Counting Triangles under Updates[⋆]

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1 Problem Setting and Contributions

We consider the problem of maintaining the result of the triangle count query $Q(\,) = \mathit{\Gamma}_{;sum}\, R(A, B) \bowtie S(B, C) \bowtie T(C, A)$ under single-tuple updates to the input relations R, S, and T. The relations are given as key-payload maps whose keys are tuples over relation schemas, payloads are tuple multiplicities, and key lookups are (amortized) $\mathcal{O}(1)$-time operations. A single-tuple update $\delta R(a, b) = \{\,(a, b) \mapsto p\,\}$ to relation R maps a key (a, b) to a nonzero payload p (positive for inserts and negative for deletes); updates to S and T are analogous.

The naïve maintenance approach recomputes the triangle count from scratch after each update. Computing this query using worst-case optimal join algorithms [5] takes $\mathcal{O}(N^{1.5})$ time, where N is the current size of the input database.

To *incrementally* maintain the triangle count under single-tuple updates, existing incremental view maintenance (IVM) approaches need linear time. For instance, under the update δR to R, the classical IVM [2] computes the delta query $\mathit{\Gamma}_{;sum}\, \delta R(a, b) \bowtie S(b, C) \bowtie T(C, a)$ in $\mathcal{O}(N)$ time because it needs to intersect two lists of possibly linearly many C-values that are paired with b in S and with a in T. The factorized IVM [6] materializes the view $V_{ST}(B, A) = \mathit{\Gamma}_{B,A;sum}\, S(B, C) \bowtie T(C, A)$ using $\mathcal{O}(N^2)$ space. It then computes the delta query $\mathit{\Gamma}_{;sum}\, \delta R(a, b) \bowtie V_{ST}(b, a)$ in $\mathcal{O}(1)$ time; however, updates to S and T still require $\mathcal{O}(N)$ time to maintain the triangle count Q and view V_{ST}.

This raises the question of whether the triangle count can be maintained in sublinear time. Recent work proves that no algorithm can maintain Q in time $\mathcal{O}(N^{0.5-\gamma})$ for any $\gamma > 0$, under reasonable complexity-theoretic assumptions [1]. An algorithm with sublinear maintenance time for Q is not yet known.

This work introduces IVM$^\epsilon$, an IVM approach that maintains the triangle count in *amortized sublinear* time. IVM$^\epsilon$ partitions each input relation into two parts, heavy and light, based on the degrees of data values, the database size, and a parameter ϵ. It then adapts the maintenance strategy to different heavy-light combinations of parts of the input relations to achieve worst-case sublinear maintenance. As the database evolves under updates, IVM$^\epsilon$ rebalances the partitions to account for a new database size and updated degrees of data values. While this rebalancing may take superlinear time, it remains sublinear per update.

Given a database of size N and $\epsilon \in [0, 1]$, IVM$^\epsilon$ maintains the triangle count in $\mathcal{O}(N^{\max\{\epsilon,1-\epsilon\}})$ amortized time while using $\mathcal{O}(N^{1+\min\{\epsilon,1-\epsilon\}})$ space. It thus defines a continuum of approaches exhibiting a space-time tradeoff based on ϵ.

[⋆] An extended version of this work is available online [3].

Materialized View Definition	Space Complexity
$Q(\,) = \bigcup_{u,v,w\in\{h,l\}} \Gamma_{;sum}\ R_u(A,B) \bowtie S_v(B,C) \bowtie T_w(C,A)$	$\mathcal{O}(1)$
$V_{RS}(A,C) = \Gamma_{A,C;sum}\ R_h(A,B) \bowtie S_l(B,C)$	$\mathcal{O}(N^{1+\min\{\epsilon,1-\epsilon\}})$
$V_{ST}(B,A) = \Gamma_{B,A;sum}\ S_h(B,C) \bowtie T_l(C,A)$	$\mathcal{O}(N^{1+\min\{\epsilon,1-\epsilon\}})$
$V_{TR}(C,B) = \Gamma_{C,B;sum}\ T_h(C,A) \bowtie R_l(A,B)$	$\mathcal{O}(N^{1+\min\{\epsilon,1-\epsilon\}})$

Fig. 1. The materialized views used by IVM$^\epsilon$ for a database of size N and $\epsilon \in [0,1]$.

Setting $\epsilon = 0.5$ gives $\mathcal{O}(N^{0.5})$ amortized *worst-case optimal* time and $\mathcal{O}(N^{1.5})$ space utilization. Existing IVM approaches are extreme points in this continuum of approaches defined by IVM$^\epsilon$. For instance, to recover classical IVM, we set $\epsilon \in \{0,1\}$ to achieve $\mathcal{O}(N)$ update time and $\mathcal{O}(N)$ space utilization; to recover factorized IVM, we set distinct parameters ϵ for each relation (cf. [3] for details). IVM$^\epsilon$ can also count all triangles in a static database in worst-case optimal time $\mathcal{O}(N^{1.5})$ by inserting N tuples, one at a time, into initially empty relations.

2 Adaptive Maintenance Strategy

We split each input relation into two disjoint parts, called heavy and light parts. Given $\epsilon_R \in [0,1]$, an A-value a is heavy in R if $|\sigma_{A=a}R| \geq N^{\epsilon_R}$, where N is the database size; otherwise, it is light. We partition R into R_h and R_l such that $R_h = \{\, t \in R \mid t.A \text{ is heavy}\,\}$ and $R_l = R \setminus R_h$; similarly, we partition S on B, and T on C. In the following, we assume that $\epsilon = \epsilon_R = \epsilon_S = \epsilon_T$ is fixed.

We decompose the query Q into skew-aware views expressed over the relation parts: $Q_{uvw}(\,) = \Gamma_{;sum}\, R_u(A,B) \bowtie S_v(B,C) \bowtie T_w(C,A)$, where $u,v,w \in \{h,l\}$. The query Q is thus a union (sum) of partial counts: $Q(\,) = \bigcup_{u,v,w\in\{h,l\}} Q_{uvw}(\,)$.

We adapt the maintenance strategy to each skew-aware view to ensure sublinear update time. While most of these views admit sublinear delta computation, few exceptions require linear-time maintenance. For these exceptions, IVM$^\epsilon$ precomputes the update-independent parts of delta queries as *materialized views* and uses them to speed up the delta evaluation. Such auxiliary views also require maintenance, yet their maintenance cost is sublinear for single-tuple updates.

Figure 1 shows the materialized views used by IVM$^\epsilon$ to maintain the triangle count query. The size of the view $V_{RS}(A,C)$ is upper-bounded by the size of the result of the join of $R_h(A,B)$ and $S_l(B,C)$ in two distinct ways. One can iterate over all (a,b) pairs in R_h and then find the C-values in S_l for each b. Since S_l contains only tuples with light B-values, there are at most N^ϵ distinct C-values for each B-value. This gives an upper bound of $\mathcal{O}(|R_h| \cdot N^\epsilon) = \mathcal{O}(N^{1+\epsilon})$. Alternatively, one can iterate over all (b,c) pairs in S_l and then find the A-values in R_h for each b. Since R_h contains only tuples with heavy A-values, there are at most $\frac{N}{N^\epsilon} = N^{1-\epsilon}$ distinct A-values. This gives an upper bound of $\mathcal{O}(|S_l| \cdot N^{1-\epsilon}) = \mathcal{O}(N^{2-\epsilon})$. The overall space complexity is the minimum of the bounds. The space analysis for V_{ST} and V_{TR} is analogous.

We explain our adaptive strategy on a single-tuple update $\delta R_*(a,b)$ to relation R. This update can affect either the heavy or light part of R, hence the $*$

Delta Evaluation Strategy	Time Complexity
$\delta Q_{*hh}(\,) = \delta R_*(a,b) \cdot \sum_C T_h(C,a) \cdot S_h(b,C)$	$\mathcal{O}(N^{1-\epsilon})$
$\delta Q_{*hl}(\,) = \delta R_*(a,b) \cdot V_{ST}(b,a)$	$\mathcal{O}(1)$
$\delta Q_{*lh}(\,) = \delta R_*(a,b) \cdot \sum_C T_h(C,a) \cdot S_l(b,C)$ or $= \delta R_*(a,b) \cdot \sum_C S_l(b,C) \cdot T_h(C,a)$	$\mathcal{O}(N^{\min\{\epsilon,1-\epsilon\}})$
$\delta Q_{*ll}(\,) = \delta R_*(a,b) \cdot \sum_C S_l(b,C) \cdot T_l(C,a)$	$\mathcal{O}(N^{\epsilon})$
$\delta Q(\,) = \delta Q_{*hh}(\,) + \delta Q_{*hl}(\,) + \delta Q_{*lh}(\,) + \delta Q_{*ll}(\,)$	$\mathcal{O}(1)$
$\delta V_{RS}(a,C) = \delta R_h(a,b) \cdot S_l(b,C)$	$\mathcal{O}(N^{\epsilon})$
$\delta V_{TR}(C,b) = \delta R_l(a,b) \cdot T_h(C,a)$	$\mathcal{O}(N^{1-\epsilon})$

Fig. 2. Computing the deltas of the views from Figure 1 for an update $\delta R_*(a,b)$ to the heavy or light part of R. The symbol $*$ stands for h or l. The delta δV_{ST} is empty since V_{ST} does not refer to R. The evaluation order of deltas is from left to right.

symbol; we assume that checking whether a is heavy or not in R is a constant-time operation. Updates to the other two relations are handled similarly.

Figure 2 shows the deltas of the views affected by the update $\delta R_*(a,b)$ and their time complexity when evaluated from left to right. In all but one case, the complexity is determined by the number of C-values that need to be iterated over. Computing the deltas involves multiplying the payloads of matching tuples and, if C is not in the target view schema, summing them over C-values.

We first analyze the access patterns of the skew-aware delta views: (1) For δQ_{*hh}, we iterate over at most $N^{1-\epsilon}$ C-values in T_h for the given a and then look up in S_h for each (b,c); (2) For δQ_{*hl}, we look up in the materialized view V_{ST} for the given (a,b); (3) For δQ_{*lh}, we either iterate over at most $N^{1-\epsilon}$ C-values in T_h for the given a and look up in S_l for each (b,c), or we iterate over at most N^{ϵ} C-values in S_l for the given b and look up in T_h for each (c,a); (4) For δQ_{*ll}, we iterate over at most N^{ϵ} C-values in S_l for the given b and then look up in T_l for each (c,a). Then, summing these partial deltas and updating Q take constant time. The views V_{RS} and V_{TR}, which facilitate updates to T and respectively to S, are maintained for updates to distinct parts of R. Computing δV_{RS} and updating V_{RS} requires iterating over at most N^{ϵ} C-values in S_l for the given b; similarly, computing δV_{TR} and updating V_{TR} involves at most $N^{1-\epsilon}$ heavy C-values in T_h. The final step of IVM$^{\epsilon}$ updates the (heavy or light) part of R that corresponds to δR_* in (amortized) $\mathcal{O}(1)$ time. Overall, IVM$^{\epsilon}$ maintains the views from Figure 1 under single-tuple updates to any of the input relations in $\mathcal{O}(N^{\max\{\epsilon,1-\epsilon\}})$ time using $\mathcal{O}(N^{1+\min\{\epsilon,1-\epsilon\}})$ space.

An insert (a,b) into R may promote a from light to heavy in R or may increase the heavy-light threshold such that some A-values change from heavy to light. Without rebalancing the partitions, our assumptions on the number of B-values paired with a or the number of heavy A-values may become invalid.

IVM$^{\epsilon}$ loosens the partition threshold to amortize the cost of rebalancing over multiple updates. Instead of the actual database size N, the threshold now depends on a variable M for which the invariant $\lfloor\frac{1}{4}M\rfloor \leq N < M$ always holds. If the database size violates one of the limits, we perform *major rebalancing* where we double or halve M to satisfy the invariant again, repartition the input relations using the new threshold M^{ϵ}, and recompute the auxiliary views. The time complexity of this operation is $\mathcal{O}(M^{1+\min\{\epsilon,1-\epsilon\}})$, which is amortized over at least $\lceil\frac{1}{4}M\rceil$ updates between two major rebalancing steps.

IVM$^{\epsilon}$ also enforces the following two invariants: The number of tuples with the same value of the partitioning attribute is less than $\frac{3}{2}M^{\epsilon}$ in each light part and at least $\frac{1}{2}M^{\epsilon}$ in each heavy part. If any of the two invariants is violated, we perform *minor rebalancing* where we move at most $\lceil\frac{3}{2}M^{\epsilon}\rceil$ tuples from one part to another and update the affected views. The time complexity of this operation is $\mathcal{O}(M^{\epsilon+\max\{\epsilon,1-\epsilon\}})$, which is amortized over at least $\lceil\frac{1}{2}M^{\epsilon}\rceil$ updates between two minor rebalancing steps for the same value of the partitioning attribute.

In conclusion, both rebalancing steps together take $\mathcal{O}(M^{\max\{\epsilon,1-\epsilon\}})$ amortized time. Since each single-tuple update can be realized in time $\mathcal{O}(M^{\max\{\epsilon,1-\epsilon\}})$ and $M = \mathcal{O}(N)$, IVM$^{\epsilon}$ needs $\mathcal{O}(N^{\max\{\epsilon,1-\epsilon\}})$ overall amortized time. The extended version of this work presents a detailed complexity analysis of IVM$^{\epsilon}$ [3].

3 Beyond the Triangle Query

IVM$^{\epsilon}$ can be applied to any query but may not always yield asymptotic improvements over existing approaches. It can achieve sublinear maintenance for the counting variants of acyclic queries, e.g., 3-path and 4-path, and cyclic queries, e.g., Loomis-Whitney and 4-cycle. Different semirings can be used to specifiy operations on the payloads [6]; we used here $(\mathbb{Z}, +, *, 0, 1)$ to express counting. An early prototype implementation of IVM$^{\epsilon}$ on top of DBToaster [4] shows several factors performance improvement over classical and factorized IVM.

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