The n-dimensional cotangent formula

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Abstract. The Laplace-Beltrami operator Δ plays a central role in geometric algorithms on curved domains. The so-called *cotangent formula* provides a convenient approximation of the Laplace-Beltrami operator for 2-dimensional triangulated surfaces. This note extends the cotangent formula to the n-dimensional case. In particular, we give an expression for the gradient of the volume of an n-dimensional simplex in terms of the volumes of its faces and the cotangents of its dihedral angles, which in turn provides an n-dimensional discrete Laplace operator. As an important special case we give a convenient expression for tetrahedral meshes ($n = 3$).

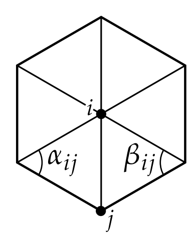
1 Introduction

To motivate the cotangent formula, consider solving the Poisson equation

$$\Delta u = f$$

on a curved surface (M, g). Here $f : M \to \mathbb{R}$ is a source term, $u : M \to \mathbb{R}$ is an unknown function, and Δ is the Laplace-Beltrami operator associated with M—or more simply, just the *Laplacian*. A common way to approximate the solution is to replace M by a triangulation K with vertices V and edges E, and solve the matrix equation

$$\mathsf{L}\mathsf{u} = \mathsf{M}\mathsf{f}.$$



Here $\mathsf{u}, \mathsf{f} \in \mathbb{R}^{|V|}$ encode the two functions, and $\mathsf{M}, \mathsf{L} \in \mathbb{R}^{|V|\times|V|}$ are known as the *mass matrix* and *stiffness matrix*, respectively. There are many possible choices for these two matrices, and significant work has gone into developing discretizations that yield an accurate solution. In practice, a very simple discretization can actually work quite well—especially on fine triangulations with "nice" elements (*e.g.*, those that satisfy the so-called *Delaunay condition*). For instance, a reasonable choice for the mass matrix M is a diagonal matrix where M_{ii} equals one-third the area of all incident triangles. A standard choice for the stiffness matrix L is to use the 2-dimensional *cotangent formula*

$$(\mathsf{L}\mathsf{u})_i := \tfrac{1}{2}\sum_{ij}(\cot\alpha_{ij} + \cot\beta_{ij})(\mathsf{u}_j - \mathsf{u}_i).$$

In other words: the Laplacian of the function u at vertex i is obtained by summing up the difference across all edges ij incident on i, weighted by the sum of the cotangents of the two interior angles α_{ij}, β_{ij} opposite ij.

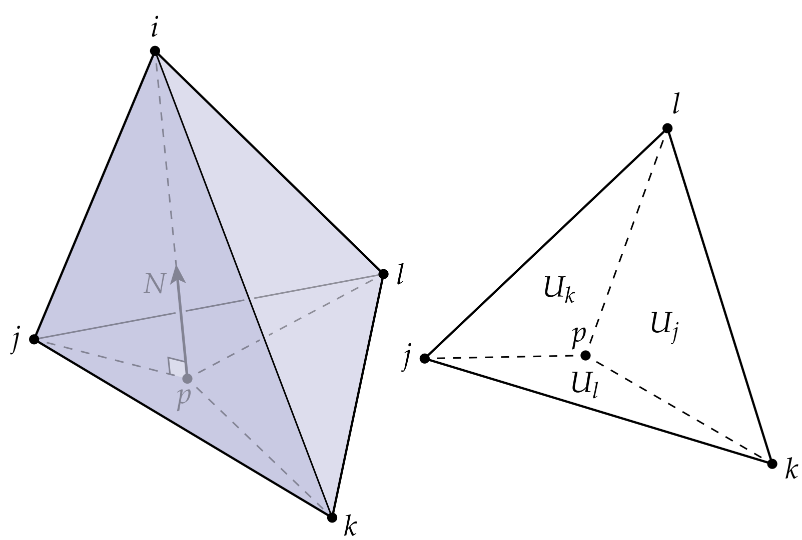
The cotangent formula can be derived in many different ways, and has been re-discovered many times over the years. The earliest known reference is MacNeal [1949, Section 3.2], where the formula is used to determine resistances in an electrical network whose voltages approximate the solution to a Laplace equation. Despite the age of this reference, it provides a rather clear description of the cotangent formula—including boundary conditions that are not described even by much later references. A footnote on page 68 credits Dr.

Stanley Frankel for the actual cotan construction, suggesting that the formula was likely known even prior to MacNeal's thesis. (Courant and others also discuss the use of triangulations to solve partial differential equations [Courant, 1943], but never appear to give the cotangent formula itself.) Duffin [1959] revisits the cotangent formula in the context of electrical networks, making again an explicit connection to the Dirichlet problem. Pinkall and Polthier [1993] apply the formula to the construction of discrete minimal surfaces, *i.e.*, "soap films" interpolating a given boundary curve; Desbrun et al. [1999] derive the formula in the context of discrete mean curvature flow; it also shows up in *discrete exterior calculus* [Desbrun et al., 2008] as a discretization of the Hodge star operator on differential 1-forms. Meyer et al. [2003] discuss 3-manifolds embedded in n dimensions, but omits a critical length term. Chao et al. [2010] and Jacobson [2013] give the correct 3-dimensional formula in the context of elasticity and function interpolation, respectively. Given the long history of the cotangent formula, it is quite possible that the n-dimensional version given here also appears somewhere in the literature—though it is difficult to track down such a formula in a concise and easily-implementable form. Formulas derived via linear finite element theory are typically expressed via the Jacobian of a mapping to some reference element; generic formulas provided via discrete exterior calculus are given abstractly in terms of primal-dual volume ratios, which must still somehow be evaluated.

In this note we take the following approach to obtain an n-dimensional cotangent formula. The first order variation of the volume of an immersed n-dimensional hypersurface $f : M \to \mathbb{R}^{n+1}$ is equal to the mean curvature $H = \kappa_1 + \cdots + \kappa_n$ times the surface normal N (where κ_i are the principal curvatures). We also have (in any dimension) the relationship $\Delta f = HN$. Therefore, we can use the discrete volume variation to obtain a "mimetic" expression for the discrete Laplacian, in the sense of discrete differential geometry [Crane and Wardetzky, 2017]. We first derive an expression for the gradient of the volume of a single n-simplex; the cotangent formula for the discrete Laplace-Beltrami operator is then obtained by summing these values over appropriate simplices. Although the calculation is extrinsic, the resulting formula is valid in the purely intrinsic setting (*e.g.*, when one has edge lengths but not vertex coordinates).

2 Volume gradient

2.1 3-dimensional case



We first consider the 3-dimensional case, since the argument easily generalizes to n dimensions (whereas the 2-dimensional argument is a bit too elementary to illustrate the general idea). Consider a tetrahedron $ijkl$ with vertex coordinates f_i, f_j, f_k, f_l in Euclidean 3-space E^3. Then the volume of the tetrahedron can be expressed as

$$V = \frac{1}{3}Ah,$$

where A is the area of any base triangle—say, jkl—and h is the corresponding height, *i.e.*, the length of the altitude passing through i. Since moving i parallel to the base does not change the volume, and a motion along the altitude changes only the height, the gradient with respect to the position of vertex i is given by

$$\nabla_{f_i} V = \frac{1}{3} AN,$$

where N is the inward unit normal at the base. If p is the foot of the altitude through i, then we can express this normal as

$$N = \frac{1}{h}(f_i - f_p).$$

The foot splits the base into three smaller triangles—letting U_j, U_k, and U_l be the areas of triangles pkl, plj, and pjk, respectively, we can express the location of p in barycentric coordinates

$$f_p = \frac{U_j}{A} f_j + \frac{U_k}{A} f_k + \frac{U_l}{A} f_l.$$

Noting that $A = U_j + U_k + U_l$, we then get

$$\begin{aligned} 3\nabla_{f_i} V &= \tfrac{A}{h}(f_i - f_p) \\ &= \tfrac{1}{h}((U_j + U_k + U_l) f_i - U_j f_j - U_k f_k - U_l f_l) \\ &= \tfrac{U_j}{h}(f_i - f_j) + \tfrac{U_k}{h}(f_i - f_k) + \tfrac{U_l}{h}(f_i - f_l). \end{aligned}$$

All that remains is to evaluate the ratio of the small triangle areas to the height h. Consider for instance triangle pkl, and let q be the foot of the altitude through p. The area of this triangle can be expressed as

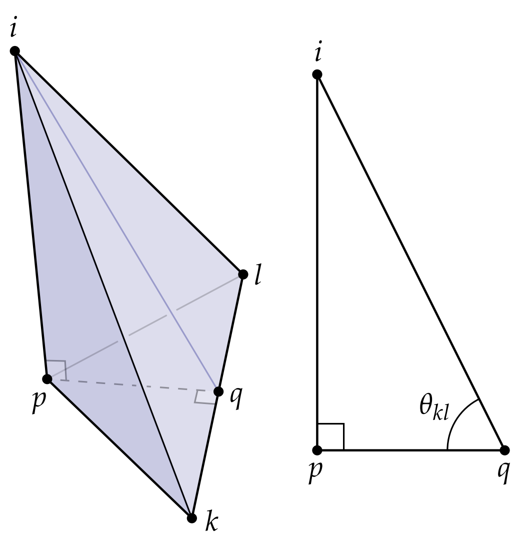
$$U_j = \frac{1}{2} \ell_{kl} |f_p - f_q|,$$

where $\ell_{kl} = |f_l - f_k|$ is the length of edge kl (*i.e.*, just one half base times height). Then since $|f_p - f_q|$ and $h = |f_i - f_p|$ are the legs of a right triangle, we get

$$\frac{U_j}{h} = \frac{1}{2} \ell_{lk} \frac{|f_p - f_q|}{h} = \frac{1}{2} \ell_{kl} \cot \theta_{kl},$$

where θ_{kl} is the interior dihedral angle at edge kl. Repeating this calculation for the other two triangles we obtain a cotangent formula for the volume gradient of a tetrahedron:

$$\nabla_{f_i} V = \frac{1}{6} \left(\ell_{kl} \cot \theta_{kl} (f_i - f_j) + \ell_{lj} \cot \theta_{lj} (f_i - f_k) + \ell_{jk} \cot \theta_{jk} (f_i - f_l) \right).$$



2.2 n-dimensional case

The argument in the n-dimensional case is essentially identical, only the notation becomes more annoying. In particular, consider an n-simplex σ with vertices

$$f_0, \ldots, f_n \in E^n.$$

Its volume can be expressed as

$$V = \frac{1}{n} V_0 h_0,$$

where V_0 denotes the volume of the $n-1$ simplex $\sigma_0 \subset \sigma$ which omits vertex, and h_0 denotes the height of the corresponding altitude. The gradient with respect to f_0 is then

$$\nabla_{f_0} V = \frac{1}{n} V_0 N_0,$$

where N_0 is the inward unit normal of σ_0. The foot p of the altitude splits σ_0 into n simplices, each of degree $n-1$; we will use U_r to denote the volume of the $n-1$ simplex that omits vertices 0 and r, but contains the foot p and all other vertices of σ. The location of p can be expressed in barycentric coordinates as

$$f_p = \sum_{r=1}^{n} \frac{U_r}{V_0} f_r.$$

Noting that

$$V_0 = \sum_{r=1}^{n} U_r,$$

we can write the volume gradient as

$$\begin{array}{rcl} n\nabla_{f_0} V & = & \frac{V_0}{h_0}(f_i - f_p) \\ & = & \frac{1}{h_0}\left(\sum_{r=1}^{n} U_r f_i - \sum_{r=1}^{n} U_r f_r\right) \\ & = & \sum_{r=1}^{n} \frac{U_r}{h_0}(f_i - f_r). \end{array}$$

The volume U_r can be expressed as

$$\frac{1}{(n-1)} V_{0,r} |f_p - f_q^r|,$$

where $V_{0,r}$ is the volume of the $(n-1)$-simplex that omits 0 and r but contains all other vertices of σ, and f_q^r is the location of the foot of this volume relative to the altitude passing through p. Just as in the tetrahedral case, then, the numerator $|f_p - f_q^r|$ and denominator $h_0 = |f_i - f_p|$ are legs of a right triangle, and we get

$$\frac{U_r}{h_0} = \frac{1}{(n-1)} V_{0,r} \frac{|f_p - f_q^r|}{h_0} = \frac{1}{(n-1)} V_{0,r} \cot\theta_{0,r},$$

where $\theta_{0,r}$ is the dihedral angle found at the face of σ that omits vertices 0 and r. All together, then, we have

$$\nabla_{f_0} V = \frac{1}{n(n-1)} \sum_{r=1}^{n} V_{0,r} \cot\theta_{0,r} (f_0 - f_r). \tag{1}$$

In other words, to get the volume gradient of a simplex σ with respect to the position of any vertex i, we take a weighted sum of edge vectors ij, where the weight is equal to the dihedral angle at the simplex $\sigma_{i,j}$ complementary to ij in σ, times the volume of $\sigma_{i,j}$ (along with the leading coefficient, which accounts for dimension). One can check that the units work out properly: the change in volume with respect to a linear motion should have units of length^{n-1}, the volume of the complementary simplex has units of length^{n-2}, the difference of vertex positions has units of length, and the dihedral angle is unitless.

One can also verify that this construction agrees with the 3-dimensional case: here, the volume $V_{0,r}$ corresponds to the length of the edge opposite $0r$, and $\theta_{0,r}$ is the dihedral angle of that edge. Likewise, in the 2-dimensional case the volume $V_{0,r}$ is the volume of the 0-simplex obtained by omitting two of the triangle vertices; by convention, the volume of every 0-simplex is 1 and we are left with just the angle times the difference of f values, *i.e.*, the ordinary cotangent formula. In the 1-dimensional case the complementary simplex is the empty set. Though it is possible to give an interpretation to Eqn. 1 in this case, it is perhaps simpler to just note that the gradient of the length ℓ_{ij} of a segment ij with respect to the position of one of its vertices is a unit vector parallel to the segment:

$$\nabla_{f_i}\ell_{ij} = \frac{1}{\ell_{ij}}(f_i - f_j).$$

In this case, the "cotan" weight is just the reciprocal of the edge length.

3 Discrete Laplace operator

To obtain a discrete Laplace operator at any vertex i, we can sum up the area gradients (with respect to f_i) of all n-simplices containing i. This sum gives us an expression for $(\mathsf{L}f)_i$, and by substituting the vertex coordinates f for a generic function u, we get a general expression for the Laplace operator. This operator will be a weighted graph Laplacian on the 1-skeleton of the triangulation (*i.e.*, the graph of edges). More explicitly, for an n-dimensional triangulation with vertices $|V|$ and edges $|E|$, the discrete Laplacian is a matrix $\mathsf{L} \in \mathbb{R}^{|V|\times|V|}$ with non-zero entries

$$\mathsf{L}_{i,j} = -w_{ij}$$

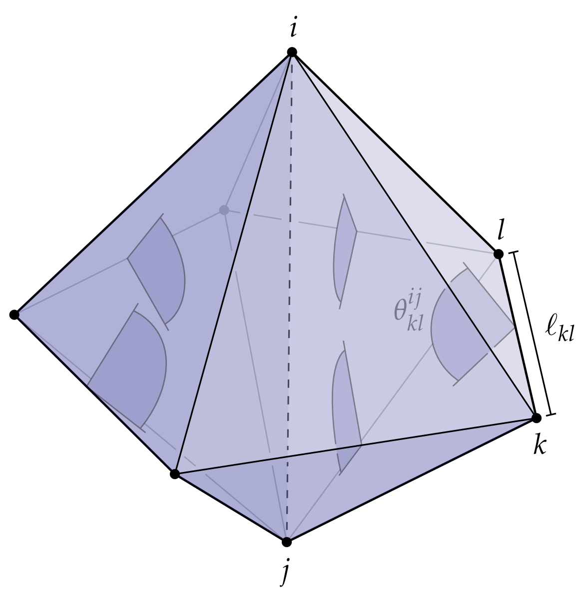
for each edge ij, where w_{ij} is an edge weight defined below, and

$$\mathsf{L}_{i,i} = -\sum_{ij}\mathsf{L}_{i,j}$$

for each vertex $i \in V$.

In the 3-dimensional case the weight w_{ij} associated with any edge ij is given by

$$w_{ij} = \frac{1}{6}\sum_{ijkl}\ell_{kl}\cot\theta^{ij}_{kl},$$



where the sum is taken over all tetrahedra $ijkl$ containing edge ij, ℓ_{kl} is the length of edge kl and θ_{kl}^{ij} is the (interior) dihedral angle at edge ij of tetrahedron $ijkl$.

In the n-dimensional case the edge weights are given by

$$w_{ij} = \frac{1}{n(n-1)} \sum_{\sigma \ni ij} V_{\overline{\sigma}_{i,j}} \cot \theta_{\overline{\sigma}_{i,j}},$$

where the sum is taken over all n-simplices σ containing edge ij, $\overline{\sigma}_{i,j}$ is the $(n-2)$-simplex obtained by removing vertices i and j from σ, $V_{\overline{\sigma}_{i,j}}$ is the volume of this simplex, and $\theta_{\overline{\sigma}_{i,j}}$ is the dihedral angle found at this simplex. (Note that for an n-simplex, dihedral angles are made by two faces of degree $(n-1)$, which always meet along some $(n-2)$-simplex.) The volume V of any n-simplex σ with vertices $f_0, \ldots, f_n \in E^n$ can be computed via

$$V = \frac{1}{n!} \det(f_1 - f_0, \ldots, f_n - f_0),$$

though in practice there may be more stable numerical algorithms.

Finally, the 1-dimensional edge weights are just the reciprocal edge lengths:

$$w_{ij} = \frac{1}{\ell_{ij}}.$$

3.1 Properties of the n-dimensional cotan Laplacian

As discussed by Wardetzky et al. [2007], not all properties of the smooth Laplace-Beltrami operator Δ immediately carry over to the discrete Laplacian L. The n-dimensional cotan Laplacian exhibits all the same properties as the 2-dimensional version: it is symmetric, has constant vectors in its null space (since, by construction, all rows sum to zero), exhibits linear precision (owing to the fact that the volume variation of an interior vertex is zero), and is positive semidefinite. For $n = 1$ the edge weights are always positive; for $n = 2$, the edge weights w_{ij} are positive if and only if the triangulation is Delaunay [Bobenko and Springborn, 2007]. For $n \geq 3$ positivity of edge weights is less clear—for instance, if all dihedral angles θ are acute, then all angle cotans (and hence edge weights) will be positive. However, acuteness is unnecessarily restrictive, especially given that acute triangulations are difficult to obtain in practice Zamfirescu [2013], and may not even exist in higher dimensions Kopczyński et al. [2012].

The full matrix L corresponds to a Laplace operator with zero-Neumann boundary conditions; removing the boundary rows/columns yields a discrete Laplace operator with zero-Dirichlet boundary conditions.

3.2 Mass matrices

As noted in Sec. 1, solving, *e.g.*, a discrete Poisson equation involves not only the stiffness matrix L, but also a mass matrix M. A simple choice in the n-dimensional case is the diagonal matrix $\mathsf{M} \in \mathbb{R}^{|V| \times |V|}$ with entries

$$\mathsf{M}_{ii} = \frac{1}{n+1} \sum_{\sigma \ni i} V_\sigma,$$

where the sum is taken over all n-simplices σ containing vertex i, and V_σ denotes the volume of σ. The factor $1/(n+1)$ accounts for the fact that the volume of each n-simplex is evenly shared by its $n+1$ vertices. This mass matrix is simple to implement and always has positive entries, but is not particularly accurate; Mullen et al. [2011] discuss several possible alternatives.

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References

Bobenko, A. I. and Springborn, B. A. (2007). A discrete laplace–beltrami operator for simplicial surfaces. *Discrete & Computational Geometry*, 38(4):740–756.

Chao, I., Pinkall, U., Sanan, P., and Schröder, P. (2010). A simple geometric model for elastic deformations. In *ACM SIGGRAPH 2010 Papers*, SIGGRAPH '10, pages 38:1–38:6, New York, NY, USA. ACM.

Courant, R. (1943). Variational methods for the solution of problems of equilibrium and vibrations. *Bull. Amer. Math. Soc.*, 49(1):1–23.

Crane, K. and Wardetzky, M. (2017). A glimpse into discrete differential geometry. *Notices of the American Mathematical Society*, 64(10):1153–1159.

Desbrun, M., Kanso, E., and Tong, Y. (2008). *Discrete Differential Forms for Computational Modeling*, pages 287–324. Birkhäuser Basel, Basel.

Desbrun, M., Meyer, M., Schröder, P., and Barr, A. H. (1999). Implicit fairing of irregular meshes using diffusion and curvature flow. In *Proceedings of the 26th Annual Conference on Computer Graphics and Interactive Techniques*, SIGGRAPH '99, pages 317–324, New York, NY, USA. ACM Press/Addison-Wesley Publishing Co.

Duffin, R. J. (1959). Distributed and lumped networks. *Journal of Mathematics and Mechanics*, 8(5):793–826.

Jacobson, A. (2013). *Algorithms and Interfaces for Real-Time Deformation of 2D and 3D Shapes*. PhD thesis, ETH Zurich.

Kopczyński, E., Pak, I., and Przytycki, P. (2012). Acute triangulations of polyhedra and n. *Combinatorica*, 32(1):85–110.

MacNeal, R. (1949). *The Solution of Partial Differential Equations by Means of Electrical Networks*. PhD thesis, California Institute of Technology.

Meyer, M., Desbrun, M., Schröder, P., and Barr, A. H. (2003). Discrete differential-geometry operators for triangulated 2-manifolds. In Hege, H.-C. and Polthier, K., editors, *Visualization and Mathematics III*, pages 35–57, Berlin, Heidelberg. Springer Berlin Heidelberg.

Mullen, P., Memari, P., de Goes, F., and Desbrun, M. (2011). Hot: Hodge-optimized triangulations. *ACM Trans. Graph.*, 30(4):103:1–103:12.

Pinkall, U. and Polthier, K. (1993). Computing discrete minimal surfaces and their conjugates. *Experiment. Math.*, 2(1):15–36.

Wardetzky, M., Mathur, S., Kaelberer, F., and Grinspun, E. (2007). Discrete Laplace operators: No free lunch. In Belyaev, A. and Garland, M., editors, *Geometry Processing*. The Eurographics Association.

Zamfirescu, C. T. (2013). Survey of two-dimensional acute triangulations. *Discrete Mathematics*, 313(1):35 – 49.