. Since the length of the longest Delaunay edge is strongly influenced by the boundary of the enclosing region, we study the problem for two cases, which we call *with boundary* and *without boundary*.

Our results include upper and lower bounds for d-dimensional bodies with and without boundaries, that hold for a parametric error probability ε and are computed up to the constant factors (they are tight only asymptotically). In comparison, the upper bounds presented in [11] are only asymptotic, are restricted to two dimensions ($d = 2$), and apply to domains with boundary (disks), although results without boundary are implicitly given, since the results are parametric in the distance to the boundary.

Lower bounds without boundary and all upper bounds apply for any $d > 1$. Lower bounds with boundary are shown for $d \in \{2, 3\}$. The results shown are asymptotically tight for $e^{-cn} \leq \varepsilon \leq n^{-c}$, for any constant $c > 0$, and $d \in O(1)$. To the best of our knowledge, this is the first comprehensive study of this problem. The results obtained are summarized in Table 1. In order to compare upper and lower bounds for bodies with boundary, it is crucial to notice that we bound the volume of a circular segment (2D) and the volume of an spherical cap (3D), which can be approximated by polynomials of third and fourth degree respectively on the diameter of the base. Upper bounds are proved exploiting the fact that, thanks to the uniform density, it is very unlikely that a "large" volume is void of points. Lower bounds, on the other hand, are proved by showing that a configuration that yields a Delaunay edge of a certain length is not very unlikely.

In the following section, some necessary notation is introduced. Upper and lower bounds for enclosing bodies without boundaries are shown in Section 3, and the case with boundaries is covered in Section 4. We conclude with some open problems.

2 Preliminaries

The following notation will be used throughout. We will restrict attention to Euclidean (L_2) spaces. A *d-sphere*, $S = S_{r,c}$, of radius r is the set of all points in a $(d + 1)$-dimensional space that are located at distance r (the *radius*) from a given point c (the *center*). A *d-ball*, $B = B_{r,c}$, of radius r is the set of all points in a d-dimensional space that are located at distance *at most* r (the *radius*) from a given point c (the *center*). The *area* of a d-sphere S (in $(d + 1)$-space) is its d-dimensional volume. The *volume* of a d-ball B (in d-space) is its d-dimensional volume. We refer to a *unit sphere* as a sphere of area 1 and a *unit ball* as a ball of volume 1. (This is in contrast with some definitions of a "unit" ball/sphere as a unit-radius ball/sphere; we find it convenient to standardize the volume/area to be 1 in all dimensions.)

Let P be a set of points on a d-sphere, S. Given two points $a, b \in P$, let $\widehat{ab}$ be the arc of a great circle between them. Let $\delta(a, b)$ be the length of the arc $\widehat{ab}$, which is also known as the *orthodromic distance* between a and b on the sphere S. Let the *orthodromic diameter* of a subset $X \subseteq S$ be the greatest orthodromic distance between a pair of points in X. A *spherical cap on S* is the set of all points at orthodromic distance at most r from some center point $c \in S$. Let $A_d(x)$ be the area (d-volume) of a spherical cap of orthodromic diameter x, on a d-sphere of surface area 1. A *ball cap of B* is the intersection of a d-ball B with a closed halfspace, bounded by a hyperplane h, in d-space; the *base* of a ball cap is the $(d - 1)$-ball that is the intersection of h with the ball B. Let $V_d(x)$ be the d-volume of a ball cap of base diameter x, of a d-ball of volume 1. For any pair of points a, b, let $d(a, b)$ be the Euclidean distance between a and b, i.e. $d(a, b) = ||\overrightarrow{ab}||_2$. Let $D(P)$ be the Delaunay graph of a set of points P.

The following definitions of a Delaunay graph, $D(P)$, of a finite set P of points in d-dimensional bodies follow the standard definitions of Delaunay graphs (see, e.g., Theorem 9.6 in [8]).

Definition 1 *Let P be a generic set of points on a d-sphere S.*

(i) A set $F \subseteq P$ of $d + 1$ points define the vertices of a Delaunay face *of $D(P)$ if and only if there is a d-dimensional spherical cap $C \subset S$ such that F is contained in the boundary, ∂C, of C and no points of P lie in the interior of C (relative to the sphere S).*

(ii) Two points $a, b \in P$ form a Delaunay edge*, an arc of $D(P)$, if and only if there is a d-dimensional spherical cap C such that $a, b \in \partial C$ and no points of P lie in the interior of C (relative to the sphere S).*

Definition 2 *Let P be a generic set of points in a d-ball B.*

(i) A set $F \subseteq P$ of $d + 1$ points define the vertices of a Delaunay face *of $D(P)$ if and only if there is a d-ball B' such that F is contained in the boundary, $\partial B'$, of B' and no points of P lie in the interior of B'.*

	d	Upper Bound: w.p. $\geq 1-\varepsilon$, $\nexists\ \widehat{ab} \in D(P)$	Lower Bound: w.p. $\geq \varepsilon$, $\exists\ \widehat{ab} \in D(P)$
Without boundary	d	$A_d(\delta(a,b)) \geq \frac{\ln\left(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon\right)}{n-d-1}$	$A_d(\delta(a,b)) \geq \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln((e-1)/(e^2\varepsilon))}$
	1	$\delta(a,b) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}$	$\delta(a,b) \geq \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln((e-1)/(e^2\varepsilon))}$
	2	$\delta(a,b) \geq \frac{\cos^{-1}\left(1-\frac{2\ln\left(\binom{n}{2}(n-2)/\varepsilon\right)}{n-3}\right)}{\sqrt{\pi}}$	$\delta(a,b) \geq \frac{\cos^{-1}\left(1-\frac{2\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)}\right)}{\sqrt{\pi}}$
With boundary	d	$V_d(d(a,b)) \geq \frac{\ln\left(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon\right)}{n-d-1}$	–
	2	$d(a,b) \geq \sqrt[3]{\frac{16}{\sqrt{\pi}}\frac{\ln\left(\binom{n}{2}(n-2)/\varepsilon\right)}{n-3}}$	$d(a,b) \geq \rho_2/2 : V_2(\rho_2) = \frac{\ln(\alpha_2/\varepsilon)}{(n-2+\ln(\alpha_2/\varepsilon))}$ $\implies d(a,b) \geq \sqrt[3]{\frac{\ln(\alpha/\varepsilon)}{2\sqrt{\pi}(n-2+\ln(\alpha/\varepsilon))}}$
	3	$d(a,b) \geq \sqrt[4]{\frac{96}{\pi^{3/2}}\frac{\ln\left(\binom{n}{2}\binom{n-2}{2}/\varepsilon\right)}{n-4}}$	$d(a,b) \geq \rho_3/2 : V_3(\rho_3) = \frac{\ln(\alpha_3/\varepsilon)}{(n-2+\ln(\alpha_3/\varepsilon))}$ $\implies d(a,b) \geq \sqrt[4]{\sqrt[3]{\frac{48}{\pi^4}}\frac{\ln(\alpha_3/\varepsilon)}{(n-2+\ln(\alpha_3/\varepsilon))}}$

Table 1: Summary of results. α_2, α_3 are constants.

(ii) *Two points $a, b \in P$ form a* Delaunay edge, *an arc of $D(P)$, if and only if there is a d-ball B' such that $a, b \in \partial B'$ and no points of P lie in the interior of B'.*

The following inequalities [15] are used throughout

$$e^{-x/(1-x)} \leq 1-x \leq e^{-x}, \text{ for } 0 < x < 1. \tag{1}$$

3 Enclosing Body without Boundary

The following theorems show upper and lower bounds on the length of arcs in the Delaunay graph on a d-sphere.

3.1 Upper Bound

Theorem 3 *Consider the Delaunay graph $D(P)$ of a set P of $n > d+1 \geq 2$ points distributed uniformly and independently at random in a unit d-sphere, S. Then, for $0 < \varepsilon \leq 1$, the probability is at least $1-\varepsilon$ that there is no arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$A_d(\delta(a,b)) \geq \frac{\ln\left(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon\right)}{n-d-1}. \tag{$*$}$$

Proof. Let E_ε be the event that "there exists an arc $\widehat{ab} \in D(P)$, $a, b \in P$, with inequality $(*)$ satisfied" Our goal is to prove that $P(E_\varepsilon) \leq \varepsilon$.

Let us consider a fixed pair of points, $a, b \in P$. We let $E_{a,b}$ be the event that $\widehat{ab} \in D(P)$. For any subset $Q \subset P$ of $d+1$ points containing a and b, let C_Q denote the spherical cap through Q and let F_Q denote the event that the interior of C_Q contains no points of P (i.e., $int(C_Q) \cap P = \emptyset$).

Thus, we can write $E_{a,b} = \bigcup_Q F_Q$ as the union, over all $\binom{n-2}{(d+1)-2} = \binom{n-2}{d-1}$ subsets $Q \subset P$ with $|Q| = d+1$ and $a, b \in Q$, of the events F_Q. Then, by the union bound, we know that $P(E_{a,b}) \leq \sum_Q P(F_Q)$. Further, in order for event F_Q to occur, all points of P *except* the $d+1$ points of Q must lie *outside* the spherical cap C_Q through Q; thus, $P(F_Q) = (1-\mu_d(C_Q))^{n-(d+1)}$, where $\mu_d(C_Q)$ denotes the d-volume of C_Q.

We see that $P(F_Q) \leq (1 - A_d(\delta(a,b)))^{n-(d+1)}$, since, for any subset $Q \supset \{a, b\}$, the d-volume $\mu_d(C_Q)$ is at least as large as the d-volume, $A_d(\delta(a,b))$, of the spherical cap having orthodromic diameter $\delta(a,b)$. (In other words, $A_d(\delta(a,b))$ is the d-volume of the smallest volume spherical cap whose boundary passes through a and b.)

Altogether, we get

$$\begin{aligned} P(E_{a,b}) &\leq \sum_Q P(F_Q) = \sum_Q (1-\mu_d(C_Q))^{n-(d+1)} \\ &\leq \binom{n-2}{d-1}(1 - A_d(\delta(a,b)))^{n-(d+1)}. \end{aligned}$$

Now, the event of interest is

$$E_\varepsilon = \bigcup_{a,b\in P:(*) \text{ holds}} E_{a,b}.$$

The inequality $(*)$ is equivalent to

$$(n-d-1)A_d(\delta(a,b)) \geq \ln\left(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon\right),$$

which is equivalent to

$$\left(e^{-A_d(\delta(a,b))}\right)^{(n-d-1)} \leq \frac{\varepsilon}{\binom{n}{2}\binom{n-2}{d-1}}.$$

Since, by Inequality 1, $e^{-x} \geq 1-x$, the above inequality implies that

$$(1-A_d(\delta(a,b)))^{(n-d-1)} \leq \frac{\varepsilon}{\binom{n}{2}\binom{n-2}{d-1}},$$

which implies that

$$\binom{n}{2}\binom{n-2}{d-1}(1-A_d(\delta(a,b)))^{(n-d-1)} \leq \varepsilon.$$

Using the union bound, we get

$$P(E_\varepsilon) = P\left(\bigcup_{a,b\in P:(*) \text{ holds}} E_{a,b}\right) \leq \sum_{a,b\in P:(*) \text{ holds}} P(E_{a,b}).$$

Since each term $P(E_{a,b})$ in the above summation is bounded above by $\binom{n-2}{d-1}(1-A_d(\delta(a,b)))^{n-(d+1)}$, and there are at most $\binom{n}{2}$ terms in the summation, we get

$$\begin{aligned} P(E_\varepsilon) &\leq \sum_{a,b\in P:(*) \text{ holds}} P(E_{a,b}) \\ &\leq \binom{n}{2}\binom{n-2}{d-1}(1-A_d(\delta(a,b)))^{(n-d-1)} \leq \varepsilon. \end{aligned}$$

□

The following corollaries for $d=1$ and $d=2$ can be obtained from Theorem 3 using the corresponding surface areas.

Corollary 4 *In the Delaunay graph $D(P)$ of a set P of $n>2$ points distributed uniformly and independently at random on a unit circle (1-sphere), with probability at least $1-\varepsilon$, for $0<\varepsilon<1$, there is no arc $\widehat{ab} \in D(P)$, $a,b\in P$, such that*

$$\delta(a,b) \geq \frac{\ln\left(\binom{n}{2}/\varepsilon\right)}{n-2}.$$

Corollary 5 *In the Delaunay graph $D(P)$ of a set P of $n>3$ points distributed uniformly and independently at random on a unit sphere (2-sphere), with probability at least $1-\varepsilon$, for $0<\varepsilon<1$, there is no arc $\widehat{ab} \in D(P)$, $a,b\in P$, such that*

$$\delta(a,b) \geq \frac{1}{\sqrt{\pi}}\cos^{-1}\left(1-\frac{2\ln\left(\binom{n}{2}(n-2)/\varepsilon\right)}{n-3}\right).$$

Proof. The surface area of a spherical cap of a 2-sphere is $2\pi Rh$, where R is the radius of the sphere and h is the height of the cap. For a unit 2-sphere is $R=1/(2\sqrt{\pi})$. Then, the perimeter of a great circle is $2\pi/(2\sqrt{\pi}) = \sqrt{\pi}$. Thus, the central angle of a cap whose orthodromic diameter is ρ is $2\pi\rho/\sqrt{\pi} = 2\sqrt{\pi}\rho$. Let the angle between the line segment $\overline{ab}$ and the radius of the sphere be α. Then,

$$\alpha = \begin{cases} \pi/2-\sqrt{\pi}\rho & \text{if } \rho \leq \sqrt{\pi}/2 \\ \sqrt{\pi}\rho-\pi/2 & \text{if } \rho > \sqrt{\pi}/2 \end{cases}$$

And the height of the cap is $h = 1/(2\sqrt{\pi}) - 1/(2\sqrt{\pi})\sin(\pi/2-\sqrt{\pi}\rho) = (1-\cos(\sqrt{\pi}\rho))/(2\sqrt{\pi})$. Therefore, the surface area of a spherical cap of a 2-sphere whose orthodromic diameter is ρ is $(1-\cos(\sqrt{\pi}\rho))/2$. Replacing in Theorem 3, the claim follows. □

3.2 Lower Bound

Theorem 6 *In the Delaunay graph $D(P)$ of a set P of $n>2$ points distributed uniformly and independently at random in a unit d-sphere, with probability at least ε, there is an arc $\widehat{ab} \in D(P)$, $a,b\in P$, such that $A_d(\delta(a,b)) \geq A_d(\rho_1)$, where*

$$A_d(\rho_1) = \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)},$$

for any $0<\varepsilon<1$ such that $A_d(2\rho_1) \leq 1-1/(n-1)$.

Proof. In order to prove this claim, we consider a configuration given by a specific pair of points and a specific empty spherical cap circumscribing them, that would yield a Delaunay arc between those points. Then, we relate the probability of existence of such configuration to the distance between the points. Finally, we relate this quantity to the desired parametric probability. The details follow.

For any pair of points $a,b\in P$, by Definition 1, for the arc $\widehat{ab}$ to be in $D(P)$, there must exist a d-dimensional spherical cap C such that a and b are located on the boundary of the cap base and the cap surface of C is void of points from P. We compute the probability of such an event as follows. Let $\rho_2 > \rho_1$ be such that $A_d(2\rho_2) - A_d(2\rho_1) = 1/(n-1)$. Consider any point $a\in P$. The probability that some other point b is located so that $\rho_1 < \delta(a,b) \leq \rho_2$ is $1-(1-1/(n-1))^{n-1} \geq 1-1/e$, by Inequality 1.

The spherical cap with orthodromic diameter $\delta(a,b)$ is empty with probability $(1-A_d(\delta(a,b)))^{n-2}$. To lower bound this probability we consider separately the spherical cap with orthodromic diameter ρ_1 and the remaining annulus of the spherical cap with orthodromic diameter $\delta(a,b)$. The probability that the latter is empty is lower bounded by upper bounding the area $A_d(\delta(a,b)) - A_d(\rho_1) \leq A_d(2\rho_2) - A_d(2\rho_1) = 1/(n-1)$. Then, $(1-1/(n-1))^{n-2} \geq 1/e$, by Inequality 1.

Finally, the probability that the spherical cap with orthodromic diameter ρ_1 is empty is, by Inequality 1,

$$\begin{aligned}(1-A_d(\rho_1))^{n-2} &\geq \exp\left(-\frac{A_d(\rho_1)(n-2)}{1-A_d(\rho_1)}\right),\\ &= \exp\left(-\ln\left(\frac{e-1}{e^2\varepsilon}\right)\right) = \frac{e^2\varepsilon}{e-1}.\end{aligned}$$

Therefore,

$$Pr\left(\widehat{ab} \in D(P)\right) \geq \left(1-\frac{1}{e}\right)\frac{1}{e}\frac{e^2\varepsilon}{e-1} = \varepsilon.$$

$\square$

The following corollaries for $d = 1$ and $d = 2$ can be obtained from Theorem 6 using the corresponding surface areas.

Corollary 7 *In the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit circle (1-sphere), with probability at least ε, for any $e^{1-n-4/n} \leq \varepsilon < 1$, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)}.$$

Corollary 8 *In the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit sphere (2-sphere), with probability at least ε, for any $e^{-n+2\sqrt{n-1}-1} \leq \varepsilon < 1$, there is an arc $\widehat{ab} \in D(P)$, $a, b \in P$, such that*

$$\delta(a,b) \geq \frac{1}{\sqrt{\pi}}\cos^{-1}\left(1-\frac{2\ln\left((e-1)/(e^2\varepsilon)\right)}{n-2+\ln\left((e-1)/(e^2\varepsilon)\right)}\right).$$

Proof. As shown in the proof of Corollary 5, the surface area of a spherical cap of a 2-sphere whose orthodromic diameter is ρ is $(1-\cos(\sqrt{\pi}\rho))/2$. Replacing in Theorem 6, the claim follows. $\square$

4 Enclosing Body with Boundary

The following theorems show upper and lower bounds on the length of edges in the Delaunay graph in a d-ball. The proofs, omitted here for brevity, can be found in the full version of this work [3].

4.1 Upper Bound

Theorem 9 *In the Delaunay graph $D(P)$ of a set P of $n > d+1 \geq 2$ points distributed uniformly and independently at random in a unit d-ball, with probability at least $1-\varepsilon$, for $0 < \varepsilon < 1$, there is no edge $(a,b) \in D(P)$, $a, b \in P$, such that*

$$V_d(d(a,b)) \geq \frac{\ln\left(\binom{n}{2}\binom{n-2}{d-1}/\varepsilon\right)}{n-d-1}.$$

The following corollaries for $d = 2$ and $d = 3$ can be obtained from Theorem 9 using the corresponding volumes.

Corollary 10 *In the Delaunay graph $D(P)$ of a set P of $n > 3$ points distributed uniformly and independently at random in a unit disk (2-ball), with probability at least $1-\varepsilon$, for $\binom{n}{2}(n-2)e^{-\sqrt{2}(n-3)/\pi} < \varepsilon < 1$, there is no edge $(a,b) \in D(P)$, $a, b \in P$, such that*

$$d(a,b) \geq \sqrt[3]{\frac{16}{\sqrt{\pi}}\frac{\ln\left(\binom{n}{2}(n-2)/\varepsilon\right)}{n-3}}.$$

Corollary 11 *In the Delaunay graph $D(P)$ of a set P of $n > 4$ points distributed uniformly and independently at random in a unit ball (3-ball), with probability at least $1-\varepsilon$, for $\binom{n}{2}\binom{n-2}{2}e^{-2(n-4)/(3\sqrt{\pi})} < \varepsilon < 1$, there is no edge $(a,b) \in D(P)$, $a, b \in P$, such that*

$$d(a,b) \geq \sqrt[4]{\frac{96}{\pi^{3/2}}\frac{\ln\left(\binom{n}{2}\binom{n-2}{2}/\varepsilon\right)}{n-4}}.$$

4.2 Lower Bound

In this section we give lower bounds for $d = 2$ and $d = 3$. As in the case without boundary, we prove our lower bounds showing a configuration given by a specific pair of points and a specific empty body circumscribing them, that would yield a Delaunay edge between those points. Then, we relate the probability of existence of such configuration to the distance between the points and to the desired parametric probability.

Theorem 12 *For $d = 2$, given the Delaunay graph $D(P)$ of a set P of $n > 2$ points distributed uniformly and independently at random in a unit d-ball, with probability at least ε, there is an edge $(a,b) \in D(P)$, $a, b \in P$, such that $d(a,b) \geq \rho_1/2$, where*

$$V_d(\rho_1) = \frac{\ln(\alpha/\varepsilon)}{(n-2+\ln(\alpha/\varepsilon))},$$

where $\alpha = (1-e^{-1/16})(1-e^{-1/32})e^{-1}$, for any $0 < \varepsilon \leq \alpha/e^2$ such that $V_d(\rho_1) \leq 1/2 - 1/n$. Which implies that

$$d(a,b) \geq \sqrt[3]{\frac{\ln(\alpha/\varepsilon)}{2\sqrt{\pi}\,(n-2+\ln(\alpha/\varepsilon))}}.$$

Theorem 13 *For $d = 3$, given the Delaunay graph $D(P)$ of a set P of $n > 4$ points distributed uniformly and independently at random in a unit d-ball, with probability at least ε, there is an edge $(a, b) \in D(P)$, $a, b \in P$, such that $d(a, b) \geq \rho_1/2$, where*

$$V_d(\rho_1) = \frac{\ln(\alpha/\varepsilon)}{(n - 2 + \ln(\alpha/\varepsilon))},$$

where $\alpha = (1 - e^{-1/6})(1 - e^{-1/12})e^{-12}$, for any $0 < \varepsilon \leq \alpha/e$ such that $V_d(\rho_1) \leq 1/2 - 1/n$. Which implies that

$$d(a, b) \geq \sqrt[4]{\sqrt[3]{\frac{48}{\pi^4}} \frac{\ln(\alpha/\varepsilon)}{(n - 2 + \ln(\alpha/\varepsilon))}}.$$

5 Future Directions, Open Problems

It would be interesting to extend this study to other norms, such as L_1 or L_∞. Also, Theorems 12 and 13 were proved by showing that the existence of a configuration that yields a Delaunay edge of some length is not unlikely. Different configurations were used for each, but a configuration that works for both cases exists (although yielding worse constants). We conjecture that (modulo some constant) the same bound can be obtained in general for any $d > 1$. Both questions are left for future work.

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