Establishing Strong Connectivity using Optimal Radius Half-Disk Antennas

Greg Aloupis* Mirela Damian† Robin Flatland‡ Matias Korman§ Özgür Özkan¶
David Rappaport‖ Stefanie Wuhrer**

Abstract

Given a set S of points in the plane representing wireless devices, each point equipped with a directional antenna of radius r and aperture angle $\alpha \geq 180°$, our goal is to find orientations and a minimum r for these antennas such that the induced communication graph is strongly connected. We show that $r = \sqrt{3}$ suffices to establish strong connectivity, assuming that the longest edge in the Euclidean minimum spanning tree for S is 1. This result is optimal in the sense that $r = \sqrt{3}$ is necessary in the worst-case for $\alpha \in [180°, 240°)$. In contrast, $r = 2$ is sometimes necessary when $\alpha < 180°$.

1 Introduction

Consider a wireless network modeled by a set of planar point sites S each equipped with a transceiver having a transmission radius r. Typically one assumes that communication is omni-directional and two nodes can directly communicate with each other if the distance separating them is r or less. Geometrically the transmission region of an antenna at a point p is modeled by a circle of radius r centered at p. The connectivity of the network can be represented by a communication graph $G(S)$, which has a node for each point and an edge between each pair of nodes separated by distance r or less.

Recently there has been interest in using directional antennas in place of their omni-directional counterparts [2, 3, 4, 5, 6, 7, 8]. Some advantages of using directional antennas are that security can be enhanced and communication interference can be reduced. Furthermore, if directional antennas are used cleverly the power consumption of the network may be reduced. The transmission region of a directional antenna at a node p is geometrically represented by the sector of a circle with its apex at p, a central angle α, and a radius r. Its orientation is determined by a rotation θ about p. We assume that all antennas have the same α and r; it is only θ that varies. Thus communication between two nodes is no longer symmetric and is best modeled by a directed communication graph in which a directed edge $\overrightarrow{pq}$ indicates that q lies in p's sector.

The *direction assignment problem* is the task of finding orientations for a set of directional antennas such that the induced communication graph has certain desired properties. In this paper we focus on obtaining a strongly connected communication graph using minimal r. We will assume S is normalized so that the length of the longest edge in a Euclidean minimum spanning tree is 1. It is not difficult to see that to achieve connectivity in the normalized point set, r must be at least 1. Caragiannis *et al.* [3] show that, for antennas with $\alpha < 240°$, an increase in r by a factor of $\sqrt{3}$ is sometimes necessary to guarantee strong connectivity in the communication graph. We show here that, for $\alpha \geq 180°$, an increase factor of $\sqrt{3}$ is always sufficient. In contrast, when $\alpha < 180$, the communication range must sometimes increase by a factor of 2 (i.e., consider points at unit intervals on a line).

We review some related results. In addition to providing lower bounds on r, Caragiannis *et al.* [3] also give an algorithm for orienting antennas with $180° \leq \alpha < 288°$ to obtain strong connectivity using $r = 2\sin(180° - \alpha/2)$. Thus the algorithm presented here (with $r = \sqrt{3}$) improves upon their result when $180° \leq \alpha < 240°$. Damian and Flatland [6] consider directional antenna angles of 120° and 90°, and provide bounds of $r = 5$ and $r = 7$ (resp.) while at the same time bounding the number of hops to 5 and 6 (resp.) for nodes within unit distance. Bose *et al.* [8] have recently shown that a connected network using omni-directional communication, can be replaced with directional antennas (with any $\alpha > 0°$) so that the increase of r and hop distance are bounded by constant factors (which depend on α).

In other related work, Nijnatten [2] also considers the problem of finding suitable orientations of α-antennas

*Université Libre de Bruxelles (ULB), Belgique, aloupis.greg@gmail.com

†Department of Computer Science, Villanova University, USA mirela.damian@villanova.edu

‡Department of Computer Science, Siena College, USA, flatland@siena.edu

§Université Libre de Bruxelles (ULB), Belgique, mkormanc@ulb.ac.be

¶Department of Computer Science and Engineering, Polytechnic Institute of NYU, USA, ozgurozkan@gmail.com. Research supported by US Department of Education grant P200A090157.

‖School of Computing, Queen's University, Canada daver@cs.queensu.ca. Research supported by NSERC Discovery grant 388-329.

**Institute for Information Technology, National Research Council, Canada, stefanie.wuhrer@nrc-cnrc.gc.ca

to form a strongly connected graph, but in his variant of the problem he allows a different r for each antenna and seeks to minimize the overall power consumption of the network. Ben-Moshe *et al.* [5] consider 90°-antennas but restrict the orientations to one of the four standard quadrant directions. Bhattacharya *et al.* [4] consider nodes with multiple directional antennas and focus on minimizing the sum of the antenna angles for a fixed r. Kranakis *et al.* [7] have recently published a survey of results pertaining to the use of directional antennas in wireless networks.

2 Orienting Antennas Using $r = \sqrt{3}$

Here we establish an upper bound of $\sqrt{3}$ for r, by means of an algorithm for orienting 180°-antennas of radius $r = \sqrt{3}$ to achieve a strongly connected communication graph. Let MST_5 be a minimum spanning tree of P with maximum degree of five, such as the one described in [1]. Our algorithm processes nodes in the order in which they are visited in a breath-first traversal of MST_5. When a node is visited, it is assigned other nodes (within distance $\sqrt{3}$) for its antenna to cover (so as to satisfy certain invariants). If a node v is assigned to cover node w, we will say that "*v points to w*," and we use the notation $v \to w$. During the traversal of MST_5, nodes are colored *white*, *gray*, or *black*. Initially all nodes are white, meaning that they have not yet been visited and do not point to any nodes. Visited nodes are black, and they point to at least one and at most two other gray or black nodes. Gray nodes are direct children of visited nodes but have not themselves been visited. They point to one other gray or black node.

Let the *gray/black communication subgraph* be the graph consisting of the gray/black nodes and having a directed edge $\overrightarrow{uw}$ between each pair of nodes where $u \to w$. Our goal is to assign/adjust what the nodes point to by inserting/updating edges of length at most $\sqrt{3}$ in the gray/black communication subgraph such that, throughout the tree traversal, the gray/black communication subgraph is strongly-connected. For each gray/black node, observe that it is trivial to determine an orientation for its 180°-antenna that covers the one or two nodes it points to. We note that the full communication graph induced by these nodes may include additional edges, since a node's 180°-antenna may (by chance) cover nodes in addition to the one or two explicitly assigned to it, but these edges are not needed for strong connectivity.

Let the root of MST_5 be any node with degree one. To get started, we color the root node black and its child gray, and we constrain them to point to each other. Starting with the root's child, we visit the nodes one by one in a breadth-first search order. When a node v is visited, it is initially gray. During the visit, we change its color from gray to black and change the color of its children from white to gray. We then locally update/insert directed edges in the gray/black communication subgraph so that the following invariants are satisfied:

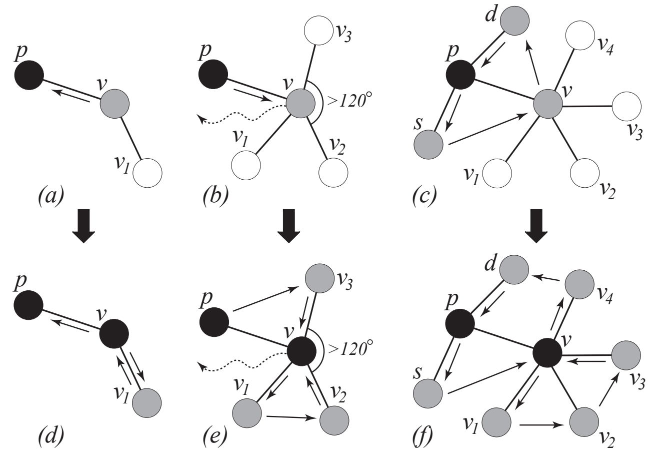

Figure 1: Solid edges are MST_5 edges; the arrows represent directed edges in the communication graph; the dotted arrow in (b,e) represents v's directed edge to some other gray/black node (by Invariant (I2)).

(I1) Each black node points to at least one and at most two gray/black nodes.

(I2) Each gray node points to exactly one gray/black node.

(I3) For each gray node v, one of the following is true:

- (I3a) v points to its parent p. (Fig. 1a)
- (I3b) p points to v. (Fig. 1b)
- (I3c) p has children s and d that are (resp.) the first child clockwise and first child counterclockwise from v, and $p \to s \to v \to d \to p$. In addition, $\widehat{spv} + \widehat{vpd} \leq 180°$, and s and d lie on opposite sides of the line passing through v and p. (Fig. 1c)

(I4) The gray/black communication subgraph has edges no longer than $\sqrt{3}$ and is strongly connected.

We describe inductively on the number of black nodes how to maintain these invariants. In the base case there is one black node, the root of MST_5, and it points to its single gray child, which points back to the root. Observe that invariant (I1) holds for the root and invariants (I2, I3a) hold for its child. Also, observe that invariant (I4) holds for the root and its child. Assume inductively that the invariants hold after visiting and coloring i nodes black. Let v be the $(i+1)$-st node visited.

We introduce some definitions so that we can process v in a uniform manner independent of its degree. Let $\mathsf{boundary}(v)$ be the children of v angularly adjacent to its parent p. More formally, let $v_0, v_1, \ldots, v_{k-1}$, for $k \leq 5$, be the nodes adjacent to v in counter-clockwise order with $v_0 = p$. Then, if $\mathsf{deg}(v) = 1$, let $\mathsf{boundary}(v) = \emptyset$; otherwise, let $\mathsf{boundary}(v) = \{v_1, v_{k-1}\}$ (e.g., see v_1 and v_3 in Fig. 1b.) An *isolated* child is a boundary child that is angularly separated from v's other children. In other words, if $\mathsf{deg}(v) < 3$, then $\mathsf{isolated}(v) = \emptyset$; otherwise if $\widehat{v_1 v v_2} > 120°$, then $v_1 \in \mathsf{isolated}(v)$, and similarly, if $\widehat{v_{k-1} v v_{k-2}} > 120°$, then $v_{k-1} \in \mathsf{isolated}(v)$ (e.g., in Fig. 1b v_3 is isolated, but v_1 is not.) Let $\mathsf{Children}(v) = \{v_1, \ldots, v_{k-1}\}$. The predicate $\mathsf{in\text{-}range}(v, w) = \mathsf{true}$ iff $dist(v, w) \leq \sqrt{3}$. For any gray node v, let $\mathsf{source}(v)$ be the node constrained to point to node v, and let $\mathsf{dest}(v)$ be the node that v is constrained to point to. We use these terms only when well-defined within the context of Invariant I3. For instance, if v satisfies (I3a), then $\mathsf{dest}(v) = p$; if v satisfies (I3b), then $\mathsf{source}(v) = p$; and if v satisfies (I3c), then $\mathsf{source}(v) = s$ and $\mathsf{dest}(v) = d$ (see Fig. 1c).

Algorithm 1 details how we update/insert edges in the gray/black communication subgraph when visiting v. We begin by describing the operation of the algorithm. The IF statement in lines 1-13 initializes the variables $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ to be two nodes with a directed edge between them such that one of the two nodes is v. The existence of such an edge is guaranteed by Invariant (I3). For example, in Fig 1b, $\mathsf{v_{from}} = p$ and $\mathsf{v_{to}} = v$. In Fig. 1c, there are two directed edges incident to v, one of which will be used to initialize $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$; in this case, there are no isolated children and we will assume v_4 is within range of d, so we set $\mathsf{v_{from}} = v$ and $\mathsf{v_{to}} = d$.

The remaining pseudocode (lines 14-20) first determines if there is a boundary child of v that is within distance $\sqrt{3}$ of both $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$. If so, then variable $\mathsf{v_{via}}$ is initialized to one such child, with preference being given in lines 17-18 to an isolated boundary child (which will be explained shortly). For example, in Fig. 1b, $\mathsf{v_{via}} = v_3$; in Fig. 1c, $\mathsf{v_{via}} = v_4$. Then the algorithm does two things. First, it replaces the edge $\mathsf{v_{from}} \to \mathsf{v_{to}}$ with the two edges, $\mathsf{v_{from}} \to \mathsf{v_{via}}$ and $\mathsf{v_{via}} \to \mathsf{v_{to}}$ (line 19). This incorporates child $\mathsf{v_{via}}$ into the strongly connected subgraph of gray/black nodes. Second, it calls the subroutine CHAIN (line 20) which inserts edges that link v and its children other than $\mathsf{v_{via}}$ into a cycle. This incorporates the other children into the strongly connected subgraph of gray/black nodes. See figure pairs 1b, 1e and 1c, 1f showing before and after edge insertions. If, however, there is no boundary child in range of $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ in line 15, then $\mathsf{v_{via}} = \emptyset$ when the call to CHAIN in line 20 is made and all of v's children are linked into a cycle, thus incorporating them into the gray/black strongly connected subgraph.

We give intuition regarding the isolated children and the algorithm's preference for them. If v has an isolated boundary child, v', then we may not be able to CHAIN

Algorithm 1: Visit(Node v)

1 **if** $v \to p$ **then** /* Invariant I3a */
2 $\quad \mathsf{v_{to}} = p$, and $\mathsf{v_{from}} = v$
3 **else if** $p \to v$ **then** /* Invariant I3b */
4 $\quad \mathsf{v_{to}} = v$, and $\mathsf{v_{from}} = p$
5 **else** /* Invariant I3c */
6 $\quad$ **if** $\exists v' \in \mathsf{isolated}(v)$ $s.t.$ $\mathsf{in\text{-}range}(v', \mathsf{dest}(v))$ **then**
7 $\quad\quad \mathsf{v_{from}} = v$, $\mathsf{v_{to}} = \mathsf{dest}(v)$
8 $\quad$ **else if** $\exists v' \in \mathsf{isolated}(v)$ $s.t.$ $\mathsf{in\text{-}range}(v', \mathsf{source}(v))$ **then**
9 $\quad\quad \mathsf{v_{from}} = \mathsf{source}(v)$, $\mathsf{v_{to}} = v$
10 $\quad$ **else if** $\exists v' \in \mathsf{boundary}(v)$ $s.t.$ $\mathsf{in\text{-}range}(v', \mathsf{dest}(v))$ **then**
11 $\quad\quad \mathsf{v_{from}} = v$, $\mathsf{v_{to}} = \mathsf{dest}(v)$
12 $\quad$ **else**
13 $\quad\quad \mathsf{v_{from}} = \mathsf{source}(v)$, $\mathsf{v_{to}} = v$
14 $\mathsf{v_{via}} = \emptyset$
15 **if** $\exists v' \in \mathsf{boundary}(v)$ $s.t.$ $\mathsf{in\text{-}range}(v', \mathsf{v_{from}}) \wedge \mathsf{in\text{-}range}(v', \mathsf{v_{to}})$ **then**
16 $\quad \mathsf{v_{via}} = v'$
17 $\quad$ **if** $\exists v' \in \mathsf{isolated}(v)$ $s.t.$ $\mathsf{in\text{-}range}(v', \mathsf{v_{from}}) \wedge \mathsf{in\text{-}range}(v', \mathsf{v_{to}})$ **then**
18 $\quad\quad \mathsf{v_{via}} = v'$
19 $\quad$ $\mathsf{REPLACE}$ $\mathsf{v_{from}} \to \mathsf{v_{to}}$ with $\mathsf{v_{from}} \to \mathsf{v_{via}}$ and $\mathsf{v_{via}} \to \mathsf{v_{to}}$
20 $\mathsf{CHAIN}(v, \mathsf{Children}(v) \setminus \{\mathsf{v_{via}}\})$

Subroutine 2: $\mathsf{CHAIN}(v, v'_1, v'_2 \ldots, v'_\ell)$

Add edges: $v \to v'_1 \to v'_2 \to \ldots v'_{\ell-1} \to v'_\ell \to v$

v' with the other children since the angle between it and the next sibling (in clockwise or counter-clockwise order) is $> 120°$, and thus the next sibling may be at a distance $> \sqrt{3}$. Observe that in the portion of the IF statement involving Invariant (I3c) (lines 5-13), there is a preference for initializing $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ such that they both are in range of an isolated child. (Since one of these two variables will be set to v which is within range of all its children, we only need to check if $\mathsf{dest}(v) = d$ or $\mathsf{source}(v) = s$ is within range.) This is done so that $\mathsf{v_{via}}$ will be set to an isolated child (in line 18), and thus an isolated child undergoes the $\mathsf{REPLACE}$ operation rather than being chained with the other children. (In Section 3 we prove that the remaining children can be chained.) If no isolated child is within range in lines 6-9, then the algorithm attempts to set $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ so that they are within range of a regular boundary child (lines 10-13). The reason for this is that to maintain our invariants, we must not chain more than 3 children.

Thus if there is a boundary child in range, then it undergoes the REPLACE operation, and the other (at most 3) children are chained.

3 Proof of Correctness

We now prove that Algorithm 1 is correct. We begin by proving that the CHAIN and REPLACE operations only add edges between nodes that are in range of each other and that they maintain Invariants (I1), (I2), and (I3). We then show that these operations also ensure that Invariant (I4) is satisfied. In what follows, let $p(w)$ denote the parent of node w.

Consider the REPLACE operation in line 19. Observe first that execution only reaches line 19 if $\text{in-range}(\mathsf{v_{from}}, \mathsf{v_{via}}) = \text{in-range}(\mathsf{v_{via}}, \mathsf{v_{to}}) = \mathsf{true}$, and therefore the edge updates are valid. We now verify that $\mathsf{v_{via}}$ satisfies the invariants after REPLACE. If v satisfies (I3a), or if v satisfies (I3c) and $\text{in-range}(\mathsf{v_{via}}, \text{dest}(v)) = \mathsf{true}$, then $\mathsf{v_{from}} = v$. After REPLACE, $\mathsf{v_{via}}$ will satisfy (I3b) since $v = \mathsf{v_{from}} \rightarrow \mathsf{v_{via}}$ and $v = p(\mathsf{v_{via}})$. Otherwise, v satisfies (I3b), or v satisfies (I3c) and $\text{in-range}(\mathsf{v_{via}}, \text{source}(v)) = \mathsf{true}$, and so $\mathsf{v_{to}} = v$. After REPLACE, $\mathsf{v_{via}}$ will satisfy (I3a) since $\mathsf{v_{via}} \rightarrow \mathsf{v_{to}} = v$ and $v = p(\mathsf{v_{via}})$. It is easy to verify that $\mathsf{v_{via}}$ satisfies (I2), and since REPLACE does not change the number of nodes pointed to by $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$, they continue to satisfy either (I1) or (I2).

We now prove the correctness of the CHAIN operation. It is easy to verify that the children involved in CHAIN satisfy (I2) afterwards, since their color changes from white to gray and CHAIN makes them each point to one node. In addition, v satisfies (I1) since v points to one gray/black node before CHAIN, and CHAIN makes it point to one more. Therefore, we focus on verifying (I3). In each case that follows, when $\mathsf{v_{via}} \neq \emptyset$, we assume $\mathsf{v_{via}}$ is intialized to boundary child $v_{deg(v)-1}$; situations in which $\mathsf{v_{via}}$ is initialized to v_1 are analogous.

- **Case 1** $(\deg(v) = 1) \vee (\deg(v) = 2 \wedge \mathsf{v_{via}} \neq \emptyset)$**.** In this case there are no points in $\text{Children}(v) \setminus \{\mathsf{v_{via}}\}$.

- **Case 2** $(\deg(v) = 2 \wedge \mathsf{v_{via}} = \emptyset) \vee (\deg(v) = 3 \wedge \mathsf{v_{via}} \neq \emptyset)$**.** v_1 is the only child in $\text{Children}(v) \setminus \{\mathsf{v_{via}}\}$, and CHAIN adds edges $v \rightarrow v_1 \rightarrow v$. Since $v = p(v_1)$, $\text{in-range}(v, v_1) = \mathsf{true}$. Thus, v_1 satisfies (I3a).

- **Case 3** $(\deg(v) = 3 \wedge \mathsf{v_{via}} = \emptyset) \vee (\deg(v) = 4 \wedge \mathsf{v_{via}} \neq \emptyset)$**.** v_1 and v_2 are the two children in $\text{Children}(v) \setminus \{\mathsf{v_{via}}\}$, and CHAIN adds edges $v \rightarrow v_1 \rightarrow v_2 \rightarrow v$. Note that since $v = p(v_1) = p(v_2)$, $\text{in-range}(v, v_1) = \text{in-range}(v, v_2) = \mathsf{true}$. Also, by Lemma 4 $\text{in-range}(v_1, v_2) = \mathsf{true}$. Thus, v_1 satisfies (I3b) and v_2 satisfies (I3a).

- **Case 4** $(\deg(v) = 4 \wedge \mathsf{v_{via}} = \emptyset) \vee (\deg(v) = 5)$**.** Note that if $\deg(v) \geq 4$ then $\mathsf{v_{via}} \neq \emptyset$ by Lemma 3. Therefore, we only need to handle the case when $\deg(v) = 5 \wedge \mathsf{v_{via}} \neq \emptyset$. In this case, v_1 , v_2, and v_3 are the three children in $\text{Children}(v) \setminus \{\mathsf{v_{via}}\}$, and CHAIN adds edges $v \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v$. Note that since $v = p(v_1) = p(v_3)$, $\text{in-range}(v, v_1) = \text{in-range}(v, v_3) = \mathsf{true}$. Also, by Lemma 4 $\text{in-range}(v_1, v_2) = \text{in-range}(v_2, v_3) = \mathsf{true}$. Thus, v_1 satisfies (I3b) and v_3 satisfies (I3a).

  To complete the proof, we show that v_2 satisfies Invariant (I3c). First we verify that $\widehat{v_1 v v_2} + \widehat{v_2 v v_3} \leq 180°$. This is true since $\mathsf{v_{via}}$ is a boundary child of v, and thus the remaining children v_1, v_2 and v_3 are radially consecutive about v. For a degree 5 node, any three radially consecutive adjacent nodes can span at most 180°, since otherwise the sum of all five angles is more than 360° (because the angle between radially consecutive adjacent edges in a MST is at least 60°). Finally, we verify that v_1 and v_3 are on opposite sides of the line through $v_2 v$. For contradiction, suppose they are on or to the same side of this line. Because v_1, v_2, v_3 are radially consecutive, this implies that all five nodes adjacent to v are on or to the same side of the line through $v_2 v$, which again is impossible in a MST.

We end by proving that (I4) is satisfied after visiting v. Let G be the gray/black communication subgraph just prior to v being visited. By the inductive hypothesis, G is strongly connected. Consider the REPLACE operation. Observe that both $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ are gray/black nodes and thus are in G. In addition, $\mathsf{v_{from}} \rightarrow \mathsf{v_{to}}$ corresponds to an edge in G. It is straightforward then to verify that adding node $\mathsf{v_{via}}$ to G and replacing edge $\mathsf{v_{from}} \rightarrow \mathsf{v_{to}}$ with $\mathsf{v_{from}} \rightarrow \mathsf{v_{via}}$ and $\mathsf{v_{via}} \rightarrow \mathsf{v_{to}}$ results in a strongly connected graph. Similarly, adding v's children, $v'_1, \ldots, v'_\ell$, involved in the CHAIN operation to G along with edges $v \rightarrow v'_1 \rightarrow \cdots \rightarrow v'_\ell \rightarrow v$ results in a strongly connected graph. This combined with the fact that v's children are all colored gray when v is visited ensures that Invariant (I4) is satisfied.

Due to space constraints, we omit the proof of the following lemma.

Lemma 1 *Let (a, b, c, d) be a path in a minimum spanning tree T such that a and d lie on or to a same side of a line through bc. Then $\widehat{abc} + \widehat{bcd} > 150°$.*

Lemma 2 *Let (a, b, c, d) be a path in MST_5 such that a and d lie on or to the same side of a line through bc. Furthermore, $60° \leq \widehat{abc} \leq 150°$, $60° \leq \widehat{bcd} \leq 150°$, and $\widehat{abc} + \widehat{bcd} \leq 210°$. Then $|ad| \leq \sqrt{3}$.*

Proof. Let $D(p, r)$ denote the open disk of radius r centered at point p, let $\partial D(p, r)$ denote its boundary,

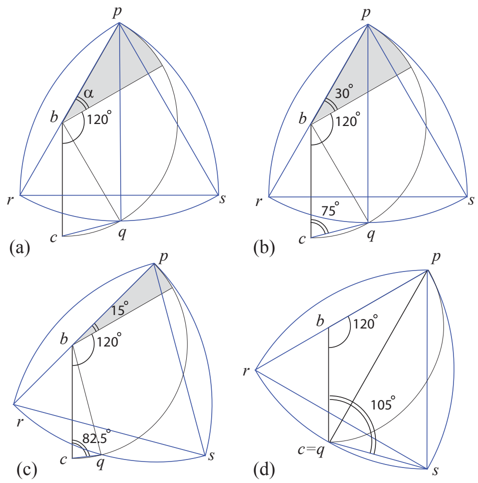

Figure 2: Lemma 2: Regions of diameter $\sqrt{3}$.

and let $D[p,r] = D(p,r) \cup \partial D(p,r)$ denote the closed disk. Rotate MST_5 so that bc is vertical (as shown in Fig. 2a), and a and d lie right of bc. For simplicity, let $|bc| = 1$, since this is the value for which $|ad|$ is maximum. Assume $\widehat{bcd} \le \widehat{abc}$. The case when $\widehat{bcd} \ge \widehat{abc}$ is symmetrical. We start with a small observation regarding certain regions of diameter $\sqrt{3}$.

Fix $0 \le \alpha \le 30°$, and define the points $p = p(\alpha)$, $q = q(\alpha)$, $r = r(\alpha)$ and $s = s(\alpha)$ as follows: p is the point above and right of b, such that $\widehat{pbc} = 120°+\alpha$ and $|bp| = 1$; q is the right intersection point between $\partial D(p,\sqrt{3})$ and $\partial D(b,1)$; r is the intersection point between the ray with origin p passing through b, and $\partial D(p,\sqrt{3})$; and s is the corner of the equilateral triangle $\triangle prs$, right of r. Refer to Figure 2a. Then the following properties hold:

(P1) $\widehat{pbc}+\widehat{bcq} = 210°+\frac{\alpha}{2}$. This follows immediately from the fact that $\widehat{pbq} = 120°$ (because $|pb| = |bq| = 1$ and $|pq| = \sqrt{3}$), thus $\widehat{cbq} = \alpha$ and $\widehat{bcq} = 90 - \alpha/2$.

(P2) The closed region formed by the intersection $D[p,\sqrt{3}] \cap D[r,\sqrt{3}] \cap D[s,\sqrt{3}]$ has diameter $\sqrt{3}$. We abuse the terminology here and denote this region by $\mathsf{lune}[p,r,s]$.

We apply these properties repeatedly, to determine regions for which the lemma holds. We start with the value $\alpha = 30°$, so that $\widehat{pbc}$ is at its maximum value of $150°$. Let points p,q,r,s be as defined above. (See Figure 2b.) By property (P1), $\widehat{bcq} = 75°$. If both a and d lie inside $\mathsf{lune}[p,r,s]$, then the lemma holds by property (P2). Suppose then that d lies outside $\mathsf{lune}[p,r,s]$. Observe that d cannot lie in $D(b,1)$, because then $|bd| < |bc|$ and $|bc|$ is not an edge in MST_5. So it must be that $\widehat{bcd} \ge \widehat{bcq} = 75°$, meaning that $\widehat{abc} \le 135°$ (because their sum does not exceed $210°$, by the lemma statement). We capture this situation by resetting $\alpha = 15°$ and redefining p,q,r,s for this new α value. (See to Figure 2c.) By property (P1), $\widehat{bcq} = 82.5°$. If both a and d lie inside $\mathsf{lune}[p,r,s]$, then the lemma holds by property (P2). If d lies outside $\mathsf{lune}[p,r,s]$, then $\widehat{bcd} \ge \widehat{bcq} = 82.5°$, meaning that $\widehat{abc} \le 127.5°$. After repeating these steps k times, we either get the result of the lemma, or we get $\widehat{bcd} \ge 60°+30°\cdot(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^k})$, and $\widehat{abc} \le 150°-30°\cdot(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^k})$. In the limit, as $k \longrightarrow \infty$, we get that $\widehat{bcd} \ge 90°$ and $\widehat{abc} \le 120°$ (otherwise the lemma holds). When $\alpha = 0$, q and c coincide: $\widehat{pbc} = 120°$ and $|pc| = \sqrt{3}$. (See Figure 2d.) Simple calculations show that $\widehat{bcp} = 30°$, $\widehat{pcs} = 75°$, therefore $\widehat{bcs} = 105°$. If both a and d lie inside $\mathsf{lune}[p,r,s]$, then the lemma holds. Otherwise, if d lies outside $\mathsf{lune}[p,r,s]$, then $\widehat{bcd} \ge 105°$ and therefore $\widehat{abc} \le 105°$. From this point on, we are in the situation $\widehat{bcd} \ge \widehat{abc}$, which is symmetric to the situation $\widehat{bcd} \le \widehat{abc}$ discussed above. This concludes the proof. $\square$

Lemma 3 *If* $\mathsf{deg}(v) \ge 4$, *then* $\mathsf{v_{via}}$ *is initialized. Furthermore, if* v *has an isolated child, then* $\mathsf{v_{via}}$ *is initialized to an isolated child.*

Proof. If v satisfies invariant (I3a) or (I3b), then $\mathsf{v_{to}}, \mathsf{v_{from}} \in \{v,p\}$. Note that at most one boundary child v' of v may satisfy $\widehat{v'vp} > 120°$, because each angle between radially consecutive children of v is at least $60°$, and the sum of all these angles is $360°$. It follows that the second boundary child (which always exists, because $\mathsf{deg}(v) \ge 4$) is within range of both p and v, therefore the condition of the IF statement on line 15 evaluates to true and $\mathsf{v_{via}}$ is initialized on line 16. By similar arguments, if v has an isolated boundary child, say v_1, then with the exception of $\widehat{v_1vv_2}$, all other angles at v must be smaller than $120°$. Thus v_1 is within range of p and v, and therefore $\mathsf{v_{via}}$ is initialized to an isolated child in line 18.

Next we discuss the more complex situation when v satisfies invariant (I3c), so v is involved in a cycle $p \to s \to v \to d \to p$. (See for example Fig. 1c.) First recall that by Invariant (I3c), s and d lie to opposite sides of the line through vp, and s, v and d are radially consecutive children of p, in counter-clockwise order. Since $\mathsf{deg}(v) \ge 4$ and radially consecutive adjacent edges in an MST form an angle of at least $60°$, boundary children v_1 and v_{k-1} (where $k = \mathsf{deg}(v)$) cannot lie on the same side of the line through vp. Also recall that v_1, p and v_{k-1} are radially consecutive neighbors of v, in clockwise order. It follows that s and v_1 are both on or to one side of the line through vp, and d and v_k are both on or to the other side. We will use this fact when applying Lemma 2 below.

Consider first the case when $\mathsf{deg}(v) = 4$. Assume first that v has no isolated children. Note that $\widehat{v_1vp} + \widehat{v_3vp} \le 240°$, because each of $\widehat{v_1vv_2}$ and $\widehat{v_2vv_3}$ is at least $60°$, and the sum of all these angles is $360°$. These together imply that $\widehat{v_1vp} + \widehat{v_3vp} + \widehat{spd} \leq 240° + 180° = 420°$, so at least one of $\widehat{v_3vp} + \widehat{vpd}$ and $\widehat{v_1vp} + \widehat{vps}$ is no greater than $210°$. For the pair whose angle sum is no more than $210°$, each individual angle is at least $60°$ and no more than $210° - 60° = 150°$. Having verified the requirements of Lemma 2 for one of the two paths, (v_1, v, p, s) or (v_3, v, p, d), we use it to show that either $\mathsf{in\text{-}range}(v_3, d) = \mathsf{true}$ or $\mathsf{in\text{-}range}(v_1, s) = \mathsf{true}$ (or both). If $\mathsf{in\text{-}range}(v_3, d) = \mathsf{true}$, then $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ are initialized in line 11 of the algorithm; otherwise, $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ are initialized in line 13 of the algorithm. In either case, the condition of the IF statement in line 15 of the algorithm evaluates to true.

Assume now that v has an isolated child, say v_1. By definition, $\widehat{v_1vv_2} > 120°$. This along with the fact that $\widehat{v_2vv_3} \geq 60°$ implies that $\widehat{v_1vp} + \widehat{v_3vp} \leq 180°$. It follows that $\widehat{v_1vp} + \widehat{v_3vp} + \widehat{spd} \leq 360°$. So by Lemma 1, $\widehat{v_3vp} + \widehat{vpd} > 150°$. These together imply that $\widehat{v_1vp} + \widehat{vps} \leq 210°$, and each of these angles has a value in the interval $[60°, 150°]$. By Lemma 2, $\mathsf{in\text{-}range}(v_1, s) = \mathsf{true}$. Then $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ are initialized in line 9 of the algorithm, and the conditions of both IF statements in lines 15 and 17 of the algorithm evaluate to true.

Consider now the case when $\mathsf{deg}(v) = 5$. In this case, v has no isolated children: each angle at v is at least $60°$, the sum of all five angles is $360°$, therefore each angle is at most $120°$. It follows that $\widehat{v_1vp} + \widehat{v_4vp} \leq 240°$. (In fact, a stronger upper bound is $180°$, but this is irrelevant to the discussion here.) This situation is identical to the degree 4, no isolated children case. $\square$

Lemma 4 *If* $\mathsf{deg}(v) \geq 3$, *let* $v'_1, \ldots, v'_\ell \in \mathsf{Children}(v) \setminus \{\mathsf{v_{via}}\}$ *be radially sorted around* v. *Then,* $\mathsf{in\text{-}range}(v'_i, v'_{i+1}) = \mathsf{true}$ *for* $i = 1, \ldots, \ell - 1$.

Proof. Recall that when $\mathsf{deg}(v) = 5$, no angle between two radially consecutive children of v exceeds $120°$, and so the lemma is clearly true. So consider the situation where $\mathsf{deg}(v) < 5$. By similar arguments, at most one angle between two radially consecutive children of v may exceed $120°$. Furthermore, one of these children is necessarily a boundary (isolated) child since all angles between radially consecutive children involve a boundary child when v is of degree 3 or 4. As noted previously, a degree 4 vertex can have at most one angle $> 120°$. So if v is of degree 4 and has an isolated child, then both its boundary children form an angle $< 120°$ with p, and thus both are within range of p. When $\mathsf{deg}(v) = 3$, if one child is isolated, then they both are (since there are only two children.) In this case, at least one of the two children must be within range of p or else the sum of the three angles at v is more than $360°$. If v satisfies invariant (I3a) or (I3b), then $\mathsf{v_{to}}, \mathsf{v_{from}} \in \{v, p\}$, therefore the conditions of both IF statements on lines 15 and 17 evaluate to true. It follows that $\mathsf{v_{via}}$ is set to an isolated child of v in line 18, and $\mathsf{Children}(v) \setminus \{\mathsf{v_{via}}\}$ contains either one child of v (the degree 3 case), or two children of v within range of each other (the degree 4 case).

It remains to discuss the more complex situation when v satisfies invariant (I3c), so v is involved in a cycle $v \to d \to p \to s \to v$, and $\widehat{spv} + \widehat{vpd} \leq 180°$. Assume without loss of generality that v_1 is isolated, and v_1 and s lie on the same side of vp (refer to Fig. 1c). If $\mathsf{deg}(v) = 3$, then $\widehat{v_1vp} + \widehat{v_2vp} \leq 240°$ (because v_1 and v_2 are both isolated, by our assumption). Arguments similar to the ones used in the proof of Lemma 3 show that in this case either $\mathsf{in\text{-}range}(v_2, d) = \mathsf{true}$, or $\mathsf{in\text{-}range}(v_1, s) = \mathsf{true}$, or both. If $\mathsf{in\text{-}range}(v_2, d) = \mathsf{true}$, $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ are initialized in line 7 of the algorithm; otherwise, $\mathsf{v_{from}}$ and $\mathsf{v_{to}}$ are initialized in line 9 of the algorithm. In either case, the conditions of both IF statements in lines 15 and 17 of the algorithm evaluate to true, and $\mathsf{Children}(v) \setminus \{\mathsf{v_{via}}\}$ contains a single child of v.

If $\mathsf{deg}(v) = 4$, Lemma 3 shows that $\mathsf{in\text{-}range}(v_1, s) = \mathsf{true}$. This guarantees that line 11 of the algorithm gets executed and $\mathsf{v_{via}} = v_1$. It follows that $\mathsf{Children}(v) \setminus \{\mathsf{v_{via}}\}$ contains two children of v within range of each other. $\square$

Acknowledgement. Many thanks to the Fields Institute of Canada for financial support, and to all participants of the Fields workshop for fruitful discussions.

References

[1] W. Wu, H. Du, X. Jia, Y. Li, and S.C.-H. Huang: Minimum connected dominating sets and maximal independent sets in unit disk graphs. *Theor. Comp. Sci.*, 352:1–7, 2006.

[2] F. van Nijnatten: Range Assignment with Directional Antennas. *Master's Thesis, Technische Universiteit Eindhoven*, 2008.

[3] I. Caragiannis, C. Kaklamanis, E. Kranakis, D. Krizanc, and A. Wiese: Communication in wireless networks with directional antennae. *Proc. of the 20th Symp. on Parallelism in Algorithms and Architectures*, *Proc. of SPAA*, pp. 344–351, 2008.

[4] B. Bhattacharya, Y. Hu, Q. Shi, E. Kranakis, and D. Krizanc: Sensor network connectivity with multiple directional antennae of a given angular sum. *Proc. of IPDPS*, pp. 1–11, 2009.

[5] B. Ben-Moshe, P. Carmi, L. Chaitman, M.J. Katz, G. Morgenstern, and Y. Stein: Direction Assignment in Wireless Networks. *Proc. of CCCG*, pp. 39–42, 2010.

[6] M. Damian, and R. Flatland: Spanning Properties of Graphs Induced by Directional Antennas. *Proc. of FWCG*, Stony Brook, NY, 2010.

[7] E. Kranakis, D. Krizanc, and O. Morales: Maintaining Connectivity in Sensor Networks Using Directional Antennae. *Theor. Aspects of Distr. Comp. in Sensor Netw.*, Part 2, pp. 59–84, 2011.

[8] P. Bose, P. Carmi, M. Damian, R. Flatland, M.J. Katz, and A. Maheshwari. Switching to Directional Antennas with Constant Increase in Radius and Hop Distance *To appear in WADS*, 2011.