A Separator Theorem for Intersecting Objects in the Plane

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Abstract

Separators in graphs are instrumental in the design of algorithms, having proven to be the key technical tool in approximation algorithms for many optimization problems. In the geometric setting, this naturally translates into the study of separators in the intersection graphs of geometric objects. Recently a number of new separator theorems have been proven for the case of geometric objects in the plane. In this paper we present a new separator theorem that unifies and generalizes some earlier results.

1 Introduction

Basic combinatorial problems such as the independent set problem and the set-cover problem are provably hard to approximate in the general setting. Therefore considerable work has focused on specific cases arising in applications. Of those, a well-studied problem is the geometric independent set problem: given a set $\mathcal{O}$ of geometric objects, compute the largest subset of objects in $\mathcal{O}$ that are pairwise disjoint. While there has been considerable progress on this problem (e.g., PTAS for the case when $\mathcal{O}$ is a set of balls [4]), many fundamental cases are still open. For example, the current best polynomial-time algorithm for computing the maximum independent set of a set of line-segments in the plane returns a $O(n^{\epsilon})$-approximation, for $\epsilon > 0$ [8].

Recently in a breakthrough result, Adamaszek-Wiese [1] showed the existence of a QPTAS for approximating the maximum independent set in the intersection graph of a set of line segments in the plane. For any $\epsilon > 0$, their algorithm returns a $(1+\epsilon)$-approximation in time $O(n^{(\log n/\epsilon)^{O(1)}})$. Building on that idea, very recently Mustafa *et al.* [12] presented QPTAS for the geometric set-cover problems for weighted pseudodisks in the plane, and weighted halfspaces in $\mathbb{R}^3$.

Geometric Separators

The main structural result on which the breakthrough of Adamaszek-Wiese [1] rests is the following:

Theorem *Given a set of weighted disjoint rectangles, with no rectangle having more than a third of the total weight W, and a parameter $\delta > 0$, there exists a closed piece-wise linear curve $\mathcal{C}$ with $O(1/\delta^4)$ vertices so that:*

1. *The total weight of the set of rectangles intersecting $\mathcal{C}$ is at most δW, and*
2. *The total weight of the set of rectangles that lie entirely in the bounded (unbounded) region defined by $\mathcal{C}$ is at most $2W/3$.*

Their long proof immediately raised the following questions: 1) Does it hold for objects more general than axis-parallel rectangles? 2) Can the bound $O(1/\delta^4)$ on the complexity of the curve $\mathcal{C}$ be improved? 3) Is there a simpler proof? 4) Can the result be generalized to situations where the objects are not disjoint?

Adamaszek-Wiese [2] answered 1) by showing that their technique can be extended to work for line segments (and thus also for regions whose boundaries have a bounded number of segments). The "complexity" of their separator went up to $O(1/\delta^c)$ for some constant $c \geq 6$. Subsequently, the work of Mustafa-Raman-Ray (first published in [13]), Adamaszek-Wiese [3] and Har-Peled [9] independently answered 2) and 3) by showing that there exists a simple proof, using cuttings, for a set of disjoint line segments in the plane, and such that the bound on the complexity of $\mathcal{C}$ can be brought down to $O(1/\delta)$:

Theorem 1 ([3, 9, 13]) *Given a set of weighted disjoint line-segments in the plane and a parameter $\delta > 0$, there exists a closed piece-wise linear curve $\mathcal{C}$ with $O(1/\delta)$ vertices so that:*

1. *The total weight of segments intersecting $\mathcal{C}$ is at most δW and*
2. *The total weight of segments that lie entirely in the bounded (unbounded) region defined by $\mathcal{C}$ is at most $2W/3$.*

In Section 3 we present an example showing that this is optimal. Given a set of objects in the plane with total weight W, a piece-wise linear closed curve $\mathcal{C}$ is *balanced* if the total weight of the objects contained in its interior (resp. exterior) is at most $2W/3$.

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Intersecting Objects

Har-Peled [9] also gave an answer to 4) showing that the result could be extended to work in situations where the intersection graph of the regions have some hereditary "sparsity" properties (furthermore, his technique also works for polygons with arbitrary number of sides). For this sparsity property, Fox-Pach [7] have shown that the number of intersections between the objects is linear. The proof proceeds by adding vertices for these linear number of intersection points and applying the separator result for disjoint segments. For the case where the number of intersections can be arbitrary, the best-previous result is:

Theorem 2 (Fox-Pach [7]) *The intersection graph of a collection L of curves in the plane with a total of m intersection points among them has a separator of size at most $O(\sqrt{m})$.*

First observe that in general if both the weights and number of intersections can be arbitrary, then no such separator exists, even when m is $o(n)$. To see this, given the parameter δ, consider L to be n line-segments in the plane where $1/2\delta$ of these segments *i*) pairwise intersect, and *ii*) each such segment has weight $2\delta W$. The remaining segments are disjoint with 0 weight, and set n large enough so that $m = O(1/\delta^2) = o(n)$. Then any balanced curve must intersect at least one segment of weight $2\delta W$, so it is not a δ-separator.

2 Our Result

If the objects are allowed to intersect arbitrarily, we need to consider the unweighted setting. In this paper, we present and prove a statement that partially unifies several previous results: it implies as a special case the Fox-Pach Theorem 2 for line segments, it implies as a special case the unweighted disjoint separator of Theorem 1, and extends the result of [9] for set of segments with arbitrary pattern of intersections.

Theorem 3 *Given a set S of n line-segments in the plane with m intersections, and a parameter r, there exists a piece-wise linear simple closed curve $\mathcal{C}$ in the plane such that the number of segments completely inside (or outside $\mathcal{C}$) is at most $2n/3$ (call any such curve* balanced*), and*

- *the number of vertices of $\mathcal{C}$ are $O\left(\sqrt{r + \frac{mr^2}{n^2}}\right)$,*
- *the number of line-segments in S intersecting $\mathcal{C}$ are $O\left(\sqrt{\frac{n^2}{r} + m}\right)$.*

Implications of Theorem 3 are:

- **Theorem 1 (unweighted case).** As the segments are disjoint, $m = 0$. Set $r = 1/\delta^2$, and apply Theorem 3.
- **Theorem 2 (for line segments).** Set $r = n^2/m$. The separator in the intersection graph of the segments then is the set of segments of S intersecting the curve $\mathcal{C}$ returned by Theorem 3.

Remark: While we consider only line segments in this paper for ease of exposition, all the results in this paper can be generalized to x-monotone curves and thus also to regions whose boundary can be decomposed into a finite number of x monotone curves. This includes convex sets, for instance, since their boundary can be decomposed into two x-monotone curves. The generalization is straightforward and no new ideas are needed.

3 Proofs

Proof of optimality of Theorem 1. First note that we can always find a balanced separator of size $O(\sqrt{n})$ which does not intersect any of the objects. This can be done by taking the trapezoidal decomposition of the arrangement of the objects and finding a cycle separator of size $O(\sqrt{n})$. Thus for the case $\delta = o(1/\sqrt{n})$, there always exists a balanced separator curve intersecting at most δn segments, and having complexity $O(\sqrt{n}) = o(1/\delta)$. We therefore consider only the case when $\delta \geq \sqrt{12/n}$.

For a fixed $\delta \geq \sqrt{12/n}$, we construct a set of n disjoint line segments (with weight one for each segment) in the plane such that there is no balanced closed curve *i*) that has less than $1/\delta$ vertices, and *ii*) intersects at most $\delta n/12$ segments in S. Since Theorem 1 is for all values of δ, this shows that the bound in the theorem cannot be improved in general.

Our construction consists of $\delta n/2$ concentric layers – each made up of $\frac{2}{\delta}$ segments. Each layer has $\frac{1}{\delta}$ *long segments* which are the sides of a regular polygon shrunk and shifted slightly so that they don't intersect and another set of $\frac{1}{\delta}$ short segments called *blockers*. See Figure 1. The blockers have very small length, and so and the layers can be packed arbitrarily close to each other.

Consider any balanced closed curve $\mathcal{C}'$ in the plane. If it contains at least one layer completely inside, and one layer completely outside, then it has at least $1/\delta$ vertices since it is 'sandwiched' between two of the layers that are arbitrarily close to each other. Otherwise, without loss of generality, say there is no layer completely inside $\mathcal{C}'$. As $\mathcal{C}'$ is balanced, it contains at least $n/3$ segments inside or intersecting $\mathcal{C}'$; these segments belong to at least $(n/3)/(2/\delta) = \delta n/6$ different layers. The curve $\mathcal{C}'$ must 'cross' each of these layers by either intersecting one of its edges or by having at least one more bend. Since $\delta n/6 \geq \delta n/12 + 1/\delta$ for $\delta \geq \sqrt{12/n}$, either the number of intersections of $\mathcal{C}'$ is at least $\delta n/12$ or the number of bends (and hence vertices) is at least $1/\delta$.

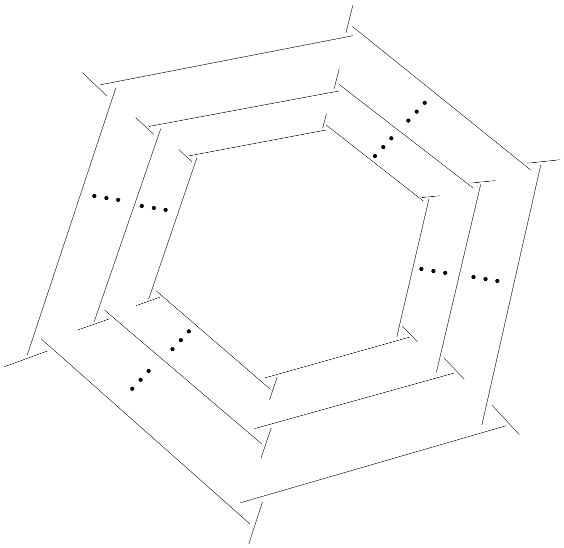

Figure 1: Lower-bound construction.

Proof of Theorem 3. The proof relies on the following technical tool, whose proof follows in a standard manner from well-known sampling techniques in the study of ϵ-nets. Very recently we became aware that it was proved in [6]; for completeness we briefly sketch our proof, which is similar.

Lemma 4 *Given a set S of n line segments in $\mathbb{R}^2$ with m intersections and a parameter $r > 0$, there exists a triangulation $\mathcal{T}$ of the plane of size $O(r + \frac{mr^2}{n^2})$ such that each triangle $\Delta \in \mathcal{T}$ intersects at most n/r line segments of S in its interior.*

where the size of a triangulation is the number of triangles in it.

Assume Lemma 4. Then given the set S of n segments with m intersections, apply Lemma 4 to S to get a triangulation $\mathcal{T}$ of $\mathbb{R}^2$ into $O(r + mr^2/n^2)$ regions (see Figure 2). $\mathcal{T}$ can be seen as a planar graph G. Give weights to each face of $\mathcal{T}$: if a segment $s \in S$ intersects t faces of $\mathcal{T}$, add weight $1/t$ to the weight of each of these t faces. Now from [11] we get a simple cycle $\mathcal{C}$ in $\mathcal{T}$ of $O(\sqrt{r + mr^2/n^2})$ vertices such that faces completely inside (and outside) have total weight at most $2n/3$, and hence so do the segments of S inside (and outside) $\mathcal{C}$. By Lemma 4, the total number of segments of S intersected by each edge of $\mathcal{T}$ is $O(n/r)$, and so the total number of segments of S intersected by $\mathcal{C}$ is at most $O(\sqrt{r + mr^2/n^2}) \cdot O(n/r) = O(\sqrt{m + n^2/r})$.

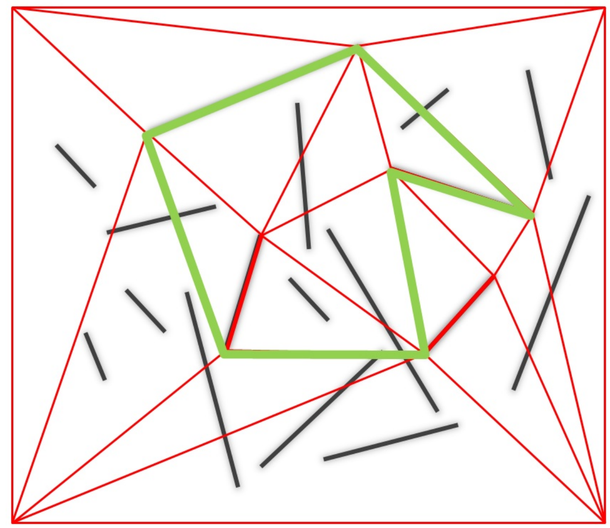

Figure 2: Segments of S in black, the triangulation $\mathcal{T}$ in red, and the curve $\mathcal{C}$ in green.

Proof of Lemma 4. All the tools used in the proof for cuttings[1] directly apply (or can be generalized in a straightforward manner) to work for this purpose. We first briefly review the basic partitioning method of using *trapezoidal decompositions*. Given a set $R \subseteq S$ of segments, one can partition the space (say inside a large-enough rectangle containing all the segments of S) as follows. For each endpoint of a segment in R or an intersection-point between segments in R, shoot a vertical ray upwards (and downwards) till it hits another segment (or the bounding rectangle). The union of all these vertical segments together with R partitions the bounding rectangle into a set of regions. A crucial fact is that each region Δ in this partition is determined by a constant (2, 3 or 4) number of segments in R. Call such regions *trapezoidal regions* (or trapezoids for brevity), and the partition is called a *trapezoidal decomposition*[2]. Denote by $\Xi(R)$ this set of trapezoidal regions in the trapezoidal decomposition of R. The size, $|\Xi(R)|$, of the trapezoidal decomposition of R is the number of trapezoids in $\Xi(R)$; it is, within a constant-factor, equal to the total number of end- and intersection- points in R. A trapezoid present in the trapezoidal decomposition of any subset R of S is called a *canonical trapezoid*. For a canonical trapezoid Δ, let S_Δ denote the set of segments of S intersected by Δ. A trapezoid Δ is present in the trapezoidal decomposition of R if and only if its determining segments are present in R, and R does not contain any of the segments of S that intersect Δ. For the rest of the proof, we only work with canonical trape-

[1] We refer the reader to Chazelle [5] and Matoušek [10] for a full account.

[2] We refer the reader to [10] for a nice exposition on trapezoidal decompositions.

zoids determined by 4 segments. The case for canonical trapezoids determined by 2 and 3 segments is similar. See Figure 3.

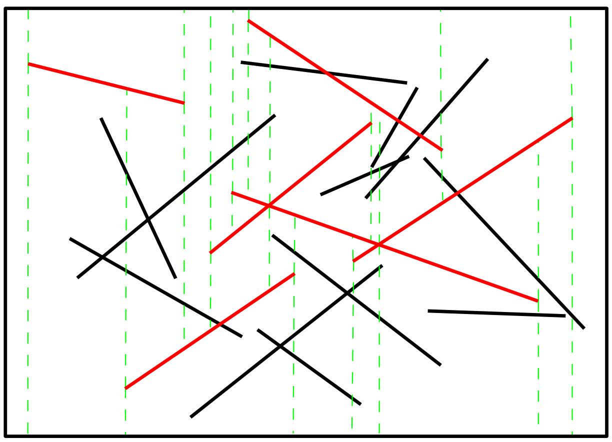

Figure 3: The trapezoidal decomposition of a set of segments (red) sampled from the set S (black).

Proof. First note that a slightly weaker bound (within logarithmic factors) follows immediately from ϵ-nets. Given S, consider the set-system $(S, \mathcal{F})$ induced by intersection with segments in the plane, i.e.,

$$F \in \mathcal{F} \text{ iff there exists a line segment } l \text{ s.t. } F = \{s \in S \mid s \cap l \neq \emptyset\}$$

Pick a random set R by uniformly adding each segment of S with probability $p = (Cr \log r)/n$, where C is a large constant. Then R is a $(1/3r)$-net for $(S, \mathcal{F})$ with probability at least 9/10 [10] (i.e., any line segment l in the plane that intersects more than $n/3r$ segments of S must intersect a segment of R; alternatively, any segment l that does not intersect any segment of R intersects at most $n/3r$ segments of S). The expected size of R is np, and the expected number of intersections of segments in R is mp^2. By Markov's inequality, the probability that the size of R is more than $10np$ is at most 1/10. Similarly the probability that the number of intersections in R is more than $10mp^2$ is also at most 1/10. Thus, with probability at least 7/10, R is a $(1/3r)$-net for $(S, \mathcal{F})$ *and* the size of the trapezoidal decomposition of R, $|\Xi(R)|$, is $O(r \log r + (mr^2 \log^2 r)/n^2)$. One can triangulate the trapezoidal decomposition of R to get a triangulation $\mathcal{T}$ with asymptotically the same number of triangles. Finally note that each triangle in $\mathcal{T}$ intersects at most n/r segments of S in the interior: any line segment l forming an edge of a triangle in this triangulation must intersect at most $n/3r$ segments of S, as otherwise the set of segments intersecting l would not be hit by a segment from R, contradicting the fact that R is a $(1/3r)$-net. Therefore each triangle in $\mathcal{T}$ intersects at most $3 \cdot n/3r = n/r$ segments in its interior.

We now show how the log factor can be shaved off. Set $p = Cr/n$ (for a large-enough constant C), and pick each segment in S with probability p to get a random sample R. Construct the trapezoidal decomposition $\Xi(R)$ of R. If all trapezoids $\Delta \in \Xi(R)$ intersect at most n/r segments in S, we are done. Otherwise we will further partition each violating trapezoid Δ (i.e., a trazepoid that intersects more than n/r segments of S), based on two ideas. First, the expected number of trapezoids in $\Xi(R)$ intersecting more than n/r segments is small. In particular, we will show (Lemma 6) that, for any $t > 0$, the expected number of trapezoids intersecting at least tn/r segments in S is *exponentially* decreasing as a function of t.

Second, consider a $\Delta \in \Xi(R)$ intersecting a set, say S_Δ, of $n_\Delta = tn/r$ segments of S. Use the weaker bound (derived earlier) on S_Δ with parameter t (i.e., compute a $(1/t)$-net for S_Δ) to get a partition inside Δ of $O(t \log t + (m_\Delta t^2 \log^2 t)/n_\Delta^2) = O(t^2 \log^2 t)$ trapezoids. By definition, each such new trapezoid inside Δ intersects at most $n_\Delta/t = n/r$ segments of S_Δ (and hence of S). Thus refining each Δ by adding new trapezoids, and taking the union of all these trapezoids for all $\Delta \in \Xi(R)$ gives the required partition on S with parameter r.

It remains to bound the overall expected size of this partition. Towards that we will need the two lemmas below.

Lemma 5 *Given a set S of n segments in the plane with m intersections, the number of canonical trapezoids defined by S that intersect at most k segments of S is $O(nk^3 + mk^2)$.*

Proof. Let $\Xi_{\leq k}$ be the set of canonical trapezoids defined by S that intersect at most k segments of S. The proof uses the Clarkson-Shor technique. Construct a sample T by adding each segment of S to T with probability p_0; the expected total number of picked segments is np_0 and the expected number of intersections between the segments of T is mp_0^2. The trick is to count the expected size of $\Xi(T)$ in two ways. On one hand, it is $O(np_0 + mp_0^2)$ (i.e., the expected number of vertices present in $\Xi(T)$). On the other hand, the probability of a canonical trapezoid Δ being in $\Xi(T)$ is $p_0^4(1-p_0)^{|\Delta \cap S|}$ – recall that a trapezoid Δ appears in $\Xi(T)$ iff its four defining segments are picked in T, and none of the segments of S intersecting Δ are picked in T. Therefore the size of $\Xi(T)$ is

$$\sum_{\Delta} p_0^4(1-p)^{|\Delta \cap S|} \geq \sum_{\Delta \in \Xi_{\leq k}} p_0^4(1-p)^{|\Delta \cap S|} \geq \sum_{\Delta \in \Xi_{\leq k}} p_0^4(1-p_0)^k$$

where the sum is over all canonical trapezoids Δ which intersect at most k segments of S. Therefore putting the two bounds together,

$$\begin{aligned}\sum_{\Delta\in\Xi_{\leq k}} p_0^4(1-p_0)^k &= |\Xi_{\leq k}|\cdot\ p_0^4(1-p_0)^k \\ &\leq E[|\Xi(T)|] = O(np_0+mp_0^2) \\ |\Xi_{\leq k}| &= O(\frac{np_0+mp_0^2}{p_0^4(1-p_0)^k}) = O(nk^3+mk^2)\end{aligned}$$

for $p_0 = 1/2k$. □

Lemma 6 *Expected number of trapezoids in* $\Xi(R)$ *which intersect at least* tn/r *segments of* S *is*

$$O\left((t^3r+\frac{mr^2t^2}{n^2})e^{-t}\right)$$

Proof. Consider first the expected number of trapezoids in $\Xi(R)$ which intersect tn/r segments of S:

$$\begin{aligned}&E[|\Delta\in\Xi(R)\ s.t.\ |\Delta\cap S| = tn/r|] \\ &\quad = |\Delta\ s.t.\ \ |\Delta\cap S| = tn/r|\cdot p^4(1-p)^{tn/r}.\end{aligned}$$

Using Lemma 5,

$$\begin{aligned}&E[|\Delta\in\Xi(R)\ s.t.\ |\Delta\cap S| = tn/r|] \\ \leq\ & O\left(n(tn/r)^3+m(tn/r)^2\right)p^4(1-p)^{tn/r} \\ =\ & O\left((t^3r+\frac{mr^2t^2}{n^2})e^{-t}\right)\end{aligned}$$

Observe that the above bound is decreasing exponentially in t, and therefore the required bound, which would follow by summing up over all trapezoids intersecting at least tn/r segments in S, will be asymptotically the same. Formally:

$$\begin{aligned}&\sum_{\Delta:|\Delta\cap S|\geq tn/r} p^4(1-p)^{|\Delta\cap S|} \\ =\ & \sum_{i=0}\ \sum_{2^itn/r\leq|\Delta\cap S|\leq 2^{i+1}tn/r} p^4(1-p)^{|\Delta\cap S|} \\ \leq\ & \sum_i\left(n(2^{i+1}tn/r)^3+m(2^{i+1}tn/r)^2\right)\cdot p^4(1-p)^{2^itn/r} \\ \leq\ & \sum_i\left(t^3r2^{3i+3}+mr^2t^22^{2i+2}/n^2\right)e^{-2^it}\end{aligned}$$

This series is geometrically decreasing, so it is asymptotically equal to the required bound. □

Now we can complete the proof of the theorem. Let $n_\Delta = t_\Delta n/r$ be the number of segments in S intersected by each trapezoid $\Delta\in\Xi(R)$ (and m_Δ the number of their intersections). Using the weaker bound, refine trapezoid Δ by adding a $(1/t_\Delta)$-net R_Δ for all the $t_\Delta n/r$ segments of S intersected by Δ. The resulting expected total size of the trapezoidal partition is:

$$\begin{aligned}=\ & |R|+\sum_\Delta Pr[\Delta\in\Xi(R)]\cdot \\ & \text{(Size of trapezoidal decomp. of } (1/t_\Delta)\text{-net within } \Delta) \\ =\ & |R|+\sum_\Delta Pr[\Delta\in\Xi(R)] \\ & \cdot O\left(t_\Delta\log t_\Delta+\frac{m_\Delta t_\Delta^2\log^2 t_\Delta}{n_\Delta^2}\right) \\ & \text{(using the weaker bound)} \\ \leq\ & |R|+\sum_\Delta Pr[\Delta\in\Xi(R)]\cdot O\left(t_\Delta^2\log^2 t_\Delta\right) \\ & \text{(as } m_\Delta\leq n_\Delta^2) \\ =\ & |R|+\sum_j\ \sum_{\substack{\Delta\ s.t.\\ 2^j\leq t_\Delta\leq 2^{j+1}}} Pr[\Delta\in\Xi(R)]\cdot O\left(t_\Delta^2\log^2 t_\Delta\right) \\ \leq\ & |R|+\sum_j E[\ \#\text{ trapezoids } \Delta \text{ in } \Xi(R) \text{ with } 2^j\leq t_\Delta\ ] \\ & \cdot\ O\left(2^{2(j+1)}\log^2 2^{j+1}\right) \\ \leq\ & |R|+\sum_j O\left((2^{3j}r+\frac{mr^22^{2j}}{n^2})e^{-2^j}\right) \\ & \cdot O\left(2^{2(j+1)}\log^2 2^{j+1}\right)\ \text{(Lemma 6)} \\ =\ & |R|+r\sum_j O\left(2^{3j}e^{-2^j}\right) \\ & \cdot O\left(2^{2(j+1)}\log^2 2^{j+1}\right) \\ & +\frac{mr^2}{n^2}\sum_j O\left(2^{2j}e^{-2^j}\right)\cdot O\left(2^{2(j+1)}\log^2 2^{j+1}\right) \\ =\ & np+mp^2+O(r)+O(\frac{mr^2}{n^2}) = O(r+\frac{mr^2}{n^2}) \\ & \text{(the summands form a geometric series)}\end{aligned}$$

as required. This finishes the proof of Lemma 4. □

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