Stable and Sequential Functions on Scott domains, dI-domains and FM-domains

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Background

- Plotkin: the full abstraction problem for a sequential functional programming language PCF: start of search for semantic characterization of sequential functions.
- Kahn, Plotkin: *sequential functions* on *concrete data structures* (and *concrete domains*), using *cell* structure. Not closed under sequential function space.
- Berry: *stable functions* on dI-domains, and *stable ordering*. A cartesian closed category, but stability does not imply sequentiality.
- Zhang: a generalized topological definition of stable functions on dI-domains and the stable ordering.
- Berry, Curien: *sequential algorithms* on concrete data structures. A cartesian closed category, but not extensional, does not solve full abstraction for PCF. Sequentiality based on cell structure.
- Bucciarelli, Ehrhard: *sequential algorithms* on *sequential structures*. A cartesian closed category, but does not solve PCF problem. Sequentiality based on extra coherence structure.
- None of these definitions permits a characterization of sequentiality in an arbitrary Scott domain.

Our Contribution

- A new definition of *sequential functions* for Scott domains, characterized by a generalized form of topology. Sequentiality defined intrinsically.
- Considerably expands the class of domains for which sequential functions may be defined.
- Our sequential functions coincide with Kahn-Plotkin sequential functions when restricted to distributive concrete domains.
- The sequential functions between two dI-domains, ordered stably, form a dI-domain.
- The category of dI-domains and sequential functions is not cartesian closed: application is not sequential. We attribute this to certain operational assumptions underlying our notion of sequentiality.
- Scott domains satisfying a "finite meet" property are closed under the pointwise-ordered stable function space, so that we obtain a new stable model based on the pointwise order.
- Towards a class of domains closed under pointwise-ordered sequential function space... and perhaps a solution to the full abstraction problem for PCF?

Generalized Topologies

A generalized topological framework Ω assigns to each domain D a family ΩD of subsets of D, called Ω-open sets, together with an ordering relation $\leq^\Omega$ on ΩD.

- We define the Ω-continuous functions from D to E to be the functions f such that the inverse image $f^{-1}(q)$ of every $q \in \Omega E$ is in ΩD.
- We will order these functions by $f \leq^\Omega g$ iff for every $q \in \Omega E$, $f^{-1}(q) \leq^\Omega g^{-1}(q)$.
- Different orders on Ω-opens will naturally induce different orders on the Ω-continuous functions.
- We obtain a category of domains and Ω-continuous functions: the identity function is always Ω-continuous, and composition preserves Ω-continuity.
- We are mainly interested in showing that a class of domains is closed under Ω-continuous function space. A necessary condition (not always sufficient) is that $(\Omega D, \leq^\Omega)$ belong to the class of domains whenever D does.

Remarks

- ΩD is a topology if
  - $\emptyset$ and D are Ω-open;
  - Ω-open sets are closed under arbitrary unions and finite intersections;
  - The order on ΩD is set inclusion.
- Equivalently, if ΩD is a sub-frame of the powerset lattice of D, ordered by inclusion.

The Scott Topology

As is well known…

- A set $p \subseteq D$ is *Scott open* iff it is upwards closed and for every directed set X, if $\vee X \in p$ then $x \in p$ for some $x \in X$.
- We write $\mathsf{Sc}D$ for the set of Scott opens of D.
- Scott opens, ordered by inclusion, determine the Scott topology.
- For every $x \in D_{\text{fin}}$, $\mathsf{up}(x)$ is Scott open.
- p is Scott open iff $p = \cup\{\mathsf{up}(x) \mid x \in p \cap D_{\text{fin}}\}$.
- A function $f : D \to E$ is Scott continuous, or just *continuous*, iff the inverse image of every Scott open is Scott open.
- Equivalently, a function $f : D \to E$ is continuous iff it is monotone and preserves directed lubs.
- Set inclusion on Scott opens induces an order on continuous functions: $f \leq g$ iff

  $$\forall q \in \mathsf{Sc}E . f^{-1}(q) \subseteq g^{-1}(q).$$

  This is the pointwise order: $f \leq g$ iff $\forall x \in D . f(x) \leq g(x)$.

Stable Opens and Stable Functions

- A set $p \subseteq D$ is *stable* iff it is closed under consistent meets, *i.e.*, $x_1, x_2 \in p$ and $x_1 \Uparrow x_2$ imply $x_1 \wedge x_2 \in p$.
- A set p is *stable open* iff it is Scott open and stable.
- We write $\mathsf{St}D$ for the set of stable opens of D.
- For any $x \in D_{\text{fin}}$, $\mathsf{up}(x)$ is stable open.
- A function $f : D \to E$ is stable continuous, or *stable*, iff the inverse image of every stable open is stable open.
- For a function $f : D \to E$, the following are equivalent:

  (1) f is stable.

  (2) f is continuous and preserves consistent meets: if $x_1 \Uparrow x_2$ then $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2)$.

  (3) f is continuous and whenever $e \leq f(d)$, the set $\{d' \in D \mid d' \leq d \; \& \; e \leq f(d')\}$ is down-directed.
- Definition (3) specializes in dI-domains to the usual "minimum point" definition of stable functions: f is stable iff it is continuous and for every $e \leq f(d)$ the set $\{d' \leq d \mid e \leq f(d)\}$ has a least element.
- Our treatment extends Zhang's characterization of "stable neighborhoods".

Scott is not always stable

- Every stable open is also Scott open, by definition.
- The converse fails. For example, the Scott open set

  $$\mathsf{up}(\{(\top,\bot),(\bot,\top)\}) \subseteq 2\times 2,$$

  is not stable, because it does not contain

  $$(\bot,\bot) = (\top,\bot)\wedge(\bot,\top),$$

  and this is a consistent meet.
- Every stable function is also Scott continuous.
- The converse fails. For example, the parallel-or function is continuous but not stable. The inverse image

  $$\mathsf{por}^{-1}(\{\mathsf{tt}\}) = \{(\mathsf{tt},\bot),(\bot,\mathsf{tt})\}$$

  is not stable open.

Lobes of a Stable Set

- A stable set p can be partitioned by identifying all pairs of points of p that have a lower bound in p.
- We call the equivalence classes the *lobes* of p.
- A lobe is downwards-directed.
- In a dI-domain every lobe has a least element.
- In a Scott domain lobes may fail to contain their glb.

Covering, covers and indices

- The *covering relation* between elements of D is: $x \prec y$ iff $x < y$ and there is no point between x and y.

- A *cover* of $x \in D$ is a stable set r such that $x < y$ for every $y \in r$ and $\Delta(x, r) = \emptyset$, where

  $$\Delta(x, r) = \{z \mid x < z \;\&\; \exists r' \in \mathsf{lobes}(r) \;.\; \forall y \in r' \;.\; z < y\}\,.$$

  We write $\mathsf{I}(x)$ for the set of covers of x.

- Equivalently, a stable set r is a cover of x iff for every lobe r' of r, either r' has a least element y and $x \prec y$, or r' has no least element and $x = \wedge r'$.

- For $x \in D$ and $s \subseteq D$, an *index* of s at x is a cover r of x such that $s \cap \mathsf{up}(x) \subseteq r$.

- Let $\mathsf{I}(x, s)$ be the set of indices of s at x:

  $$\mathsf{I}(x, s) = \{r \in \mathsf{I}(x) \mid s \cap \mathsf{up}(x) \subseteq r\}\,.$$

Intuition

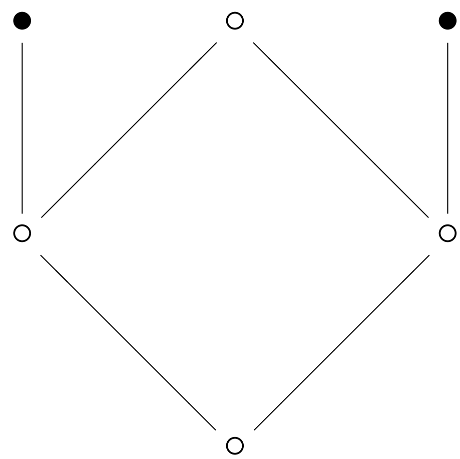
- A stable set s represents a choice between its lobes.
- If the current state of information is x, a cover of x represents an atomic increase in information content, with atomicity captured by the condition $\Delta(x, r) = \emptyset$.
- A cover r of x provides a way of locally decomposing the domain at x into a flat domain, with x as the least element and the lobes of r as the proper elements.
- Covers may be used to reason about the progress of an incremental computation, generalizing the notion of cell in a concrete data structure.
- The existence of an index $r \in \mathsf{I}(x, s)$ indicates that the choice represented by s may be decomposed, with the index r serving as a first step from x towards s.

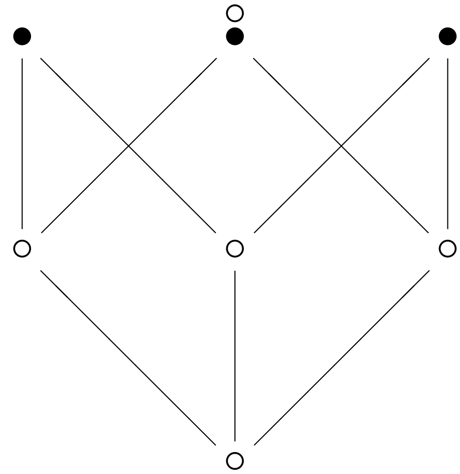
Some Obvious Properties

- $\Delta(x, \emptyset) = \emptyset$.
- $\Delta(x, r) = \cup\{\Delta(x, r') \mid r' \in \mathsf{lobes}(r)\}$.
- $\emptyset \in \mathsf{I}(x, \emptyset)$.
- $\mathsf{I}(x, s) = \mathsf{I}(x, s \cap \mathsf{up}(x))$.

Stable is not always sequential

- In these domains the shaded points form a stable open set with no index at $\perp$, since the shaded points are not contained in any cover of $\perp$.





- Another example of a stable open with no index at $\perp$:

  $\mathsf{up}(\{(\mathsf{tt}, \mathsf{ff}, \perp), (\perp, \mathsf{tt}, \mathsf{ff}), (\mathsf{ff}, \perp, \mathsf{tt})\}) \subseteq \mathsf{Bool}\times\mathsf{Bool}\times\mathsf{Bool}.$

- Absence of an index implies non-sequentiality…

Sequential Opens

- A set $p \subseteq D$ is *sequential at* $x \in D$ iff $x \in p$, or $x \notin p$ and for every finite $s \subseteq p$, $\mathsf{I}(x, s) \neq \emptyset$.
- A set p is *sequential* iff it is sequential at every $x \in D_{\text{fin}}$.
- A *sequential open* is a stable open that is sequential.
- We write $\mathsf{Sq}D$ for the set of sequential opens of D.
- For any $x \in D_{\text{fin}}$, $\mathsf{up}(x)$ is sequential open.
- If $x < y$ then $\mathsf{I}(x, \mathsf{up}(y)) \neq \emptyset$.

Sequential Functions

- A function $f : D \to E$ is *sequential* iff the inverse image of every sequential open is sequential open.

Properties

- Every sequential function is Scott-continuous.
- Every sequential function is stable.

Examples

- The doubly-strict-or function $\mathsf{sor} : \mathsf{Bool}^2 \to \mathsf{Bool}$ is sequential (and stable).
  - The inverse image of the sequential open set $\{\mathsf{tt}\}$ is the sequential open set $p = \{(\mathsf{tt}, \mathsf{tt}), (\mathsf{tt}, \mathsf{ff}), (\mathsf{ff}, \mathsf{tt})\}$.
  - There are two indices of p at $(\bot, \bot)$: $\mathsf{up}(\{(\mathsf{tt}, \bot), (\mathsf{ff}, \bot)\})$ and $\mathsf{up}(\{(\bot, \mathsf{tt}), (\bot, \mathsf{ff})\})$.
  - These two indices at $(\bot, \bot)$ correspond to the fact that this function is strict in both arguments.
- The left-strict-or function lor is also sequential. There is a single index $\mathsf{up}(\{(\mathsf{tt}, \bot), (\mathsf{ff}, \bot)\})$ for $\mathsf{lor}^{-1}(\{\mathsf{tt}\})$ at $(\bot, \bot)$.
- The parallel-or function $\mathsf{por} : \mathsf{Bool}^2 \to \mathsf{Bool}$ is not sequential, since the inverse image of $\{\mathsf{tt}\}$ is not sequential open (and not even stable).

Stable is not always sequential

- Let $\mathsf{gf} : \mathsf{Bool}^3 \to \mathsf{Bool}$ be the least continuous function such that

$$\begin{aligned} \mathsf{gf}(\mathsf{tt},\mathsf{ff},\perp) &= \mathsf{tt} \\ \mathsf{gf}(\perp,\mathsf{tt},\mathsf{ff}) &= \mathsf{tt} \\ \mathsf{gf}(\mathsf{ff},\perp,\mathsf{tt}) &= \mathsf{tt} \\ \mathsf{gf}(\mathsf{ff},\mathsf{ff},\mathsf{ff}) &= \mathsf{ff}. \end{aligned}$$

  This function is stable but not sequential. The stable open set $\mathsf{gf}^{-1}(\{\mathsf{tt}\}) = \mathsf{up}(\{(\mathsf{tt},\mathsf{ff},\perp),(\mathsf{ff},\perp,\mathsf{tt}),(\perp,\mathsf{tt},\mathsf{ff})\})$ is not sequential open, since it has no index at $(\perp,\perp,\perp)$.

- Let $\mathsf{gf}_1, \mathsf{gf}_2, \mathsf{gf}_3 : \mathsf{Bool}^3 \to \mathsf{Bool}$ map $(\mathsf{ff},\mathsf{ff},\mathsf{ff})$ to ff, and satisfy

$$\begin{aligned} \mathsf{gf}_1(\mathsf{tt},\mathsf{ff},\perp) &= \mathsf{tt} \\ \mathsf{gf}_2(\perp,\mathsf{tt},\mathsf{ff}) &= \mathsf{tt} \\ \mathsf{gf}_3(\mathsf{ff},\perp,\mathsf{tt}) &= \mathsf{tt}. \end{aligned}$$

  Let their pairwise lubs be $\mathsf{gf}_{1,2} = \mathsf{gf}_1 \vee \mathsf{gf}_2$, $\mathsf{gf}_{1,3} = \mathsf{gf}_1 \vee \mathsf{gf}_3$, and $\mathsf{gf}_{2,3} = \mathsf{gf}_2 \vee \mathsf{gf}_3$. All of these functions are sequential.

- Since $\mathsf{gf} = \mathsf{gf}_1 \vee \mathsf{gf}_2 \vee \mathsf{gf}_3$, this shows that a pairwise consistent set of sequential functions need not have a sequential lub. This works with either stable or pointwise order, since the orders coincide in this case. As a corollary, concrete domains are not closed under sequential function space.

Products

- The categories of Scott domains and (respectively) continuous, stable and sequential functions are cartesian.

- The projection functions $\pi_i : D_1 \times D_2 \to D_i$, for $i = 1, 2$, are sequential.

- For Scott domains D_1 and D_2,

$$\begin{aligned} \mathsf{Sc}(D_1 \times D_2) &= \{p_1 \times p_2 \mid p_1 \in \mathsf{Sc}\, D_1 \ \&\ p_2 \in \mathsf{Sc}\, D_2\} \\ \mathsf{St}\,(D_1 \times D_2) &\supseteq \{p_1 \times p_2 \mid p_1 \in \mathsf{St}\, D_1 \ \&\ p_2 \in \mathsf{St}\, D_2\} \\ \mathsf{Sq}(D_1 \times D_2) &\supseteq \{p_1 \times p_2 \mid p_1 \in \mathsf{Sq} D_1 \ \&\ p_2 \in \mathsf{Sq} D_2\} \end{aligned}$$

- Stable or sequential opens of $D_1 \times D_2$ may not be formed by a product of stable or sequential opens of D_1 and D_2.

- For example, let $p = \mathsf{up}\,\{((\mathsf{tt}, \bot), \mathsf{tt}), ((\bot, \mathsf{tt}), \mathsf{ff})\}$. While p is stable and sequential, $\pi_1(p) = \mathsf{up}\,\{(\mathsf{tt}, \bot), (\bot, \mathsf{tt})\}$ is neither stable nor sequential.

Relationship to Kahn-Plotkin

In a distributive concrete domain D,

(1) Every non-empty cover r of x corresponds to a unique cell c accessible from x and filled in all elements of r.

(2) For every Scott open p and $x \notin p$, every finite subset s of p has an index at x iff p itself has an index at x.

(3) For every sequential open p the set C of cells that are filled in all elements of p is finite. If $p \neq \emptyset$ and $p \neq \mathsf{up}(\bot)$, C is non-empty.

For every finite set of cells C, the set of states that fill all cells in C is sequential open.

(4) A Scott open p is sequential at every isolated point iff it is sequential at every point.

Theorem

For distributive concrete domains D and E, a function $f : D \to E$ is sequential iff it is sequential in the Kahn-Plotkin sense.

In other words. . .

- That is, f is sequential iff it is continuous and for every state x of D, either no cell is accessible from x, or for every cell c' accessible from $f(x)$ there is a cell c accessible from x such that c is filled in all states $y \supseteq x$ such that c' is filled in $f(y)$.

The Pointwise Order

Stable

- Set inclusion on stable opens induces the pointwise order on stable functions.
- The union of a (set inclusion) directed family of stable opens is stable open.
- The pointwise lub of a (pointwise) directed family of stable functions is a stable function.

Sequential

- Set inclusion on sequential opens induces the pointwise order on sequential functions.
- The union of a (set inclusion) directed family of sequential opens is sequential open.
- The pointwise lub of a (pointwise) directed family of sequential functions is a sequential function.

Problem

Berry: application fails to be stable (or sequential) under the pointwise order, but is stable wrt the stable order.

The Stable Order

- The *lobe inclusion* order on stable opens is given by: $p_1 \sqsubseteq p_2$ iff $\mathsf{lobes}(p_1) \subseteq \mathsf{lobes}(p_2)$.
- This induces the *stable order* on stable functions, defined by: $f \sqsubseteq g$ iff for every $q \in \mathsf{St}E$, $f^{-1}(q) \sqsubseteq g^{-1}(q)$.
- We write $(D \to^{\mathrm{st}} E, \sqsubseteq)$ for the stably-ordered stable function space.
- For any stable functions $f, g : D \to E$, the following are equivalent:

  (1) $f \sqsubseteq g$.

  (2) $f \leq g$ and $f(x) = g(x) \wedge f(y)$ for every $x \leq y$.

  (3) $f \leq g$ and $f(x) \wedge g(y) = g(x) \wedge f(y)$ for every $x \Uparrow y$.

  (4) $f \leq g$ and, for every $d \in D$ and $e \leq f(d)$,

  $$\{d' \leq d \mid e \leq f(d')\} = \{d' \leq d \mid e \leq g(d')\}\,.$$

- Thus our stable order generalizes Berry's and Zhang's definition of stable order, which were based on dI-domains.

Sequential Functions and Stable Order

- If p is stable open, p' is sequential open, and $p \sqsubseteq p'$, then p is sequential open.
- If f is stable, g is sequential, and $f \sqsubseteq g$, then f is sequential.
- The isolated elements of $(D \to^{\mathrm{sq}} E, \sqsubseteq)$ are the isolated elements of $(D \to^{\mathrm{st}} E, \sqsubseteq)$ that are also sequential.
- dI-domains are closed under the stably-ordered sequential function space.
- This improves on earlier results for KP-sequentiality:
  - KP-sequential functions only defined on concrete domains.
  - Concrete domains not closed under stably-ordered sequential function space.

Application is not Sequential

- $\mathsf{app} : (\mathsf{Bool}^3 \to \mathsf{Bool}) \times \mathsf{Bool}^3 \to \mathsf{Bool}$
- Not sequential: $p = \mathsf{app}^{-1}(\{\mathsf{tt}\})$ has no index at $x = (\mathsf{gf}_1, \bot, \bot, \bot)$.
  - Any cover r of x must have one of the forms:

$$\begin{aligned} r &= r_1 \times \mathsf{up}(\bot) \times \mathsf{up}(\bot) \times \mathsf{up}(\bot) \\ r &= \mathsf{up}(\mathsf{gf}_1) \times r_2 \times \mathsf{up}(\bot) \times \mathsf{up}(\bot) \\ r &= \mathsf{up}(\mathsf{gf}_1) \times \mathsf{up}(\bot) \times r_2 \times \mathsf{up}(\bot) \\ r &= \mathsf{up}(\mathsf{gf}_1) \times \mathsf{up}(\bot) \times \mathsf{up}(\bot) \times r_2, \end{aligned}$$

    where r_1 covers gf_1 and r_2 covers $\bot$ in Bool.
  - In first case, the element $(\mathsf{gf}_1, \mathsf{tt}, \mathsf{ff}, \bot)$ of $p \cap \mathsf{up}(x)$ is not in r.
  - In the other cases we can also find elements of $p \cap \mathsf{up}(x)$ that are not contained in r.
  - Hence $\mathsf{I}(x, p)$ is empty and p is not sequential open.
- Application is not sequential since when we know that the function is at least gf_1 we can't tell what needs to be evaluated further.
- Failure seems caused by assumption that functions are computed incrementally, as in Kahn-Plotkin.

FM-domains

- A Scott domain has the *finite meet* property (FM) iff the meet of every pair of isolated elements is isolated.
- An FM-domain is a Scott domain with property FM.
- dI-domains are FM-domains.
- The converse is not generally true, and FM-domains are a proper intermediate notion, between Scott domains and dI-domains.
- The following are equivalent in an FM-domain:
  (1) p is sequential open.
  (2) p is Scott open and is sequential at every finite point.

Theorem

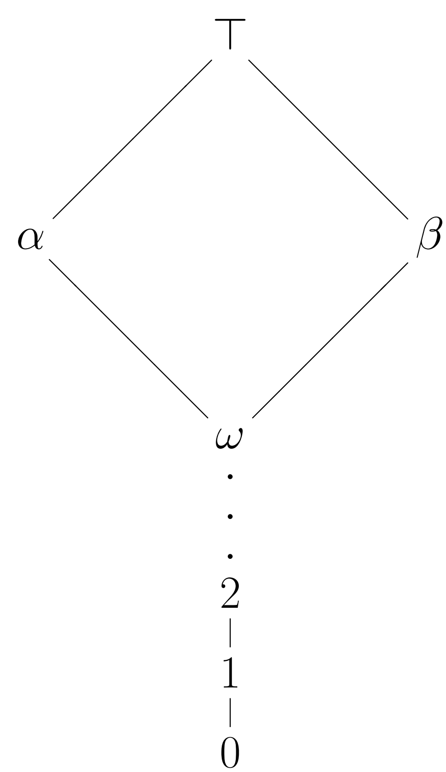
- FM-domains are closed under product and under continuous function space, so FM-domains and continuous functions are a sub-ccc of the ccc of Scott domains and continuous functions.
- All domains occurring in the Scott continuous functions model of PCF are FM-domains.

Stable Functions on FM-domains

- FM-domains are closed under the pointwise-ordered stable function space.
- This improves on a result that the pointwise-ordered stable function space between dI-domains is a Scott domain (Berry).
- We restrict to FM-domains, because the poset of stable opens, ordered by inclusion, is not bounded complete for general Scott domains.

Example

- For example, consider the following Scott domain, where ω is the limit of an infinite ascending chain, and all other elements are isolated. The stable opens $\mathsf{up}(\alpha)$ and $\mathsf{up}(\beta)$ are upper-bounded under inclusion, but have no lub.



Stable Completion in FM-domains

- For a Scott-open set p in an FM-domain D, define

$$\begin{aligned}
\mathsf{stc}(p) &= \mathsf{up}\,\{x_1 \wedge x_2 \mid x_1, x_2 \in p \ \&\ x_1 \Uparrow x_2\} \\
\mathsf{stc}^0(p) &= p \\
\mathsf{stc}^{n+1}(p) &= \mathsf{stc}(\mathsf{stc}^n(p)) \\
\mathsf{stc}^*(p) &= \cup\,\{\mathsf{stc}^n(p) \mid n \geq 0\}\,.
\end{aligned}$$

- For any Scott-open p,
  - $\mathsf{stc}(p)$ is Scott-open;
  - $p \subseteq \mathsf{stc}(p)$;
  - $\mathsf{stc}^*(p)$ is the least stable open that contains p.

- For a function $f : D \to E$ and $x \in D$, define

$$\begin{aligned}
\mathsf{stc}(f)(x) &= \vee\{f(z_1) \wedge f(z_2) \mid z_1, z_2 \in D_{\mathrm{fin}} \ \&\\
&\qquad\qquad z_1 \Uparrow z_2 \ \&\ z_1 \wedge z_2 \leq x\} \\
\mathsf{stc}^0(f) &= f \\
\mathsf{stc}^{n+1}(f) &= \mathsf{stc}(\mathsf{stc}^n(f)) \\
\mathsf{stc}^*(f) &= \vee\,\{\mathsf{stc}^n(f) \mid n \geq 0\}\,.
\end{aligned}$$

- If $f : D \to E$ is continuous and f is dominated by a stable function h, then
  - $\mathsf{stc}(f)$ is a continuous function;
  - $f \leq \mathsf{stc}(f) \leq h$;
  - $\mathsf{stc}^*(f)$ is the least stable function that dominates f.

Properties

- The lub of a bounded set F of stable functions is $\mathsf{stc}^*(\vee F)$, where $\vee F$ is the pointwise lub.
- If f is isolated in $D \rightarrow^{\mathrm{ct}} E$ then $\mathsf{stc}(f)$ and $\mathsf{stc}^*(f)$ are isolated, and $\mathsf{stc}^*(f) = \mathsf{stc}^n(f)$ for some n.
- The isolated elements of $D \rightarrow^{\mathrm{st}} E$ are the isolated elements of $D \rightarrow^{\mathrm{ct}} E$ that are stable.
- The pointwise meet of two stable functions is stable.
- For any FM-domains D and E, $D \rightarrow^{\mathrm{st}} E$ is an FM-domain.

Sequential Functions on FM-domains

- If D is an FM-domain and E is a flat domain then the sequential functions from D to E, ordered pointwise, forms an FM-domain.

Further Research

- Our notion of sequentiality works well at first-order types.
- Would like to develop an extension to deal adequately with higher-order types. A suitable higher-order notion of sequentiality must not rely on the Kahn-Plotkin operational assumption.
- It seems essential that the syntactic *type* of a function be used in defining sequentiality, not just the domain structure.
- We are currently working out the details of a definition of sequentiality at type $\tau \to \tau'$ using the above definition at first-order types. This would make application sequential.
- We conjecture that there is a (non-trivial) sub-class of the FM-domains that is closed under the pointwise-ordered sequential function space.
- These developments may lead to a fully abstract sequential model…?