Dyson's Rank Function and Andrews's SPT-Function

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ABSTRACT

Let spt(n) denote the number of smallest parts in the partitions of n. In 2008, Andrews found surprising congruences for the spt-function mod 5, 7 and 13. We discuss new congruences for spt(n) mod powers of 2.
We give new generating function identities for the spt-function and Dyson's rank function. Recently with Andrews and Liang we found a spt-crank function that explains Andrews spt-congruences mod 5 and 7. We extend these results by finding spt-cranks for various overpartition-spt-functions of Ahlgren, Bringmann, Lovejoy and Osburn. This most recent work is joint with Chris Jennings-Shaffer.

PLAN

* Ramanujan Partition Congruences
* The Rank and the Crank
* The Andrews SPT Function and Congruences
* The SPT-CRANK for Vector Partitions
* Overpartitions and Congruences
* Other SPT Functions and Congruences
* Other SPT Crank Function Functions and Idea of Proof
* SPT mod powers of 2
* New Identities for the Generating Functions of the SPT function,
  SPT-CRANK function and the Dyson Rank Function
* Sketch of Proof if time permits

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```
>
currentdir("C:\\cygwin\\home\\fgarvan\\math\\talks\\TIANJIN-2013
");
 "C:\\cygwin\\home\\fgarvan\\math\\talks\\TIANJIN-2013"
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```


PARTITIONS

A partition of n is a finite nonincreasing sequence of positive integers whose sum is n.

Let $p(n)$ = # of partitions of n:

n		$p(n)$
1	1	1
2	2, 1+1	2
3	3, 2+1, 1+1+1	3
4	4, 3+1, 2+2, 2+1+1, 1+1+1+1	5
⋮		
10		42
⋮		
100		190 569 292

```
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```

p. 2

Generating Function

$p(0) = 1$

$$\sum_{n=0}^{\infty} p(n)\, q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \qquad |q|<1$$

?

```
> with(qseries):
> P:=series(1/etaq(q,1,1000),q,1001):
> findcong(P,1000);
```

$$[4, 5, 5]$$
$$[5, 7, 7]$$
$$[6, 11, 11]$$
$$[24, 25, 25]$$

$\{[6, 11, 11], [4, 5, 5], [5, 7, 7], [24, 25, 25]\}$

\>

\>

p. 3

Ramanujan Congruences

$$p(5n+4) \equiv 0 \pmod 5$$

$$p(7n+5) \equiv 0 \pmod 7$$

$$p(11n+6) \equiv 0 \pmod{11}.$$

\>

DYSON RANK

largest part − # of parts

		rank mod 5
4	$4-1=3$	$\equiv 3$
$3+1$	$3-2=1$	$\equiv 1$
$2+2$	$2-2=0$	$\equiv 0$
$2+1+1$	$2-3=-1$	$\equiv 4$
$1+1+1+1$	$1-4=-3$	$\equiv 2$

Dyson Conjectures: Let $N(r,t,n)$ = # of partitions of n with rank $\equiv r \pmod t$

$$N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4) = \frac{1}{5}p(5n+4)$$

$$N(0,7,7n+5) = N(1,7,7n+5) = \cdots = N(6,7,7n+5) = \frac{1}{7}p(7n+5)$$

>

Let $N(m,n)$ = # of partitions of n with rank m.

Then

$$\sum_{n=0}^{\infty}\sum_{m} N(m,n)\, z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}$$

q-notation

$$(a)_n = (a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

$$(a)_\infty = \lim_{n\to\infty} (a)_n, \quad |q|<1.$$

$$(a_1,a_2,\ldots,a_k;q)_n = (a_1,a_2,\ldots,a_k) = \prod_{j=1}^{k}(a_j;q)_n$$

>
>

ANDREWS – G – CRANK largest part if no ones

# of parts > # of ones − # of ones if ones

		crank mod 5
4	4	$\equiv 4$
3+1	$1-1=0$	$\equiv 0$
2+2	2	$\equiv 2$
2+1+1	$0-2=-2$	$\equiv 3$
1+1+1+1	$0-4=-4$	$\equiv 1$

Let $M(r,t,n)$ = # of partitions of n with crank $\equiv r \pmod t$

A – G (1988)

$M(0,5,5n+4) = M(1,5,5n+4) = \dots = M(4,5,5n+4) = \frac{1}{5}p(5n+4)$

$M(0,7,7n+5) = M(1,7,7n+5) = \dots = M(6,7,7n+5) = \frac{1}{7}p(7n+5)$

$M(0,11,11n+6) = M(1,11,11n+6) = \dots = M(10,11,11n+6) = \frac{1}{11}p(11n+6)$

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VECTOR PARTITIONS

Let $M(m,n)$ = # of partitions of n with crank m. $(n \neq 1)$

Then

$$\sum_{n=0}^{\infty} \sum_{m} M(m,n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}.$$

Let

$$V = \mathcal{D} \times \mathcal{P} \times \mathcal{P}$$

For $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V$. Define

$$|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$$

$$\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}$$

$$\text{crank}(\vec{\pi}) = \#(\pi_2) - \#(\pi_3).$$

Then

$$M(m,n) = \sum_{\substack{\vec{\pi} \in V \\ |\vec{\pi}| = n \\ \text{crank}(\vec{\pi}) = m}} \omega(\vec{\pi})$$

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ANDREWS - SPT-FONCTION

Let $spt(n)$ = # of smallest parts in the partitions of n.

n		$spt(n)$
1	1	1
2	2, 1+1	3
3	3, 2+1, 1+1+1	5
4	4, 3+1, 2+2, 2+1+1, 1+1+1+1	10
5		14
⋮		⋮
10		119
⋮		⋮
100		1 545 832 615

>

Generating Function

$$\sum_{n=1}^{\infty} spt(n)\, q^n = q + 3q^2 + 5q^3 + 10q^4 + \cdots$$

$$= \sum_{n=1}^{\infty} (1 \cdot q^n + 2 \cdot q^{2n} + 3 \cdot q^{3n} + \cdots) \frac{1}{(q^{n+1};q)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \frac{1}{(q^{n+1};q)_\infty}$$

$$spt(n) = \frac{1}{2}\left(M_2(n) - N_2(n)\right) \qquad \text{(ANDREWS)}$$

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(p.10)

RANK & CRANK MOMENTS

$$N_k(n) := \sum_m m^k N(m,n) \quad [\text{k-th rank moment}]$$

$$M_k(n) := \sum_m m^k M(m,n) \quad [\text{k-th crank moment}]$$

```
> with(rank): with(crank):
> M2:=n->add(m^2*M(m,n),m=1..n)*2:
> N2:=n->add(m^2*N(m,n),m=1..n)*2:
> spt:=n->(M2(n)- N2(n))/2:
> SGEN:=add(spt(n)*q^n,n=1..500):
> seq(spt(n),n=1..15);
```

1, 3, 5, 10, 14, 26, 35, 57, 80, 119, 161, 238, 315, 440, 589

TRY SLOANE'S ONLINE ENC. OF SEQUENCES: http://oeis.org/

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Integer Sequences

1,3,5,10,14,26,35,57 Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,3,5,10,14,26,35,57**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created   Format: long | short | data

A092269 Spt function: total number of smallest parts in all partitions of n.

1, 3, 5, 10, 14, 26, 35, 57, 80, 119, 161, 238, 315, 440, 589, 801, 1048, 1407, 1820, 2399, 3087, 3998, 5092, 6545, 8263, 10486, 13165, 16562, 20630, 25773, 31897, 39546, 48692, 59960, 73423, 89937, 109553, 133439, 161840, 196168, 236843, 285816, 343667, 412950, 494702, 592063, 706671 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS Row sums of triangle A220504. - Omar E. Pol, Jan 19 2013

LINKS Joerg Arndt, Table of n, a(n) for n = 1..550

F. G. Garvan, Table of a(n) for n=1..10000 (Coefficients of Andrews spt-function)

G. E. Andrews, The number of smallest parts in the partitions of n

George E. Andrews, Song Heng Chan and Byungchan Kim, The Odd Moments of Ranks and Cranks, 2012. - From N. J. A. Sloane, Sep 04 2012

G. E. Andrews, F. G. Garvan, and J. Liang, Combinatorial interpretation of congruences for the spt-function

G. E. Andrews, F. G. Garvan, and J. Liang, Self-conjugate vector partitions and the parity of the spt-function

A. Folsom and K. Ono, The spt-function of Andrews

F. G. Garvan, Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank

F. G. Garvan, Congruences for Andrews' spt-function modulo powers of 5, 7 and 13

F. G. Garvan, Congruences for Andrews' spt-function modulo 32760 and extension of Atkin's Hecke-type partition congruences

F. G. Garvan, Higher Order Spt-functions, Adv. Math. 228 (2011), no. 1, 241-265; . - From N. J. A. Sloane, Jan 02 2013

F. G. Garvan, The smallest parts partition function, 2012

K. Ono, Congruences for the Andrews spt-function

O. E. Pol, Illustration of initial terms

Wikipedia, Spt function

FORMULA G.f.: sum(n>=1, x^n/(1-x^n) * prod(k>=n, 1/(1-x^k)).

a(n) = A000070(n-1) + A195820(n). - Omar E. Pol, Oct 19 2011

a(n) = n*A000041(n) - A220908(n)/2 = A066186(n) - A220907(n) = (A220909(n) - A220908(n))/2 = A211982(n)/2. (from Andrews's paper and Garvan's paper). - Omar E. Pol, Jan 03 2013

a(n) = A000041(n) + A000070(n-2) + A220479(n), n>=2. - Omar E. Pol, Feb 16 2013

EXAMPLE Partitions of 4 are [1,1,1,1], [1,1,2], [2,2], [1,3], [4]. 1 appears 4 times in the first, 1 twice in the second, 2 twice in the third, etc.; thus a(4)=4+2+2+1+1=10.

MAPLE
```
b:= proc(n, i) option remember; `if`(n=0 or i=1, n,
      `if`(irem(n, i, 'r')=0, r, 0)+add(b(n-i*j, i-1), j=0..n/i))
    end:
a:= n-> b(n, n):
seq (a(n), n=1..60);  # Alois P. Heinz, Jan 16 2013
```

MATHEMATICA
```
terms = 47; gf = Sum[x^n/(1 - x^n)*Product[1/(1 - x^k), {k, n, terms}], {n, 1, terms}]; CoefficientList[ Series[gf, {x, 0, terms}], x] // Rest (* Jean-François Alcover, Jan 17 2013 *)
```

PROG
```
(PARI)
N = 66;  x = 'x + O('x^N);
gf = sum(n=1, N, x^n/(1-x^n) * prod(k=n, N, 1/(1-x^k) )  );
v = Vec(gf)
/* Joerg Arndt, Jan 12 2013 */
```

CROSSREFS	Cf. A092314, A092322, A092309, A092321, A092313, A092310, A092311, A092268, A006128, A195053. For higher-order spt functions see A221140-A221144.
KEYWORD	nonn
AUTHOR	Vladeta Jovovic, Feb 16 2004
EXTENSIONS	More terms from Pab Ter (pabrlos(AT)yahoo.com), May 25 2004
STATUS	approved

```
--> findcong(SGEN,500);
```

[4, 5, 5]
[5, 7, 7]
[6, 13, 13]

{[5, 7, 7], [4, 5, 5], [6, 13, 13]}

```
> findcong(SGEN,500,25);
```

[4, 5, 5]
[5, 7, 7]
[6, 13, 13]
[4, 25, 2]
[9, 25, 4]
[14, 25, 4]
[19, 25, 2]

{[4, 25, 2], [19, 25, 2], [5, 7, 7], [14, 25, 4], [9, 25, 4], [4, 5, 5], [6, 13, 13]}

```
>
>
>
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```

SPT - CONGRUENCES

$$spt(5n+4) \equiv 0 \pmod 5$$

$$spt(7n+5) \equiv 0 \pmod 7$$ [ANDREWS]

$$spt(13n+6) \equiv 0 \pmod{13}$$

>

SPT- CRANK [ANDREWS - G- LIANG]

$$\sum_{n=1}^{\infty} spt(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \frac{1}{(q^{n+1};q)_\infty}$$

Define

$$\sum_{n=1}^{\infty} \sum_{m} N_S(m,n) z^m q^n = \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_\infty}{(zq^n;q)_\infty (z^{-1}q^n;q)_\infty}$$

Then

$$\sum_{m} N_S(m,n) = spt(n)$$

>

S - PARTITIONS

$$V = \mathcal{D} \times \mathcal{P} \times \mathcal{P}$$

$$S := \left\{ \vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 < s(\pi_1) < \infty \text{ and } s(\pi_1) \leq \min(s(\pi_2), s(\pi_3)) \right\}$$

Let $\omega_1(\vec{\pi}) = (-1)^{\#(\pi_1) - 1}$

$$\text{crank}(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$$

$$|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$$

Then

$$N_S(m, n) = \sum_{\substack{\vec{\pi} \in S,\ |\vec{\pi}| = n \\ \text{crank}(\vec{\pi}) = m}} \omega_1(\vec{\pi})$$

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S- PARTITIONS

$$V = \mathcal{D} \times \mathcal{P} \times \mathcal{P}$$

$$S := \left\{ \vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 < s(\pi_1) < \infty \text{ and } s(\pi_1) \leq \min\left(s(\pi_2), s(\pi_3)\right) \right\}$$

Let $\omega_1(\vec{\pi}) = (-1)^{\#(\pi_1) - 1}$

$$\operatorname{crank}(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$$

$$|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$$

Then

$$N_S(m, n) = \sum_{\substack{\vec{\pi} \in S,\ |\vec{\pi}| = n \\ \operatorname{crank}(\vec{\pi}) = m}} \omega_1(\vec{\pi})$$

```
> with(sptcrank):
> for n from 1 to 10 do
> print(seq(NS(m,n),m=0..n));
> od;
```

1, 0
1, 1, 0
1, 1, 1, 0
2, 2, 1, 1, 0
2, 2, 2, 1, 1, 0
4, 4, 3, 2, 1, 1, 0
5, 4, 4, 3, 2, 1, 1, 0
7, 7, 6, 5, 3, 2, 1, 1, 0
10, 9, 8, 6, 5, 3, 2, 1, 1, 0
13, 13, 11, 10, 7, 5, 3, 2, 1, 1, 0

```
>
>
```

THEOREM (A-G-L)

Let $N_S(r, t, n) = \sum_{m \equiv r \pmod{t}} N_S(m, n)$.

Then

$$N_S(0, 5, 5n+4) = N_S(1, 5, 5n+4) = \ldots = N_S(4, 5, 5n+4) = \frac{1}{5}\,spt(5n+4)$$

$$N_S(0, 7, 7n+5) = N_S(1, 7, 7n+5) = \ldots = N_S(6, 7, 7n+5) = \frac{1}{7}\,spt(7n+5)$$

THEOREM

$$N_S(m, n) \geqslant 0$$

PROBLEM: Find what $N_S(m, n)$ is counting in terms of partitions.

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(p.15)

	weight	crank
$\vec{\pi}_1 = (1, 1+1+1, -)$	$+1$	3
$\vec{\pi}_2 = (1, -, 1+1+1)$	$+1$	-3
$\vec{\pi}_3 = (1, 1+1, 1)$	$+1$	1
$\vec{\pi}_4 = (1, 1, 1+1)$	$+1$	-1
$\vec{\pi}_5 = (1, 1+2, -)$	$+1$	2
$\vec{\pi}_6 = (1, -, 1+2)$	$+1$	-2
$\vec{\pi}_7 = (1, 2, 1)$	$+1$	0
$\vec{\pi}_8 = (1, 1, 2)$	$+1$	0
$\vec{\pi}_9 = (1, 3, -)$	$+1$	1
$\vec{\pi}_{10} = (1, -, 3)$	$+1$	-1
$\vec{\pi}_{11} = (1+2, 1, -)$	-1	1
$\vec{\pi}_{12} = (1+2, -, 1)$	-1	-1
$\vec{\pi}_{13} = (1+3, -, -)$	-1	0
$\vec{\pi}_{14} = (2, 2, -)$	$+1$	1
$\vec{\pi}_{15} = (2, -, 2)$	$+1$	-1
$\vec{\pi}_{16} = (4, -, -)$	$+1$	0

From the table, we have

$$\begin{aligned} N_S(0,5,4) &= \omega_1(\vec{\pi}_7) + \omega_1(\vec{\pi}_8) + \omega_1(\vec{\pi}_{13}) + \omega_1(\vec{\pi}_{16}) \\ &= 1 + 1 - 1 + 1 = 2. \end{aligned}$$

Similarly,

$$N_S(0,5,4) = N_S(1,5,4) = N_S(2,5,4) = N_S(3,5,4) = N_S(4,5,4) = 2 = \frac{\mathrm{spt}(4)}{5}.$$

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(p. 18) ②

<u>Theorem</u> (A - G - L)

$$\sum_{n=1}^{\infty} \sum_{m} N_s(m,n) z^m q^n$$

$$= \frac{-1}{(1-z)(1-z^{-1})} \sum_{n=1}^{\infty} \sum_{m} \Big(M(m,n) - N(m,n) \Big) z^m q^n$$

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OVERPARTITIONS

An *overpartition of n* is a partition of n in which the first occurence of a part may be overlined.

n		$\overline{p}(n)$
1	$1, \bar{1}$	2
2	$2, \bar{2}, 1+1, 1+\bar{1}$	4
3	$3, \bar{3}, 2+1, 2+\bar{1}, \bar{2}+1, 1+1+1, 1+1+\bar{1}, \bar{2}+\bar{1}$	8
⋮		⋮
10		232
⋮		
100		53 287 424 374

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Generating Function

$$\sum_{n=0}^{\infty} \bar{p}(n) q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)}{(1-q^n)}$$

Congruences

$$\bar{p}(3n+2) \equiv 0 \pmod 4$$

$$\bar{p}(4n+3) \equiv 0 \pmod 8$$

$$\bar{p}(8n+7) \equiv 0 \pmod{64}$$

Hirsch
Sella

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OTHER SPT-FUNCTIONS (Bringmann, Lovejoy, Osburn 2010)

$$\mathrm{Spt}(d,e;q) = \frac{(-dq,-eq)_\infty}{(deq;q)_\infty} \sum_{n=1}^{\infty} \frac{(q,deq)_n \, q^n}{(1-q^n)^2(-dq,-eq)_n}$$

$d=1, e=0$
$$\sum_{n=1}^{\infty} \overline{spt}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^n \, (-q^{n+1};q)_\infty}{(1-q^n)^2 \, (q^{n+1};q)_\infty}$$

$$\sum_{n=1}^{\infty} \overline{spt}_1(n)\, q^n = \sum_{n=0}^{\infty} \frac{q^{2n+1} \, (-q^{2n+2};q)_\infty}{(1-q^{2n+1})^2 \, (q^{2n+2};q)_\infty}$$

$d=1, e=1/q, q\to q^2$
$$\sum_{n=1}^{\infty} \overline{spt}_2(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^{2n} \, (-q^{2n+1};q)_\infty}{(1-q^{2n})^2 \, (q^{2n+2};q)_\infty}$$

$d=0, e=1/q, q\to q^2$
$$\sum_{n=1}^{\infty} M2spt(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} \frac{(-q^{2n+1};q^2)_\infty}{(q^{2n+2};q^2)_\infty}$$

>

$\overline{spt}(n)$ = # of smallest parts in the overpartitions of n with smallest part not overlined.

$\overline{spt}_1(n)$ = # of smallest parts in the overpartitions of n with smallest part not overlined & odd.

$\overline{spt}_2(n)$ = # of smallest parts in the overpartitions of n with smallest part not overlined & even.

$M2spt(n)$ = # of smallest parts in the partitions of n with odd parts distinct.

```
> SBAR:=etaq(q,2,100)/etaq(q,1,100)^2*add(q^n/(1-q^n)^2*aqprod(q,q,n)/aqprod(-q,q,n),n=1..100):
> SBAR:=series(SBAR,q,101):
> findcong(SBAR,100,10,{2});
```

$$[0, 3, 3]$$
$$[7, 8, 4]$$

$$\{[0, 3, 3], [7, 8, 4]\}$$

>

(p. 20)

Congruences for SPT-Functions

$\overline{\mathrm{spt}}(3n) \equiv 0 \pmod 3$

$\overline{\mathrm{spt}}_1(3n) \equiv 0 \pmod 3$

$\overline{\mathrm{spt}}_1(5n) \equiv 0 \pmod 5$

$\overline{\mathrm{spt}}_1(n)$ is odd iff n is an odd square

$\overline{\mathrm{spt}}_2(3n) \equiv 0 \pmod 3$

$\overline{\mathrm{spt}}_2(3n+1) \equiv 0 \pmod 3$

$\overline{\mathrm{spt}}_2(5n+3) \equiv 0 \pmod 5$

$\mathrm{M2spt}(3n+1) \equiv 0 \pmod 3$

$\mathrm{M2spt}(5n+1) \equiv 0 \pmod 5$

$\mathrm{M2spt}(5n+3) \equiv 0 \pmod 5$

>

SPT-CRANK FUNCTIONS

$$\overline{S}(z,q) = \sum_{n=1}^{\infty} \sum_{m} N_{\overline{S}}(m,n) z^m q^n = \sum_{n=1}^{\infty} \frac{q^n(-q^{n+1};q)_\infty (q^{n+1};q)_\infty}{(zq^n;q)_\infty (z^{-1}q^n;q)_\infty}$$

$$\overline{S}_1(z,q) = \sum_{n=1}^{\infty} \sum_{m} N_{\overline{S}_1}(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{2n+1}(-q^{2n+2};q)_\infty (q^{2n+2};q)_\infty}{(zq^{2n+1};q)_\infty (z^{-1}q^{2n+1};q)_\infty}$$

$$\overline{S}_2(z,q) = \sum_{n=1}^{\infty} \sum_{m} N_{\overline{S}_2}(m,n) z^m q^n = \sum_{n=1}^{\infty} \frac{q^{2n}(-q^{2n+1};q)_\infty (q^{2n+1};q)_\infty}{(zq^{2n};q)_\infty (z^{-1}q^{2n};q)_\infty}$$

$$M2S(z,q) = \sum_{n=1}^{\infty} \sum_{m} N_{M2S}(m,n) z^m q^n = \sum_{n=1}^{\infty} \frac{q^{2n}(-q^{2n+1};q^2)_\infty (q^{2n+1};q^2)_\infty}{(zq^{2n};q^2)_\infty (z^{-1}q^{2n};q^2)_\infty}$$

NOTE: TYPO in last equation. (q^(2n+1);q^2)oo should be (q^(2n+2);q^2)oo.

```
>
SBARZ:=(1-z)*etaq(q,1,30)*etaq(q,2,30)/tripleprod(z,q,30)*add(q^
n*aqprod(z*q,q,n-1)*aqprod(q/z,q,n-1)/aqprod(q^2,q^2,n),n=1..30)
:
> SBARZ:=series(SBARZ,q,31):
> SBARZ:=normal(SBARZ):
> for n from 1 to 10 do
> print(n,factor(coeff(SBARZ,q,n)));
> od;
                                  1, 1
```

$$2, \frac{z^2+z+1}{z}$$

$$3, \frac{(z^2+z+1)(z^2+1)}{z^2}$$

$$4, \frac{z^6+z^5+3z^4+3z^3+3z^2+z+1}{z^3}$$

$$5, \frac{(z^2+1)(z^6+z^5+2z^4+3z^3+2z^2+z+1)}{z^4}$$

$$6, \frac{(z^2+z+1)(z^2+1)(z^6+z^4+3z^3+z^2+1)}{z^5}$$

$$7, \frac{(z^8+z^7+z^6+3z^5+5z^4+3z^3+z^2+z+1)(z^2+1)^2}{z^6}$$

$$8, \frac{(z^4+z^3+z^2+z+1)(z^{10}+2z^8+2z^7+4z^6+5z^5+4z^4+2z^3+2z^2+1)}{z^7}$$

$$9, ((z^2+z+1)$$
$$(z^{14}+2z^{12}+3z^{11}+4z^{10}+7z^9+9z^8+9z^7+9z^6+7z^5+4z^4+3z^3+2z^2+1)) /$$
$$z^8$$

$$10, ((z^2+1)(z^4+z^3+z^2+z+1)$$
$$(z^{12}+z^{10}+2z^9+3z^8+5z^7+5z^6+5z^5+3z^4+2z^3+z^2+1)) / z^9$$

>
>

Theorem (G – Jennings-Shaffer)

These SPT-crank functions give combinatorial refinements for the congruences for $\overline{spt}(n)$, $\overline{spt}_1(n)$, $\overline{spt}_2(n)$, $M2spt(n)$ mod 2, 3 and 5.

Let $\overline{V} = \mathcal{D} \times \mathcal{P} \times \mathcal{P} \times \mathcal{D}$

Let $\overline{S} = \{ \vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4) \in \overline{V} : \quad 1 \le s(\pi_1) < \infty,$
$s(\pi_1) \le s(\pi_2), \quad s(\pi_1) \le s(\pi_3),$
and $s(\pi_1) < s(\pi_4) \}$

Then

$$N_{\overline{S}}(m,n) = \sum_{\substack{\vec{\pi} \in \overline{S} \\ \text{crank}(\vec{\pi}) = m \\ |\vec{\pi}| = n}} \omega_1(\vec{\pi})$$

```
> with(osptcrank):
> for n from 1 to 10 do
> print(seq(ONS(m,n),m=0..n));od;
```

1, 0
1, 1, 0
2, 1, 1, 0
3, 3, 1, 1, 0
4, 4, 3, 1, 1, 0
8, 7, 5, 3, 1, 1, 0
12, 10, 8, 5, 3, 1, 1, 0
17, 17, 13, 9, 5, 3, 1, 1, 0

27, 25, 20, 14, 9, 5, 3, 1, 1, 0
40, 37, 31, 23, 15, 9, 5, 3, 1, 1, 0
>
> **seq(ONS(0,n),n=1..10);**
1, 1, 2, 3, 4, 8, 12, 17, 27, 40

TRY SLOANE'S ONLINE ENC. OF SEQUENCES: http://oeis.org/

1, 1, 2, 3, 4, 8, 12, 17, 27, 40 Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,1,2,3,4,8,12,17,27,40**

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

> **seq(ONS(m,100),m=85..100);**
611, 429, 299, 205, 139, 93, 61, 39, 25, 15, 9, 5, 3, 1, 1, 0
> **seq(ONS(100-m,100),m=1..20);**
1, 1, 3, 5, 9, 15, 25, 39, 61, 93, 139, 205, 299, 429, 611, 861, 1201, 1663, 2285, 3115

1, 1, 3, 5, 9, 15, 25, 39, 61, 93, 139, 205, 299, 429, 611, 8 Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,1,3,5,9,15,25,39,61,93,139,205,299,429,611,861,1201,1663,2285,3115**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A207641 G.f.: Sum_{n>=0} x^n * Product_{k=1..n} (1+x^k)/(1-x^k).

1, 1, 3, 5, 9, 15, 25, 39, 61, 93, 139, 205, 299, 429, 611, 861, 1201, 1663, 228

5683, 7605, 10123, 13405, 17661, 23163, 30245, 39323, 50925, 65699, 84445, 108167, 138089, 175719, 222921, 281965, 355627, 447309, 561139, 702133, 876395, 1091301 (list; graph; refs; listen; history; text; internal format)

OFFSET	0,3
LINKS	Table of n, a(n) for n=0..42.
EXAMPLE	G.f.: A(x) = 1 + x + 3*x^2 + 5*x^3 + 9*x^4 + 15*x^5 + 25*x^6 + 39*x^7 +... such that, by definition, A(x) = 1 + x*(1+x)/(1-x) + x^2*(1+x)*(1+x^2)/((1-x)*(1-x^2)) + x^3*(1+x)*(1+x^2)*(1+x^3)/((1-x)*(1-x^2)*(1-x^3)) +...
PROG	(PARI) {a(n)=polcoeff(sum(m=0, n, x^m*prod(k=1, m, (1+x^k)/(1-x^k +x*O(x^n)))), n)} for(n=0, 50, print1(a(n), ", "))
KEYWORD	nonn
AUTHOR	Paul D. Hanna, Feb 19 2012
STATUS	approved

>
>
>
>
>

Theorem $\quad N_{\bar{S}}(m,n) \geq 0.$

PROBLEM: What is $N_{\bar{S}}(m,n)$ counting interms of overpartitions?

Let

$$N_{\bar{S}}(r,t,n) = \sum_{m \equiv r \pmod{t}} N_{\bar{S}}(m,n).$$

Then

Theorem:

$$N_{\bar{S}}(0,3,3n) = N_{\bar{S}}(1,3,3n) = N_{\bar{S}}(2,3,3n) = \frac{\overline{spt}(3n)}{3}.$$

>

Example $\overline{S}$-partitions of 3

	ω_1	crank	(mod 3)
$(1, -, -, 2)$	1	0	$\equiv 0$
$(1, -, 2, -)$	1	-1	$\equiv 2$
$(1, 2, -, -)$	1	1	$\equiv 1$
$(1, -, 1+1, -)$	1	-1	$\equiv 2$
$(1, 1+1, -, -)$	1	1	$\equiv 1$
$(1, 1, 1, -)$	1	0	$\equiv 0$
$(2+1, -, -, -)$	-1	0	$\equiv 0$
$(3, -, -, -)$	1	0	$\equiv 0$

$$N_{\overline{S}}(0, 3, 3) = 1 + 1 - 1 + 1 = 2$$
$$N_{\overline{S}}(1, 3, 3) = 1 + 1 = 2$$
$$N_{\overline{S}}(2, 3, 3) = 1 + 1 = 2$$

>

Note: $\overline{spt}(3) = 6$

Overpartitions of 3 (with smallest part not overlined)

3

$2 + 1$

$\overline{2} + 1$

$1 + 1 + 1$

>

Rank and Crank of Overpartitions (Lovejoy 2005)

Let $\overline{N}(m,n)$ = # of overpartitions of n with Dyson rank m.
Define
$\overline{M}(m,n)$ by

$$\sum_{n=0}^{\infty}\sum_{m} \overline{M}(m,n) z^m q^n = (-q;q)_\infty \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}$$

Then

$$\sum_{n=0}^{\infty}\sum_{m} \overline{N}(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-1)_n q^{n(n+1)/2}}{(zq)_n (z^{-1}q)_n}$$

$$= \frac{(-q)_\infty}{(q)_\infty}\left(1 + 2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)}}{(1-zq^n)(1-z^{-1}q^n)}\right).$$

```
> with(qseries):
> with(bailey);
```

[*alphadown*, *alphafind*, *alphaup*, *betadown*, *betafind*, *betaup*]

```
> beta:=(a,q,n)->1/aqprod(q^2,q^2,n);
```

$$\beta := (a,q,n) \to \frac{1}{\mathrm{aqprod}(q^2,q^2,n)}$$

```
> alphafind();
```

```
------------------------------------------------------------
alphafind(a,q,beta,n)
   This proc is used to find alpha given beta so that
   (alpha,beta) is a Bailey pair.
   It returns alpha[n].
```

```
> for n from 1 to 5 do alphafind(1,q,beta,n);od;
```

$$-2\,q$$

$$2\,q^{4}$$

$$-2\,q^{9}$$

$$2\,q^{16}$$

$$-2\,q^{25}$$

\>

\>

THEOREM (G – Jennings-Shaffer)

$$\sum_{n=1}^{\infty}\sum_{m} N_{\bar{S}}(m,n) z^m q^n = \frac{-1}{(1-z)(1-z^{-1})}\sum_{n=1}^{\infty}\sum_{m}\left(\bar{M}(m,n)-\bar{N}(m,n)\right) z^m q^n$$

PROOF: $\alpha_n = \begin{cases} 1 & n=0 \\ 2(-1)^n q^{n^2} & n\geq 1\end{cases}$ $\quad \beta_n = \dfrac{1}{(q^2;q^2)_n}$ BAILEY PAIR

Bailey's Lemma

$$\sum_{n=0}^{\infty}\frac{(z)_n (z^{-1})_n q^n}{(-q)_n (q)_n} = \frac{(zq)_\infty (z^{-1}q)_\infty}{(q)_\infty^2}\left(1 + 2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right)$$

$$\bar{S}(z,q) = \sum_{n=1}^{\infty}\frac{q^n(-q^{n+1};q)_\infty (q^{n+1};q)_\infty}{(zq^n;q)_\infty (z^{-1}q^n;q)_\infty} = \frac{(q)_\infty(-q)_\infty}{(z)_\infty (z^{-1})_\infty}\sum_{n=0}^{\infty}\frac{q^n (z)_n (z^{-1})_n}{(-q)_n (q)_n}$$

$$- \frac{(q)_\infty (-q)_\infty}{(z)_\infty (z^{-1})_\infty}$$

```
> with(orank);
```

[*NBAR, orankgen, oranknum, orankresgen, orankresgenb, orankresnum, orankresnumb, oranktablemake*]

```
> X3:=add((NBAR(0,3,n)-NBAR(1,3,n))*q^n,n=0..120):
> with(qseries):
> X30:=sift(X3,q,3,0,120):
> X31:=sift(X3,q,3,1,120):
> X32:=sift(X3,q,3,2,120):
> series(X30,q,10);
```

$1+2q+4q^2+4q^3+6q^4+8q^5+12q^6+16q^7+20q^8+26q^9+O(q^{10})$

```
> prodmake(X30,q,10,list);
```

[-2, -1, 2, -1, -2, 1, -2, -1, 2]

```
> etamake(X30,q,40);
```

$$\frac{\eta(3\,\tau)^4\,\eta(2\,\tau)}{\eta(6\,\tau)^2\,\eta(\tau)^2}$$

```
> series(X31,q,10);
```

$2+2q+4q^2+4q^3+8q^4+10q^5+12q^6+16q^7+22q^8+28q^9+O(q^{10})$

```
> prodmake(X31/2,q,10,list);
```

[-1, -1, 0, -1, -1, 1, -1, -1, 0]

```
> etamake(X31,q,39);
```

$$2\,\frac{\eta(6\,\tau)\,\eta(3\,\tau)}{q^{(1/3)}\,\eta(\tau)}$$

```
> series(X32,q,10);
```

$-2-6q-8q^2-10q^3-16q^4-22q^5-26q^6-38q^7-48q^8-58q^9+O(q^{10})$

```
> prodmake(X32/(-2),q,10,list);
```

[-3, 2, -1, -1, 0, 0, -3, 5, -8]

```
>
>
>
>
>
```

Let $z = z_3 = \exp(2\pi i/3)$:

$$\sum_{n=1} \left(N_{\bar{S}}(0,3,n) - N_{\bar{S}}(1,3,n)\right) q^n = -\tfrac{1}{3}\left(\sum_{n=1}^{\infty} \left(\bar{M}(0,3,n) - \bar{M}(1,3,n)\right.\right.$$
$$\left.\left. - \bar{N}(0,3,n) + \bar{N}(1,3,n)\right) q^n\right)$$

Lovejoy & Osburn (2008)

$$\sum_{n=0}^{\infty} \left(\bar{N}(0,3,3n) - \bar{N}(1,3,3n)\right) q^n = \frac{(q^3;q^3)_\infty (-q)_\infty}{(q)_\infty (-q^3,q^3)_\infty^2}$$

$$\sum_{n=0}^{\infty} \left(\bar{M}(0,3,n) - \bar{M}(1,3,n)\right) q^n = \frac{(q^2;q^2)_\infty (q)_\infty}{(q^3;q^3)_\infty}$$

$$\bar{N}(0,3,3n) - \bar{N}(1,3,3n) = \bar{M}(0,3,3n) - \bar{M}(1,3,3n)$$

$$N_{\bar{S}}(0,3,3n) = N_{\bar{S}}(1,3,3n) = N_{\bar{S}}(2,3,3n).$$

```
> Y1:=etaq(q,2,120)*etaq(q,1,120)/etaq(q,3,120):
> Y10:=sift(Y1,q,3,0,120):
> Y11:=sift(Y1,q,3,1,120):
> Y12:=sift(Y1,q,3,2,120):
> series(Y10,q,10);
```

$$1 + 2q + 4q^2 + 4q^3 + 6q^4 + 8q^5 + 12q^6 + 16q^7 + 20q^8 + 26q^9 + O(q^{10})$$

```
> etamake(Y10,q,40);
```

$$\frac{\eta(3\tau)^4\,\eta(2\tau)}{\eta(6\tau)^2\,\eta(\tau)^2}$$

```
> series(Y11,q,10);
```

$$-1-q-2\,q^{2}-2\,q^{3}-4\,q^{4}-5\,q^{5}-6\,q^{6}-8\,q^{7}-11\,q^{8}-14\,q^{9}+\mathrm{O}(q^{10})$$

```
> etamake(Y11,q,39);
```

$$-\frac{\eta(6\,\tau)\,\eta(3\,\tau)}{q^{(1/3)}\,\eta(\tau)}$$

```
> series(Y12,q,10);
```

$$-2-2\,q^{2}-4\,q^{3}-4\,q^{4}-4\,q^{5}-8\,q^{6}-8\,q^{7}-12\,q^{8}-16\,q^{9}+\mathrm{O}(q^{10})$$

```
> etamake(Y12,q,39);
```

$$-2\,\frac{\eta(6\,\tau)^{4}}{q^{(2/3)}\,\eta(3\,\tau)^{2}\,\eta(2\,\tau)}$$

```
>
>
>
>
```

SPT-FUNCTIONS mod 2^α

Theorem (Andrews - G - Liang 2012; Folsom - Ono 2008)
$\mathrm{spt}(n)$ is odd iff $24n-1 = p^{4a+1} m^2$
for some prime $p \equiv 23 \pmod{24}$ & some integers a, m where
$(p, m) = 1$.

Theorem (G - Jennings-Shaffer) Let $\ell \geq 5$ be prime. Then

$$\mathrm{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right)\mathrm{spt}(n) + \ell\,\mathrm{spt}\left(\frac{n+s_\ell}{\ell^2}\right)$$

$$\equiv \chi_{12}(\ell)(1+\ell)\,\mathrm{spt}(n) \mod 2^\beta$$

where $s_\ell = (\ell^2-1)/24$ &

$$\beta = \begin{cases} 3 & \text{if } \ell \equiv 5, 7 \pmod{24} \\ 4 & \text{if } \ell \equiv 13, 23 \pmod{24} \\ 5 & \text{if } \ell \equiv 1, 11, 17, 19 \pmod{24} \end{cases}$$

>

Theorem (Andersen 2012) Let $\ell \geq 3$ be prime. The

$$\overline{spt}_1(\ell^2 n) + \left(\frac{-n}{\ell}\right)\overline{spt}_1(n) + \ell\,\overline{spt}_1\left(\frac{n}{\ell^2}\right) \equiv (1+\ell)\,\overline{spt}_1(n) \mod 2^\gamma,$$

$$M2spt(\ell^2 n - t_\ell) + \left(\frac{2}{\ell}\right)\left(\frac{1-8n}{\ell}\right) M2spt(n) + \ell\, M2spt\left(\frac{n+t_\ell}{\ell^2}\right)$$

$$\equiv \left(\frac{2}{\ell}\right)(1+\ell)\, M2spt(n) \pmod{2^\delta}$$

where $t_\ell = (\ell^2-1)/8$,

$$\gamma = \begin{cases} 5 & \ell \equiv 3 \pmod 8 \\ 6 & \ell \equiv 5,7 \pmod 8 \\ 7 & \ell \equiv 1 \pmod 8 \end{cases}$$

$$\delta = \begin{cases} 1 & \ell \equiv 3 \pmod 8 \\ 2 & \ell \equiv 5 \pmod 8 \\ 3 & \ell \equiv 1,7 \pmod 8. \end{cases}$$

>

Theorem (G)

$$(q)_\infty^3 \sum_{n=1}^{\infty} spt(n) q^n =$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} -\frac{1}{4}(n-j)^2 \chi_4(n) \chi_{12}(j) q^{\frac{1}{12}\left(\frac{1}{2}(3n^2-j^2)-1\right)}$$

$$(z^{-1})_\infty (z)_\infty (q)_\infty S(z,q)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \left(1 - z^{-\frac{1}{2}(n-j)}\right)^2 z^{\frac{1}{2}(j-n)} \chi_4(n) \chi_{12}(j) q^{\frac{1}{12}\left(\frac{1}{2}(3n^2-j^2)-1\right)}$$

>

Theorem (G.)

$$(zq)_\infty (z^{-1}q)_\infty (q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}$$

$$= \frac{1}{2}\left(\sum_{m=0}^{\infty} \left(\sum_{k=0}^{[m/2]} (-1)^{m+k} \left(z^{m-3k} + z^{3k-m}\right) q^{\frac{1}{2}(m^2-3k^2)+\frac{1}{2}(m-k)} + \sum_{k=1}^{[m/2]} (-1)^{m+k} \left(z^{m-3k+1} + z^{3k-m-1}\right) q^{\frac{1}{2}(m^2-3k^2)+\frac{1}{2}(m+k)} \right)\right)$$

Corollary (Hecke-Rogers)

$$(q)_\infty^2 = \sum_{m=0}^{\infty} \sum_{2|k| \leq m} (-1)^{m+k} q^{\frac{1}{2}(m^2-3k^2)+\frac{1}{2}(m-k)}$$

>
?