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Conditional independence in propositional logic

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Abstract

Independence—the study of what is relevant to a given problem of reasoning—is an important AI topic. In this paper, we investigate several notions of conditional independence in propositional logic: Darwiche and Pearl's conditional independence, and some more restricted forms of it. Many characterizations and properties of these independence relations are provided. We show them related to many other notions of independence pointed out so far in the literature (mainly formula-variable independence, irrelevance and novelty under various forms, separability, interactivity). We identify the computational complexity of conditional independence and of all these related independence relations.

Keywords: Conditional independence; Relevance; Novelty; Separability

1. Introduction

1.1. Motivations

Focusing on what is relevant is a natural approach to design efficient knowledge-based engines. Indeed, as a preliminary step to various intelligent tasks (e.g., planning, decision making, reasoning), it is reasonable to discard everything but what is relevant. For instance, I do not need to remember the date of birth of Arthur Rimbaud when my objective is to cook noodles. The idea of focusing on what is relevant is strongly related to many AI notions, like local computation and micro-theories [16]. Irrelevance is also a central topic in probabilistic reasoning [29]. Furthermore, the complementary notion of relevance is a key notion for defining information filtering policies [13], and cooperative answering techniques [6]. For instance, when a database user is unable to express her queries in a formal way, an approach consists in determining her topics of interest, then to return

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in a structured way all what the database tells about such topics. Obviously, relevance relations are needed to characterize precisely what "tells about" means. This explains why (ir)relevance, under various names as independence, irredundancy, influenceability, novelty, separability, and interactivity is nowadays considered as an important notion in many AI fields [1,15,24,32].

In the following, relevance is captured by relations in the metalanguage of propositional logic. Arguments of such relations are propositional formulas encoding knowledge bases and pieces of knowledge (including queries), and sets of propositional variables or literals that represent, for instance, subject matters or topics of interest.

To what extent is the goal of improving inference reachable through (ir)relevance? To address this point, a key issue is *computational complexity*. Indeed, assume that we know that the resolution of some reasoning problems can be speeded up once some relevance information has been elicited. In the situation where it is computationally harder to point out such information from the input than to reason directly from it, computational benefits are hard to be expected. If so, alternative uses of relevance for reasoning are to be investigated. For instance, searching for relevance information can be limited by considering only those pieces of knowledge that can be generated in a tractable way. In the case where such information depend only on the knowledge base, another approach consists in (tentatively) compensating the computational resources spent in deriving the relevance information through many queries (computing the relevance information can then be viewed as a form of compilation). Clearly enough, the computational issue is also central when relevance relations are not used to improve inference but for other purposes, like defining information filtering policies or designing cooperative answering approaches. Thus, a cooperative answering approach relying on a highly intractable relevance relation could hardly be used on large instances.

Unfortunately, little is known about the computational complexity of relevance. This paper, together with a companion paper [20], contributes to fill this gap. The complexity of various logic-based relevance relations is identified in a propositional setting. By logic-based we mean that the notions of relevance we focus on are not extra-logical but built inside the logic: they are defined using the standard logical notions of (classical) formula, model, logical deduction, etc.

1.2. Scope and organization of the paper

In the companion paper [20], several forms of relevance bearing between a piece of information (a propositional formula) and a set of literals or variables have been investigated (some of these notions are briefly recalled in Section 2). Here, we consider *conditional independence*, introduced as a logical counterpart to probabilistic independence in [7,8]. Intuitively, two sets of variables X and Y are conditionally independent given a set of variables Z and a formula Σ if and only if, whichever full information about Z we consider, the addition of information about X in Σ does not enable us telling anything new about Y. Darwiche [8] intensively shows how the exploitation of conditional independence can prove computationally valuable for several forms of inference (including deduction, abduction, and diagnosis). Basically, through the exploitation of conditional independence, a global computation can be replaced by a number of efficient, local computations.

According to Darwiche [8], there are two main positions in the literature with respect to irrelevance: (1) a “philosophical” position where we start with some intuitive properties of independence, and some independence relations satisfying these properties are then exhibited, and (2) a “pragmatic” position where independence is not an absolute notion but a task-specific one and its utility is measured at the light of the improvement it offers when taken into account.

In this paper, we adhere to both positions. We first focus on Darwiche’s conditional independence. We complete the investigation reported in [8] by showing close connections with probabilistic independence (the philosophical side), by identifying the computational complexity of conditional independence and by suggesting additional applications in the context of reasoning about actions (the practical side). In addition, we introduce a useful restriction of conditional independence, namely strong conditional independence. For this restriction, any conjunctive information (not necessarily complete) about Z is acceptable. From the philosophical side, we present several semantical characterizations of strong conditional independence and some of its metatheoretic properties. Especially, we show that strong conditional independence satisfies all graphoid axioms. From the practical side, we identify the computational complexity of strong conditional independence in the general case and in some restricted ones. Then, we successively consider several forms of (ir)relevance already pointed out so far in the literature, and show them closely connected to conditional independence: formula-variable independence [20], strict relevance, explanatory relevance, relevance between two subject matters [19], novelty under various forms (positive and negative, novelty-based independence) [14,26], separability [23], causal independence [9], and interactivity [4]. As additional results, we identify the complexity of all these independence relations.

The rest of the paper is organized as follows. Some formal preliminaries are given in Section 2. Conditional independence relations and some metatheoretic properties are presented in Section 3. Complexity results are reported in Section 4. Close connections of both notions of conditional independence with existing irrelevance relations are exhibited in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

2.1. Propositional logic

Let PS be a finite set of propositional variables. $PROP_{PS}$ is the propositional language built up from PS, the connectives, and the Boolean constants *true* and *false* in the usual way. For every $X \subseteq PS$, $PROP_X$ denotes the sublanguage of $PROP_{PS}$ generated from the variables of X only. A *literal* of $PROP_X$ is either a variable of X (positive literal) or the negation of a variable of X (negative literal). A clause δ (respectively a term γ) of $PROP_X$ is a (possibly empty) disjunction (respectively conjunction) of literals of $PROP_X$. Often clauses and terms are considered as the sets of their literals. A CNF (respectively a DNF) formula of $PROP_X$ is a conjunction of clauses (respectively a disjunction of terms) of $PROP_X$.

From now on, Σ denotes a propositional formula, i.e., a member of $PROP_{PS}$. $Var(\Sigma)$ is the set of propositional variables appearing in Σ. Elements of PS are denoted x, y, etc. Subsets of PS are denoted X, Y, etc. In order to simplify notations, we will assimilate every singleton $X = \{x\}$ with its unique element x. The size $|\Sigma|$ of a propositional formula Σ is the number of symbols used to write it.

Formulas of $PROP_{PS}$ are interpreted in the usual way. Especially, every finite set of formulas is identified with the conjunction of its elements. Full instantiations of variables of $X \subseteq PS$ are called X-worlds and denoted by ω_X; their set is noted Ω_X. Every X-world ω_X will be identified with the term containing x as a literal when x is interpreted as true in ω_X, and $\neg x$ when x is false in ω_X for every $x \in X$. Equivalently, ω_X will also be identified with the (conjunctively-interpreted) set of these literals. Whenever ω_X is an X-world and ω_Y is a Y-world s.t. $X \cap Y = \emptyset$, (ω_X, ω_Y) denotes the $X \cup Y$-world which coincides with ω_X on X and with ω_Y on Y. In order to simplify notations, we assume that every ω_X represents an X-world (even when $\omega_X \in \Omega_X$ is not stated explicitly). PS-worlds are the usual interpretations over PS; their set is noted Ω. When Σ is true in an interpretation ω, ω is a model of Σ. When Σ has a model, it is said to be consistent or satisfiable; otherwise, it is said to be inconsistent, contradictory, or unsatisfiable. When every interpretation of Ω is a model of Σ, Σ is said to be valid, or a tautology. As usual, $\models$ denotes classical entailment, and $\equiv$ denotes logical equivalence. ω_X is a partial model of Σ whenever there exists a model of Σ that coincides with ω_X on X; stated otherwise, ω_X is a partial model of Σ whenever $\omega_X \wedge \Sigma$ is consistent (here, ω_X is viewed as a term).

Given a set of interpretations $S \subseteq \Omega$, we denote $for(S)$ a formula that has S as a set of models. Of course, there are many equivalent formulas having S as models, but *for* will be used only when this does not matter. When $S = \{\omega\}$, i.e., S is composed of a single interpretation, we write $for(\omega)$ instead of $for(\{\omega\})$. Conversely, given a formula Σ, we denote $\mathrm{Mod}(\Sigma)$ the set of models of Σ.

In this paper we use the concepts of implicates and prime implicates.

Definition 1. The set of implicates of a formula Σ, denoted by $IS(\Sigma)$, is defined as:

$$IS(\Sigma) = \{\text{clause } \delta \mid \Sigma \models \delta\}.$$

The set of prime implicates of a formula Σ, denoted by $IP(\Sigma)$, is defined as:

$$IP(\Sigma) = \left\{\delta \in IS(\Sigma) \mid \nexists \delta' \in IS(\Sigma) \text{ s.t. } \delta' \models \delta \text{ and } \delta \not\models \delta'\right\}.$$

It is well known that a clause δ is a logical consequence of a formula Σ if and only if it is entailed by at least one prime implicate π of Σ. This can be checked efficiently since a clause δ is a logical consequence of a clause π if and only if δ is a tautology or every literal of π is a literal of δ. Accordingly, the prime implicates form of Σ can be considered as a compilation of Σ [30].

Implicants and prime implicants will also be considered in the following.

Definition 2. The set of implicants of a formula Σ, denoted by $SI(\Sigma)$, is defined as:

$$SI(\Sigma) = \{\text{term } \gamma \mid \gamma \models \Sigma\}.$$

The set of prime implicants of a formula Σ, denoted by $PI(\Sigma)$, is defined as:

$$PI(\Sigma) = \{\gamma \in SI(\Sigma) \mid \nexists \gamma' \in SI(\Sigma) \text{ s.t. } \gamma \models \gamma' \text{ and } \gamma' \not\models \gamma\}.$$

Often, we will not be interested in all prime implicants (respectively prime implicates) of Σ but only in the subset $IP^X(\Sigma)$ (respectively $PI^X(\Sigma)$) containing those built up from variables of X, only.

Of course, the set of implicants/ates, prime implicants/ates may contain equivalent terms/clauses. We can restrict our attention to one term/clause for each set of equivalent terms/clauses. Stated otherwise, in both $IP(\Sigma)$, $PI(\Sigma)$, $IP^X(\Sigma)$, $PI^X(\Sigma)$, only one representative per equivalence class is kept.

2.2. *Formula-variable independence and forgetting*

Let us first recall the definitions and results about formula-variable independence and variable forgetting [20] needed in this paper.

Let Σ be a formula from $PROP_{PS}$ and X be a subset of PS. Σ is *semantically V-independent from* X if and only if there exists a formula Φ s.t. $\Phi \equiv \Sigma$ holds and Φ is syntactically V-independent from X, i.e., $Var(\Phi) \cap X = \emptyset$. When X is a singleton $\{x\}$ we say that Σ is V-independent from x (instead of $\{x\}$). It can be easily shown [20] that Σ is (semantically) V-independent from X if and only if Σ is V-independent from each variable of X. The set of variables on which a formula Σ depends is denoted by $DepVar(\Sigma)$.

For instance, $\Sigma = (a \wedge (b \vee \neg b))$ is V-dependent on a and V-independent from b, and $DepVar(\Sigma) = \{a\}$.

For every formula Σ and every variable x, $\Sigma_{x \leftarrow 0}$ (respectively $\Sigma_{x \leftarrow 1}$) is the formula obtained by replacing every occurrence of x in Σ by the constant *false* (respectively *true*). The next four statements are equivalent [20]:

(1) Σ is V-independent from x;
(2) $\Sigma_{x \leftarrow 0} \equiv \Sigma_{x \leftarrow 1}$;
(3) $\Sigma \equiv \Sigma_{x \leftarrow 0}$;
(4) $\Sigma \equiv \Sigma_{x \leftarrow 1}$.

Variable independence can be determined in an efficient way when Σ is given in some specific normal forms, namely its prime implicates form or its prime implicants form. For such normal forms, V-independence comes down to its syntactical form. Namely, the next statements are equivalent [20]:

(1) Σ is V-independent from X;
(2) $PI(\Sigma) \subseteq PROP_{PS \setminus X}$;
(3) $IP(\Sigma) \subseteq PROP_{PS \setminus X}$.

The problem of determining whether Σ is V-independent from X is coNP-complete [20].

A basic way to render a formula Σ V-independent from a given set X of variables consists in *forgetting* X in Σ.

Let Σ be a formula from $PROP_{PS}$ and let X be a subset of PS. $ForgetVar(\Sigma, X)$ is the formula inductively defined as follows:

- $ForgetVar(\Sigma, \emptyset) = \Sigma$,
- $ForgetVar(\Sigma, \{x\}) = \Sigma_{x \leftarrow 1} \vee \Sigma_{x \leftarrow 0}$,
- $ForgetVar(\Sigma, \{x\} \cup Y) = ForgetVar(ForgetVar(\Sigma, Y), \{x\})$.

For instance, with $\Sigma = (\neg a \vee b) \wedge (a \vee c)$, we have $ForgetVar(\Sigma, \{a\}) \equiv (b \vee c)$.

As a direct consequence of the definition, $ForgetVar(\Sigma, \{x_1, \ldots, x_n\})$ is equivalent to the quantified Boolean formula (usually with free variables!) noted $\exists x_1 \ldots \exists x_n \Sigma$.

It can be shown that $ForgetVar(\Sigma, X)$ is the logically strongest consequence of Σ that is V-independent from X (up to logical equivalence). Thus, if a formula φ is V-independent from X, then $\Sigma \models \varphi$ if and only if $ForgetVar(\Sigma, X) \models \varphi$. Accordingly, Σ is V-independent from X if and only if $\Sigma \equiv ForgetVar(\Sigma, X)$ holds.

It turns out that forgetting is a fundamental operation for many AI tasks [20,21,25].

2.3. Computational complexity

The complexity results we give in this paper refer to some complexity classes which deserve some recalls. More about them can be found in Papadimitriou's textbook [28]. Given a problem A, we denote by $\overline{\text{A}}$ the complementary problem of A. We assume that the classes P, NP and coNP are known to the reader. The following classes will also be considered:

- BH_2 (also known as DP) is the class of all languages L such that $L = L_1 \cap L_2$, where L_1 is in NP and L_2 in coNP. The canonical BH_2-complete problem is SAT–UNSAT: a pair of formulas $\langle \varphi, \psi \rangle$ is in SAT–UNSAT if and only if φ is satisfiable and ψ is not. The complementary class coBH_2 is the class of all languages L such that $L = L_1 \cup L_2$, where L_1 is in NP and L_2 in coNP. The canonical coBH_2-complete problem is SAT-OR-UNSAT: a pair of formulas $\langle \varphi, \psi \rangle$ is in SAT–UNSAT if and only if φ is satisfiable or ψ is not.
- $\Sigma_2^p = \mathsf{NP}^{\mathsf{NP}}$ is the class of all languages recognizable in polynomial time by a nondeterministic Turing machine using an NP oracle, where an NP oracle solves any instance of an NP or a coNP problem in unit time. The canonical Σ_2^p-complete problem 2-QBF is the set of all triples $\langle A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_n\}, \Phi \rangle$ where A and B are two disjoint sets of propositional variables and Φ is a formula of $PROP_{A \cup B}$. A positive instance of it is such that there exists an A-world ω_A such that for every B-world ω_B we have $(\omega_A, \omega_B) \models \Phi$.
- $\Pi_2^p = \mathsf{co}\Sigma_2^p = \mathsf{coNP}^{\mathsf{NP}}$. The canonical Π_2^p-complete problem 2-$\overline{\text{QBF}}$ is the set of all triples $\langle A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_n\}, \Phi \rangle$ where A and B are two disjoint sets of propositional variables and Φ is a formula of $PROP_{A \cup B}$. A positive instance of it is such that for every A-world ω_A there exists a B-world ω_B such that $(\omega_A, \omega_B) \models \Phi$. Both Σ_2^p and Π_2^p are complexity classes located at the so-called second level of the polynomial hierarchy [31] which plays a prominent role in knowledge representation and reasoning.

3. Conditional independence

Conditional independence can be seen as a generalization of formula-variable independence. Given three sets of propositional variables X, Y and Z, and a propositional formula Σ, we want to express the fact that, given Σ and some knowledge about Z, the truth value of the variables in X may affect the truth value of variables in Y (and *vice versa*).

3.1. Simple conditional independence

Darwiche and Pearl's conditional independence [8,9] (often referred as "simple conditional independence" or "conditional independence" in the following) is defined as follows:

Definition 3 (*conditional independence*). Let Σ be a propositional formula and X, Y, Z be disjoint subsets of *PS*. X and Y are *independent* given Z with respect to Σ (denoted by $X \sim_{\Sigma}^{Z} Y$) if and only if $\forall \omega_X \in \Omega_X$, $\forall \omega_Y \in \Omega_Y$, $\forall \omega_Z \in \Omega_Z$, the consistency of both $\omega_X \wedge \omega_Z \wedge \Sigma$ and $\omega_Y \wedge \omega_Z \wedge \Sigma$ implies the consistency of $\omega_X \wedge \omega_Y \wedge \omega_Z \wedge \Sigma$.

Example 1. Let $\Sigma = \{\neg a \vee \neg b \vee c, \neg a \vee b \vee d, a \vee \neg c, \neg a \vee c \vee d, b \vee \neg c \vee d\}$. We have the following:

- $c \sim_{\Sigma}^{\emptyset} d$. Indeed, $(c \wedge d) \wedge \Sigma$, $(c \wedge \neg d) \wedge \Sigma$, $(\neg c \wedge d) \wedge \Sigma$ and $(\neg c \wedge \neg d) \wedge \Sigma$ are all consistent.
- $c \not\sim_{\Sigma}^{\{a\}} d$. Indeed, $(a \wedge \neg c \wedge \neg d) \wedge \Sigma$ is inconsistent while $(a \wedge \neg c) \wedge \Sigma$ and $(a \wedge \neg d) \wedge \Sigma$ are both consistent. Intuitively speaking, when a is true, learning $\neg c$ tells that d is true.
- $c \sim_{\Sigma}^{\{a,b\}} d$. Indeed, the set of $\{a, b, c\}$-worlds that are consistent with Σ is

  $$S_1 = \{a \wedge b \wedge c,\ a \wedge \neg b \wedge c,\ a \wedge \neg b \wedge \neg c,\ \neg a \wedge b \wedge \neg c,\ \neg a \wedge \neg b \wedge \neg c\};$$

  the set of $\{a, b, d\}$-worlds that are consistent with Σ is

  $$S_2 = \{a \wedge b \wedge d,\ a \wedge b \wedge \neg d,\ a \wedge \neg b \wedge d,\ \neg a \wedge b \wedge d,\ \neg a \wedge b \wedge \neg d,\ \neg a \wedge \neg b \wedge d\};$$

  and the set of $\{a, b, c, d\}$-worlds that are consistent with Σ (in other terms, the models of Σ) is

  $$S_3 = \{a \wedge b \wedge c \wedge d,\ a \wedge b \wedge c \wedge \neg d,\ a \wedge \neg b \wedge c \wedge d,\ a \wedge \neg b \wedge \neg c \wedge d,\ \neg a \wedge b \wedge \neg c \wedge d,\ \neg a \wedge b \wedge \neg c \wedge \neg d,\ \neg a \wedge \neg b \wedge \neg c \wedge d,\ \neg a \wedge \neg b \wedge \neg c \wedge \neg d\};$$

  it can be checked that for each $\omega_1 \in S_1$ and each $\omega_2 \in S_2$ such that ω_1 and ω_2 give the same truth values to a and b, then $\omega_1 \wedge \omega_2 \in S_3$.

As explained by Darwiche and Pearl [9], $X \sim_{\Sigma}^{Z} Y$ holds if and only if for any possible *full information* about Z, adding some information about Y does not tell us anything new about X. Intuitively, if in the context ω_Z, adding ω_X gives some information about Y, then some partial models of Σ over Y, i.e., those in contrast with the new information obtained on Y, should not remain partial models any longer. As a result, X and Y are independent if, for any "consistent" choice (with Σ) of ω_X, ω_Y, and ω_Z, the formula $\omega_X \wedge \omega_Y \wedge \omega_Z \wedge \Sigma$ is consistent.

Clearly enough, conditional independence given Z with respect to Σ satisfies the following properties [8]:

Proposition 1.

(1) $X \sim_\Sigma^Z Y$ *if and only if* $Y \sim_\Sigma^Z X$.
(2) *If* $\Sigma \equiv \Sigma'$, *then* ($X \sim_\Sigma^Z Y$ *if and only if* $X \sim_{\Sigma'}^Z Y$).
(3) *If* $X' \subseteq X$, $Y' \subseteq Y$ *and* $X \sim_\Sigma^Z Y$, *then* $X' \sim_\Sigma^Z Y'$.

Proof. (1) and (2) are straightforward. As to (3), assume $X' \subseteq X$, $Y' \subseteq Y$ and $X \sim_\Sigma^Z Y$ and let $\omega_{X'}$, $\omega_{Y'}$ and ω_Z s.t. $\omega_{X'} \wedge \omega_Z$ and $\omega_{Y'} \wedge \omega_Z$ are both consistent. Since $\omega_{X'} \equiv \bigvee_{\omega_X \supseteq \omega_{X'}} \omega_X$, there is an $\omega_X \supseteq \omega_{X'}$ s.t. $\omega_X \wedge \omega_Z$ is consistent, and similarly, there is an $\omega_Y \supseteq \omega_{Y'}$ s.t. $\omega_Y \wedge \omega_Z$ is consistent. Now, because $X \sim_\Sigma^Z Y$, we get that $\omega_X \wedge \omega_Y \wedge \omega_Z \wedge \Sigma$ is consistent, which in turn implies the consistency of $\omega_{X'} \wedge \omega_{Y'} \wedge \omega_Z \wedge \Sigma$. □

However, conditional independence is stable neither by contraction nor by expansion of Z. For instance, taking the same Σ as in Example 1, we have $c \sim_\Sigma^\emptyset d$ and however $c \not\sim_\Sigma^{\{a\}} d$; we have $c \not\sim_\Sigma^{\{a\}} d$ and however $c \sim_\Sigma^{\{a,b\}} d$. Conditional independence is also not stable by weakening or strengthening Σ in the general case. Thus, while we have $c \sim_\Sigma^\emptyset d$, we also have $c \not\sim_{\Sigma \cup \{c \Leftrightarrow d\}}^\emptyset d$; while we have $c \sim_\Sigma^{\{a,b\}} d$, we also have $c \not\sim_{\Sigma \setminus \{\neg a \vee \neg b \vee c\}}^{\{a,b\}} d$.

The two limit cases when Z is respectively empty or equal to $Var(\Sigma) \setminus (X \cup Y)$, are of particular interest, especially when computational complexity is investigated.

Definition 4 (*marginal independence*). X and Y are *marginally independent* with respect to Σ if and only if $X \sim_\Sigma^\emptyset Y$.

Definition 5 (*ceteris paribus independence*). X and Y are *ceteris paribus independent* with respect to Σ (denoted by $X \sim_\Sigma^{\text{ceteris paribus}} Y$) if and only if $X \sim_\Sigma^{Var(\Sigma) \setminus (X \cup Y)} Y$.

Darwiche showed [8] that conditional independence satisfies all *semi-graphoid axioms*, which are considered reasonable postulates for conditional independence relations. We recall here these axioms, more so because we will need them further on. Let $Ind(X, Z, Y)$ be an independence relation between X and Y given Z (where X, Y and Z are pairwise disjoint sets of variables).

Symmetry	$Ind(X, Z, Y) \Leftrightarrow Ind(Y, Z, X)$.	(S)
Decomposition	$Ind(X, Z, Y \cup W) \Rightarrow Ind(X, Z, Y)$.	(D)
Weak union	$Ind(X, Z, Y \cup W) \Rightarrow Ind(X, Z \cup W, Y)$.	(WU)
Contraction	$Ind(X, Y \cup Z, W)$ and $Ind(X, Z, Y) \Rightarrow Ind(X, Z, Y \cup W)$.	(C)

The *graphoid axioms* are composed by all the above ones plus the following one.

Intersection $\quad Ind(X, Z \cup W, Y)$ and $Ind(X, Z \cup Y, W) \Rightarrow Ind(X, Z, Y \cup W)$. (I)

Simple conditional independence does not satisfy Intersection. Indeed, let

$$\Sigma = \{y \Leftrightarrow w, z \Rightarrow x \vee y\};$$

$\neg y \wedge \neg w \wedge z \wedge \Sigma$ and $\neg x \wedge z \wedge \Sigma$ are both consistent while $\neg x \wedge \neg y \wedge \neg w \wedge z \wedge \Sigma$ is inconsistent. Hence $x \not\sim_{\Sigma}^{\{z\}} \{y, w\}$, while $x \sim_{\Sigma}^{\{z,w\}} y$ and $x \sim_{\Sigma}^{\{z,y\}} w$ both hold.

Hereafter, we complete Darwiche's characterization of conditional independence by establishing a clear link between simple conditional independence and probabilistic independence. This shows that there is more than an analogy between these notions but a concrete mathematical connection.

Definition 6. Let pr be a probability distribution on Ω, and $X, Y, Z \subseteq PS$.

- X and Y are independent given Z according to pr, denoted by $X \sim_{pr}^{Z} Y$, if and only if $\forall \omega_X \in \Omega_X, \forall \omega_Y \in \Omega_Y, \forall \omega_Z \in \Omega_Z$, we have
$$pr(\omega_X \wedge \omega_Y | \omega_Z) = pr(\omega_X | \omega_Z) \cdot pr(\omega_Y | \omega_Z).$$
- pr is strictly compatible with a propositional formula Σ if and only if $\forall \omega \in \Omega$, $\omega \models \Sigma$ is equivalent to $pr(\omega) > 0$.

Proposition 2. *$X \sim_{\Sigma}^{Z} Y$ if and only if there is a probability distribution pr strictly compatible with Σ such that $X \sim_{pr}^{Z} Y$.*

Proof. ($\Rightarrow$) For any $A \subseteq Var(\Sigma)$, let $Cons_A(\Sigma) = \{\omega_A \mid (\omega_A \wedge \Sigma)$ is consistent$\}$. Assume that $X \sim_{\Sigma}^{Z} Y$. Let us define the probability distribution pr by:

$$\forall \omega = (\omega_X, \omega_Y, \omega_Z, \omega_{Var(\Sigma) \setminus (X \cup Y \cup Z))}),$$
if $\omega \models \Sigma$ then
$$pr(\omega) = \frac{1}{|Cons_Z(\Sigma)|.|Cons_X(\Sigma \wedge \omega_Z)|.|Cons_Y(\Sigma \wedge \omega_Z)|.|\mathrm{Mod}(\omega_X \wedge \omega_Y \wedge \omega_Z \wedge \Sigma)|};$$
if $\omega \models \neg\Sigma$ then $pr(\omega) = 0$.

First, pr is a probability distribution:

$$\begin{aligned}
&\sum_{\omega \in \Omega} pr(\omega) \\
&\quad = \sum_{\omega \models \Sigma} pr(\omega) \\
&\quad = \sum_{\omega_Z \in Cons_Z(\Sigma)} \sum_{\omega_X} \sum_{\omega_Y} \sum_{\omega \supseteq (\omega_X, \omega_Y, \omega_Z)} pr(\omega) \\
&\quad = \sum_{\omega_Z \in Cons_Z(\Sigma)} \sum_{\omega_X \in Cons_X(\omega_Z \wedge \Sigma)} \sum_{\omega_Y \in Cons_Y(\omega_Z \wedge \Sigma)} \frac{1}{|Cons_Z(\Sigma)|.|Cons_X(\Sigma \wedge \omega_Z)|.|Cons_Y(\Sigma \wedge \omega_Z)|} \\
&\quad = \sum_{\omega_Z \in Cons_Z(\Sigma)} \frac{1}{|Cons_Z(\Sigma)|} \\
&\quad = 1.
\end{aligned}$$

It is obvious that pr is strictly compatible with Σ.

Lastly, let us check that $X \sim^{Z}_{pr} Y$. For all $\omega_X, \omega_Y, \omega_Z$ we have

$$pr(\omega_X \wedge \omega_Y \wedge \omega_Z) = \frac{1}{|Cons_Z(\Sigma)|.|Cons_X(\Sigma \wedge \omega_Z)|.|Cons_Y(\Sigma \wedge \omega_Z)|};$$

$$pr(\omega_X \wedge \omega_Z) = \frac{1}{|Cons_Z(\Sigma)|.|Cons_X(\Sigma \wedge \omega_Z)|};$$

$$pr(\omega_Z) = \frac{1}{|Cons_Z(\Sigma)|};$$

$$pr(\omega_X|\omega_Z) = \frac{1}{|Cons_X(\Sigma \wedge \omega_Z)|};$$

similarly,

$$pr(\omega_Y|\omega_Z) = \frac{1}{|Cons_Y(\Sigma \wedge \omega_Z)|};$$

$$pr(\omega_X \wedge \omega_Y|\omega_Z) = \frac{1}{|Cons_X(\Sigma \wedge \omega_Z)|.|Cons_Y(\Sigma \wedge \omega_Z)|} = pr(\omega_X|\omega_Z).pr(\omega_Y|\omega_Z).$$

Hence, $X \sim^{Z}_{pr} Y$ holds.

($\Leftarrow$) Let pr be a probability distribution strictly compatible with Σ such that $X \sim^{Z}_{pr} Y$. For all $\omega_X, \omega_Y, \omega_Z$, the consistencies of $\omega_X \wedge \omega_Z \wedge \Sigma$ and of $\omega_Y \wedge \omega_Z \wedge \Sigma$ imply, respectively, that $pr(\omega_X \wedge \omega_Z) > 0$ and $pr(\omega_Y \wedge \omega_Z) > 0$ by strict compatibility with Σ, therefore $pr(\omega_X \wedge \omega_Y \wedge \omega_Z) > 0$ by probabilistic independence of X and Y given Z. Then, using again strict compatibility of pr with Σ, we get the consistency of $\omega_X \wedge \omega_Y \wedge \omega_Z \wedge \Sigma$. □

Darwiche shows [8] how conditional independence can prove a valuable notion to improve many forms of reasoning (including consistency, entailment, diagnosis and abduction). It can also be helpful in the context of reasoning about actions, when dealing about concurrency: indeed, the following definition of compatibility between actions consisting in saying that *two actions a and b are compatible if and only if for each initial situation where both actions a and b are separately applicable* (*i.e., without producing an inconsistency*), *then a and b are jointly applicable* [12], can be mapped easily into a conditional independence problem.

Conditional independence may also be helpful for computing ramifications of an action: if $Dep(a)$ is the set of variables that are *directly* influenced by action a (i.e., appearing in its effects), and if Σ is the set of static laws (or integrity constraints), then any variable y such that $Dep(a) \sim^{\emptyset}_{\Sigma} y$ is guaranteed to be "ramification-free" (the converse, however, is not true).

Altogether, this explains why conditional independence is an important notion and motivates the investigation of its computational complexity.

3.2. Strong conditional independence

Simple conditional independence does not apply to contexts where the new information that can be learned about Z is incomplete, i.e., the truth value of some variables of Z is not available, or, more generally, many partial (and possibly mutually exclusive) Z-worlds are possible. For instance, if Z represents a set of possibly measurable variables, associated to a set of sensors (one for each $z \in Z$), it can be the case that some measurements fail, i.e., the value of z is not always available.

The following notion, *strong conditional independence*, strengthens Darwiche and Pearl's conditional independence by taking into account the case in which the information about Z is *any conjunctive information*, i.e., any term of $PROP_Z$. Namely, X and Y are strongly independent given Z with respect to Σ if and only if, *whichever conjunctive information* (*i.e., a set of facts*) we may learn about Z, then the addition of information about Y does not enable one to tell anything new about X.

Definition 7 (*strong conditional independence*). Let Σ be a propositional formula and X, Y, Z be disjoint subsets of PS. X and Y are *strongly independent* given Z with respect to Σ (denoted $X \approx_\Sigma^Z Y$) if and only if, for every term γ_Z of $PROP_Z$, $\forall \omega_X \in \Omega_X$, $\forall \omega_Y \in \Omega_Y$, the consistency of both $\omega_X \wedge \gamma_Z \wedge \Sigma$ and $\omega_Y \wedge \gamma_Z \wedge \Sigma$ implies the consistency of $\omega_X \wedge \omega_Y \wedge \gamma_Z \wedge \Sigma$.

Strong conditional independence has the same metatheoretic properties as conditional independence, plus the preservation by contraction of Z (which is a trivial consequence of the definition).

Marginal strong conditional independence obviously coincides with marginal conditional independence. *Ceteris paribus* strong conditional independence is defined by imposing $Z = Var(\Sigma) \setminus (X \cup Y)$ and denoted by $X \approx_\Sigma^{\textit{ceteris paribus}} Y$.

Since the set of all possible choices for γ_Z corresponds to the set of all partial assignments of the variables of Z, we get:

Proposition 3. *$X \approx_\Sigma^Z Y$ if and only if $X \sim_\Sigma^{Z'} Y$ for every $Z' \subseteq Z$.*

Proof. Comes straightforwardly from the fact that each term γ_Z can be uniquely identified with a Z'-world $\omega_{Z'}$ for some $Z' \subseteq Z$ and conversely. □

Obviously, $X \approx_\Sigma^Z Y$ entails $X \sim_\Sigma^Z Y$. The converse generally does not hold since conditional independence is not stable by contraction of Z. Indeed, stepping back to the previous example, we have $c \sim_\Sigma^{\{a,b\}} d$ but $c \not\approx_\Sigma^{\{a,b\}} d$ since $c \sim_\Sigma^{\{a\}} d$ does not hold.

The following results characterize strongly conditionally independent sets of variables. They both express that $X \approx_\Sigma^Z Y$ holds if and only if any set of simple facts (i.e., literals) we may learn about Z never enables one to deduce a nontrivial *disjunctive information* involving both X and Y.

Proposition 4 (consequence decomposability). $X \approx^Z_\Sigma Y$ *if and only if, for any term* γ_Z *of* $PROP_Z$*, and* $\forall \varphi_X \in PROP_X$, $\forall \varphi_Y \in PROP_Y$, $\gamma_Z \wedge \Sigma \models \varphi_X \vee \varphi_Y$ *implies* $\gamma_Z \wedge \Sigma \models \varphi_X$ *or* $\gamma_Z \wedge \Sigma \models \varphi_Y$.

Proof. ($\Rightarrow$) Assume that $X \approx^Z_\Sigma Y$ and let γ_Z be a term of $PROP_Z$, $\varphi_X \in PROP_X$ and $\varphi_Y \in PROP_Y$.

γ_Z can be identified with a unique Z'-world $\omega_{Z'}$ for a unique subset Z' of Z. We now have to prove that $\omega_{Z'} \wedge \Sigma \models \varphi_X \vee \varphi_Y$ implies $\omega_{Z'} \wedge \Sigma \models \varphi_X$ or $\omega_{Z'} \wedge \Sigma \models \varphi_Y$.

So, assume that $\omega_{Z'} \wedge \Sigma \not\models \varphi_X$ and $\omega_{Z'} \wedge \Sigma \not\models \varphi_Y$. Since $\omega_{Z'} \wedge \Sigma \wedge \neg\varphi_X$ and $\omega_{Z'} \wedge \Sigma \wedge \neg\varphi_Y$ are both consistent, there exist two extensions ω and ω' of $\omega_{Z'}$ s.t. $\omega \models \omega_{Z'} \wedge \Sigma \wedge \neg\varphi_X$ and $\omega' \models \omega_{Z'} \wedge \Sigma \wedge \neg\varphi_Y$. Let ω_X be the restriction of ω to X and ω'_Y the restriction of ω' to Y. Then $\omega_X \models \neg\varphi_X$, $\omega'_Y \models \neg\varphi_Y$. Now, $\omega \models \omega_{Z'} \wedge \Sigma \wedge \neg\varphi_X$ implies that $\omega_X \wedge \omega_{Z'} \wedge \Sigma$ is consistent; similarly, $\omega_X \wedge \omega'_Y \wedge \Sigma$ is consistent. These two facts, together with the assumption that $X \approx^Z_\Sigma Y$, entail that $\omega_X \wedge \omega'_Y \wedge \omega_{Z'} \wedge \Sigma$ is consistent, and thus, since $\omega_X \models \neg\varphi_X$ and $\omega'_Y \models \neg\varphi_Y$, we get $\omega_X \wedge \neg\varphi_X \wedge \neg\varphi_Y \wedge \Sigma$ consistent, i.e., $\omega_X \wedge \Sigma \not\models \varphi_X \vee \varphi_Y$.

($\Leftarrow$) Assume that, for any term γ_Z of $PROP_Z$, $\forall \varphi_X \in PROP_X$, $\forall \varphi_Y \in PROP_Y$: $\gamma_Z \wedge \Sigma \models \varphi_X \vee \varphi_Y$ implies $\gamma_Z \wedge \Sigma \models \varphi_X$ or $\gamma_Z \wedge \Sigma \models \varphi_Y$, and let ω_X, ω_Y and ω_Z s.t. $\omega_X \wedge \omega_Z \wedge \Sigma$ is consistent and $\omega_Y \wedge \omega_Z \wedge \Sigma$ is consistent. Let $\gamma_Z = \mathit{for}(\omega_Z)$, $\varphi_X = \neg\mathit{for}(\omega_X)$ and $\varphi_Y = \neg\mathit{for}(\omega_Y)$. The consistency of $\omega_X \wedge \omega_Z \wedge \Sigma$ implies that $\gamma_Z \wedge \Sigma \not\models \varphi_X$; similarly, $\gamma_Z \wedge \Sigma \not\models \varphi_Y$. Together with the initial assumption, this implies that $\gamma_Z \wedge \Sigma \not\models \varphi_X \vee \varphi_Y$, hence the consistency of $\omega_X \wedge \omega_Y \wedge \omega_Z \wedge \Sigma$. □

The following property expresses strong conditional independence in terms of prime implicates. Indeed, if a formula Σ is expressed as its set of prime implicates, checking strong conditional independence with respect to Σ can be done by checking whether there are clauses that contain both variables from X and from Y.

Proposition 5. $X \approx^Z_\Sigma Y$ *if and only if no* $\delta \in IP^{X\cup Y\cup Z}(\Sigma)$ *includes both a variable of* X *and a variable of* Y.

Proof. ($\Rightarrow$) Suppose that $X \approx^Z_\Sigma Y$ and let $\delta \in IP^{X\cup Y\cup Z}(\Sigma)$. Let us note $\delta \equiv \delta_X \vee \delta_Y \vee \delta_Z$, where δ_X (respectively δ_Y, δ_Z) is a clause of $PROP_X$ (respectively of $PROP_Y$, $PROP_Z$). Using Proposition 4, $\Sigma \wedge \neg\delta_Z \models \delta_X \vee \delta_Y$ implies that $\Sigma \wedge \neg\delta_Z \models \delta_X$ or $\Sigma \wedge \neg\delta_Z \models \delta_Y$, which is equivalent to $\Sigma \models \delta_X \vee \delta_Z$ or $\Sigma \models \delta_Y \vee \delta_Z$. Thus, if δ contains both a variable from X and a variable from Y, i.e., δ_X and δ_Y are not empty, then δ is not minimal among the clauses of $PROP_{X\cup Y\cup Z}$ entailed by Σ and thus it is not in $IP^{X\cup Y\cup Z}(\Sigma)$. Therefore, either δ_X or δ_Y is empty.

($\Leftarrow$) Suppose that $X \not\approx^Z_\Sigma Y$. Then, due to Proposition 4, there is a term γ_Z in $PROP_Z$, $\exists \varphi_X \in PROP_X$, $\exists \varphi_Y \in PROP_Y$ s.t. $\Sigma \wedge \gamma_Z \models \varphi_X \vee \varphi_Y$ and $\Sigma \wedge \gamma_Z \not\models \varphi_X$, $\Sigma \wedge \gamma_Z \not\models \varphi_Y$. Let δ_Z be a clause s.t. $\delta_Z \equiv \neg\gamma_Z$. Since $\Sigma \wedge \gamma_Z \not\models \varphi_X$, and because the propositional formula φ_X is equivalent to the conjunction of its prime implicates, there is a prime implicate δ_X of φ_X s.t. $\Sigma \wedge \gamma_Z \not\models \delta_X$, or equivalently $\Sigma \not\models \delta_Z \vee \delta_X$. Similarly, there is a prime implicate δ_Y of φ_Y s.t. $\Sigma \not\models \delta_Z \vee \delta_Y$. Now, since $\varphi_X \models \delta_X$ and $\varphi_Y \models \delta_Y$, we have $\varphi_X \vee \varphi_Y \models \delta_X \vee \delta_Y$ and therefore $\Sigma \wedge \gamma_Z \models \delta_X \vee \delta_Y$, or equivalently, $\Sigma \models \delta_X \vee \delta_Y \vee \delta_Z$. Consequently, there is a prime implicate δ of Σ s.t. δ is a subclause of $\delta_X \vee \delta_Y \vee \delta_Z$. If δ were a subclause of $\delta_X \vee \delta_Z$ it would be the case that $\Sigma \models \delta_X \vee \delta_Z$, which is not possible; and similarly for $\delta_Y \vee \delta_Z$. Thus δ contains at least a variable of X and a variable of Y. □

As a consequence of Proposition 5, strong conditional independence can be reduced to the problem of checking strong conditional independence in the case in which both X and Y are composed of a single variable.

Proposition 6. *$X \approx^Z_\Sigma Y$ if and only if $\forall x \in X\ \forall y \in Y$, $x \approx^{Z\cup(X\setminus\{x\})\cup(Y\setminus\{y\})}_\Sigma y$.*

Proof. ($\Rightarrow$) Assume that $X \approx^Z_\Sigma Y$, and let $x \in X$, $y \in Y$. Remarking that

$$X \cup Y \cup Z = \{x\} \cup \{y\} \cup \big(Z \cup \big(X \setminus \{x\}\big) \cup \big(Y \setminus \{y\}\big)\big) \qquad (*)$$

the characterization given by Proposition 5 can be rewritten this way: $X \approx^Z_\Sigma Y$ if and only if $\forall \delta \in IP^{\{x\}\cup\{y\}\cup(Z\cup(X\setminus\{x\})\cup(Y\setminus\{y\}))}(\Sigma)$, δ does not mention both x and y, which, using again Proposition 5, means that $x \approx^{Z\cup(X\setminus\{x\})\cup(Y\setminus\{y\})}_\Sigma y$.

($\Leftarrow$) If $X \not\approx^Z_\Sigma Y$ then there is a δ in $IP^{X\cup Y\cup Z}(\Sigma)$ mentioning both an $x_i \in X$ and a $y_j \in Y$; then, using again identity $(*)$, we get $\delta \in IP^{Z\cup(X\setminus\{x_i\})\cup(Y\setminus\{y_j\})}(\Sigma)$, which, using Proposition 5, implies $x_i \not\approx^{Z\cup(X\setminus\{x_i\})\cup(Y\setminus\{y_j\})}_\Sigma y_j$. □

This result is useful for the practical computation of strong conditional independence relations. Note that there is no similar result for (standard) conditional independence. Things become even simpler with *ceteris paribus* strong independence, since Proposition 6 becomes: $X \approx^{\textit{ceteris paribus}}_\Sigma Y$ if and only if $\forall x \in X\ \forall y \in Y$, $x \approx^{\textit{ceteris paribus}}_\Sigma y$ (cf. Lemma 15 in [19]).

According to Proposition 5, $x \not\approx^Z_\Sigma y$ holds if and only if there is a prime implicate δ in $IP^{Z\cup\{x,y\}}(\Sigma)$ mentioning both x and y. This is equivalent to saying that there is a prime *implicant* γ in $PI^{Z\cup\{x\}}(\Sigma \Rightarrow y)$ or in $PI^{Z\cup\{x\}}(\Sigma \Rightarrow \neg y)$, *consistent with* Σ and mentioning x. The consistency condition is necessary; indeed, let us consider $\Sigma = \{c \Rightarrow a, d \Rightarrow b\}$ and $Z = \{c, d\}$; $PI^{Z\cup\{a\}}(\Sigma \Rightarrow b) = \{c \wedge \neg a, d\}$ mentions a but nevertheless $a \approx^{\{c,d\}}_\Sigma b$ holds; this is because $c \wedge \neg a$ is not consistent with Σ or, in other words, $c \wedge \neg a$ is a prime implicant of $\Sigma \Rightarrow b$ only because it is a prime implicant of $\neg\Sigma$. Thus, the set of prime implicants of interest is $PI^{Z\cup\{x\}}(y)$ filtered by removing those containing a prime implicant of $\neg\Sigma$, which corresponds exactly to the set of minimal abductive explanations for y with respect to Σ, where the set of possible individual hypotheses is the set of literals built up from $Z \cup \{x\}$ [11]. Equivalently, this set is the *label* of y according to the ATMS literature [30]. This leads to the following characterization:

Proposition 7. *Let $PI^{Z\cup\{x\}}_\Sigma(\varphi)$ be the disjunction of all prime implicants γ in $PI^{Z\cup\{x\}}(\Sigma \Rightarrow \varphi)$ such that $\gamma \wedge \Sigma$ is consistent. Then $x \approx^Z_\Sigma y$ if and only if both $PI^{Z\cup\{x\}}_\Sigma(y)$ and $PI^{Z\cup\{x\}}_\Sigma(\neg y)$ are V-independent from x.*

We first prove the following lemma:

Lemma 1. *$x \not\approx^Z_\Sigma y$ if and only if $\exists\gamma \in PI^{Z\cup\{x\}}(\Sigma \Rightarrow y) \cup PI^{Z\cup\{x\}}(\Sigma \Rightarrow \neg y)$ s.t. γ mentions x and $\gamma \wedge \Sigma$ is consistent.*

Proof. ($\Rightarrow$) Assume that $x \not\approx^Z_\Sigma y$; from Proposition 5, we know that there is a prime implicate $\delta \in IP^{Z\cup\{x,y\}}(\Sigma)$ mentioning x and y. Without loss of generality, let $\delta \equiv x \vee y \vee \delta_Z$ where $\delta_Z \in PROP_Z$. Let $\gamma \equiv \neg x \wedge \neg\delta_Z$.

- if $\gamma \wedge \Sigma$ were inconsistent, then $\Sigma \wedge \neg x \wedge \neg\delta_Z$ would be inconsistent, i.e., $\Sigma \models \delta_Z \vee x$; thus, $\delta_Z \vee x \vee y$ would not be a prime implicate of Σ (because it would not be minimal). Thus, $\gamma \wedge \Sigma$ is consistent;
- γ mentions x;
- $\gamma \wedge \Sigma \models y$ (because $\Sigma \models x \vee y \vee \delta_Z$, i.e., $\neg x \wedge \neg\delta_Z \wedge \Sigma \models y$);
- if there were a $\gamma' \models \gamma$ s.t. $\gamma \not\models \gamma'$ and $\gamma' \wedge \Sigma \models y$, then we would have $\Sigma \models y \vee \neg\gamma'$ with $y \vee \neg\gamma' \models \delta$ and $\delta \not\models y \vee \neg\gamma'$ thus δ would not be in $IP^{Z\cup\{x,y\}}(\Sigma)$. Therefore, $\gamma \in PI^{Z\cup\{x\}}(\Sigma \Rightarrow y)$.

($\Leftarrow$) Assume, without loss of generality, that $\exists\gamma \in PI^{Z\cup\{x\}}(\Sigma \Rightarrow y)$ s.t. γ mentions x and y and $\gamma \wedge \Sigma$ is consistent; again without loss of generality, assume that γ has the form $x \wedge \gamma_Z$; let $\delta \equiv \gamma_Z \vee \neg x \vee y$.

- $\Sigma \models \delta$, because $\Sigma \wedge \gamma \models y$, i.e., $\Sigma \wedge x \wedge \gamma_Z \models y$, i.e., $\Sigma \models \neg x \vee \neg\gamma_Z \vee y$;
- if there were a $\delta' \models \delta$ s.t. $\delta \not\models \delta'$ and $\Sigma \models \delta'$ then γ would not be minimal, thus $\delta \in IP^{Z\cup\{x,y\}}(\Sigma)$;
- δ mentions x and y. □

Proof. Since $\gamma \wedge \Sigma$ is inconsistent if and only if $\gamma \models \neg\Sigma$ and thus if and only if $\exists\gamma' \subseteq \gamma$ s.t. $\gamma' \in PI(\neg\Sigma)$ we have that $x \approx^Z_\Sigma y$ if and only if neither $PI^{Z\cup\{x\}}_\Sigma(y)$ nor $PI^{Z\cup\{x\}}_\Sigma(\neg y)$ mention x or, equivalently, they are V-independent from x. □

In other words, $x \approx^Z_\Sigma y$ if and only if both x and $\neg x$ are irrelevant hypotheses for (minimally) explaining y and $\neg y$, i.e., neither x nor $\neg x$ participates in any minimal explanation of y and neither x nor $\neg x$ participates in any minimal explanation of $\neg y$ [11].

This gives us an algorithm for computing strong independence relations using a basic ATMS (or an algorithm for computing abductive explanations). Let

$$SIV^Z_\Sigma(y) = \left\{x \in Var(\Sigma) \setminus \bigl(Z \cup \{y\}\bigr) \mid x \approx^Z_\Sigma y\right\}.$$

A set of variables S is initialized to $Var(\Sigma) \setminus (Z \cup \{y\})$, and each time a new consistent environment of y (i.e., one of the disjuncts of $PI^{Z\cup\{x\}}_\Sigma(y)$) or of $\neg y$ is computed, then all variables appearing in it are removed from S. At any step, S contains $SIV^Z_\Sigma(y)$ and the algorithm reaches $SIV^Z_\Sigma(y)$ when it ends up (this shows a possible "anytime" use of this algorithm).

Another interesting feature of strong conditional independence is that it satisfies *all* graphoid axioms (including intersection, unlike simple conditional independence):

Proposition 8. *$\approx_\Sigma$ satisfies all graphoid axioms.*

Proof.

- **(S)** Obvious.
- **(D)** Obvious.
- **(WU)** Assume that $X \approx_{\Sigma}^{Z} Y \cup W$ and let $\delta \in IP(\Sigma)$ such that $Var(\delta) \subseteq X \cup Y \cup Z \cup W$, if any (note that if no such δ exists, the claim trivially holds, thanks to Proposition 5). Let us write $\delta \equiv \delta_X \vee \delta_Y \vee \delta_Z \vee \delta_W$. $X \approx_{\Sigma}^{Z} Y \cup W$ entails that either $\delta_X \equiv \mathit{false}$ or $(\delta_Y \equiv \delta_W \equiv \mathit{false})$ by Proposition 5, which implies $\delta_X \equiv \mathit{false}$ or $\delta_Y \equiv \mathit{false}$. hence there cannot exist a prime implicate of Σ over $X \cup Y \cup Z \cup W$ mentioning both a variable from X and a variable from Y, which (by Proposition 5) means that $X \approx_{\Sigma}^{Z \cup W} Y$.
- **(C)** Assume that $X \approx_{\Sigma}^{Y \cup Z} W$ and $X \approx_{\Sigma}^{Z} Y$, and let $\delta \in IP(\Sigma)$ such that

  $$Var(\delta) \subseteq X \cup Y \cup Z \cup W.$$

  Let us write $\delta \equiv \delta_X \vee \delta_Y \vee \delta_Z \vee \delta_W$.
  $X \approx_{\Sigma}^{Y \cup Z} W$ entails that $\delta_X \equiv \mathit{false}$ or $\delta_Y \equiv \mathit{false}$ (1).
  $X \approx_{\Sigma}^{Z} Y$ entails that $\delta_X \equiv \mathit{false}$ or $\delta_Y \equiv \mathit{false}$ or $\delta_W \not\equiv \mathit{false}$ (2).
  (1) and (2) together imply $\delta_X \equiv \mathit{false}$ or $(\delta_Y \equiv \delta_W \equiv \mathit{false})$ (3).
  Therefore, $X \approx_{\Sigma}^{Z} Y \cup W$ holds.
- **(I)** Assume that $X \approx_{\Sigma}^{Z \cup W} Y$ and $X \approx_{\Sigma}^{Z \cup Y} W$ hold and, again, let $\delta \in IP(\Sigma)$ such that $\delta \equiv \delta_X \vee \delta_Y \vee \delta_Z \vee \delta_W$.
  $X \approx_{\Sigma}^{Z \cup W} Y$ entails that $\delta_X \equiv \mathit{false}$ or $\delta_Y \equiv \mathit{false}$ (1);
  $X \approx_{\Sigma}^{Z \cup Y} W$ entails that $\delta_X \equiv \mathit{false}$ or $\delta_W \equiv \mathit{false}$ (2);
  (1) and (2) imply $\delta_X \equiv \mathit{false}$ or $(\delta_Y \equiv \delta_W \equiv \mathit{false})$ which means that $X \approx_{\Sigma}^{Z} Y \cup W$ holds. □

This confirms the particular interest of strong conditional independence, which not only can be nicely characterized by means of prime implicates (in contrast to simple conditional independence), but also satisfies all graphoid axioms. Furthermore, in Section 5 we show that strong conditional independence is closely related to other notions such as relevance or novelty.

We can define a last notion of conditional independence, stronger than the two previous ones, that we call *perfect conditional independence*. While the definition of strong independence takes into account information over the variables Z that is represented as terms (conjunction of literals), here we remove this assumption, and consider the case in which any information may be available, that is, any possible propositional formula. Namely, X and Y are perfectly independent given Z with respect to Σ if and only if *whichever information*, i.e., any formula, we may learn about Z, then the addition of information about Y does not enable one to tell anything new about X. This intuitively means that no significant relationship between X and Y can be inferred when learning *any* information, including disjunctive information, about Z.

As an illustration, let $Z = \{n(orth), s(outh), e(ast), w(est)\}$ and $Z' = \{ne, nw, se, sw\}$ where Σ contains $s \Leftrightarrow (se \vee sw)$, $e \Leftrightarrow (ne \vee se)$, etc. and mutual exclusivity statements between *ne*, *nw*, *se* and *sw* (such as $sw \Rightarrow \neg se$, etc.). Z and Z' define each other, because Σ entails $ne \Leftrightarrow (n \wedge e)$, etc. Let us now add to Σ the two formulas $se \Rightarrow rain$ and $sw \Rightarrow wind$, which imply $s \Rightarrow (rain \vee wind)$. Then *rain* and *wind* are strongly independent given Z' with respect to Σ while they are not given Z. In both cases, perfect independence between *rain* and *wind* does not hold. This is because we may later discover that the variables *se* and *sw* can be redefined in terms of the variables s, e, and w: in this new representation, there is a clear link between *wind* and *rain*.

This example shows that the lack of perfect independence between X and Y corresponds intuitively to a *potential dependence* given the topic corresponding to Z.

However, perfect conditional independence is an extremely strong notion and is more of theoretical interest than of practical use, therefore we do not devote much space to it. It can be shown that perfect independence is less sensitive to the granularity of the representation than the two weaker forms of independence, and that it satisfies all graphoid axioms except (WU). The interested reader may read a longer version of our article, accessible by anonymous ftp at `ftp://ftp.irit.fr/pub/IRIT/RPDMP/CIPL.ps.gz`. This long version not only contains a detailed study of perfect independence, but also a study of how conditional independence extends when we relax the assumption that the sets of variables X, Y and Z are disjoint.

4. Complexity results

We investigate now computational complexity issues. We start by analyzing in depth the complexity of simple conditional independence. We consider a number of restrictions on X, Y, Z and Σ which may lower the complexity level, namely: $|X| = 1$ and/or $|Y| = 1$ (checking whether a variable is independent from a variable / a set of variables), $X \cup Y = Var(\Sigma)$ (*twofold partition independence*), $Z = \emptyset$ (*marginal independence*) and $Z = Var(\Sigma) \setminus (X \cup Y)$ (*ceteris paribus independence*). Note that, for twofold partition independence, the distinctions on Z are irrelevant; therefore, all three problems of the last row of Table 1 are identical.

4.1. Simple conditional independence

Proposition 9 (complexity of conditional independence). *The results are synthesized in Table* 1.

Table 1
Complexity of conditional independence

$X \sim^{Z}_{\Sigma} Y$	any Z	$Z = \emptyset$ (marginal independence)	$Z = Var(\Sigma) \setminus (X \cup Y)$ (ceteris paribus)
any X, Y	Π^p_2-complete	Π^p_2-complete	coNP-complete
$X = \{x\}$ or $Y = \{y\}$	Π^p_2-complete	Π^p_2-complete	coNP-complete
$X = \{x\}$ and $Y = \{y\}$	Π^p_2-complete	coBH$_2$-complete	coNP-complete
$X \cup Y = Var(\Sigma)$ (twofold partition)	coNP-complete	coNP-complete	coNP-complete

The numerous results contained in Proposition 9 are proved in the following order, which tries to minimize the number of proofs:

1. CONDITIONAL INDEPENDENCE is in Π_2^p;
2. MARGINAL INDEPENDENCE OF A VARIABLE FROM A SET OF VARIABLES is Π_2^p-hard;
3. CONDITIONAL INDEPENDENCE OF SINGLE VARIABLES is Π_2^p-hard;
4. CETERIS PARIBUS INDEPENDENCE is in coNP;
5. TWOFOLD PARTITION INDEPENDENCE is coNP-hard;
6. CETERIS PARIBUS INDEPENDENCE OF SINGLE VARIABLES is coNP-hard;
7. MARGINAL VARIABLE INDEPENDENCE is coBH_2-complete.

Lemma 2. CONDITIONAL INDEPENDENCE *is in* Π_2^p.

Proof. The following nondeterministic algorithm with NP-oracles proves membership of $\overline{\text{CONDITIONAL INDEPENDENCE}}$ to Σ_2^p:

(1) guess $\omega_X, \omega_Y, \omega_Z$;
(2) check that $\omega_X \wedge \omega_Z \wedge \Sigma$ is consistent;
(3) check that $\omega_Y \wedge \omega_Z \wedge \Sigma$ is consistent;
(4) check that $\omega_X \wedge \omega_Y \wedge \omega_Z \wedge \Sigma$ is inconsistent.

Hence, CONDITIONAL INDEPENDENCE belongs to Π_2^p. □

Lemma 3. MARGINAL INDEPENDENCE OF A VARIABLE FROM A SET OF VARIABLES (i.e., *checking that* $X \sim_\Sigma^\emptyset y$ *holds*) *is* Π_2^p*-hard.*

Proof. We abbreviate this decision problem by MIVSV. The proof is done by exhibiting a polynomial reduction from 2-$\overline{\text{QBF}}$ to $\overline{\text{MIVSV}}$.

Let $I = \langle\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_p\}, \Phi\rangle$ be a triple s.t. the a_i's and b_j's are propositional variables and Φ is a propositional formula from the language generated by the a_i's and b_j's. I is a positive instance of 2-$\overline{\text{QBF}}$ if and only if $\forall\omega_A\exists\omega_B$ s.t. $(\omega_A, \omega_B) \models \Phi$, or equivalently, if and only if $\forall\omega_A$ $(\omega_A \wedge \Phi)$ is satisfiable.

Now, let us define the mapping F by $F(I) = \langle X, Y, \Sigma\rangle$ where

$$X = \{a_1, \ldots, a_n, x'\},$$
$$Y = \{c\},$$
$$\Sigma = c \Rightarrow (x' \vee \Phi)$$

in which c and x' are new variables appearing nowhere else.

F is obviously a polynomial transformation. In order to prove that it reduces 2-$\overline{\text{QBF}}$ to MIVSV, we first note that $\forall\omega_X \in \Omega_X$, $\omega_X \wedge \Sigma$ is satisfiable (because assigning c to false satisfies Σ whatever the rest of the assignment) and that $\forall\omega_Y \in \Omega_Y$, $\omega_Y \wedge \Sigma$ is satisfiable; indeed, if ω_Y assigns c to true, then $c \wedge \Sigma$ is satisfiable because $x' \vee \Phi$ is satisfiable assigning x' to true; and if ω_Y assigns c to false, then $\omega_Y \wedge \Sigma$ is satisfied.

Let us now show that I is a positive instance of 2-$\overline{\text{QBF}}$ if and only if $F(I)$ is a positive instance of MIVSV, i.e., X and Y are independent with respect to Σ.

(1) Assume that I is a positive instance of 2-$\overline{\text{QBF}}$. It remains to be checked that $\forall\omega_X, \forall\omega_Y$, we have $\omega_X \wedge \omega_Y \wedge \Sigma$ is satisfiable. If ω_Y assigns c to false, $\omega_X \wedge \omega_Y \wedge \Sigma$ is equivalent to $\omega_X \wedge \neg c$ and is satisfiable. If ω_Y assigns c to true, $\omega_X \wedge \omega_Y \wedge \Sigma$ is equivalent to $\omega_X \wedge c \wedge (x' \vee \Phi)$ and is satisfiable (because I is a positive instance of 2-$\overline{\text{QBF}}$).
(2) Assume that I is not a positive instance of 2-$\overline{\text{QBF}}$. Then there is a ω_X s.t. $(x' \vee \Phi) \wedge \omega_X$ is unsatisfiable (ω_X is obtained from the assignment over $a_1, \ldots, a_n$ adding $x' = \mathit{false}$) and therefore s.t. $\omega_X \wedge \Sigma \models \neg c$; hence X and $Y = \{c\}$ are not marginally independent. $\square$

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Lemmas 2 and 3 together enable us to prove the Π_2^p-completeness of the four problems located at the left-up corner of Table 1.

We let V stands for $\mathit{Var}(\Sigma)$ in the following lemmata.

Lemma 4. CONDITIONAL INDEPENDENCE OF SINGLE VARIABLES (i.e., *checking that* $x \sim_\Sigma^Z y$ *holds*) *is* Π_2^p-*hard.*

Proof. We abbreviate this decision problem by CIV. Let us exhibit a polynomial reduction from 2-$\overline{\text{QBF}}$ to $\overline{\text{CIV}}$. Let G be the following reduction: If $I = \langle\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_p\}, \Phi\rangle$ then $G(I) = \langle V, x, y, Z, \Sigma\rangle$ where

- $Z = \{a_1, \ldots, a_n\}$;
- $\Sigma = \Phi \vee (x \Leftrightarrow y)$.

G is obviously a polynomial transformation. Let us now show that I is a positive instance of 2-$\overline{\text{QBF}}$ if and only if $G(I)$ is a positive instance of CIV.

(1) Assume that I is a positive instance of 2-$\overline{\text{QBF}}$. Then $\forall\omega_Z, \omega_Z \wedge \Phi$ is satisfiable, hence, $\omega_Z \wedge x \wedge y \wedge \Sigma$, $\omega_Z \wedge x \wedge \neg y \wedge \Sigma$, $\omega_Z \wedge \neg x \wedge y \wedge \Sigma$ and $\omega_Z \wedge \neg x \wedge \neg y \wedge \Sigma$ are all satisfiable (since x and y do not appear in Φ nor in Z). This is sufficient to conclude that $x \sim_\Sigma^Z y$.
(2) Assume that I is not a positive instance of 2-$\overline{\text{QBF}}$. Then there exists a ω_Z s.t. $\omega_Z \wedge \Phi$ is unsatisfiable. For this ω_Z we have thus $\omega_Z \wedge \Sigma \models x \Leftrightarrow y$ and hence $x \not\sim_\Sigma^Z y$ (take, for example, $\omega_x = x$ and $\omega_y = \neg y$). $\square$

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Together with the previous lemmata, we have now proven all Π_2^p-completeness results of Table 1.

Lemma 5. CETERIS PARIBUS INDEPENDENCE *is in* coNP.

Proof. Let us abbreviate this problem by CPI. Let $\langle\Sigma, V, X, Y\rangle$ be a positive instance of CPI. We show that the complementary problem $\overline{\text{CPI}}$ belongs to NP using the following nondeterministic algorithm:

(1) guess $\omega_X, \omega_Y, \omega_{V\setminus(X\cup Y)}$;
(2) check that $\omega_X \wedge \omega_{V\setminus(X\cup Y)} \wedge \Sigma$ is satisfiable;
(3) check that $\omega_Y \wedge \omega_{V\setminus(X\cup Y)} \wedge \Sigma$ is satisfiable;
(4) check that $\omega_X \wedge \omega_Y \wedge \omega_{V\setminus(X\cup Y)} \wedge \Sigma$ is unsatisfiable.

Hence $\overline{\text{CPI}}$ is in NP and therefore CPI is in coNP. □

We turn now into the problem of twofold partition independence, which consists in checking whether $X \sim_\Sigma^\emptyset Y$ holds, where $X \cup Y = V$. Note that when $X \cup Y = V$ (i.e., the fourth line of Table 1), we know that $Z = \emptyset$ so that the distinctions on Z (the columns) are irrelevant. This comes down to saying that twofold partition independence is both a subproblem of marginal independence and of *ceteris paribus* independence.

Lemma 6. TWOFOLD PARTITION INDEPENDENCE *is* coNP*-hard.*

Proof. We consider the following polynomial reduction H: if φ is a propositional formula then $H(\varphi) = \langle X, V, \Sigma\rangle$ where

$$X = Var(\varphi) \cup \{x'\},$$
$$\Sigma = (x' \wedge \varphi) \vee v.$$

H is a polynomial reduction. Now, it is easy to see that $X \sim_\Sigma^\emptyset \{v\}$ if and only if φ is unsatisfiable. Hence H is a polynomial reduction from UNSAT to TWOFOLD PARTITION INDEPENDENCE. □

Now we prove the coNP-hardness in the case $Z = V\setminus(X \cup Y)$, when both X and Y are singletons.

Lemma 7. CETERIBUS PARIBUS INDEPENDENCE OF SINGLE VARIABLES *is* coNP*-hard.*

Proof. Let φ be a formula. We prove that φ is unsatisfiable if and only if X and Y are *ceteris paribus* independent with respect to Σ, where

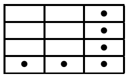
$$X = \{x\},$$
$$Y = \{y\}, \quad \text{s.t. } x \text{ and } y \text{ do not appear in } \varphi,$$
$$\Sigma = \varphi \wedge (x \Leftrightarrow y).$$

Then it can be easily verified that x and y are *ceteris paribus* independent with respect to Σ if and only if φ is unsatisfiable:

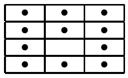
(1) Assume φ satisfiable. Let $\omega_{Var(\varphi)}$ be a model of φ. Let ω_X be the X-world that maps x into true, and ω_Y be the Y-world that maps y into false. Then $\omega_X \wedge \omega_{Var(\varphi)} \wedge \Sigma$ and $\omega_Y \wedge \omega_{Var(\varphi)} \wedge \Sigma$ are both satisfiable while $\omega_X \wedge \omega_Y \wedge \omega_{Var(\varphi)} \wedge \Sigma$ is not, hence X and Y are not independent given $Var(\varphi)$, i.e., they are not *ceteris paribus* independent with respect to Σ.

(2) Assume φ unsatisfiable. Then Σ is unsatisfiable as well, and both $\omega_X \wedge \omega_{Var(\varphi)} \wedge \Sigma$ and $\omega_y \wedge \omega_{Var(\varphi)} \wedge \Sigma$ are unsatisfiable whatever $\omega_{Var(\varphi)}$ is, hence X and Y are *ceteris paribus* independent with respect to Σ. $\square$

Lemmas 5, 6, and 7 prove all coNP-completeness results concerning conditional independence.



As a result, only one result of Table 1 is left to be proven, namely marginal independence of single variables.



Lemma 8. MARGINAL VARIABLE INDEPENDENCE *is* coBH_2*-complete.*

Proof. Membership comes from the fact that $x \sim_{\Sigma}^{\emptyset} y$ if and only if (i) $x \wedge \Sigma$ satisfiable and $y \wedge \Sigma$ satisfiable imply that $x \wedge y \wedge \Sigma$ satisfiable; (ii) idem with $\neg x$ instead of x; (iii) idem with $\neg y$; (iv) idem with $\neg x$ and $\neg y$. Now, for instance, (i) does not hold if and only if $x \wedge \Sigma$ and $y \wedge \Sigma$ are both satisfiable and $x \wedge y \wedge \Sigma$ is not satisfiable, which proves that (i) considered as an individual problem—and also (ii) to (iv)—is in coBH_2.

As to hardness, let us exhibit a polynomial reduction from SAT-OR-UNSAT to MARGINAL VARIABLE INDEPENDENCE. We define $J(\langle\varphi, \psi\rangle) = \langle x, y, \Sigma\rangle$ where:

- $\Sigma = (x \vee y \Rightarrow \mathit{rename}(\psi)) \wedge (x \wedge y \Rightarrow \varphi)$, where $\mathit{rename}(\psi)$ is obtained from ψ by renaming all variables appearing in ψ—thus φ and $\mathit{rename}(\psi)$ do not share any variables. Obviously, ψ is unsatisfiable if and only if $\mathit{rename}(\psi)$ is.
- x and y are new variables which do not appear in φ and in $\mathit{rename}(\psi)$.

Now, $x \not\sim_{\Sigma}^{\emptyset} y$ if and only if at least one of the four statements (i) to (iv) above does not hold. We get easily that (i) does not hold if and only if $x \wedge \Sigma$ is satisfiable, $y \wedge \Sigma$ is satisfiable and $x \wedge y \wedge \Sigma$ is unsatisfiable, i.e., if and only if $\mathit{rename}(\psi)$ is satisfiable and $\varphi \wedge \mathit{rename}(\psi)$ is unsatisfiable, which together with the fact that φ and $\mathit{rename}(\psi)$ do not share variables, is equivalent to $\mathit{rename}(\psi)$ satisfiable and φ unsatisfiable, i.e., ψ satisfiable and φ unsatisfiable. Then, it is easy to check that (ii), (iii) and (iv) cannot be violated. Thus, $x \not\sim_{\Sigma}^{\emptyset} y$ if and only if ψ is satisfiable and φ is unsatisfiable, or equivalently, $x \sim_{\Sigma}^{\emptyset} y$ if and only if $\langle\varphi, \psi\rangle$ is a positive instance of SAT-OR-UNSAT. $\square$

4.2. Strong conditional independence

We now turn to the corresponding results concerning *strong* conditional independence. Note that the case $Z = \emptyset$ is useless to study because when $Z = \emptyset$, strong and (simple) conditional independence coincide. A fortiori, the case $X \cup Y = V$, which entails $Z = \emptyset$, is useless as well.

Proposition 10. *The complexity results of strong independence are reported in Table* 2.

Proof. It is sufficient to prove the two following lemmata:

Table 2
Complexity of strong conditional independence

$X \approx_{\Sigma}^{Z} Y$	any Z	$Z = Var(\Sigma) \setminus (X \cup Y)$
any X,Y	Π_2^p-complete	Π_2^p-complete
$X = \{x\}$ or $Y = \{y\}$	Π_2^p-complete	Π_2^p-complete
$X = \{x\}$ and $Y = \{y\}$	Π_2^p-complete	Π_2^p-complete

1. STRONG CONDITIONAL INDEPENDENCE is in Π_2^p;
2. CETERIS PARIBUS CONDITIONAL INDEPENDENCE OF SINGLE VARIABLES is Π_2^p-hard.

Lemma 9. STRONG CONDITIONAL INDEPENDENCE *is in* Π_2^p.

Proof. Membership of the complementary problem to Σ_2^p is shown by the following nondeterministic algorithm using an NP-oracle:

1. guess $Z' \subseteq Z$, $\omega_{Z'} \in \Omega_{Z'}$, $\omega_X \in \Omega_X$ and $\omega_Y \in \Omega_Y$.
2. check that $\omega_X \wedge \omega_{Z'} \wedge \Sigma$ is satisfiable, that $\omega_Y \wedge \omega_{Z'} \wedge \Sigma$ is satisfiable and that $\omega_X \wedge \omega_Y \wedge \omega_{Z'} \wedge \Sigma$ is unsatisfiable. □

Note that Π_2^p-hardness of this case (that we do not actually have to prove since the following lemma will imply it) is a corollary of Π_2^p-hardness of MARGINAL INDEPENDENCE which is a subproblem of STRONG CONDITIONAL INDEPENDENCE (recovered when $Z = \emptyset$). Moreover, because of Proposition 6, STRONG CONDITIONAL INDEPENDENCE remains Π_2^p-complete when X or Y is a singleton and when both are singletons (these results being subsumed as well by the next lemma).

Lemma 10. CETERIS PARIBUS STRONG CONDITIONAL INDEPENDENCE OF SINGLE VARIABLES *is* Π_2^p*-hard.*

Proof. We exhibit a polynomial reduction from 2-$\overline{\text{QBF}}$ to CETERIS PARIBUS STRONG CONDITIONAL INDEPENDENCE OF SINGLE VARIABLES. Let Φ be a propositional formula over the alphabet $\{a_1, \ldots, a_n, b_1, \ldots, b_p\}$; let $K(\langle\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_p\}, \Phi\rangle) = \langle\Sigma, X, Y\rangle$ where

- $\Sigma = (x \wedge b_1 \wedge \cdots \wedge b_p) \vee (\neg x \wedge \neg b_1 \wedge \cdots \wedge \neg b_p) \vee \Phi$;
- $X = \{x\}$;
- $Y = \{y\}$.

Let $A = \{a_1, \ldots a_n\}$, $B = \{b_1, \ldots, b_p\}$ and $Z = Var(\Sigma) \setminus (\{x, y\}) = A \cup B$. We note ω_x, ω_y instead of $\omega_{\{x\}}$, $\omega_{\{y\}}$. We use the notation $C(\omega_x, \omega_y, \gamma_Z)$ for [$\Sigma \wedge \omega_x \wedge \gamma_Z$ consistent and $\Sigma \wedge \omega_y \wedge \gamma_Z$ consistent implies $\Sigma \wedge \omega_x \wedge \omega_y \wedge \gamma_Z$ consistent]. Since $Z = A \cup B$, for any Z-term γ_Z we let $\gamma_Z = \gamma_A \wedge \gamma_B$. We now have to show that $x \approx_{\Sigma}^{Z} y$ if and only if $\forall a_1 \ldots \forall a_n \exists b_1 \ldots \exists b_p \Phi$ is valid. We start by studying in detail the cases in which the condition $C(\omega_x, \omega_y, \gamma_Z)$ holds.

Case 1: γ_B is not empty and contains only positive literals.

(i) $\Sigma \wedge \omega_x \wedge \gamma_Z$

$$\equiv (x \wedge b_1 \wedge \cdots \wedge b_p \wedge \omega_x \wedge \gamma_Z) \vee (\Phi \wedge \omega_x \wedge \gamma_Z)$$

is consistent if and only if $\omega_x = x$ or $\gamma_Z \wedge \Phi$ is consistent.

(ii) $\Sigma \wedge \omega_y \wedge \gamma_Z$

$$\equiv (x \wedge b_1 \wedge \cdots \wedge b_p \wedge \omega_y \wedge \gamma_Z) \vee (\Phi \wedge \omega_y \wedge \gamma_Z)$$

is always consistent because the first disjunct, equivalent to $x \wedge b_1 \wedge \cdots \wedge b_p \wedge \omega_y \wedge \gamma_A$, is always consistent.

(iii) $\Sigma \wedge \omega_x \wedge \omega_y \wedge \gamma_Z$

$$\equiv (x \wedge b_1 \wedge \cdots \wedge b_p \wedge \omega_x \wedge \omega_y \wedge \gamma_Z) \vee (\Phi \wedge \omega_x \wedge \omega_y \wedge \gamma_Z)$$

is consistent if and only if $\omega_x = x$ or $\gamma_Z \wedge \Phi$ is consistent.

Thus, $\Sigma \wedge \omega_x \wedge \omega_y \wedge \gamma_Z$ is consistent if and only if $\Sigma \wedge \omega_x \wedge \gamma_Z$ and $\Sigma \wedge \omega_y \wedge \gamma_Z$ both are, which entails that $C(\omega_X, \omega_Y, \gamma_Z)$ holds for any ω_X, ω_Y.

Case 2: γ_B is not empty and contains only negative literals.

This case is symmetrical to Case 1 and a similar proof enables us to show that $C(\omega_X, \omega_Y, \gamma_Z)$ holds for any ω_X, ω_Y.

Case 3: γ_B contains both positive and negative literals.

$\Sigma \wedge \omega_x \wedge \omega_y \wedge \gamma_Z$ is now equivalent to $\Phi \wedge \omega_x \wedge \omega_y \wedge \gamma_Z$ and is consistent if and only if $\Phi \wedge \gamma_Z$ is consistent, independently of ω_x, ω_y. Similarly, both $\Sigma \wedge \omega_x \wedge \gamma_Z$ and $\Sigma \wedge \omega_y \wedge \gamma_Z$ are consistent if and only if $\Phi \wedge \gamma_Z$ is consistent, which shows that $C(\omega_X, \omega_Y, \gamma_Z)$ holds for any ω_X, ω_Y.

Case 4: $\gamma_B = \emptyset$.

$\Sigma \wedge \omega_x \wedge \omega_y \wedge \gamma_Z$ is equivalent to

$$\big((x \wedge b_1 \wedge \cdots \wedge b_p) \vee (y \wedge \neg b_1 \wedge \cdots \wedge \neg b_p) \vee \Phi\big) \wedge \omega_x \wedge \omega_y \wedge \gamma_A,$$

i.e., to

$$(x \wedge b_1 \wedge \cdots \wedge b_p \wedge \omega_x \wedge \omega_y \wedge \gamma_A) \vee$$
$$(y \wedge \neg b_1 \wedge \cdots \wedge \neg b_p \wedge \omega_x \wedge \omega_y \wedge \gamma_A) \vee (\Phi \wedge \omega_x \wedge \omega_y \wedge \gamma_A),$$

and is consistent if and only if one of the disjuncts is consistent, i.e., at least one of these three conditions holds:

(i) $\omega_x = x$,
(ii) $\omega_y = y$,
(iii) $\Phi \wedge \omega_x \wedge \omega_y \wedge \gamma_A$ is consistent.

Condition (iii) is equivalent to the consistency of $\Phi \wedge \gamma_A$, because x and y do not appear in Φ. Now, $\Sigma \wedge \omega_x \wedge \gamma_A$ is consistent if and only if $\omega_x = x$ is consistent or $y \wedge \neg b_1 \wedge \cdots \wedge b_p \wedge \omega_x \wedge \gamma_A$ is consistent or $\Phi \wedge \gamma_A$ is consistent, which is always satisfied because

$y \wedge \neg b_1 \wedge \cdots \wedge b_p \wedge \omega_x \wedge \gamma_A$ is always consistent. Hence, $\Sigma \wedge \omega_x \wedge \gamma_A$ is always consistent; similarly, $\Sigma \wedge \omega_y \wedge \gamma_A$ is always consistent. This means that $C(\omega_X, \omega_Y, \gamma_A \wedge \gamma_B)$ holds if and only if $\omega_x = x$ or $\omega_y = y$ or $\Phi \wedge \gamma_A$ is consistent.

Finally,

$$\begin{aligned} x \approx^Z_\Sigma y \quad & \text{if and only if} \quad \forall\omega_x\forall\omega_y\forall\gamma_A\forall\gamma_B,\ C(\omega_x, \omega_y, \gamma_A \wedge \gamma_B) \text{ holds} \\ & \text{if and only if} \quad \forall\omega_x\forall\omega_y\forall\gamma_A\ C(\omega_x, \omega_y, \gamma_A) \text{ holds} \\ & \text{if and only if} \quad \forall\gamma_A\ C(\neg x, \neg y, \gamma_A) \text{ holds} \\ & \text{if and only if} \quad \forall\gamma_A,\ \Phi \wedge \gamma_A \text{ is consistent.} \end{aligned}$$

It is not hard to see that this is equivalent to $\forall\omega_A \in \Omega_A$, $\Phi \wedge \omega_A$ is consistent; indeed, for the ($\Rightarrow$) direction, an A-world ω_A is a special case of an A-term γ_A; for the ($\Leftarrow$) direction, the consistency of $\Phi \wedge \omega_A$ implies the consistency of $\Phi \wedge \gamma_A$ for any $\gamma_A \supseteq \omega_A$, and any A-term γ_A contains at least an A-world ω_A.

Therefore, we have

$$\begin{aligned} x \approx^Z_\Sigma y \quad & \text{if and only if} \quad \forall\omega_A \in \Omega_A,\ \Phi \wedge \omega_A \text{ is consistent} \\ & \text{if and only if} \quad \forall\omega_A \in \Omega_A\ \exists\omega_B \in \Omega_B \text{ s.t. } (\omega_A, \omega_B) \models \Phi \\ & \text{if and only if} \quad \Phi \in 2\text{-}\overline{\text{QBF}}. \qquad \square \end{aligned}$$

Let us now briefly comment on these results. The Π^p_2-completeness of STRONG CONDITIONAL INDEPENDENCE coheres with the Σ^p_2-completeness of checking whether an individual hypothesis is relevant (for minimal explanation) [11]. More interestingly, the abductive characterization (Proposition 15) of strong conditional independence enables us to take advantage of some restrictions (especially restricting Σ to a set of Horn clauses) for which the computational complexity of checking irrelevance for minimal explanation falls down to the first level of the polynomial hierarchy, carrying with it the complexity of strong conditional independence. Considering DNF formulas is another restriction that makes the complexity of STRONG CONDITIONAL INDEPENDENCE falling down to the first level of the polynomial hierarchy. To be more precise:

Proposition 11. *When Σ is in DNF,* STRONG CONDITIONAL INDEPENDENCE *is* coNP-*complete.*

Proof. From Proposition 6 it follows that it suffices to consider the case where both X and Y are singletons, i.e., $X = \{x\}$ and $Y = \{y\}$. Let us consider the complementary problem of checking whether x is not strongly conditionally independent from y given Z with respect to Σ and let us prove it NP-complete. As an easy consequence of Proposition 7, x is not strongly conditionally independent from y given Z with respect to Σ if and only if there exists a prime implicate of Σ built up from $Z \cup \{x, y\}$ that contains both x and y.

- *Membership*. Guess a clause δ and check (1) that it contains both x and y, (2) that it does not contain any variable outside $Z \cup \{x, y\}$, (3) that it contains a literal from each any consistent term from the given DNF of Σ, and (4) that any proper subclause of δ violates (3). Since (2), (3), (4) can be checked in time polynomial in the size of the input, this algorithm runs in nondeterministic polynomial time.

- *Hardness*. Let us consider the following reduction from NON-TAUT, the problem of checking whether a DNF Σ is not a tautology (it is obviously NP-complete since Σ is not a tautology if and only if the CNF $\neg\Sigma$ is satisfiable). Let $M(\Sigma) = \langle new_1, new_2, \neg\Sigma \wedge (new_1 \vee new_2)\rangle$ where new_1, new_2 are new variables (from $PS \setminus Var(\Sigma)$). $M(\Sigma)$ can easily be computed in time polynomial in $|\Sigma|$. Moreover, Σ is not a tautology if and only if new_1 is not *ceteris paribus* strongly independent from new_2 with respect to $\neg\Sigma \wedge (new_1 \vee new_2)$. □

We do not investigate in detail the complexity of perfect conditional independence due to the fact that this notion is only marginal. It can be proven that PERFECT CONDITIONAL INDEPENDENCE is Π_2^p-complete (the proof is in `ftp://ftp.irit.fr/pub/IRIT/RPDMP/CIPL.ps.gz`).

5. Independence, relevance, novelty, separability and non-interactivity

In this section, we show how conditional independence is related to many other forms of independence pointed out so far in the literature.

5.1. Formula-variable independence

As evoked before, conditional independence can be viewed as a generalization of formula-variable independence. Formally, we can reduce the problem of checking formula-variable independence to the problem of checking strong conditional independence.

Proposition 12. *Let new be a variable of $(PS \setminus Var(\Sigma)) \setminus X$. Then Σ is V-independent from X if and only if $X \approx_{\Sigma\Leftrightarrow new}^{Var(\Sigma)\setminus X}$ new.*

Proof. Let $Z = Var(\Sigma) \setminus X$. Let us first remark that since

$$\Sigma \Leftrightarrow new \equiv (\Sigma \wedge new) \vee (\neg\Sigma \wedge \neg new),$$

the following equivalence holds: $\gamma \in PI(\Sigma \Leftrightarrow new)$ if and only if:

(1) $new \wedge \gamma_1 \in PI(\Sigma \Leftrightarrow new)$ and $\gamma_1 \in PI(\Sigma)$; or
(2) $\neg new \wedge \gamma_2 \in PI(\Sigma \Leftrightarrow new)$ and $\gamma_2 \in PI(\neg\Sigma)$.

Let us now prove Proposition 12:

($\Rightarrow$) If $X \not\approx_{\Sigma\Leftrightarrow new}^{Var(\Sigma)\setminus X}$ *new* then, due to Proposition 5, there is a $\gamma \in PI(\Sigma \Leftrightarrow new)$ mentioning both *new* and some $x \in X$. Using the above equivalence, either (1) $\gamma \equiv new \wedge \gamma_1$ with $\gamma_1 \in PI(\Sigma)$ or (2) $\gamma \equiv \neg new \wedge \gamma_2$ with $\gamma_2 \in PI(\neg\Sigma)$.

In case (1), there is a $\gamma_1 \in PI(\Sigma)$ mentioning $x \in X$ and thus Σ is V-dependent on X. In case (2), there is a $\gamma_2 \in PI(\neg\Sigma)$ mentioning $x \in X$, thus we have again $\neg\Sigma$ is V-dependent on X, or equivalently, Σ is V-dependent on X.

($\Leftarrow$) If Σ is V-dependent on X then there is a $\gamma' \in PI(\Sigma)$ mentioning some $x \in X$ [20]. Now, let $\gamma = new \wedge \gamma'$. Using the above equivalence, $\gamma \in PI(\Sigma \Leftrightarrow new)$. Furthermore, γ mentions both an $x \in X$ and *new*, so, due to Proposition 5, we have $X \not\approx_{\Sigma\Leftrightarrow new}^{Var(\Sigma)\setminus X}$ *new*. □

This result means that, in any state of knowledge regarding $Var(\Sigma) \setminus X$, knowing the truth values of variables in X cannot help us knowing the truth value of *new* and hence of Σ. The converse, i.e., expressing strong conditional independence from formula-variable independence, is possible as well (see Proposition 7). However, the exhibited transformation is not a polynomial one and thus will not be helpful when investigating computational complexity issues.

Conditional independence is also related to formula-variable independence through the notion of variable forgetting [20,21,25]. Especially, as a direct consequence of Theorem 5 in [8], $X \sim_{\Sigma}^{Z} Y$ holds if and only if $\forall \omega_X \in \Omega_X$, $\forall \omega_Y \in \Omega_Y$, $\forall \omega_Z \in \Omega_Z$, we have

$$\begin{aligned} & \mathit{ForgetVar}\big(\Sigma \wedge \mathit{for}(\omega_X) \wedge \mathit{for}(\omega_Y) \wedge \mathit{for}(\omega_Z), PS \setminus Z\big) \\ \equiv\; & \mathit{ForgetVar}\big(\Sigma \wedge \mathit{for}(\omega_X) \wedge \mathit{for}(\omega_Z), PS \setminus Z\big) \wedge \\ & \mathit{ForgetVar}\big(\Sigma \wedge \mathit{for}(\omega_Y) \wedge \mathit{for}(\omega_Z), PS \setminus Z\big). \end{aligned}$$

As an illustration, let us consider Example 1 again. We have

$$\Sigma = \{\neg a \vee \neg b \vee c, \neg a \vee b \vee d, a \vee \neg c, \neg a \vee c \vee d, b \vee \neg c \vee d\}.$$

We have seen that $c \not\sim_{\Sigma}^{\{a\}} d$. This can be explained by the fact that $\omega_{\{c\}} = \{\neg c\}$, $\omega_{\{d\}} = \{\neg d\}$ and $\omega_{\{a\}} = \{a\}$ are such that

$$\mathit{ForgetVar}\big(\Sigma \wedge \mathit{for}(\omega_{\{c\}}) \wedge \mathit{for}(\omega_{\{d\}}) \wedge \mathit{for}(\omega_{\{a\}}), PS \setminus \{a\}\big)$$

is inconsistent, while

$$\begin{aligned} & \mathit{ForgetVar}\big(\Sigma \wedge \mathit{for}(\omega_{\{c\}}) \wedge \mathit{for}(\omega_{\{a\}}), PS \setminus \{a\}\big) \wedge \\ & \mathit{ForgetVar}\big(\Sigma \wedge \mathit{for}(\omega_{\{d\}}) \wedge \mathit{for}(\omega_{\{a\}}), PS \setminus \{a\}\big) \end{aligned}$$

is equivalent to a.

5.2. Relevance

Lakemeyer [18,19] introduces several forms of relevance, which can be used to characterize what "tells about" means. We show how these forms of relevance are strongly related to conditional independence. We also complete the results given in [19], by exhibiting the computational complexity of each form of relevance introduced in [19].

Lakemeyer's notion of irrelevance of a formula to a subject matter (Definition 9 in [19]) is studied in [20,21] (where it is related to formula-variable independence).

5.2.1. Strict relevance of a formula to a subject matter

Lakemeyer has introduced two forms of *strict relevance*. The first (chronologically) one has been given in [18], as follows.

Definition 8 (*strict relevance to a subject matter* [18]). Let Σ be a formula from $PROP_{PS}$ and V a subset of PS. Σ is *strictly relevant to* V if and only if every prime implicate of Σ contains a variable from V.

Lakemeyer has also introduced another notion of strict relevance [19], more demanding than the original one. Here we consider an equivalent definition.

Definition 9 (*strict relevance to a subject matter* [19]). Let Σ be a formula from $PROP_{PS}$ and V a subset of PS. Σ is *strictly relevant to* V if and only if there exists a prime implicate of Σ mentioning a variable from V, and every prime implicate of Σ mentions only variables from V.

Both definitions prevent tautologies and contradictory formulas from being strictly relevant to any set of variables. The basic difference between these two definitions is that, in the first one, we want that every prime implicate of Σ contains *at least* a variable from V, while in the second case we impose that every prime implicate of Σ must contain *only* variables from V.[1] As the following example shows, there are formulas for which the two definitions of strict relevance do not coincide.

Example 2. Let $\Sigma = (a \vee b)$ and $V = \{a\}$. There is only one prime implicate of Σ, namely $a \vee b$. Since it contains at least a variable of V, it follows that Σ is strictly relevant to V with respect to [18]. However, since the prime implicate $a \vee b$ is not composed *only of* variables of V (because $b \notin V$), it follows that Σ is not strictly relevant to V with respect to [19].

Through formula-variable independence, we can derive an alternative characterization of the notion of *strict relevance* introduced by Lakemeyer in [19]. Indeed, as a straightforward consequence of the definition, we have that Σ is strictly relevant to V if and only if Σ is V-dependent on V and V-independent from $Var(\Sigma) \setminus V$ (see [20]).

We have identified the complexity of both definitions of strict relevance, and they turn out to be different, as the first definition is easier than the second one. Namely, STRICT RELEVANCE OF A FORMULA TO A SUBJECT MATTER [19] is BH_2-complete [20] while we have the following:

Proposition 13 (complexity of strict relevance as in [18]). STRICT RELEVANCE OF A FORMULA TO A SUBJECT MATTER *as in* [18] *is* Π_2^p*-complete.*

Proof.

• Membership. Let us consider the complementary problem. Guess a clause δ, check that it does not contain any variable from V (this can be achieved in time polynomial in $|\delta| + |V|$, hence in time polynomial in $|\Sigma| + |V|$ since no prime implicate of Σ can include a variable that does not occur in Σ). Then check that it is an implicate of Σ (one call to an NP-oracle) and check that every subclause of δ obtained by removing from it one of its k literals is not an implicate of Σ (k calls to an NP-oracle). Since only $k+1$ calls to such an oracle are required to check that δ is a prime implicate of Σ, the complementary problem of STRICT RELEVANCE belongs to Σ_2^p. Hence, STRICT RELEVANCE belongs to Π_2^p.

• Hardness by polynomial reduction from 2-$\overline{\text{QBF}}$: we have that $\forall A \exists B\, \Sigma(A, B)$ is valid if and only if every prime implicate of Σ that contains a variable from A also contains a variable from B (see [12], Proposition 1), i.e., if and only if every prime implicate of Σ contains a variable from B (since $Var(\Sigma) = A \cup B$), i.e., if and only if Σ is strictly relevant to B. □

[1] Strict relevance as in [19] could also be shown to be strongly related to controllability [5,22].

5.2.2. Explanatory relevance

Lakemeyer [19] also introduces a notion of relevance of a formula Φ to a subject matter V with respect to a formula Σ that can be abductively characterized (see Definition 20 in [19]):

Definition 10 (*explanatory relevance*). Let Σ and Φ be formulas from $PROP_{PS}$ and V a subset of PS. Φ is (*explanatory*) *relevant to* V with respect to Σ if and only if there exists a minimal abductive explanation for Φ with respect to Σ that mentions a variable from V.

Example 3. Let $\Sigma = (a \Rightarrow b)$ and $\Phi = b$. Φ is explanatory relevant to $\{a\}$ with respect to Σ since a is an abductive explanation for b with respect to Σ.

The next result shows that explanatory relevance can be rewritten using strong conditional independence:

Proposition 14. *Φ is explanatory relevant to V with respect to Σ if and only if $new \not\approx^{ceteris\ paribus}_{\Sigma \wedge (\Phi \Rightarrow new)} V$ where $new \in PS \setminus (V \cup Var(\Sigma))$ is a new variable.*

Proof. ($\Rightarrow$) Assume that Φ is explanatory relevant to V with respect to Σ. Then, there is a $\gamma \in PI_\Sigma(\Phi) = PI(\Sigma \Rightarrow \Phi) \setminus PI(\neg\Sigma)$ such that $Var(\gamma) \cap V \neq \emptyset$. Let δ be a clause s.t. $\delta \equiv \neg\gamma$. Since $\gamma \in PI_\Sigma(\Phi)$, we have that $\delta \in IP(\Sigma \wedge \neg\Phi) \setminus IP(\Sigma)$; and since $Var(\gamma) \cap V \neq \emptyset$ we have $Var(\delta) \cap V \neq \emptyset$. Let $\Sigma' = \Sigma \wedge (\Phi \Rightarrow new)$. Let us show that $\delta \vee new \in IP(\Sigma')$.

- (i) $\Sigma' \wedge \neg\delta \vee new$ is equivalent to $\Sigma \wedge (\Phi \Rightarrow new) \wedge \neg\delta \wedge \neg new$, i.e., to $\Sigma \wedge \neg\delta \wedge \neg new \wedge \neg\Phi$, which is inconsistent since $\Sigma \wedge \neg\Phi \models \delta$. Hence, $\Sigma' \models \delta \vee new$.
- (ii) Suppose that $\delta \vee new$ is not a *prime* implicate of Σ'. Then there exists a prime implicate of Σ' strictly contained in $\delta \vee new$. This implicate has either the form (a) δ' with $\delta' \subseteq \delta$ or the form (b) $\delta'' \vee new$ with δ'' strictly contained in δ. In case (a), we have $\Sigma' \models \delta'$, implies that $\Sigma \models \delta'$, which entails $\Sigma \models \delta$ and thus contradicts $\delta \in IP(\Sigma \wedge \neg\Phi) \setminus IP(\Sigma)$. In case (b), we have $\Sigma' \models \delta'' \vee new$, which entails $\Sigma' \wedge \neg new \models (\delta'' \vee new) \wedge \neg new$, i.e., $\Sigma \wedge \neg new \wedge \neg\Phi \models \delta'' \wedge \neg new$, which entails $\Sigma \wedge \neg\Phi \models \delta''$, which contradicts $\delta \in IP(\Sigma \wedge \neg\Phi) \setminus IP(\Sigma)$.

Thus, $\delta \vee new$ is a prime implicate of Σ' mentioning both *new* and a variable from V, which means that $new \not\approx^{ceteris\ paribus}_{\Sigma'} V$.

($\Leftarrow$) Assume that $new \not\approx^{ceteris\ paribus}_{\Sigma'} V$. Then, there is a prime implicate δ' of Σ' containing *new* and a variable of V (cf. Proposition 5). Let δ be the subclause of δ' containing every literal of δ' except *new*. We have $\Sigma \wedge (\Phi \Rightarrow new) \models \delta \vee new$. Thus, we get $\Sigma \wedge (\Phi \wedge new) \wedge \neg new \models \delta$, i.e., $\Sigma \wedge \neg new \wedge \neg\Phi \models \delta$; subsequently, we get $\Sigma \wedge \neg\Phi \models \delta$, i.e., $\Sigma \wedge \neg\delta \models \Phi$, which means that $\neg\delta$ is an explanation for Φ with respect to Σ mentioning a variable from V; its minimality comes from the abovementioned minimality of δ'. □

This result is helpful for studying the complexity of this form of relevance.

Proposition 15 (complexity of explanatory relevance). EXPLANATORY RELEVANCE *is* Σ_2^p*-complete.*

Proof. Membership is a direct consequence of the above result together with Proposition 10. Its Σ_2^p-hardness is a direct consequence of Theorem 4.2.1 from [11] (that establishes the Σ_2^p-completeness of the problem of checking whether an individual hypothesis is relevant for minimally explaining Φ with respect to Σ, i.e., belongs to at least one of its minimal abductive explanations). □

5.2.3. Relevance between two subject matters relative to a knowledge base

Lakemeyer [19] also introduces a notion of relevance between two subject matters relative to a knowledge base.

Definition 11 (*relevance between two subject matters*). Let Σ be a formula from $PROP_{PS}$ and X, Y be subsets of PS. X is *relevant to* Y with respect to Σ if and only if there exists a prime implicate δ of Σ s.t. $Var(\delta) \cap X \neq \emptyset$ and $Var(\delta) \cap Y \neq \emptyset$.

Example 4. Let $\Sigma = (a \Rightarrow b)$, $X = \{a\}$ and $Y = \{b\}$. X is relevant to Y with respect to Σ since the prime implicate $\neg a \vee b$ of Σ contains both variables a and b.

Clearly enough, such a notion of relevance is symmetric: X is relevant to Y with respect to Σ if and only if Y is relevant to X with respect to Σ. The corresponding notion of irrelevance coincides with *ceteris paribus* strong conditional independence:

Proposition 16. *Let Σ be a formula from $PROP_{PS}$ and X, Y be subsets of PS. X is irrelevant to Y with respect to Σ if and only if X and Y are* ceteris paribus *strongly independent with respect to Σ.*

Proof. Easy consequence from Theorem 31 in [19] which states that X is relevant to Y with respect to Σ if and only if there is a Z such as $X \not\rightsquigarrow^{Z}_{\Sigma} Y$, plus the definition of *ceteris paribus* strong independence. □

Using then Proposition 10, we get the following corollary:

Proposition 17 (complexity of relevance between two subject matters). RELEVANCE BETWEEN TWO SUBJECT MATTERS RELATIVE TO A KNOWLEDGE BASE *is* Σ_2^p*-complete.*

Proof. Trivial from the fact that two subject matters are relevant with respect to a knowledge base if and only if they are not *ceteris paribus* strongly independent, and checking this form of strong independence is Π_2^p-complete. □

5.3. Novelty

Novelty is a form of relevance between two formulas given some background knowledge. Introduced in [14], this notion has been analyzed in more details in the propositional case in [26]. Closely related to Lakemeyer's relevance (see [19]), it can be used to define information filtering policies and cooperative answering techniques [13].

Definition 12 (*novelty*). Let Σ, Φ and Ψ be formulas from $PROP_{PS}$. Φ is *new to* Ψ with respect to Σ if and only if there is a minimal abductive explanation for Ψ with respect to $\Sigma \wedge \Phi$ that is not a minimal abductive explanation for Ψ with respect to Σ, or there is a minimal abductive explanation for $\neg\Psi$ with respect to $\Sigma \wedge \Phi$ that is not a minimal abductive explanation for $\neg\Psi$ with respect to Σ.

Intuitively, Φ is new to Ψ with respect to Σ if and only if expanding Σ with Φ gives rise to new contexts in which the semantics of Ψ is determined (as true or false).

Example 5. Let $\Sigma = (b \Rightarrow c)$, $\Phi = (a \Rightarrow b)$, and $\Psi = c$. Φ is new to Ψ with respect to Σ since $\gamma = a$ is a minimal explanation for Ψ with respect to $\Sigma \wedge \Phi$, but not a minimal explanation for Ψ with respect to Σ. Thus, in the context where a is interpreted as true, expanding Σ with Φ enables deriving the truth value of Ψ, while it remains undetermined when Φ is not taken into account.

More refined notions of novelty have been pointed out in [26], by considering separately Ψ and $\neg\Psi$.

Definition 13 (*positive novelty, negative novelty*). Let Σ, Φ and Ψ be formulas from $PROP_{PS}$.

- Φ is *new positive to* Ψ with respect to Σ if and only if there is a minimal abductive explanation for Ψ with respect to $\Sigma \wedge \Phi$ that is not a minimal abductive explanation for Ψ with respect to Σ.
- Φ is *new negative to* Ψ with respect to Σ if and only if there is a minimal abductive explanation for $\neg\Psi$ with respect to $\Sigma \wedge \Phi$ that is not a minimal abductive explanation for $\neg\Psi$ with respect to Σ.

Thus, Φ is new to Ψ with respect to Σ if and only if Φ is new positive to Ψ or Φ is new negative to $\neg\Psi$. This simple result, as well as many characterization results for novelties, can be found in [26]. Especially, it is easy to see that Φ is new negative to Ψ with respect to Σ if and only if Φ is new positive to $\neg\Psi$ with respect to Σ. Among the results given in [26] also is a prime implicate characterization of positive novelty and negative novelty:

Proposition 18. *Let Σ, Φ and Ψ be formulas from $PROP_{PS}$.*

- *Φ is new positive to Ψ with respect to Σ if and only if there exists a prime implicate of $\Sigma \wedge \Phi \wedge \neg\Psi$ that is neither a prime implicate of $\Sigma \wedge \Phi$ nor a prime implicate of $\Sigma \wedge \neg\Psi$.*
- *Φ is new negative to Ψ with respect to Σ if and only if there exists a prime implicate of $\Sigma \wedge \Phi \wedge \Psi$ that is neither a prime implicate of $\Sigma \wedge \Phi$ nor a prime implicate of $\Sigma \wedge \Psi$.*

Proof.

• *Positive novelty*. By definition, Φ is new positive to Ψ with respect to Σ if and only if there is a minimal abductive explanation for Ψ with respect to $\Sigma \wedge \Phi$ that is not a minimal abductive explanation for Ψ with respect to Σ. This is equivalent to state that there exists a clause π for which the following three conditions hold.

$$\pi \in PI(\Sigma \wedge \Phi \wedge \neg\Psi),$$
$$\pi \notin PI(\Sigma \wedge \Phi),$$
$$\pi \notin PI(\Sigma \wedge \neg\Psi) \quad \text{or} \quad \pi \in PI(\Sigma).$$

What is left to prove is that the first two conditions implies $\pi \notin PI(\Sigma)$. Indeed, the first one implies that $\Sigma \wedge \Phi \wedge \neg\Psi \models \pi$, while the second one is equivalent to:

(1) $\Sigma \wedge \Phi \not\models \pi$; or
(2) $\Sigma \wedge \Phi \models \pi$ and there exists a clause $\pi' \models \pi$ such that $\Sigma \wedge \Phi \models \pi'$.

Let us assume that the first condition holds. Then, $\Sigma \not\models \pi$ and thus π cannot be a prime implicate of Σ. If the second condition holds, then π' is also an implicate of $\Sigma \wedge \Phi \wedge \neg\Psi$: as a result, π cannot be a prime implicate of that formula.

• *Negative novelty*. Immediate from the fact that Φ is new negative to Ψ with respect to Σ if and only if Φ is new positive to $\neg\Psi$ with respect to Σ, and the fact that the proposition holds for positive novelty. □

From this proposition, it is easy to prove that focusing on prime implicates is unnecessary (implicates are sufficient):

Corollary 1. *Let Σ, Φ and Ψ be formulas from $PROP_{PS}$.*

- *Φ is new positive to Ψ with respect to Σ if and only if there exists an implicate of $\Sigma \wedge \Phi \wedge \neg\Psi$ that is neither an implicate of $\Sigma \wedge \Phi$ nor an implicate of $\Sigma \wedge \neg\Psi$.*
- *Φ is new negative to Ψ with respect to Σ if and only if there exists an implicate of $\Sigma \wedge \Phi \wedge \Psi$ that is neither an implicate of $\Sigma \wedge \Phi$ nor an implicate of $\Sigma \wedge \Psi$.*

As an immediate consequence, considering minimal abductive explanations in the definitions above is useless (considering abductive explanations is sufficient).

We are now making precise the relationship between the various forms of novelty and strong conditional independence.

Proposition 19. *Let Σ, Φ and Ψ be propositional formulas, and let v_Φ and v_Ψ be two new propositional variables* (*not appearing in Φ, Ψ and Σ*), *and let*

$$\Sigma^+ = \Sigma \wedge (v_\Phi \Rightarrow \Phi) \wedge (v_\Psi \Rightarrow \Psi);$$
$$\Sigma^- = \Sigma \wedge (v_\Phi \Rightarrow \Phi) \wedge (\Psi \Rightarrow v_\Psi);$$
$$\Sigma' = \Sigma^+ \wedge \Sigma^- \equiv \Sigma \wedge (v_\Phi \Rightarrow \Phi) \wedge (v_\Psi \Leftrightarrow \Psi).$$

(1) *Φ is new positive to Ψ with respect to Σ if and only if $v_\Phi \not\approx_{\Sigma^+}^{\text{ceteris paribus}} v_\Psi$.*
(2) *Φ is new negative to Ψ with respect to Σ if and only if $v_\Phi \not\approx_{\Sigma^-}^{\text{ceteris paribus}} v_\Psi$.*
(3) *Φ is new to Ψ with respect to Σ if and only if $v_\Phi \not\approx_{\Sigma'}^{\text{ceteris paribus}} v_\Psi$.*

Proof. From Proposition 18 we get easily the following equivalences:

- Φ is new positive to Ψ with respect to Σ if and only if $\exists\delta \in IP(\Sigma^+)$ containing the literals $\neg v_\Phi$ and $\neg v_\Psi$.
- Φ is new positive to Ψ with respect to Σ if and only if $\exists\delta \in IP(\Sigma')$ containing the literals $\neg v_\Phi$ and $\neg v_\Psi$.
- Φ is new negative to Ψ with respect to Σ if and only if $\exists\delta \in IP(\Sigma^-)$ containing the literals $\neg v_\Phi$ and v_Ψ.
- Φ is new negative to Ψ with respect to Σ if and only if $\exists\delta \in IP(\Sigma')$ containing the literals $\neg v_\Phi$ and v_Ψ.
- Φ is new to Ψ with respect to Σ if and only if $\exists\delta \in IP(\Sigma')$ containing the literal $\neg v_\Phi$ and mentioning the variable v_Ψ.
- $\forall\delta \in IP(\Sigma')$, δ does not contain v_Φ.

The proof is then completed easily using Proposition 5. □

The situation where Σ is valid and negative novelty is not satisfied gives rise to a form of independence called *novelty-based independence*. Intuitively, Φ and Ψ are (novelty-based) independent if and only if every context that is possible for Φ (i.e., consistent with Φ) or with Ψ also is possible for $\Phi \wedge \Psi$. In other words, Φ and Ψ do not conflict, in any possible context. A definition based on prime implicate can be easily established:

Definition 14 (*novelty-based independence*). Let Φ and Ψ be formulas from $PROP_{PS}$. Φ and Ψ are (*novelty-based*) *independent* if and only if every prime implicate of $\Phi \wedge \Psi$ is either a prime implicate of Φ or a prime implicate of Ψ.

Several alternative characterizations exist. Thus, Φ and Ψ are novelty-based independent if and only if every implicate of $\Phi \wedge \Psi$ is either an implicate of Φ or an implicate of Ψ if and only if Φ is not new negative to Ψ with respect to *true*.

Interestingly, it has been shown in [27] that this form of independence characterizes exactly the formulas that are preserved under change in Winslett's Possible Models Approach to update.

We have derived the following complexity results for novelty:

Proposition 20 (complexity of novelty). NOVELTY, POSITIVE NOVELTY *and* NEGATIVE NOVELTY *are* Σ_2^p*-complete and* NOVELTY-BASED INDEPENDENCE *is* Π_2^p*-complete.*

In order to minimize our efforts, we first prove that novelty-based independence is Π_2^p-complete. An additional lemma is needed.

Lemma 11. *Let* Φ_1, Φ_2, Ψ_1, *and* Ψ_2 *be four satisfiable formulas from* $PROP_{PS}$ *s.t.* $(Var(\Phi_1) \cup Var(\Psi_1)) \cap (Var(\Phi_2) \cup Var(\Psi_2)) = \emptyset$. ($\Phi_1$ *and* Ψ_1 *are novelty-based independent and* Φ_2 *and* Ψ_2 *are novelty-based independent*) *if and only if* $\Phi_1 \wedge \Phi_2$ *and* $\Psi_1 \wedge \Psi_2$ *are novelty-based independent.*

Proof. ($\Rightarrow$) Assume that there exists a clause γ s.t. $\Phi_1 \wedge \Phi_2 \wedge \Psi_1 \wedge \Psi_2 \models \gamma$ holds and $\Phi_1 \wedge \Phi_2 \not\models \gamma$ holds and $\Psi_1 \wedge \Psi_2 \not\models \gamma$ holds. Since $(Var(\Phi_1) \cup Var(\Psi_1)) \cap (Var(\Phi_2) \cup Var(\Psi_2)) = \emptyset$, it is obvious that $\Phi_1 \wedge \Psi_1$ and $\Phi_2 \wedge \Psi_2$ are novelty-based independent. As a consequence, since $\Phi_1 \wedge \Phi_2 \wedge \Psi_1 \wedge \Psi_2 \models \gamma$ holds, we have $\Phi_1 \wedge \Psi_1 \models \gamma$ or $\Phi_2 \wedge \Psi_2 \models \gamma$. If Φ_1 and Ψ_1 are novelty-based independent and Φ_2 and Ψ_2 are novelty-based independent, this implies that $\Phi_1 \models \gamma$ holds or $\Psi_1 \models \gamma$ holds or $\Phi_2 \models \gamma$ holds or $\Psi_2 \models \gamma$ holds. This contradicts the fact that $\Phi_1 \wedge \Phi_2 \not\models \gamma$ holds and $\Psi_1 \wedge \Psi_2 \not\models \gamma$ holds.

($\Leftarrow$) Assume that Φ_1 and Ψ_1 are not novelty-based independent (the remaining case where Φ_2 and Ψ_2 would not be novelty-based independent is similar). Then, there exists a prime implicate π of $\Phi_1 \wedge \Psi_1$ that is neither a prime implicate of Φ_1 nor a prime implicate of Ψ_1. Clearly enough, $\Phi_1 \wedge \Psi_1 \wedge \Phi_2 \wedge \Psi_2 \models \pi$ holds. If $\Phi_1 \wedge \Phi_2$ and $\Psi_1 \wedge \Psi_2$ are novelty-based independent, this is equivalent to saying that $\Phi_1 \wedge \Phi_2 \models \pi$ holds or $\Psi_1 \wedge \Psi_2 \models \pi$ holds. Since $Var(\Phi_1) \cap Var(\Phi_2) = \emptyset$ and $Var(\Psi_1) \cap Var(\Psi_2) = \emptyset$, this is also equivalent to saying that $\Phi_1 \models \pi$ holds or $\Psi_1 \models \pi$ holds or $\Phi_2 \models \pi$ holds or $\Psi_2 \models \pi$ holds. We have assumed that π neither is a prime implicate of Φ_1 nor a prime implicate of Ψ_1. Actually, we can prove that π neither is an implicate of Φ_1 nor an implicate of Ψ_1. Indeed, if π were an implicate of Φ_1 (respectively Ψ_1), a prime implicate π' of Φ_1 (respectively Ψ_1) would exist s.t. $\pi' \models \pi$ holds. Since $\Phi_1 \wedge \Psi_1 \models \Phi_1$ (respectively Ψ_1) holds, there exists a prime implicate π'' of $\Phi_1 \wedge \Psi_1$ s.t. $\pi'' \models \pi'$ holds. This implies that $\pi'' \models \pi'$ holds and since π'' and π are prime implicates of the same formula, we have $\pi'' \equiv \pi$. Hence, $\pi' \equiv \pi$ holds as well. This would contradict the fact that π is not a prime implicate of Φ_1 (respectively Ψ_1). Now, since π neither is an implicate of Φ_1 nor an implicate of Ψ_1, it must be the case that $\Phi_2 \models \pi$ holds or $\Psi_2 \models \pi$ holds. Since $\Phi_1 \not\models \pi$ holds, we know that π is not a tautology. Because π is a prime implicate of $\Phi_1 \wedge \Psi_1$, it must be the case that $Var(\pi) \subseteq Var(\Phi_1 \wedge \Psi_1)$ holds, i.e., $Var(\pi) \subseteq Var(\Phi_1) \cup Var(\Psi_1)$ holds. Since $(Var(\Phi_1) \cup Var(\Psi_1)) \cap (Var(\Phi_2) \cup Var(\Psi_2)) = \emptyset$, $\Phi_2 \models \pi$ holds or $\Psi_2 \models \pi$ holds if and only if Φ_2 is unsatisfiable or Ψ_2 is unsatisfiable, contradiction (this is an easy consequence of Craig's interpolation theorem in the propositional case). $\square$

Lemma 12. NOVELTY-BASED INDEPENDENCE *is* Π_2^p*-complete.*

Proof. Membership comes from Proposition 19. Π_2^p-hardness comes from the following observations:

• Let x, y be two variables from PS and Σ a formula from $PROP_{PS}$. Then x and y are *ceteris paribus* strongly independent with respect to Σ if and only if for every term γ over $Var(\Sigma) \setminus \{x, y\}$, the four following statements are true:

- $x \wedge y \wedge \Sigma \wedge \gamma$ is satisfiable if and only if $x \wedge \Sigma \wedge \gamma$ is satisfiable and $y \wedge \Sigma \wedge \gamma$ is satisfiable.
- $\neg x \wedge y \wedge \Sigma \wedge \gamma$ is satisfiable if and only if $\neg x \wedge \Sigma \wedge \gamma$ is satisfiable and $y \wedge \Sigma \wedge \gamma$ is satisfiable.
- $x \wedge \neg y \wedge \Sigma \wedge \gamma$ is satisfiable if and only if $x \wedge \Sigma \wedge \gamma$ is satisfiable and $\neg y \wedge \Sigma \wedge \gamma$ is satisfiable.
- $\neg x \wedge \neg y \wedge \Sigma \wedge \gamma$ is satisfiable if and only if $\neg x \wedge \Sigma \wedge \gamma$ is satisfiable and $\neg y \wedge \Sigma \wedge \gamma$ is satisfiable.

This is equivalent to saying that for every clause δ over $Var(\Sigma) \setminus \{x, y\}$, the four following statements are true:

- $x \wedge y \wedge \Sigma \models \delta$ if and only if $x \wedge \Sigma \models \delta$ or $y \wedge \Sigma \models \delta$.
- $\neg x \wedge y \wedge \Sigma \models \delta$ if and only if $\neg x \wedge \Sigma \models \delta$ or $y \wedge \Sigma \models \delta$.
- $x \wedge \neg y \wedge \Sigma \models \delta$ if and only if $x \wedge \Sigma \models \delta$ or $\neg y \wedge \Sigma \models \delta$.
- $\neg x \wedge \neg y \wedge \Sigma \models \delta$ if and only if $\neg x \wedge \Sigma \models \delta$ or $\neg y \wedge \Sigma \models \delta$.

Clearly enough, if the four statements above are satisfied for every clause, they are also satisfied for the clauses that do not contain x or y as a variable. Conversely, let us show that if x and y are *ceteris paribus* strongly independent, then the four statements above are satisfied by every clause δ. Let us now consider a clause δ s.t. $Var(\delta) \cap \{x, y\} \neq \emptyset$ and δ is not a tautology (tautologies trivially satisfy the four statements above). For simplicity, assume that the variable x occurs positively in δ. Then, it is clear that the first and the third statements above are satisfied by such clauses δ. For the remaining cases (second and fourth statements), let δ' be the clause obtained by removing every occurrence of x in δ. We have $\neg x \wedge y \wedge \Sigma \models \delta$ if and only if $\neg x \wedge y \wedge \Sigma \models \delta'$. If δ' contains y as a positive literal, then $\neg x \wedge y \wedge \Sigma \models \delta'$ and $y \wedge \Sigma \models \delta'$ holds as well. Hence, $y \wedge \Sigma \models \delta$ also holds. This shows that the second statement is satisfied by δ. Otherwise, let δ'' be the clause obtained by removing every occurrence of $\neg y$ in δ'. We have $\neg x \wedge y \wedge \Sigma \models \delta'$ if and only if $\neg x \wedge y \wedge \Sigma \models \delta''$. Because δ'' does not contain any occurrence of x or y, if x and y are *ceteris paribus* strongly independent, then it must be the case that if $\neg x \wedge y \wedge \Sigma \models \delta''$ holds, then $\neg x \wedge \Sigma \models \delta''$ holds or $y \wedge \Sigma \models \delta''$ holds. This implies that $\neg x \wedge \Sigma \models \delta$ holds or $y \wedge \Sigma \models \delta$ holds, hence the second statement is satisfied. The remaining cases, i.e., δ contains a negative occurrence of x, δ contains a positive occurrence of y, δ contains a negative occurrence of y, can be handled in a similar way, *mutatis mutandis* (clearly, both x and y and x and $\neg x$ play symmetric roles with respect to the conjunction of the four statements). Thus, x and y are *ceteris paribus* strongly independent with respect to Σ if and only if:

- $\Sigma \wedge x$ and $\Sigma \wedge y$ are novelty-based independent, and
- $\Sigma \wedge \neg x$ and $\Sigma \wedge y$ are novelty-based independent, and
- $\Sigma \wedge x$ and $\Sigma \wedge \neg y$ are novelty-based independent, and
- $\Sigma \wedge \neg x$ and $\Sigma \wedge \neg y$ are novelty-based independent.

• Several instances of novelty-based independence can be gathered into a single one in polynomial time through renaming as long as all the formulas that are considered are satisfiable. This is stated formally by Lemma 11.

As a consequence of Lemma 11, we can state that x and y are *ceteris paribus* strongly independent with respect to Σ iff $rename_1(\Sigma \wedge x) \wedge rename_2(\Sigma \wedge \neg x) \wedge rename_3(\Sigma \wedge x) \wedge rename_4(\Sigma \wedge \neg x)$ and $rename_1(\Sigma \wedge y) \wedge rename_2(\Sigma \wedge \neg y) \wedge rename_3(\Sigma \wedge y) \wedge rename_4(\Sigma \wedge \neg y)$ are novelty-based independent, provided that $\Sigma \not\models x$ holds, $\Sigma \not\models \neg x$ holds, $\Sigma \not\models y$ holds, and $\Sigma \not\models \neg y$ holds. This equivalence is obtained by applying three times the lemma above; each $rename_i$ ($i \in 1, \ldots, 4$) is a renaming, i.e., a substitution from variables to variables s.t. $rename_i(x) = x_i$, that is extended to formulas in an obvious compositional way; clearly enough, renaming a formula preserves its satisfiability.

• The next observation is that in the proof of Π_2^p-hardness of *ceteris paribus* strong conditional independence of single variables given above (Lemma 10), we can assume that $\Sigma \not\models x$ holds, $\Sigma \not\models \neg x$ holds, $\Sigma \not\models y$ holds, and $\Sigma \not\models \neg y$ holds without loss of generality as soon as the matrix Φ of the 2-$\overline{\text{QBF}}$ formula $\forall A \exists B \Phi[A, B]$ used in the proof is satisfiable (we have $Var(\Phi) \cap \{x, y\} = \emptyset$). So it remains to prove that this restriction does not question the Π_2^p-hardness of checking whether a 2-$\overline{\text{QBF}}$ formula is valid. Let us consider the mapping M that associates to every 2-$\overline{\text{QBF}}$ formula $\forall A \exists B \Phi[A, B]$ the 2-$\overline{\text{QBF}}$ formula $\forall A \cup \{new\} \exists B(\Phi[A, B] \vee new)$, where $new \notin (A \cup B)$. Clearly enough, $\Phi[A, B] \vee new$ always is satisfiable. It is easy to check that $\forall A \exists B \Phi[A, B]$ is valid if and only if $M(\forall A \exists B \Phi[A, B])$ is valid as well.

Lemma 13. POSITIVE NOVELTY *is* Σ_2^p*-complete.*

Proof. Membership comes from Proposition 19. Σ_2^p-hardness is an immediate consequence of the Π_2^p-hardness of novelty-based independence. Indeed, Φ and Ψ are novelty-based independent if and only if Φ is not new positive to $\neg\Psi$ with respect to *true*. □

Corollary 2. NEGATIVE NOVELTY *is* Σ_2^p*-complete.*

Lemma 14. NOVELTY *is* Σ_2^p*-complete.*

Proof. Membership comes from Proposition 19. As to hardness, let us consider the application M that maps $\langle \Phi, \Psi \rangle$ to $\langle \Psi \vee new, \Phi, new \rangle$, where *new* is a variable from $PS \setminus (Var(\Phi) \cup Var(\Psi))$. M can be easily computed in time polynomial in the input size. The point is that Φ and Ψ are not novelty-based independent if and only if Φ is new to *new* with respect to $\Psi \vee new$. Then, the Π_2^p-hardness of novelty-based independence completes the proof. For simplicity, let us recall that Φ and Ψ are not novelty-based independent if and only if Φ is new positive to $\neg\Psi$ with respect to *true*. Let us first show that if Φ is new positive to Ψ with respect to *true*, then Φ is new positive to *new* with respect to $\neg\Psi \vee new$, hence new to *new* with respect to $\neg\Psi \vee new$.

($\Rightarrow$) Assume that there exists a term γ s.t. (1) $\Phi \wedge \gamma \models \Psi$, (2) $\Phi \wedge \gamma$ is satisfiable and (3) $\gamma \not\models \Psi$ holds.

- (1) implies that $\Phi \wedge \gamma \equiv \Phi \wedge \gamma \wedge \Psi$. Hence, $\Phi \wedge \gamma \wedge (\neg\Psi \vee new) \equiv \gamma \wedge \Phi \wedge \Psi \wedge new$. Consequently, $\gamma \wedge \Phi \wedge (\neg\Psi \vee new) \models new$ holds.
- (2) implies that $\Phi \wedge \gamma \wedge (\neg\Psi \wedge new)$ is satisfiable: if there exists a model of $\Phi \wedge \gamma$ than there exists a model of $\Phi \wedge \gamma \wedge new$, hence a model of $\Phi \wedge \gamma \wedge (\neg\Psi \wedge new)$.
- (3) implies that $\gamma \wedge (\neg\Psi \vee new) \not\models new$. Indeed, if it were not the case, we should have $\gamma \wedge \neg\Psi \models new$. Since *new* does not occur in $\gamma \wedge \neg\Psi$, it should be the case that $\gamma \wedge \neg\Psi$ is unsatisfiable, which contradicts (3).

($\Leftarrow$) It remains to show that whenever Φ is new to *new* with respect to $\neg\Psi \vee new$, then Φ is new positive to Ψ with respect to *true*. In order to prove it, let us first show that when Φ is new to *new* with respect to $\neg\Psi \vee new$, we necessarily have Φ new positive to *new* with respect to $\neg\Psi \vee new$ (in other words, Φ new negative to *new* with respect to $\neg\Psi \vee new$ is impossible). By *reductio ad absurdum*, let us assume that there exists a term γ s.t. (1) $\Phi \wedge \gamma \wedge (\neg\Psi \vee new) \models \neg new$ holds, (2) $\Phi \wedge \gamma \wedge (\neg\Psi \vee new)$ is satisfiable, and (3) $\gamma \wedge (\neg\Psi \vee new) \not\models \neg new$ holds. (1) is equivalent to saying that $\Phi \wedge \gamma \wedge new$ is unsatisfiable, i.e., $\Phi \wedge \gamma \models \neg new$ holds. Since (2) requires that $\Phi \wedge \gamma$ is satisfiable and since *new* does not occur in Φ, it must be the case that $\gamma \models \neg new$. This prevents (3) from being satisfied. This shows that each time Φ is new to *new* with respect to $\neg\Psi \vee new$, then Φ is new positive to *new* with respect to $\neg\Psi \vee new$. Then, we have to show that, in this situation, Φ is new positive to Ψ. Stating that Φ is new positive to *new* with respect to $\neg\Psi \vee new$ is equivalent to state that there exists a term γ s.t. (1) $\Phi \wedge \gamma \wedge (\neg\Psi \vee new) \models new$ holds, (2) $\Phi \wedge \gamma \wedge (\neg\Psi \vee new)$ is satisfiable, and (3) $\gamma \wedge (\neg\Psi \vee new) \not\models new$ holds.

- (1) is equivalent to saying that $\Phi \wedge \gamma \wedge \neg\Psi \wedge \neg new$ is unsatisfiable. When (3) is satisfied, it must be the case that $\gamma \not\models new$. Since *new* does not occur neither in Φ nor in Ψ, (1) is equivalent to saying that $\Phi \wedge \gamma \wedge \neg\Psi$ is unsatisfiable, i.e., $\Phi \wedge \gamma \models \Psi$ holds.
- (2) implies that $\Phi \wedge \gamma$ is satisfiable.
- (3) is equivalent to saying that $\gamma \wedge (\neg\Psi \vee new) \wedge \neg new$ is satisfiable. This is equivalent to saying that $\gamma \wedge \neg\Psi \wedge \neg new$ is satisfiable. As a consequence, $\gamma \wedge \neg\Psi$ must be satisfiable, i.e., $\gamma \not\models \Psi$ holds.

Thus, γ is a certificate showing Φ new positive to Ψ with respect to *true*, and this completes the proof. □

5.4. Separability

Levesque [23] introduces a notion of formula separability that proves helpful for the purpose of characterizing queries that can be soundly answered, using an efficient (but incomplete in the general case) evaluation-based inference engine. In the propositional case, separability can be defined as follows:

Definition 15 (*Σ-separability*). Let $\Sigma, \Phi_1, \ldots, \Phi_n$ be formulas from $PROP_{PS}$. $\Phi_1, \ldots, \Phi_n$ are *Σ-separable* if and only if for every clause δ, we have $\Sigma \wedge \Phi_1 \wedge \cdots \wedge \Phi_n \models \delta$ if and only if $\Sigma \wedge \Phi_1 \models \delta$ or $\cdots$ or $\Sigma \wedge \Phi_n \models \delta$. When Σ is valid and $\Phi_1, \ldots, \Phi_n$ are *Σ-separable*, they are said to be *separable* for simplicity.

Example 6. Let $\Sigma = (b \Rightarrow c)$, $\Phi = (a \Rightarrow b)$ and $\Psi = (c \Rightarrow d)$. Φ and Ψ are not Σ-separable since $\delta = (\neg a \vee d)$ is a logical consequence of $\Sigma \wedge \Phi \wedge \Psi$ but is neither a consequence of $\Sigma \wedge \Phi$ nor a consequence of $\Sigma \wedge \Psi$. Contrastingly, Φ and Ψ are separable.

Determining Σ-separable formulas can prove valuable for query answering in a computational perspective. To be more precise, while the complexity of query answering from a set of Σ-separable formulas remains coNP-complete, it is often advantageous from the practical side to replace one large instance of the query answering problem by a linear number of smaller instances. This is what Σ-separability enables to do.

Interestingly, the background information Σ can be incorporated into the formulas checked for separability, so that Σ-separability can always be mapped to separability.

Proposition 21. *Let $\Sigma, \Phi_1, \ldots, \Phi_n$ be formulas from $PROP_{PS}$. $\Phi_1, \ldots, \Phi_n$ are Σ*-separable *if and only if $\Sigma \wedge \Phi_1, \ldots, \Sigma \wedge \Phi_n$ are separable.*

Proof. Trivial. □

This proposition also shows that Σ-separability and separability have the same complexity in the sense that each of them can be polynomially many-one reduced to the other.

As a direct consequence of Corollary 1, in the case where $n = 2$, separability coincides with novelty-based independence:

Corollary 3. *Let Φ and Ψ be two formulas from $PROP_{PS}$. Φ and Ψ are separable if and only if Φ and Ψ are novelty-based independent.*

As a consequence, the complexity of separability and Σ-separability can be easily established:

Proposition 22 (complexity of (Σ-)separability). Σ-SEPARABILITY *and* SEPARABILITY *are Π_2^p-complete.*

Proof. It is sufficient to consider the separability situation (i.e., Σ is a tautology) since Σ-separability can be polynomially many-one reduced to separability, and *vice versa*.

• *Membership*. Consider the following algorithm for the complement problem: guess a clause δ and check that $\Phi_1 \wedge \cdots \wedge \Phi_n \models \delta$ holds, while, for any $i \in 1 \ldots n$, $\Phi_i \models \delta$ does not hold. Clearly enough, the check step of this algorithm can be achieved in time polynomial in the size of the input using an NP-oracle (only $n + 1$ calls to the oracle are required), and the algorithm returns "yes" if and only if $\Phi_1, \ldots, \Phi_n$ are not separable.

• *Hardness*. Trivial from the fact that checking novelty-based independence is Π_2^p-complete, and separability coincides with novelty-based independence in the restricted case where $n = 2$. □

5.5. Causal independence

The notion of causal independence in symbolic causal networks has been proposed by Darwiche and Pearl in [9].

Definition 16 (*causal structure*). A *causal structure* is an ordered pair $\langle \Delta, \mathcal{G} \rangle$, where Δ is a propositional formula and $\mathcal{G}$ is a directed acyclic graph on a subset of $Var(\Delta)$. The parents of a variable v are called its *direct causes* and denoted $Causes(v)$, its descendents are called its *effects*, and its non-descendents are called its *non-effects* and denoted $Noneffects(v)$.

The variables of $Var(\Delta)$ that do not appear in $\mathcal{G}$ are called the *exogenous propositions*. $EXO(\Delta, \mathcal{G})$, or EXO for short, denotes the set of exogenous propositions.

Independence for causal structure is closely related to conditional independence:

Definition 17 (*causal independence*). A causal structure $\langle \Delta, \mathcal{G} \rangle$ is *causally independent* if and only if

(a) Δ is satisfiable and
(b) for every EXO-world ω_{EXO} consistent with Δ, and $\forall v \in \mathcal{G}$, we have

$$v \sim^{Causes(v)}_{\Delta \wedge \omega_{EXO}} Noneffects(v).$$

Accordingly, its computational complexity can be derived from some of the previous results:

Proposition 23 (complexity of causal independence). CAUSAL INDEPENDENCE *is* Π_2^p-*complete.*

Proof. First of all, we will make use of the following equivalence, obtained as a direct rewriting of the definition of conditional independence:

$\langle \Delta, \mathcal{G} \rangle$ is causally independent if and only if Δ is satisfiable and
$\forall v \in \mathcal{G},\ v \sim^{EXO \cup Causes(v)}_{\Delta} Noneffects(v)$.

Now, checking causal independence comes down to a satisfiability test (in NP) and a conditional independence test (in Π_2^p). The intersection of a language in NP and a language in Π_2^p is in Π_2^p, hence the membership of CAUSAL INDEPENDENCE in Π_2^p. As to hardness, we exhibit a polynomial reduction from CONDITIONAL INDEPENDENCE OF SINGLE VARIABLES (which has been shown to be Π_2^p-complete) to CAUSAL INDEPENDENCE. Let $\langle \Sigma, x, y, Z \rangle$ such that $x, y \in Var(\Sigma)$ and $Z \subseteq Var(\Sigma)$, $x \neq y$, $x \notin Z$, $y \notin Z$. Let $M(\langle \Sigma, x, y, Z \rangle) = \langle \Delta, \mathcal{G} \rangle$ where

- $\Delta = \Sigma \wedge new$ where *new* is a new variable;
- $\mathcal{G}$ contains an edge from *new* to x and an edge from *new* to y, and nothing else.

Now, $\langle \Delta, \mathcal{G} \rangle$ is causally independent if and only if $x \sim^{Z \cup \{new\}}_{\Delta} y$, or equivalently if and only if $x \sim^{Z}_{\Sigma} y$. □

5.6. Non-interactivity

We have already shown how conditional independence relates to the classical notion of probabilistic independence (Proposition 2). Several authors [4,10] have proposed notions of independence in uncertainty calculi that are "less quantitative" than the probabilistic one. Especially, possibilistic independence can be expressed using purely ordinal notions such as min and max: if $\pi : \Omega \to [0,1]$ is a normalized possibility distribution (which imposes the constraint $\max_{\omega \in \Omega} \pi(\omega) = 1$), from which a possibility measure $\Pi : PROP_{PS} \to [0,1]$ defined by $\Pi(\varphi) = \max_{\omega \models \varphi} \pi(\omega)$ is induced (with the convention $\max \emptyset = 0$), then X and Y are *non-interactive* with respect to π given Z [4] if and only if $\forall \omega_Z \in \Omega_Z$, $\Pi(\omega_X \wedge \omega_Y \wedge \omega_Z) = \min(\Pi(\omega_X \wedge \omega_Z), \Pi(\omega_Y \wedge \omega_Z))$ (where X, Y and Z are pairwise disjoint). The ordinal nature of this definition makes the connection to conditional independence possible *in both directions*: not only conditional independence is obviously a particular case of possibilistic non-interactivity, but we can also prove the following: let $Cut(\pi, \alpha) = for(\{\omega \in \Omega \mid \pi(\omega) \geqslant \alpha\})$ where $\alpha \in [0,1]$; we have

> X and Y are non-interactive with respect to π given Z
> if and only if $\forall \alpha \in [0,1]$, $X \sim^{Z}_{Cut(\pi,\alpha)} Y$ holds.

Once remarked that the number of distinct α's used in π is finite (because Ω is finite), this establishes a useful connection, especially when it comes to computational considerations. In practice, a possibility distribution is not specified explicitly but by means of a *stratified knowledge base* $B = (B_{\alpha_0}, \ldots, B_{\alpha_n})$ where the B_i's are propositional formulas and $\alpha_0 = 1 \geqslant \alpha_1 \geqslant \cdots \geqslant \alpha_n > 0$ (B_{α_0} denotes thus the most entrenched formulas and B_{α_n} the less entrenched ones); B induces the possibility distribution π_B defined by $\pi_B(\omega) = \min\{1 - \alpha_i \mid \omega \models \neg B_i\}$ (with the convention $\min \emptyset = 1$). Then, using the equivalence above and the property $Cut(\pi, \alpha) \equiv \bigwedge_{\beta \geqslant 1-\alpha} B_\beta$, it holds

> X and Y are non-interactive with respect to π_B given Z
> if and only if $\forall \alpha \in [0,1]$, $X \sim^{Z}_{\bigwedge_{\beta \geqslant 1-\alpha} B_\beta} Y$ holds.

The latter transformation being polynomial, *all complexity results established in our paper carry on to possibilistic non-interactivity when the input is a stratified knowledge base*.

6. Concluding remarks

This paper is centered on conditional independence and its stronger form (strong conditional independence) that we have introduced. Our main contribution is related to both the "philosophical" position and the "pragmatic" position with respect to irrelevance.

On the one hand, we have investigated structural properties for both forms of independence. Simple conditional independence was known to satisfy all properties of semi-graphoids, but not intersection; the latter is also satisfied by strong conditional independence, while the former ones still hold, which mean that strong conditional independence satisfy the properties of graphoids. These results are synthesized on Table 3.

Table 3
Conditional independence vs. graphoid axioms

	$\sim^{Z}_{\Sigma}$	$\approx^{Z}_{\Sigma}$
symmetry	yes	yes
decomposition	yes	yes
weak union	yes	yes
contraction	yes	yes
intersection	no	yes

We have also characterized (simple) conditional independence in probabilistic terms (cf. Proposition 2); this confirms that conditional independence is a good logical counterpart to probabilistic independence, as Darwiche says [8]. From this result, analogous characterizations for strong independence follow.

On the other hand, we have identified the complexity of the various (in)dependence relations considered in this paper, and a number of characterizations have been given as well. In light of the results established, it appears that *most (in)dependence relations have a high complexity*. The three forms of conditional independence (and the notions connected to them) are in complexity classes located at the second level of the polynomial hierarchy. This is not so surprising since this is where a large part (if not the majority) of important problems in knowledge representation[2] is located.

According to Darwiche [8], conditional independence can be useful for improving many forms of inference, including satisfiability, entailment, abduction and diagnosis. In optimal cases, for example, a satisfiability problem can be decomposed into a small number of satisfiability problems on easier knowledge bases (with less variables). We have also briefly mentioned how conditional independence can prove valuable in the context of reasoning about actions. For all these applications, the computational value of conditional independence lies in the fact that a global computation can be (soundly) decomposed into a number of local computations (which can be performed efficiently), whenever some independence relations are satisfied. Similar ideas have been developed in [2,17].

The complexity results given in this paper show that it is not always a good idea to search in an intensive way for independence relations as a preliminary step to inference. Especially, it may be paradoxical (and sometimes dangerous) to preliminarily compute a Π_2^p-hard independence relation to help solving a NP- or coNP-complete problem (given that the input sizes of both problems are polynomially related). Fortunately, this negative comment has only a general scope (worst case complexity results have been considered), and for many instances, taking advantage of (ir)relevance information can prove quite efficient. Indeed, from the practical side, our complexity results show only that the exploitation of relevance information to improve inference must be done in a careful way. A way to escape from intractability consists in assuming a representation of the knowledge base from which some independence relations can be obtained "for free", or at least in an efficient way. This is what Darwiche achieves with the notion of structured database. While it is not the case that every propositional knowledge base satisfies the locality and modularity conditions of a structured database (see [8] for details), several independence relations can be directly read off from a structured database, and some other ones can be inferred efficiently thanks to the notion of d-separation. As Darwiche states in [8], it is not the case that all the conditional independence relations with respect to a structured database can be found this way. In some sense, our complexity results confirm that focusing on some independence relations, easy to be found, is the good way to do. The same conclusion can be drawn for relevance relations used to characterize what "tells about" means.

[2] Such as abduction, nonmonotonic inference, belief revision, belief update, some forms of planning and decision making.

Last but not least, our paper shows how closely many independence relations pointed out so far in the literature are related to conditional independence. Thus, strong conditional independence, stronger than Darwiche and Pearl's conditional independence, can easily be rewritten using the latter notion (Proposition 3). Formula-variable independence can be viewed as a special case of strong conditional independence (Proposition 12). Simple and strong conditional independence coincide on marginal independence. At the other

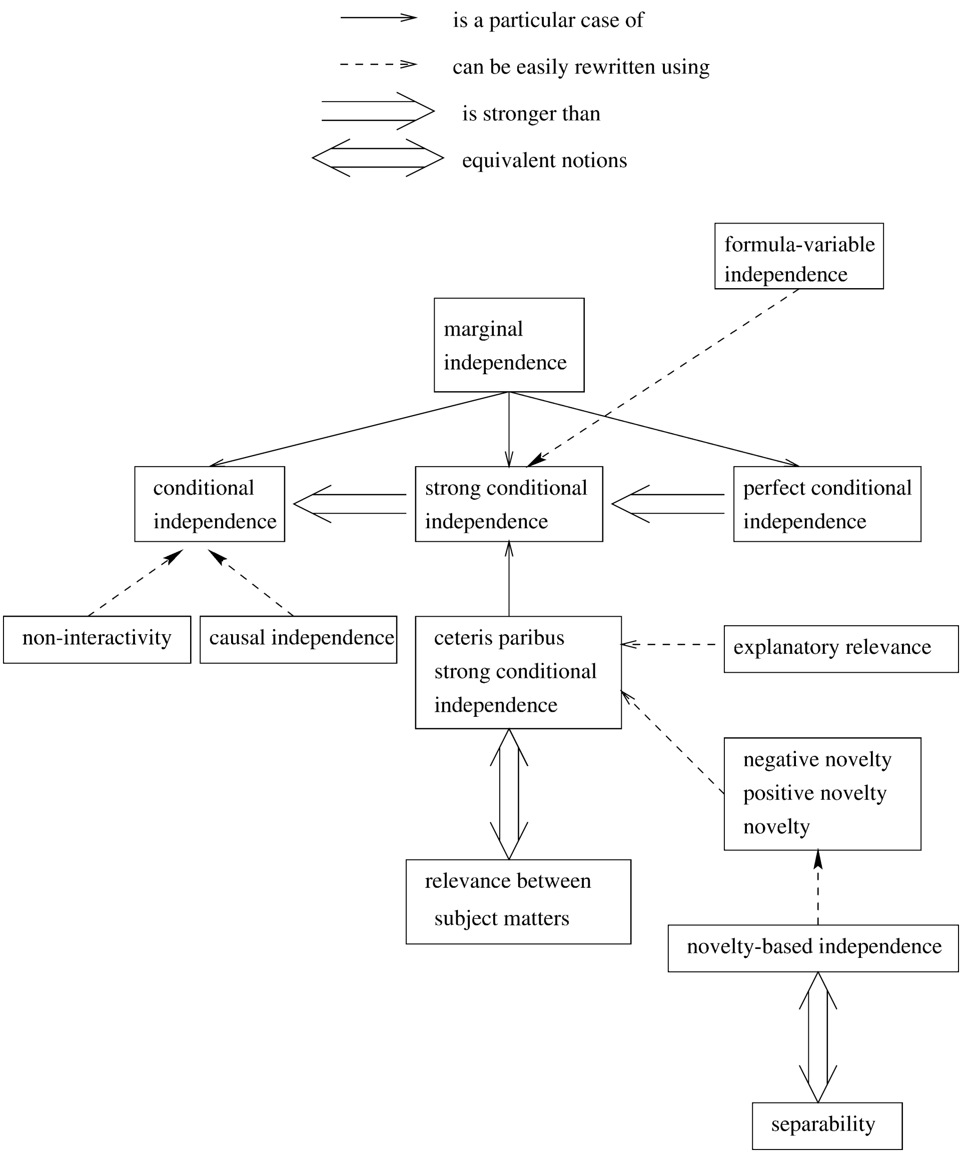


Fig. 1. Connections between (in)dependence relations.

extreme, strong *ceteris paribus* independence is a particularly interesting notion which is equivalent to Lakemeyer's irrelevance between subject matters (Proposition 16). The three notions of novelty are special cases of strong *ceteris paribus* dependence (Proposition 19) and novelty-based independence is a special case of strong *ceteris paribus* independence, which proves to be a special case of Levesque's separability (both coincide for the case of two formulas, see Proposition 21). Finally, there is also a close link between conditional independence and non-interactivity. A synthetic description of the relationships between various definitions is depicted on Fig. 1.

We think that pointing out such close connections is important because (1) babelism is always a bad thing, and (2) known results may appear synergetic. Thus, it is possible to take advantage of results about conditional independence to achieve a better understanding of the other forms of independence considered in this paper. Specifically, we have been able to identify their computational complexity knowing the complexity of conditional independence. Similar synergetic roles can emerge for other concerns, including algorithms and applications. Thus, though the practical computation of many of the independence relations considered in this paper has not been investigated in depth, our results show that it is possible to benefit from Darwiche's computational framework for conditional independence, at least as a starting point.

This work also opens several ways for further research. Especially, it would be interesting to know how the connections between logical conditional independence and conditional independence in ordinal uncertainty calculi could be transposed to the notions of utility independence and preferential independence, as defined in multicriteria decision making and studied from a knowledge representation perspective by Bacchus and Grove [3].

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