Abstract Interpretation Using Laziness: Proving Conway's Lost Cosmological Theorem

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In partial fulfillment
of the speaking skills requirement

?

2111

1231

11121311

3112111321

13211231131211

1113122112132113111221

3113112221121113122113312211

13211321322112311311222123112221

Look and say

- Read string out loud
  - 2111 → "one two, three ones" → 1231
  - 1231 → "one one, one two, one three, one one" → 11121311
  - ... etc.
- Invented by John Conway
  - paper: *The Weird and Wonderful Chemistry of Audioactive Decay*, 1987
  - also invented Life and surreal numbers
- So what happens?
  - asymptotics of string length
  - random looking? patterns in strings?
  - answered by the *Cosmological Theorem* ...

“Cosmological Theorem”

Characterizes how strings evolve

- split strings into *elements*
- elements *decay* into other elements
- *classify* all elements surviving in arbitrarily late strings

Proofs:

- J. H. Conway and friends, 1987 (by hand, lost)
- D. Zeilberger, 1997 (Maple)
- Litherland, 2003 (C)
- Watkins, 2006 (Haskell)

All proofs by exhaustion of cases

What is a proof?

Are these proofs?

- hand proof: case left out? too much paper?
- computer proof: buggy code? cosmic ray?

My strategy

- semantics of Haskell is my deductive system
- proof correct by construction!
- most code needn't be checked!
- key technique: Haskell lazy data as abstract interpretation

Vs deductive engine (e.g. Coq, HOL Light)

- their pros: correctness entirely formally verified
- my pros: *much* less effort; easy to experiment in Haskell interpreter

Other proofs with similar structure

In general, must:

- enumerate cases
- verify a property for each case

Examples:

- Four color theorem (Appel et al., 1976)
- Kepler's conjecture (Hales, 1998)

My method:

- *oracle strategy* simplifies showing sufficiency of enumeration
- *abstract interpretation via laziness* simplifies verifying property of each case
- This talk will illustrate both aspects

My contributions

New proof presentation strategy

- i.e. *oracle strategy* and *abstract interpretation via laziness* from previous slide
- may apply to similar proofs in other domains

Verify Conway's result

- Simple code: presented and justified in its entirety in my technical report
- Code written in a language with a simple semantics

Simplify prior proofs of Conway's result

- via my *marked sequences* (see technical report)

Talk outline

- Introduction
- Overview of Cosmological Theorem
- About Haskell and laziness
- Applying my method
- Conclusions

Cosmological Theorem

Recall:

Characterizes how strings evolve

- split strings into *elements*
- elements *decay* into other elements
- *classify* all elements surviving in arbitrarily late strings

Split strings into elements

Split string into parts that evolve independently

Formally:

- let xs_n be nth string in evolution
- xs splits into ys . zs if and only if:

  $xs_n = ys_n ++ zs_n$ for all $n \geq 0$

- (++) means append strings

Element: string that doesn't split into smaller ones

- Theorem (Conway, easy): every string splits into finitely many elements in unique way

Decision procedure for splitting?

- I'll tell you in a few minutes ...

Split strings into elements

Split up strings into parts that evolve independently

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... etc.

Split strings into elements

Split up strings into parts that evolve independently

2 . 111

12 . 31

1112 . 1311

3112 . 111321

132112 . 31131211

1113122112 . 132113111221

311311222112 . 1113122113312211

13211321322112 . 3113112221231112221

... etc.

Split strings into elements

Example:

- 2111 splits into 2 . 111 which are elements
- first step: 2111 → 1231, and 2 . 111 → 12 . 31. OK!
- I claim happens for nth step, all n ≥ 0 (proof later!)

Counterexample:

- 111 *does not* split into 1 . 11:
- 111 → 31, but 1 . 11 → 11 . 21. BAD!

Analogy: factoring integer into primes?

- *Note* substring of element can be element
- e.g. 111 is element, and also 1
- so splitting into elements is context dependent

Cosmological Theorem

Recall:

Characterizes how strings evolve

- split strings into *elements*
- elements *decay* into other elements
- *classify* all elements surviving in arbitrarily late strings

Audioactive decay

Start with string

- split into elements
- do look and say
- split result into elements
- do look and say
- ... repeat ad infinitum

Audioactive decay

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1112 . 1311

3112 . 111321

132112 . 31131211

1113122112 . 132113111221

311311222112 . 1113122113312211

1321132 : 1322112 . 311311222 : 12 : 3112221

... etc.

Cosmological Theorem

Recall:

Characterizes how strings evolve

- split strings into *elements*
- elements *decay* into other elements
- *classify* all elements surviving in arbitrarily late strings

Two special sets of elements:

- 92 *common elements*
- 2 infinite families of *transuranic elements*

Cosmological Theorem (Conway, proof lost): every string eventually decays into a compound of common and transuranic elements

Common elements

92 special elements

- Conway assigned them symbols H-U from chemistry
- involve only integers 1, 2, 3

Ubiquity

- Theorem (Conway): every common element eventually shows up in the decay of any *interesting* string (proved in technical report, not needed for Cosmological Theorem)

Two special cases

- empty string, 22 just repeat themselves
- call these *boring*, any other string *interesting*

Common elements example

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“holmium-silicon-erbium-calcium-antinide!”

Common elements example

2 . 111

Ca . 31

K . 1311

Ar . 111321

Cl . 31131211

S . 132113111221

P . 111312211312211

Ho : Si . Er : Ca : Sb

"holmium-silicon-erbium-calcium-antinide!"

Transuranic elements

What about integers other than 1 2 3?

- extra *transuranic elements*
- ^{n}Pu = 31221132221222112112322211n
- ^{n}Np = 1311222113321132211221121332211n

Cosmological Theorem (Conway, proof lost): Every string eventually decays into a compound of common and transuranic elements

Using the Cosmological Theorem

- How long do strings get?
  - make transition matrix on 92 common elements
  - find principal eigenvalue $\lambda = 1.3035772690...$
  - Theorem (linear algebra, not hard): length of any *interesting* string tends to $c\lambda^n$ on nth step, as $n \to \infty$
- Can also compute asymptotic relative abundance of elements
  - abundances of transuranic elements tend to 0

My proof

First step proving Cosmological Theorem: decision procedure for splitting strings into elements

This talk:

- develop correct decision procedure:
  - use *oracle strategy* to enumerate cases
  - use *abstract interpretation* to prove each case correct

See paper for the rest, leading ultimately to the proof of the Cosmological Theorem

- two more distinct uses of oracles and abstract interpretation

Splitting into elements

2111 splits into 2 and 111

But 111 doesn't split into, say, 1 and 11. Why?

- Split point is in the middle of a run of 1s
- Run 111 coded as 31 in original string
- Pieces 1 and 11 coded as 11 and 21 in split parts
- $31 \neq 1121$

Compare 2 and 111...

Splitting into elements

Compare 2 and 111...

2 . 111

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1112 . 1311

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1113122112 . 132113111221

... etc.

Split point never lands in the middle of a run

Splitting into elements

... Split point never lands in the middle of a run

Otherwise said:

- last number of left part ≠ first number of right part, forever
- Theorem (Conway, easy): necessary and sufficient for splitting
- note last number of left part never changes

Plan: see what happens to first number of arbitrary string

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Lists in Haskell

List is either:

- Empty list, written [], read “nil”
- Non-empty list, written (x:xs), read “x cons xs”
  - First element x (head)
  - List of remaining elements xs (tail)

Other syntax

- Cons associates to right: (1:2:3:[]) = (1:(2:(3:[])))
- Bracket abbreviation: [1,2,3] = (1:(2:(3:[])))

Haskell programming

Defining functions

- write equations characterizing function
- when function is called:
  - match pattern on left side of equation
  - result is right hand side of equation

Example: length of list

```
length [] = 0
length (x:xs) = length xs + 1
```

This talk: patterns always mutually exclusive

- e.g. [] and (x:xs) never both match an input

Haskell reasoning

Reasoning with functions by substitution

- can always replace (instance of) left hand side with right hand side, or vice versa
- no state, memory, etc. to screw things up
- relies on convention about mutually exclusive patterns

Example:

length (1:2:xs) = length (2:xs) + 1 = (length xs + 1) + 1 = length xs + 2

Derive properties by doing algebra

Haskell reasoning???

But can't you write inconsistent equations?

- e.g. f [] = 1 + f []

Solution:

- every Haskell type has special undefined element ⊥, read "bottom"
- have: f [] = ⊥
- therefore: ⊥ = 1 + ⊥

When running program, ⊥ means "infinite loop"

Laziness: suspending computations

What if:

g [] = 1:(g [])

Cons (:) and plus (+) work differently:

- 1+f [] evals 1 and (f []) then adds
- 1:(g []) created without eval’ing 1 and (g [])

More generally:

- (:) makes data structure, puts suspended computations in slots of structure

Classifying Haskell lists

What happens when you look for [] at end of list?

- *finite*: terminate
- *infinite*: get more and more conses forever
- *partial*: get ⊥ after seeing finitely many conses

Examples:

- *finite*, e.g. (1:(2:(3:[]))) = [1,2,3]
- *infinite*, e.g. g [] where g [] = 1 : g []
- *partial*, e.g. (1:(2:(3:⊥)))

Mutually exclusive and exhaustive

- any nonterminating expression = ⊥

Refinements of data

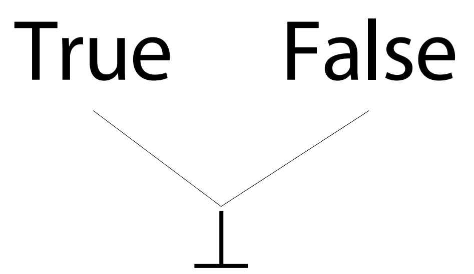
Every Haskell type has a *refinement order*:

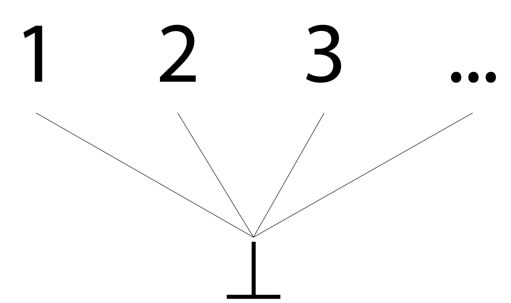
- read $x \leq y$ as “y at least as defined as x”

Pictorially ...

Integer and boolean refinements

All elements but ⊥ incomparable

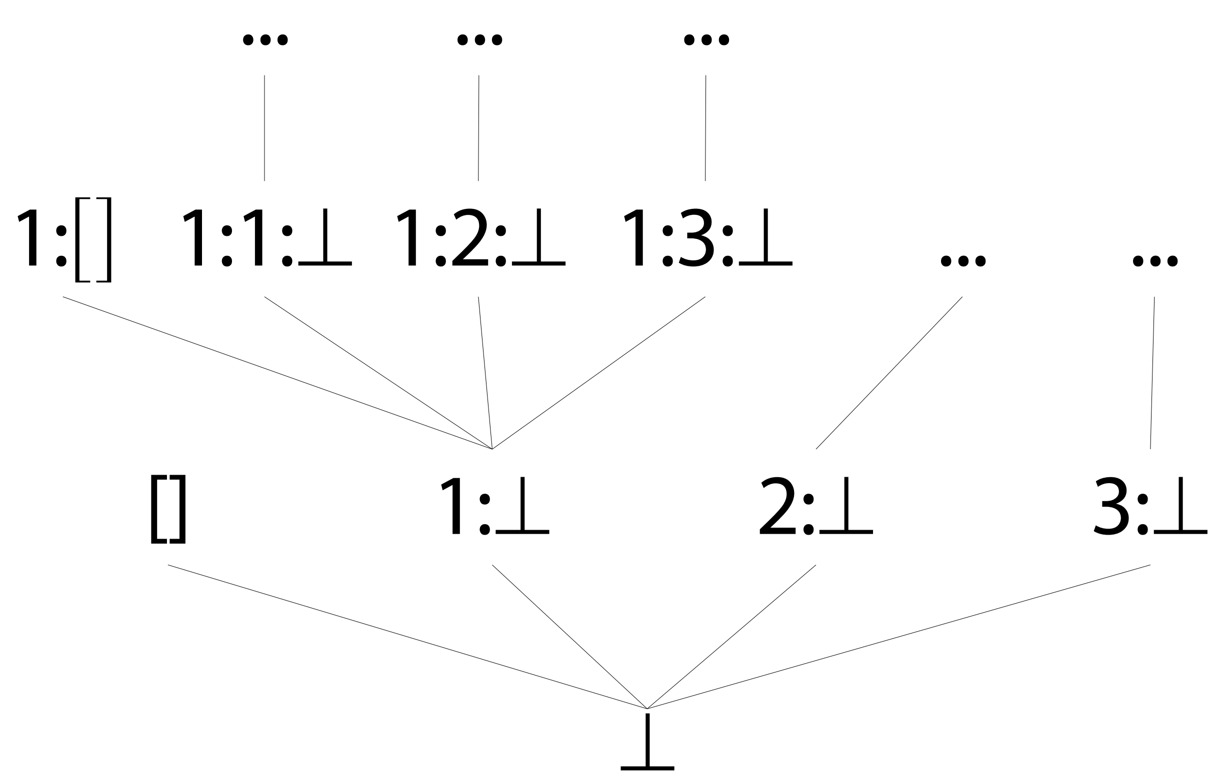




List refinements

[] and (:) incomparable; (:) monotone in both args

This talk: restrict lists to 1 2 3; all members defined for lists we consider



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Using monotonicity

Fact: every definable Haskell function is monotone

- xs ≤ ys implies f xs ≤ f ys (in refinement order)
- i.e., as arg gets more defined, result gets more defined

Suppose:

- Given f such that f (1:⊥) = True

Then:

- f (1:xs) = True for any xs, by monotonicity of f

Don't need to see code for f!

Abstractly interpreting look and say

Define Haskell function *say* that does look and say

e.g. say [2,1,1,1] = [1,2,3,1]

Want *say* to be as lazy as possible

e.g. say (2:1:1:1:3:⊥) = (1:2:3:1:⊥)

e.g. say (2:1:⊥) = (1:2:⊥)

e.g. say (2:⊥) = ⊥

definition from my paper is indeed as lazy as possible

Covering all lists

Simulate f on all lists using abstract interpretation

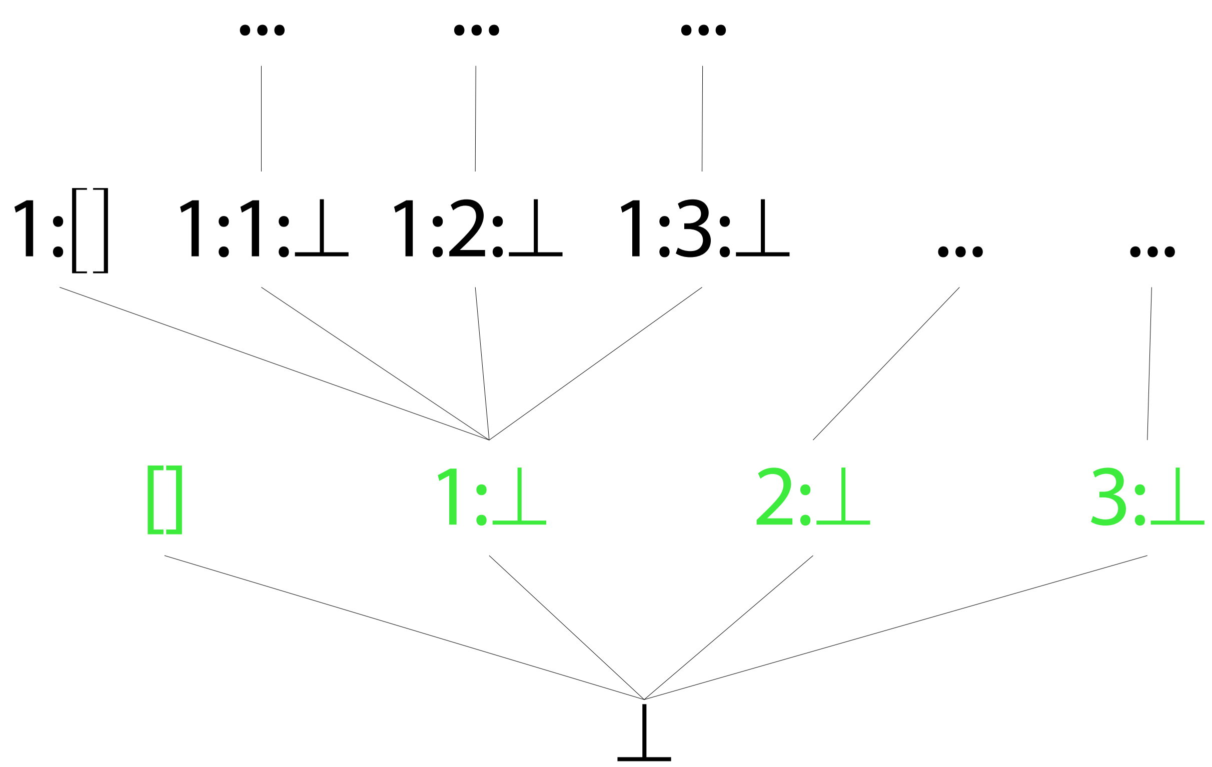
- pick (somehow) set C of partial (or finite) lists
- C must *cover* every list: $\forall xs \;\; \exists ys \in C$ such that $ys \leq xs$
- eval f on every list in C

Will explain how to pick C in a moment

Pictorially ...

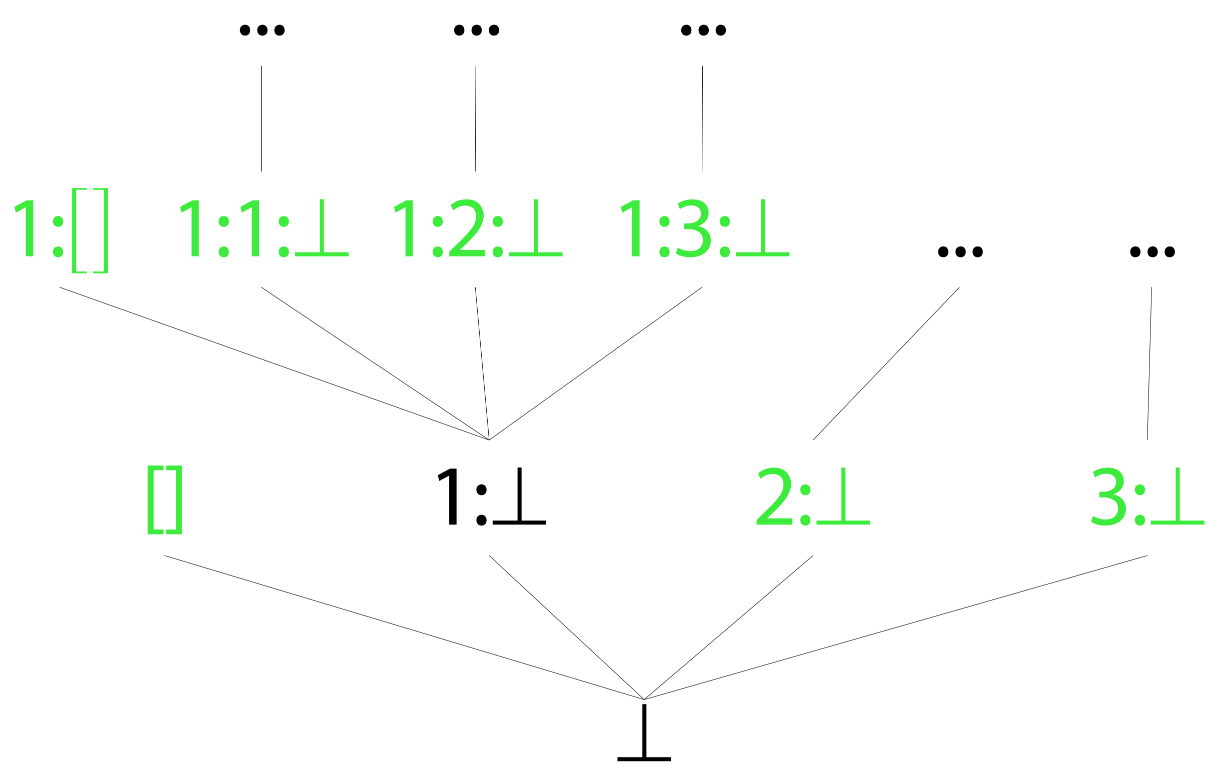
Covering all lists

A fringe in the tree of refinements



Covering all lists

A more refined cover



Applying the method

Recall: Splitting 2111...

2 . 111

12 . 31

1112 . 1311

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... etc.

2 always appears on left; want to show evolution of 111 always starts with 1 or 3

Finding the limit cycles

Pick covering set C (will say how later)

Execute the following Haskell code:

nub (map (take 20 · say^{30}) C)

nub removes duplicates from list

Constants 20, 30 chosen by trial and error

We get ...

Finding the limit cycles

? nub (map (take 20 · say^{30}) C)

[],
[31131122211311123113],
[13211321322113311213],
[13111213122112132113],
[31232112311321322112],
[11131221131211132221],
[22],
[22132113213221133112],
[22131112131221121321],
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By 30^{th} step we've reached a limit cycle!

Finding the limit cycles

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By 30^{th} step we've reached a limit cycle!

By 32^{nd} step we've seen every starting number!

Decision procedure for splitting

By 30th step we've reached a limit cycle!

By 32nd step we've seen every starting number!

Define algorithm for starting numbers:

$\text{starts } xs = [\ \text{head } (\text{say}^n\ xs) \mid n \leftarrow [0..32]\ ]$

Define decision procedure for splitting:

$\text{splits } xs\ ys = \text{null } xs \lor \text{null } ys \lor \neg (\text{last } xs \in \text{starts } ys)$

Needed to pick C. How?

Picking a covering set

Use *oracle predicate* p to decide how far to refine

Call (cover p):

```
cover p = if p [] then [⊥] else
              [] :
              [ 1:xs | xs ← cover (λys. p (1:ys)) ] ++
              [ 2:xs | xs ← cover (λys. p (2:ys)) ] ++
              [ 3:xs | xs ← cover (λys. p (3:ys)) ]
```

Example:

cover ((== 2) · length) = { [], [1], 1:1:⊥, 1:2:⊥, 1:3:⊥, [2], ... }

Claim: if (cover p) terminates, result is covering set

– Don't have to look at p!

Putting it together

Determine appropriate oracle by experiment:

$p = ((\geq 12) \cdot \text{length} \cdot \text{say})$

Generate covering set using oracle:

$C = \text{cover } p$

Find limit cycles using covering set:

? nub (map (take 20 · say^{30}) C)

[[], [3113112221131112311 3], ...

Conclude that decision procedure is correct:

$\text{starts } xs = [\ \text{head } (\text{say}^n\ xs) \mid n \leftarrow [0..32]\ ]$

$\text{splits } xs\ ys = \text{null } xs \vee \text{null } ys \vee \neg\,(\text{last } xs \in \text{starts } ys)$

About the code

- Two more applications of the method
  - proving that a lazier version of *splits* is correct
  - finding all decay products of arbitrary strings using *splits*
- Literate Haskell program
  - 1311 lines (181 code + 1130 latex)
  - 98 LOC verified; 83 LOC in oracles, needn't be verified
- Compare:
  - Zeilberger (Maple): 2234 LOC (incl. self-documentation)
  - Litherland (C): 1650 LOC (less than half comments)

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Proofs my method targets

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- enumerate cases
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Examples:

- Four color theorem (Appel et al., 1976)
- Kepler's conjecture (Hales, 1998)

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Simplify prior proofs of Conway's result

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Questions?