Compact Categories as †-Frobenius Pseudoalgebras

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Extended abstract

Introduction

In this work, we present a new understanding of compact monoidal categories. These categories are the workhorses of categorical approaches to quantum theory: the category of finite-dimensional Hilbert spaces is one of the prototypical compact categories, and examples also come from the braided fusion categories used to model topological quantum excitations [8, 11] and the Doplicher-Haag-Roberts category arising from an algebraic quantum field theory [7]. Compact categories also play important roles in topics closer to pure mathematics, such as the study of topological quantum field theories [2, 3, 15] and the representation theory of Hopf algebras [10, 13].

Our main theorem is that compact categories can be classified in terms of †-Frobenius pseudoalgebras in **Prof**, the bicategory with categories as objects, 'relations' between categories as 1-cells, and natural transformations as 2-cells. A pseudoalgebra is an algebraic structure similar to an algebra for which the associativity and unit laws do not hold exactly, but only up to a specified isomorphism. Our result is close in spirit to a theorem of Day, McCrudden and Street [6, 12] which characterizes $*$-autonomous categories in terms of Frobenius pseudoalgebras in **Prof**, but the gap between the $*$-autonomous case that they study and the compact case described here is quite significant.

It is striking that just as †-Frobenius algebras play an important role in describing structures which are important for quantum information [4, 5], so the †-Frobenius pseudoalgebras that we study here can play a similar role in describing the structures of importance for *topological* quantum computation. This opens up the possibility of studying whether techniques employing Frobenius algebras in the study of algorithm design, entanglement and complementary observables could be extended to Frobenius pseudoalgebras, and hence made relevant to topological quantum computation.

Our results also have significant relevance for the study of higher-dimensional extended topological quantum field theories. It is well-known that the category **2Cob**, which has circles as objects and diffeomorphism classes of compact 2-manifolds with boundary as

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morphisms, can be defined in terms of a commutative Frobenius algebra [1, 9], and whilst it has been conjectured that the same might be possible for higher-dimensional cobordism categories, and some partial attempts made [14], no full constructions have yet been proposed. A significant difficulty is that no finite presentations are known, or even conjectured, for the relevant higher-dimensional cobordism categories. Our axioms for a Frobenius pseudoalgebra with extra structure provide a finite presentation of exactly the correct kind, and there are compelling reasons to believe that they should at least be close to the correct axioms for constructing the 2-category $\mathbf{3Cob_2}$, which has circles as objects, 2-manifolds with boundary as morphisms, and diffeomorphism classes of 3-manifolds with boundaries and corners as 2-morphisms. Such a presentation would make the connection between 3-dimensional topological quantum field theories and compact categories much more transparent.

Technical discussion

To explain our result, we begin by considering the structure on a monoidal category that arises from an object X having a right dual X^*. It is relatively straightforward to show that this gives rise to an isomorphism

$$\phi_{A,B} \colon \mathrm{Hom}(X \otimes A, B) \to \mathrm{Hom}(A, X^* \otimes B), \quad A, B \in C, \tag{1}$$

which satisfies the following equation for all A and B, where 1 is the monoidal unit object:

$$\phi_{A,B}(f) = (\mathrm{id}_{X^*} \otimes f) \circ (\phi_{1,X}(\rho_X^{-1}) \otimes \mathrm{id}_A) \circ \lambda_A \tag{2}$$

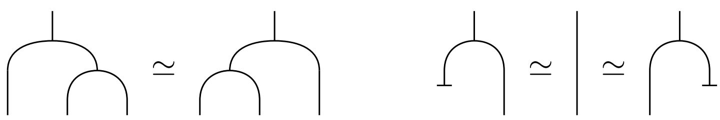
Conversely, if these isomorphisms exist satisfying these equations, then it can be shown that X^* is right dual to X. The existence of the isomorphism ϕ guarantees exactly that $X \otimes (-) \dashv X^* \otimes (-)$, but this alone is not sufficient for the right dual to exist. This is not very difficult to prove, but is a frequent source of confusion, as the requirement that ϕ should satisfy the specified equation is often overlooked in the literature.

Suppose that our monoidal category is compact, meaning that every object has a left dual and a right dual. Then, in particular, for every object X we will have a right dual object X^* such that $X \otimes (-) \dashv X^* \otimes (-)$, and a left dual object ${}^*X$ such that ${}^*X \otimes (-) \dashv X \otimes (-)$. By the results of Day, McCrudden and Street [6, 12] these are exactly the conditions to obtain a *Frobenius pseudoalgebra* in **Prof**, for which the comultiplication 1-cell is right-adjoint to the multiplication 1-cell.

A Frobenius pseudoalgebra is a higher-dimensional algebraic structure built from the following components, which are morphisms in a monoidal bicategory:

Multiplication Unit Comultiplication Counit

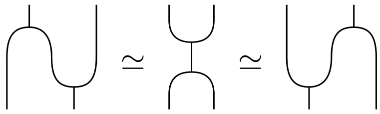
We can compose these in various ways, and require that the following isomorphisms exist:



(3)

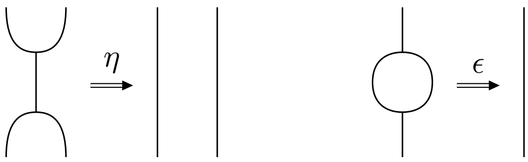
Associativity Left and right unit

Coassociativity — Left and right counit (4)



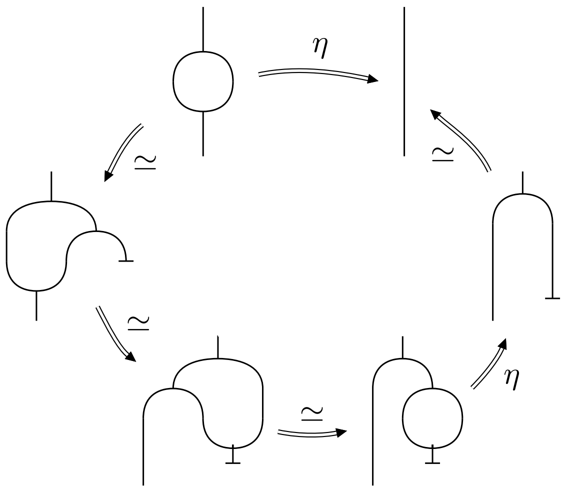
Frobenius (5)

We require these isomorphisms to be *coherent*, meaning that any two transformations of the same type which can be built out of them are equal. When the comultiplication is adjoint to the multiplication, we also have the following processes available:



These satisfy equations called the *adjunction equations*.

However, as discussed above, to demonstrate the existence of duals we require more than the existence of the adjunctions $X \otimes (-) \dashv X^* \otimes (-)$ and ${}^*X \otimes (-) \dashv X \otimes (-)$; we also require that analogues of equation (2) should hold. Implementing these as equational properties of the Frobenius pseudoalgebra gives rise to interesting extra requirements, such as the following commutative diagram:



(6)

This diagram gives an equation between the structures that we have just described. Suppose that we have a Frobenius pseudoalgebra such that this diagram, and a partner diagram, both hold: then we call the resulting structure a *†-Frobenius pseudoalgebra*.

The significance of †-Frobenius pseudoalgebras is demonstrated by the following theorem.

Theorem. *When the underlying category is Cauchy-complete, a pseudoalgebra in* **Prof** *is a compact monoidal category iff it can be made into the multiplicative part of a †-Frobenius pseudoalgebra.*

So, in this sense, the †-Frobenius pseudoalgebras that we have defined completely characterize compact monoidal categories, at least when the underlying category is

Cauchy-complete, which in the standard **Set**-enriched setting reduces to the requirement that all idempotents split.

There are many kinds of extra structure that can be placed on a compact category, including even-handed, balanced, pivotal, spherical and ribbon structures. We will describe how these extra structures manifest themselves as extra structures on the corresponding Frobenius pseudoalgebra.

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