Catmull-Rom splines

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1 Definition

Catmull-Rom splines are a family of cubic interpolating splines formulated such that the tangent at each point $\mathbf{p}_i$ is calculated using the previous and next point on the spline, $\tau(\mathbf{p}_{i+1} - \mathbf{p}_{i-1})$. The geometry matrix is given by

$$\mathbf{p}(s) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\tau & 0 & \tau & 0 \\ 2\tau & \tau - 3 & 3 - 2\tau & -\tau \\ -\tau & 2 - \tau & \tau - 2 & \tau \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-2} \\ \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}$$

Catmull-Rom splines have C^1 continuity, local control, and interpolation, but do *not* lie within the convex hull of their control points.

Note that the tangent at point $\mathbf{p}_0$ is not clearly defined; oftentimes we set this to $\tau(\mathbf{p}_1 - \mathbf{p}_0)$ although this is not necessary for the assignment (you can just assume the curve does not interpolate its endpoints).

The parameter τ is known as "tension" and it affects how sharply the curve bends at the (interpolated) control points (figure 2). It is often set to 1/2 but you can use any reasonable value for this assignment.

2 Derivation

Consider a single Catmull-Rom segment, $\mathbf{p}(s)$. Suppose it is defined by 4 control points, $\mathbf{p}_{i-2}$, $\mathbf{p}_{i-1}$, $\mathbf{p}_i$, and $\mathbf{p}_{i+1}$, as in figure 3. We know that since it is cubic, it can be expressed by the polynomial form,

$$\mathbf{p}(s) = \mathbf{c}_0 + \mathbf{c}_1 u + \mathbf{c}_2 u^2 + \mathbf{c}_3 u^3 \tag{1}$$

$$= \sum_{k=0}^{3} \mathbf{c}_k u^k \tag{2}$$

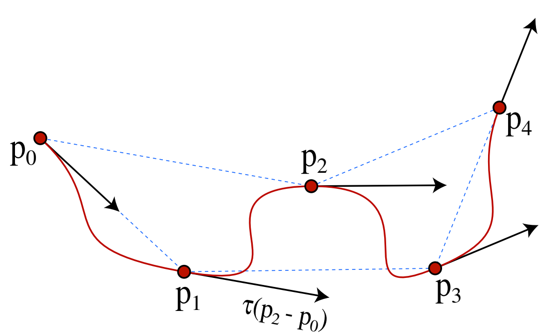

Figure 1: A Catmull-Rom spline

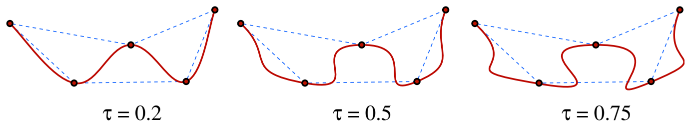

Figure 2: The effect of τ

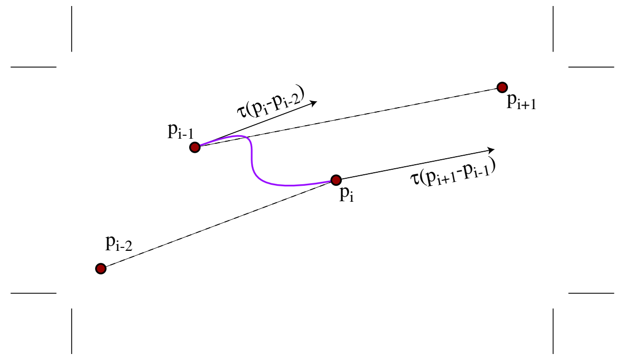

Figure 3: Catmull-Rom spline derivation

Now, we need to express some constraints. Examining figure 3, we find the following,

$$\begin{aligned}
\mathbf{p}(0) &= \mathbf{p}_{i-1} \\
\mathbf{p}(1) &= \mathbf{p}_i \\
\mathbf{p}'(0) &= \tau(\mathbf{p}_i - \mathbf{p}_{i-2}) \\
\mathbf{p}'(1) &= \tau(\mathbf{p}_{i+1} - \mathbf{p}_{i-1})
\end{aligned}$$

(you should make sure you understand where these come from). Now, we can combine these constraints with (2) to get the following,

$$\mathbf{c}_0 = \mathbf{p}_{i-1} \tag{3}$$

$$\mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 = \mathbf{p}_i \tag{4}$$

$$\mathbf{c}_1 = \tau(\mathbf{p}_i - \mathbf{p}_{i-2}) \tag{5}$$

$$\mathbf{c}_1 + 2\mathbf{c}_2 + 3\mathbf{c}_3 = \tau(\mathbf{p}_{i+1} - \mathbf{p}_{i-1}) \tag{6}$$

Now, we can substitute (3) and (5) into (4) and (6) to get equations in where $\mathbf{c}_2$ and $\mathbf{c}_3$ are the only variables.

$$\begin{aligned}
\mathbf{c}_2 + \mathbf{c}_3 &= (\mathbf{p}_i - \mathbf{p}_{i-1}) - \tau(\mathbf{p}_i - \mathbf{p}_{i-2}) \\
2\mathbf{c}_2 + 3\mathbf{c}_3 &= \tau(\mathbf{p}_{i+1} - \mathbf{p}_{i-1}) - \tau(\mathbf{p}_i - \mathbf{p}_{i-2})
\end{aligned}$$

Now, it should be clear that by subtracting these two equations we can easily solve for $\mathbf{c}_2$ and $\mathbf{c}_3$ in terms of the $\mathbf{p}_j$ and τ (which is also user-specified). We get

$$\begin{aligned}
\mathbf{c}_0 &= \mathbf{p}_{i-1} \\
\mathbf{c}_1 &= \tau(\mathbf{p}_i - \mathbf{p}_{i-2}) \\
\mathbf{c}_2 &= 3(\mathbf{p}_i - \mathbf{p}_{i-1}) - \tau(\mathbf{p}_{i+1} - \mathbf{p}_{i-1}) - 2\tau(\mathbf{p}_i - \mathbf{p}_{i-2}) \\
\mathbf{c}_3 &= -2(\mathbf{p}_i - \mathbf{p}_{i-1}) + \tau(\mathbf{p}_{i+1} - \mathbf{p}_{i-1}) + \tau(\mathbf{p}_i - \mathbf{p}_{i-2})
\end{aligned}$$

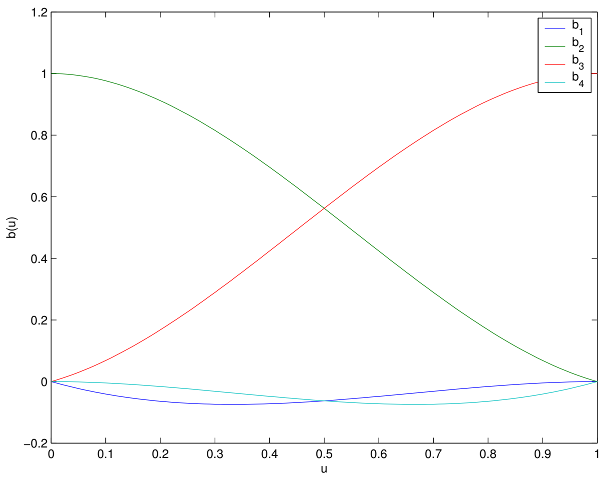

Figure 4: Catmull-Rom blending functions for $\tau = \frac{1}{2}$

Now, if we want to form a basis matrix out of this, we should group all the $\{\mathbf{p}_j\}$ terms together,

$$\begin{aligned}
\mathbf{c}_0 &= \mathbf{p}_{i-1} \\
\mathbf{c}_1 &= (-\tau)\mathbf{p}_{i-2} + (\tau)\mathbf{p}_i \\
\mathbf{c}_2 &= (2\tau)\mathbf{p}_{i-2} + (\tau-3)\mathbf{p}_{i-1} + (3-2\tau)\mathbf{p}_i + (-\tau)\mathbf{p}_{i+1} \\
\mathbf{c}_3 &= (-\tau)\mathbf{p}_{i-2} + (2-\tau)\mathbf{p}_{i-1} + (\tau-2)\mathbf{p}_i + (\tau)\mathbf{p}_{i+1}
\end{aligned}$$

From this it should be fairly clear how to fill out the geometry matrix,

$$\mathbf{p}(s) = \mathbf{u}^T\mathbf{M}\mathbf{p} = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\tau & 0 & \tau & 0 \\ 2\tau & \tau-3 & 3-2\tau & -\tau \\ -\tau & 2-\tau & \tau-2 & \tau \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-2} \\ \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}$$

From this we can also derive our blending functions (see figure 4),

$$\mathbf{M}\mathbf{u} = \begin{bmatrix} -\tau u + 2\tau u^2 - \tau u^3 \\ 1 + (\tau-3)u^2 + (2-\tau)u^3 \\ \tau u + (3-2\tau)u^2 + (\tau-2)u^3 \\ -\tau u^2 + \tau u^3 \end{bmatrix}$$

References

[1] Catmull, E., and Rom, R. A class of local interpolating splines. In *Computer Aided Geometric Design*, R. E. Barnhill and R. F. Reisenfeld, Eds. Academic Press, New York, 1974, pp. 317–326.

[2] Watt, A., and Watt, M. *Advanced Animation and Rendering Techniques: Theory and Practice*. ACM Press, 1992.