Profunctor Semantics for Linear Logic

Lawrence H. Dunn III

University of Oxford

A thesis submitted for the degree of

M.Sc Mathematics and the Foundations of Computer Science

Trinity Term 2015

Abstract

Linear logic is a sort of "resource-aware" logic underlying intuitionistic logic. Investigations into the proof theory of this logic typically revolve around proof nets, certain two-dimensional graphical representations of proofs. Where sequent calculus deductions enforce arbitrary distinctions between proofs and complicate these investigations, proof nets represent sequent calculus deductions up to a notion of equivalence suggested by the semantics provided by $*$-autonomous categories, at least for restricted fragments of the logic. Indeed, proof nets may be considered the string diagrams of these categories. However, for fragments of the logic containing units for the multiplicative connectives, the coherence of $*$-autonomous categories implies an equivalence of proofs which prevents a canonical extension of proof nets to the full system. This obstruction is what is meant by "the problem of units for proof nets."

In this paper we begin to develop a geometric approach to the proof theory of linear logic which depicts proofs as three-dimensional surfaces. This line of research is made possible by Street's observation that $*$-autonomous categories are particular Frobenius pseudoalgebras in a monoidal bicategory **PROF**. In some sense, proof nets can be embedded into these surfaces, but with a sort of "extra dimension." The central idea being developed is that coherence can be seen by visually exposing this extra structure and considering the result up to a suitable topological equivalence. These ideas are presented up to the frontiers of current work regarding the coherence of Frobenius pseudoalgebras, which proceeds in two directions. First, it is thought that the coherence of $*$-autonomous categories may be fully subsumed by that of Frobenius pseudoalgebras. Second, based on the known properties of Frobenius algebras (proper), the coherence properties of Frobenius pseudoalgebras ought to have an interpretation as some intrinsically homotopical notion of equivalence of surfaces. This work suggests there exists a principled geometrical proof theory which, among other things, exposes the problem of units as a mere observation about the topology of surfaces.

Contents

Chapter 1

Introduction

Linear logic is a "resource-sensitive" refinement of usual logic, introduced by Girard in [6]. Seely showed in [17] that a denotational semantics can be given by Barr's $*$-autonomous categories (pronounced "star-autonomous") [1]. The structure of such categories, and the structure of linear logic's cut-elimination procedure, underly a notion of equivalence of proofs which is coarser and better behaved than syntactic equality of sequent calculus deductions. The "bureaucracy of syntax" of sequent calculus (to use Girard's phasing) is partially mitigated by Girard's theory of proof nets, a sort of graphical parallel natural deduction. Proof nets are particularly satisfactory for the fragment MLL$^-$, the multiplicative fragment without units for the connectives, within which proof nets are fully canonical – equality of proofs is syntactical. However, various attempts to extend the theory to include the units have resulted in structures which are not canonical and must be considered up to a notion of equivalence [11] [13]. Moreover, the decision problem for proof equivalence in the full multiplicative fragment MLL has been found to be PSPACE-complete [8], indicating that no entirely suitable extension of proof nets can exist for this fragment.

In separate work, Street demonstrated that $*$-autonomous categories, considered as objects of a monoidal bicategory **PROF** of (small) categories, profunctors, and natural transformations, have a natural Frobenius structure [18]. The central idea in this paper is to take this approach to $*$-autonomy seriously and capture linear logic through the surface calculus of monoidal bicategories. This strategy offers a principled approach to the geometrical foundations of proof theory, and through it we can see that both proof nets and the sequent calculus seem to be approximations to the full three-dimensional presentation.

This paper proceeds in several chapters:

> **Chapter 2** We review linear logic through a sequent calculus and the denotational semantics provided by $*$-autonomous categories. Motivated by the desire to avoid cumbersome commutative conversions in cut-elimination, and the identifications between proofs suggested by the categorical semantics, we are led into a discussion of proof nets for MLL as an parallel syntax for proof theory. A brief literature survey concludes that proof nets are unable to cope

with multiplicative units in a fully satisfactory way.

Chapter 3 The surface calculus of symmetric monoidal bicategories is introduced alongside **PROF**. The Frobenius law is examined. The visualization of $*$-autonomous categories under the embedding of **CAT** into **PROF** leads to the observation that $*$-autonomous categories are certain Frobenius pseudoalgebras.

Chapter 4 By chaining together the categorical semantics and the surface calculus, we uncover a three-dimensional presentation of proofs in linear logic. Through a series of examples, we see that proof nets and sequent calculus are both approximations to the full three-dimensional presentation.

An appendix provides background material regarding monoidal bicategories.

Chapter 2

Linear Logic and its Categorical Semantics

2.1 Introduction

Linear logic was introduced by Girard in [6] as a refinement of usual logic (classical and intuitionistic). By carefully controlling the scope of the usual structural rules, the usual binary connectives bifurcate into two systems; the resulting logic can be seen to be "resource-aware."

We begin by surveying classical propositional linear logic (CLL), though we later restrict our attention to one fragment of this system. We assume a set of literals: $p, q, r, p^\perp, q^\perp, r^\perp$, etc., and special constants $\mathbf{0}, \mathbf{1}, \top, \bot$. Note that each literal x is paired with a dual, $x^\perp$, both of which we take to be atomic. There are four binary connectives: $\otimes, \&, \parr, \oplus$ (often pronounced, respectively, "times," "with," "parr," "plus"). There are two unary connectives (modalities) $?, !$ ("why not," "of course"). Especially for notational convenience, we define a negation operation $(-)^\perp$ as given in Table 2.1. Note that negation is involutive.

Linear logic is usually defined through a sequent calculus. A generic sequent has the form

$$A, B, C, \ldots \vdash X, Y, Z, \ldots$$

with a finite (possibly empty) list of formulas on each side, and we read this sequent loosely as "The conjunction of $A, B, C \ldots$ yields the disjunction of $X, Y, Z \ldots$". If the left hand side is empty, it may be read as "true" – a kind of left-hand unit. Correspondingly, an empty right-hand side is loosely read as "false." The duality between the left and right sides plays a central role in classical linear logic.

The rules of the system are often divided into thematic groups:

Identity group Rules which essentially codify that A is equal to itself (Table 2.2).

Negation group Rules governing the classical nature of negation (Table 2.3).

$(p)^{\perp} := p^{\perp}$	$(p^{\perp})^{\perp} := p$
$\mathbf{1}^{\perp} := \perp$	$\perp^{\perp} := \mathbf{1}$
$\mathbf{0}^{\perp} := \top$	$\top^{\perp} := \mathbf{0}$
$(A \otimes B)^{\perp} := A^{\perp} \parr B^{\perp}$	$(A \parr B)^{\perp} := A^{\perp} \otimes B^{\perp}$
$(A \oplus B)^{\perp} := A^{\perp} \& B^{\perp}$	$(A \& B)^{\perp} := A^{\perp} \oplus B^{\perp}$
$(!A)^{\perp} := ?(A^{\perp})$	$(?A)^{\perp} := !(A^{\perp})$
$\Gamma := A, B, C \ldots$	$\Gamma^{\perp} := A^{\perp}, B^{\perp}, C^{\perp} \ldots$

Table 2.1: Definition of the negation operation

$$\frac{}{A \vdash A}\ (\mathrm{id}) \qquad \frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\mathrm{cut})$$

Table 2.2: Identity rule group

Structure group Rules for manipulating sequents while preserving the formulae on both sides of the turnstile. In linear logic this consists merely of the exchange rule (Table 2.4), since weakening and contraction are not accepted as general principles of the logic.

Logic group Rules for introducing the connectives. They can be further divided into subgroups: additive (Table 2.5), multiplicative (Table 2.6), and exponential (Table 2.7).

Remark 2.1. We use capital Greek letters $\Gamma, \Delta, \ldots$ to stand for arbitrary finite ordered lists of formula, possible empty. Sometimes these are called "contexts," but the same word is also used to mean the slightly different notion of non-principal (i.e. unaffected) formula in a particular rule application. We simply call them lists.

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta} \perp \mathrm{L} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta} \perp \mathrm{R}$$

Table 2.3: Negation group

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta} \text{(exch. L)} \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2} \text{(exch. R)}$$

Table 2.4: Structural rule group

	Conjunction	Disjunction	Units
Left rules	$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \&L_1$	$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \&\mathrm{R}$	$\frac{}{\Gamma, \mathbf{0} \vdash \Delta} (\mathbf{0})$
	$\frac{\Gamma, A \vdash \Delta}{\Gamma, B \,\&\, A \vdash \Delta} \&L_2$		
Right rules	$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta} \parr\mathrm{R}$	$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash B \oplus A, \Delta} \oplus R_1$	$\frac{}{\Gamma \vdash \top, \Delta} \top$
		$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \oplus R_2$	

Table 2.5: Additive subset of the logical group

	Conjunction	Disjunction	Units
Left	$\dfrac{\Gamma_1 A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, A \otimes B, \Gamma_2 \vdash \Delta} \otimes L$	$\dfrac{\Gamma_1, A \vdash \Delta_1 \qquad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, A \parr B, \Gamma_2 \vdash \Delta_1, \Delta_2} \parr L$	$\dfrac{}{\bot \vdash} \bot L$
			$\dfrac{\Gamma \vdash \Delta}{\Gamma, \mathbf{1} \vdash \Delta} \mathbf{1} L$
Right	$\dfrac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, A \otimes B, \Delta_2} \otimes R$	$\dfrac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta} \parr R$	$\dfrac{}{\vdash \mathbf{1}} \mathbf{1} R$
			$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot} \bot \mathrm{R}$

Table 2.6: Multiplicative subset of the logical group

$\dfrac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta} ?L$	$\dfrac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta} !R$	
$\dfrac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} !W$	$\dfrac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} !C$	$\dfrac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} !D$
$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta} ?W$	$\dfrac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta} ?C$	$\dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta} ?D$

Table 2.7: Exponential subset of the logical group

Remark 2.2. In classical linear logic, linear implication is commonly defined

$$A \multimap B := A^{\perp} \parr B.$$

(In intuitionistic linear logic, linear implication is usually taken as a primitive construct.) The resource-sensitive nature of the logic is witnessed, for example, in the fact that we can prove $X \multimap X$ for any X, but we cannot generally prove $X \multimap X \otimes X$ – in effect, linear "implication" is about converting resources into others, and in general we cannot duplicate (or destroy) resources.

Our presentation of the rules is not minimal. For instance, the left rules of the connective $\otimes$ can be deduced from the right rules of $\parr$, taking into account the deMorgan duality between the connectives (which can be seen in Table 2.1) and the negation laws from Table 2.3.

We shall restrict our attention to MLL – the multiplicative fragment of CLL. This system consists of Tables 2.2, 2.3, 2.4, and 2.6, and acts as the primary fragment on which the theory of proof nets is founded.

2.2 Proof Nets and Parallelism

One feature of the multiplicatives that becomes immediately apparent is their parallel nature. Considering the deductions in Figure 2.1.

$$\dfrac{\dfrac{}{A \vdash A}\,(\text{id}) \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,(\text{id}) \qquad \dfrac{}{C \vdash C}\,(\text{id})}{B, C \vdash B \otimes C}\,\otimes\text{R}}{B \otimes C \vdash B \otimes C}\,\otimes\text{L}}{A, B \otimes C \vdash A \otimes (B \otimes C)}\,\otimes\text{R}$$

(a)

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\text{id}) \qquad \dfrac{\dfrac{}{B \vdash B}\,(\text{id}) \qquad \dfrac{}{C \vdash C}\,(\text{id})}{B, C \vdash B \otimes C}\,\otimes\text{R}}{A, B, C \vdash A \otimes (B \otimes C)}\,\otimes\text{R}}{A, B \otimes C \vdash A \otimes (B \otimes C)}\,\otimes\text{L}$$

(b)

Figure 2.1: Two proofs featuring "parallel" rule applications.

Notice that proofs differ by the order of application of the rules $\otimes L$ and $\otimes R$. Notice also that the two rule applications are, in some sense, independent of one another. It is not apparent why we ought to distinguish between the two proofs. Instead, we would prefer a syntax which makes it clear that particular rule instances can be applied in parallel.

The situation gets worse when we look at the (cut) rule. One of the most essential theorems of linear logic (indeed, of essentially any logic of mathematical interest) is that cuts can be eliminated.

Theorem 2.1. *Any proof which can be proved from the given rules of linear logic, can be proved without invoking the (cut) rule.*

Proof. A proof is found in [7]. □

In fact there are effective procedures for rewriting sequent calculus deductions into ones which prove the same sequent, without using the (cut). The standard approach in these procedures is to push (cut) applications up the deduction tree as high as possible. Now consider the situation of a (cut) applied after two independent rule instances (say, R_1 and R_2).

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash \Gamma, A}}{\vdash \Gamma', A}(R_1) \qquad \dfrac{\dfrac{\pi_2}{\vdash A^{\perp}, \Delta}}{\vdash A^{\perp}, \Delta'}(R_2)}{\vdash \Gamma', \Delta'}(\text{cut})$$

The (cut) application is applied to the A and corresponding $A^{\perp}$. The two generic rule applications shown do not act on these formulas. Therefore we can push the (cut) up the tree. However, there are two branches of the tree to choose, resulting in an arbitrary choice to prefer the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\vdash \Gamma, A} \qquad \dfrac{\pi_2}{\vdash A^{\perp}, \Delta}}{\vdash \Gamma, \Delta}(\text{cut})}{\vdash \Gamma', \Delta}(R_1)}{\vdash \Gamma', \Delta'}(R_2)$$

or the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\vdash \Gamma, A} \qquad \dfrac{\pi_2}{\vdash A^{\perp}, \Delta}}{\vdash \Gamma, \Delta}(\text{cut})}{\vdash \Gamma, \Delta'}(R_2)}{\vdash \Gamma', \Delta'}(R_1)$$

This situation is a prototypical example of a "commutative conversion." From the sequent calculus perspective, cut-elimination (normalization, from a λ -calculus perspective), becomes "extremely complex and awkward" [7] – and highly non-canonical

– due to constant commutations of parallel rule applications, complicating the resulting theory.

In fact, the doctrine of denotational semantics (discussed in Section 2.3) requires, as a central tenet, that denotations of proofs be invariant under cut-elimination, meaning that however we rewrite proofs in carrying out cut-elimination, each rewrite must preserve the denotation of the proof. Again, the natural desideratum here is a syntax which is able to cope with the parallel nature of the multiplicatives, offers a canonical form for deductions which are identified in the semantics, and moreover might allow for a simple cut-elimination procedure. Such a syntax is found in Girard's theory of proof nets, introduced in [6] alongside linear logic itself.

First we invoke the rules of negation to present the logic in a one-sided fashion. We also make the assumption that our lists are multi-sets, making the exchange rule unnecessary. We are left with just a few rules, shown in Table 2.8.

$$\frac{}{\vdash A, A^{\perp}}\ (\mathrm{id}) \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{cut})$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

Table 2.8: Rules for one-sided MLL$^-$, treating sequents as multi-sets

To each of these rules we associate a graphical "link." This is a somewhat ad-hoc device which we could formalize in terms of graphs, but we will present the ideas more intuitively. The four links are given in Figure 2.2.

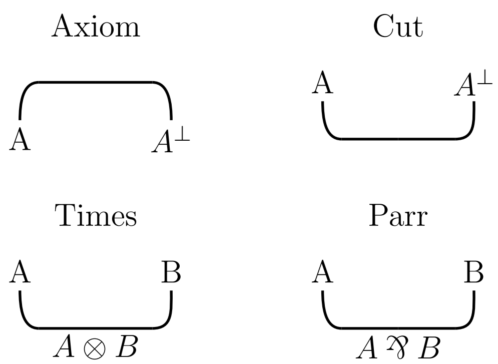


Figure 2.2: The links for multiplicative proof nets, without units

For each link, we consider the top to represent its "input ports." The bottom represents its "output ports." The Times link takes two inputs and yields one output, for instance. We connect these links in an intuitive fashion suggested by this idea of input and output. Now return to the example deductions given in Figure 2.1. Under the scheme of forcing all sequents to be one-sided, we arrive at the proofs

$$\cfrac{\cfrac{}{\vdash A^\perp, A}\,(\text{id}) \qquad \cfrac{\cfrac{}{\vdash B^\perp, B}\,(\text{id}) \qquad \cfrac{}{\vdash C^\perp, C}\,(\text{id})}{\vdash B^\perp, C^\perp, B \otimes C}\,\otimes}{\cfrac{\vdash A^\perp, B^\perp, C^\perp, A \otimes (B \otimes C)}{\vdash A^\perp, B^\perp \parr C^\perp, A \otimes (B \otimes C)}\,\parr}\,\otimes$$

and

$$\cfrac{\cfrac{}{\vdash A^\perp, A}\,(\text{id}) \qquad \cfrac{\cfrac{\cfrac{}{\vdash B^\perp, B}\,(\text{id}) \qquad \cfrac{}{\vdash C^\perp, C}\,(\text{id})}{\vdash B^\perp, C^\perp, B \otimes C}\,\otimes}{\vdash B^\perp \parr C^\perp, B \otimes C}\,\parr}{\vdash A^\perp, B^\perp \parr C^\perp, A \otimes (B \otimes C)}\,\otimes$$

At each rule application, we introduce the relevant graphical link. If the link takes inputs, we wire it to the corresponding outputs of previous links. Applying this process shows that both proofs have the same proof net, given in Figure 2.3. Notice there are three "free" outputs $A^\perp$, $A \otimes (B \otimes C)$, and $B^\perp \parr C^\perp$– free in the sense that they are not being fed to the input ports of any other link. We consider the proof net to represent a proof of the disjunction of its free outputs. Notice that the geometry of how we draw the proof net is not as important as how we have arranged the connections. In fact this seems to be evidence already that these proofs are topological entities.

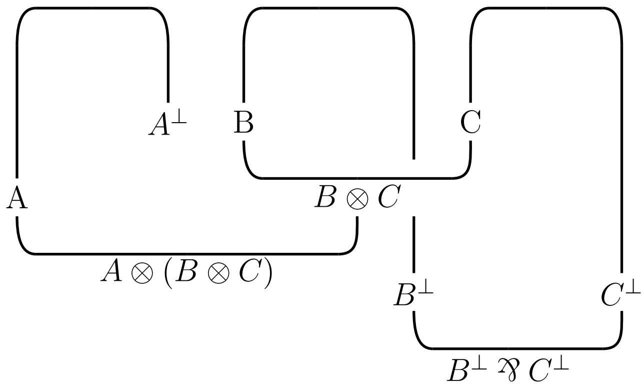


Figure 2.3: The common proof net associated to the proofs above

Not every graph built from these links is a proof, however. The nature of the links does not forbid diagrams which "plug into themselves.' For instance, Figure 2.4 results from wiring a Cut link to an Axiom link. But this cannot be a proof – there is not even a conclusion.

The general term for the figures obtained by arranging occurrences of links is "proof structure." It is only subset of these, by definition the proof *nets*, which can be put into correspondence with sequential deductions in MLL$^-$. A variety of criteria exist to distinguish between an arbitrary proof structure and a net, one of the most widely used being the Danos-Regnier criterion of [3]. The several criteria for proof

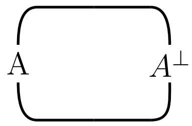


Figure 2.4: A degenerate proof structure – not a proof net

structure correctness are slightly ad-hoc – one of the goals of our work is to analyze their content from a surface calculus perspective. Work in this direction is ongoing.

Of the many nice properties that proof nets enjoy, there is the fact that cut-elimination can be performed on proof nets directly – there is no need to bother with the sort of commutative conversions we encounter in a sequent calculus presentation. For instance, the proof in Figure 2.5 has a proof net given in Figure 2.6. While cut-elimination for the former case is complicated, cut-elimination for the proof net involves a simple rewrite to the proof net in Figure 2.7. For proof nets, cut-elimination is of a local nature.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,\text{id} \qquad \dfrac{}{\vdash B^\perp, B}\,\text{id}}{\vdash A \otimes B, A^\perp, B^\perp}\,\otimes}{\vdash (A \otimes B) \parr B^\perp, A^\perp}\,\parr \qquad \dfrac{\dfrac{}{\vdash A, A^\perp}\,\text{id} \qquad \dfrac{}{\vdash C, C^\perp}\,\text{id}}{\vdash A, A^\perp \parr C, C^\perp}\,\parr}{\vdash (A \otimes B) \parr B^\perp, A^\perp \parr C, C^\perp}\,\text{cut}$$

Figure 2.5: A proof involving a cut link between occurrences of $A^\perp$ and A

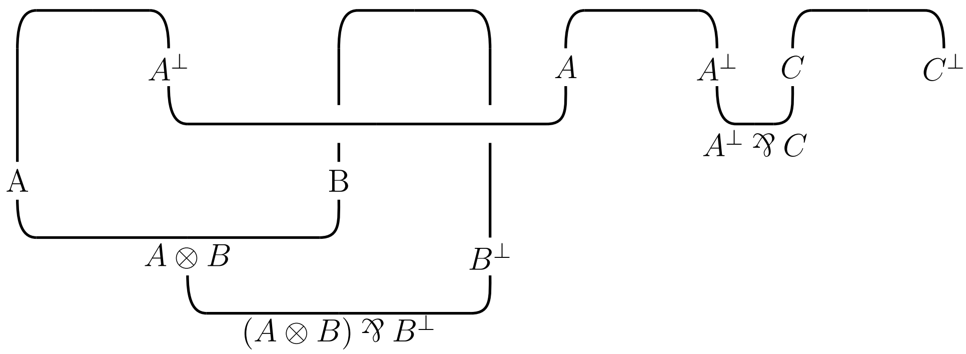


Figure 2.6: A proof net involving a cut link between occurrences of $A^\perp$ and A

When proof structures are correct (i.e., they are nets and correspond to sequent calculus deductions), they are canonical. Two deductions have the same denotation (in $*$-autonomous categories, Section 2.3) precisely when they have the same proof net. Moreover, two different proof nets must correspond to different (equivalence classes of) proofs – the net for a given proof is unique. Additionally, proof nets are tractable entities – translating from deductions to nets requires very little computation, giving a simple method of determining proof equivalence. We shall see that introducing the multiplicative units breaks these properties.

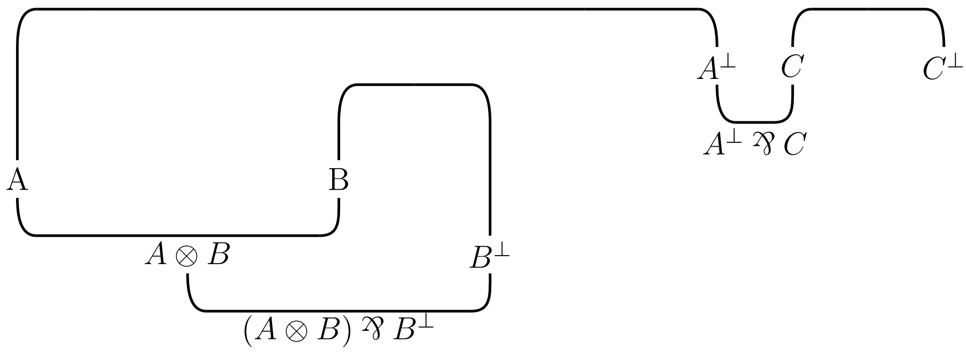


Figure 2.7: A proof net resulting from eliminating the cut in Figure 2.6

2.3 Categorical Semantics

It is well understood that there is a tight correspondence between proof theory and programming (certainly for intuitionist logic, if not immediately apparent in the classical case, but see [5]). These ideas extend to linear logic. Along those lines, one seeks a denotational semantics: To each formula A we seek an interpretation as some object $[\![A]\!]$, its *denotation*, and we extend this to consider denotations of lists of formulas. Sequents $\Gamma \vdash \Delta$ ought to correspond to some notion of a function between the denotations of the Γ and Δ – essentially a function type. A deduction of a sequent ought to describe the construction of a term of the type of the conclusion, and moreover this assignment of denotations ought to be invariant under cut-elimination – we are interested in a denotational, rather than operational semantics, and this choice corresponds to seeking a semantics for which the denotation is static throughout the execution of the program (see [12], especially Chapter 3).

One particular choice for giving a denotational semantics to linear logic, and the one we shall consider throughout this paper, is through $*$-autonomous categories. Without extra structure, theses categories model the fragment MLL, with additional connectives modeled by further structure imposed on the category.

Remark 2.3. For a category $\mathcal{A}$, we will find it convenient to write $\mathcal{A}(x, y)$ in place of $\mathrm{Hom}_{\mathcal{A}}(x, y)$. We will also name the identity morphism on an object after the object itself, e.g. $A \xrightarrow{A} A$.

Definition 2.1 ($*$-autonomous category)**.** A category $(\mathcal{A}, \otimes, \mathbf{1}, S, \alpha, \lambda, \rho, \sigma)$ given by a symmetric monoidal category $(\mathcal{A}, \otimes, \mathbf{1}, \alpha, \lambda, \rho, \sigma)$ (where α is the associator, λ and ρ the unitors, and σ is the braiding) equipped with an equivalence $S \colon \mathcal{A} \longrightarrow \mathcal{A}^{op}$ with natural isomorphisms

- $A \simeq SSA$
- $\mathcal{A}(A \otimes B, SC) \simeq \mathcal{A}(A, S(B \otimes C))$

By a slight but common abuse of notation, we simply refer to a $*$-autonomous category by the name of the underlying category. However we truly mean that we have in mind a particular choice of symmetric monoidal structure, a particular equivalence S, and particular natural isomorphisms as above.

Remark 2.4. It is a theorem that if a $*$-autonomous category $\mathcal{A}$ satisfies $S(A \otimes B) \simeq (SA \otimes SB)$, then it is compact-closed. Thus, what we are asking for from $\mathcal{A}$ is a notion of tensor product and a notion akin to that of dual objects, but such that the "duality" functor S does not (necessarily) distribute over the tensor.

This minimal definition of $*$-autonomy hides the full structure of such categories. For a given such category $\mathcal{A}$, there is much we can say about it:

- S is self-adjoint (or rather, $A \xrightarrow{S} A^{op}$ is adjoint with the functor $A^{op} \xrightarrow{S^{op}} A$ with the same action on objects and morphisms). This gives us isomorphisms like $\mathcal{A}^{op}(Sa, b) \simeq \mathcal{A}(a, Sb)$.
- $\mathcal{A}$ is monoidal-closed via a functor $\multimap$ given on objects by

  $$A \multimap B := S(A \otimes SB).$$

  The family of adjunctions arises from the isomorphisms

  $$\begin{aligned} \mathcal{A}(A \otimes B, C) &\simeq \mathcal{A}^{op}(S(A \otimes B), SC) \\ &\simeq \mathcal{A}(A \otimes B, SSC) \\ &\simeq \mathcal{A}(A, S(B \otimes SC)) \end{aligned}$$

  natural in the three variables A, B, C, where we have used the self-adjointness of S.
- $\mathcal{A}$ features an additional symmetric monoidal structure, a tensor we denote $\bullet$, defined on objects as

  $$A \bullet B := S(SA \otimes SB).$$

  The full monoidal and coherence structure is inherited from $\otimes$ (we shall distinguish between the natural isomorphisms with superscripts). The object $S\mathbf{I}$ acts as a unit, and we define $\bot := S\mathbf{I}$ to emphasize the logical correspondence. Notice $A \bullet B = SA \multimap B \simeq SB \multimap A$.
- The two tensors $\otimes$, $\bullet$ make $\mathcal{A}$ into a linearly distributive category. We shall return to this subject shortly below, since it is the source of the "problem of units" with proof nets.

The correspondence between the objects of linear logic and those of $*$-autonomous categories is straightforward. The key ideas are summarized in Table 2.9. Assuming some map $[\![-]\!]$ which associates to each atomic proposition p an object $[\![p]\!] \in \mathcal{A}$, we inductively extend this map to all formula by taking tensors of atomic denotations. For a list $\Gamma = A, B, C, \ldots X$ we write $\otimes[\![\Gamma]\!]$ for $[\![A]\!] \otimes ([\![B]\!] \otimes (\ldots [\![X]\!])$, of A is Γ is a singleton, or $\mathbf{1}$ if Γ is empty. Similarly we write $\bullet[\![\Gamma]\!]$ to mean the disjunction of the items of Γ.

Remark 2.5. One particular feature of this interpretation is that we ignore the syntactical distinction between the commas in a sequent, and the corresponding multiplicative connective: e.g., a left-hand list A, B, C and the formula $A \otimes (B \otimes C)$ are both simply $[\![A]\!] \otimes ([\![B]\!] \otimes [\![C]\!])$ in the category. One approach to removing this discrepancy is through multi-categories. For our purposes, maintaining this distinction is not particularly important.

The notation for denotations quickly becomes clumsy. Therefore we shall not usually write the double brackets $[\![-]\!]$ when examining the denotation of a given sequent. In the case of a list Γ, whether we mean $\otimes[\![\Gamma]\!]$ or $\bullet[\![\Gamma]\!]$ must be inferred. However, this will be clear from context.

The interpretation of the rules of MLL largely rests on the two families of natural transformations which exist in any $*$-autonomous category. They are the "linear distribution" morphisms

$$\delta_L \colon A \otimes (B \bullet C) \to (A \otimes B) \bullet C$$

$$\delta_R \colon (A \bullet B) \otimes C \to A \bullet (B \otimes C)$$

Remark 2.6. This distribution is "linear" in the sense of being somewhat resource-aware, like linear logic in general. Compare it to the distributive laws in other contexts, e.g.

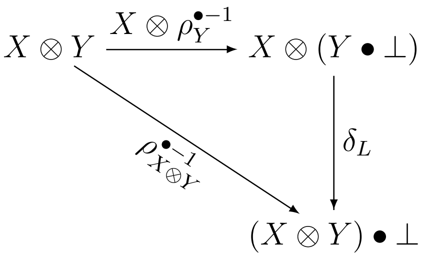
$$A \otimes (B \bullet C) \simeq (A \otimes B) \bullet (A \otimes C)$$

in which the resource A appears in the right-hand side twice, but the left-hand side just once.

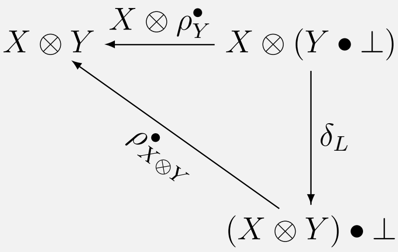
(eliding subscripts for objects). The structure of linear distribution is interesting in its own right – [15] defines linearly distributive categories (there called "weakly" distributive, apparently an older name) as categories $\mathcal{C}$ consisting of two symmetric-monoidal structures, with natural transformations as above, which moreover satisfy a quite large number of coherence conditions, such as the following:

In Linear Logic	In $*$-aut. categories
Atomic p	Object $[\![p]\!]$
$\otimes$ unit $\mathbf{1}$	$\otimes$ unit $\mathbf{I}$
$\parr$ unit $\perp$	$\bullet$ unit $\perp := S\,\mathbf{I}$
Multiplicative Conjuction $A \otimes B$	Tensor $[\![A]\!] \otimes [\![B]\!]$
Multiplicative Disjuction $A \bullet B$	"Dual" Tensor $[\![A]\!] \bullet [\![B]\!]$
Linear Implication $A \multimap B$	Internal Hom $[\![A]\!] \multimap [\![B]\!]$
Negation $A^\perp$	Object $S\,[\![A]\!]$
Right-sided sequent $\vdash \Delta$	Sequent $\mathbf{I} \to \bullet[\![\Gamma]\!]$
Two-sided sequent $\Gamma \vdash \Delta$	Sequent $\otimes[\![\Gamma]\!] \to \bullet[\![\Delta]\!]$
Left-sided sequent $\Gamma \vdash$	Sequent $[\![\Gamma]\!] \to \perp$

Table 2.9: Categorical interpretation of linear logic



Remark 2.7. Actually, the coherence axioms listed in [15] feature the diagram above with the two “unitor” natural isomorphisms going in the other direction, like so:



It is not hard to show that one of these diagrams commutes if and only if the other one does, since

$$\begin{aligned} \rho^{\bullet}_{X\otimes Y} \circ \delta_L = (X \otimes \rho^{\bullet}_Y) \iff & \quad \rho^{\bullet}_{X\otimes Y} \circ \delta_L \circ (X \otimes \rho^{\bullet -1}_Y) = id_{X\otimes Y} \\ \iff & \quad \delta_L \circ (X \otimes \rho^{\bullet -1}_Y) = \rho^{\bullet -1}_{X\otimes Y} \end{aligned}$$

We have chosen our diagram because it more directly corresponds to the unit introduction rule of linear logic.

Theorem 2.2 (Cockett-Seely). *$*$-autonomous categories are particular linearly distributive categories.* □

This theorem is not an entirely trivial verification. Proving that $*$-autonomous categories satisfy the coherence requirements of linearly distributive categories is an exercise in applying, besides the naturality equations resulting from the adjunctions, also the resulting extranaturality equations. A proof is not given in the original paper [15] (noting that the diagrams involved are “pretty horrid”). A full proof of a theorem relating linearly distributive categories to $*$-autonomous ones is presented in [4], mostly confined to pages 51–75. It is hoped that coherence conditions of $*$-autonomous categories can be recovered from those satisfied by Frobenius pseudoalgebras, although verifying the relevant equalities in the latter case has also been found to be less than obvious. We shall have more to say about the subject later.

The authors of [15] do indicate how they derive one of the linear distribution natural transformations. At first glance, one might presume we can use the morphism of type

$$\dfrac{\dfrac{\mathcal{A}(SX, S(Y \otimes Z))}{\mathcal{A}(SX \otimes Y, SZ)}\,(\ast\text{-autonomy})}{\mathcal{A}(Y \otimes SX, SZ)}\,(\text{Braiding})$$

Figure 2.8

$$\begin{array}{c}
\mathcal{A}(A \otimes X, Y \multimap B) \\ \hline
\mathcal{A}((A \otimes X) \otimes Y, B) \\ \hline
\mathcal{A}(A \otimes (X \otimes Y), B) \\ \hline
\mathcal{A}(A \otimes (Y \otimes X), B) \\ \hline
\mathcal{A}((A \otimes Y) \otimes X, B) \\ \hline
\mathcal{A}(A \otimes Y, X \multimap B)
\end{array}
\quad
\begin{array}{l}
(\text{Monoidal Closure}) \\
(\text{Associator}) \\
(\text{Braiding}) \\
(\text{Associator}) \\
(\text{Monoidal Closure})
\end{array}$$

Figure 2.9

$$(SB \multimap C) \otimes ((A \multimap SB) \otimes A) \to C,$$

built from the monoidal-closure counits and, after precomposing with braiding and associator transformations, take the transpose across the monoidal-closure isomorphism to arrive at a morphism of type

$$A \otimes (SB \multimap C) \to (A \multimap SB) \multimap C,$$

which combined with the isomorphism $A \multimap SB \simeq S(A \otimes B)$ would yield a morphism of the correct type. However, we have not verified the coherence conditions for this morphism. Therefore we shall indicate how the authors of [15] derive a linear distribution transformation. (It is probably reasonable to suspect their morphism is in fact equal to the one we defined above, but we have not investigated this thoroughly).

Using the adjunction S we can derive a natural isomorphism as in Figure 2.8. Specializing to the case $X = (A \otimes B)$, $Y = A$, $Z = B$, and taking the identity morphism in the first set yields a morphism of type $A \otimes S(A \otimes B) \to SB$.

Using the monoidal closure defined earlier we can define the evaluation arrow $S(SB \otimes SC) \otimes SB \to C$. Precomposing with the braiding and then taking the image of the result across the closure yields a morphism of type $SB \to S(S(SB \otimes SC) \otimes SC)$, i.e. $SB \to (SB \multimap C) \multimap C$. Composing this morphism with the one given earlier yields one of type

$$A \otimes S(A \otimes B) \to (SB \multimap C) \multimap C$$

Finally there is a natural isomorphism of hom-sets as in Figure 2.9 which, combined with the above morphism, yields a morphism of the correct type.

We are now in a position to define the categorical interpretation of the sequent calculus rules. For convenience we shall simply ignore the associator natural transformation and presume the tensor product is strictly associative.

Identity

$$\frac{}{A \vdash A}\ (\text{identity})$$

From no morphism at all, we must define a morphism of type $A \to A$. For this we simply use the identity, id_A.

Cut

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\text{cut})$$

From two arrows

$$\Gamma_1 \xrightarrow{f} \Delta_1 \bullet A$$

and

$$A \otimes \Gamma_2 \xrightarrow{g} \Delta_2,$$

we take

$$\Gamma_1 \otimes \Gamma_2 \xrightarrow{f \otimes \Gamma_2} (\Delta_1 \bullet A) \otimes \Gamma_2 \xrightarrow{\delta_R} \Delta_1 \bullet (A \otimes \Gamma_2) \xrightarrow{id_{\Delta_1} \bullet g} \Delta_1 \bullet \Delta_2$$

Exchange

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ (\text{exchange left}) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ (\text{exchange right})$$

From

$$\Gamma_1 \otimes A \otimes B \otimes \Gamma_2 \xrightarrow{f} \Delta$$

We construct

$$\Gamma_1 \otimes B \otimes A \otimes \Gamma_2 \xrightarrow{\Gamma_1 \otimes \sigma^{\otimes}_{B,A} \otimes \Gamma_2} \Gamma_1 \otimes A \otimes B \otimes \Gamma_2 \xrightarrow{f} \Delta$$

Likewise, from

$$\Gamma \xrightarrow{f} \Delta_1 \bullet A \bullet B \bullet \Delta_2$$

we define

$$\Gamma \xrightarrow{f} \Delta_1 \bullet A \bullet B \bullet \Delta_2 \xrightarrow{\Delta_1 \bullet \sigma^{\bullet}_{A,B} \bullet \Delta_2} \Delta_1 \bullet B \bullet A \bullet \Delta_2$$

$\otimes L$ and $\invamp R$

$$\frac{\Gamma_1 A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, A \otimes B, \Gamma_2 \vdash \Delta}\otimes L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \invamp B, \Delta}\invamp R$$

Under the categorical interpretation, both rules are trivial, because we do not interpret the comma separately from the tensors.

$\otimes R$

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, A \otimes B, \Delta_2}\otimes R$$

Given two arrows

$$\Gamma_1 \xrightarrow{f} \Delta_1 \bullet A$$
$$\Gamma_2 \xrightarrow{g} B \bullet \Delta_2$$

we define

$$\begin{array}{l}
\Gamma_1 \otimes \Gamma_2 \\
\quad \downarrow f \otimes g \\
(\Delta_1 \bullet A) \otimes (B \bullet \Delta_2) \xrightarrow{\delta_R} \Delta_1 \bullet (A \otimes (B \bullet \Delta_2)) \xrightarrow{\Delta_1 \bullet \delta_L} \Delta_1 \bullet ((A \otimes B) \bullet \Delta_2)
\end{array}$$

$\parr L$

$$\frac{\Gamma_1, A \vdash \Delta_1 \qquad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, A \parr B, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \parr L$$

Given two arrows

$$\Gamma_1 \otimes A \xrightarrow{f} \Delta_1$$
$$B \otimes \Gamma_2 \xrightarrow{g} \Delta_2$$

we define

$$\Gamma_1 \otimes (A \bullet B) \otimes \Gamma_2 \xrightarrow{\Gamma_1 \otimes \delta_R} \Gamma_1 \otimes (A \bullet (B \otimes \Gamma_2)) \xrightarrow{\delta_L} (\Gamma_1 \otimes A) \bullet (B \otimes \Gamma_2) \xrightarrow{f \bullet g} \Delta_1 \bullet \Delta_2$$

Units

$$\frac{}{\bot \vdash} \bot L \qquad \frac{}{\vdash \mathbf{1}} \mathbf{1}R$$

Under the categorical interpretation, these rules merely correspond to the identity arrow on $\bot$ and $\mathbf{1}$, respectively.

$$\frac{\Gamma \vdash \Delta}{\Gamma, \mathbf{1} \vdash \Delta} \mathbf{1}L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot} \bot\mathrm{R}$$

These rules correspond to the unitors for the two units. Namely, for any

$$\Gamma \xrightarrow{f} \Delta$$

there is an arrow

$$\Gamma \otimes \mathbf{1} \xrightarrow{\rho^{\otimes}_{\Gamma}} \Gamma \xrightarrow{f} \Delta$$

and likewise there is an arrow

$$\Gamma \xrightarrow{f} \Delta \xrightarrow{\rho^{\bullet -1}_{\Delta}} \Delta \bullet \bot$$

The problem of units in linear logic results from equalities induced by these denotations.

Theorem 2.3. *The following proofs have the same denotation*

(1) $$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1}}{\Gamma_1 \vdash \Delta_1, \bot}\,\bot \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1 \otimes \Delta_2, \bot}\,\otimes$$

(2) $$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1 \otimes \Delta_2}\,\otimes}{\Gamma_1, \Gamma_2 \vdash \Delta_1 \otimes \Delta_2, \bot}\,\bot$$

(3) $$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma_1 \vdash \Delta_1} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma_2 \vdash \Delta_2}}{\Gamma_2 \vdash \Delta_2, \bot}\,\bot}{\Gamma_1, \Gamma_2 \vdash \Delta_1 \otimes \Delta_2, \bot}\,\otimes$$

Proof. Suppose $\Gamma_1 \xrightarrow{f} \Delta_1$ is the morphism constructed by the deduction π_1, and that $\Gamma_2 \xrightarrow{g} \Delta_2$ is the morphism from π_2. Then (1) corresponds to the morphism

$$\Gamma_1 \otimes \Gamma_2 \xrightarrow{f \otimes (\rho^{\bullet -1}_{\Delta_2} \circ g)} \Delta_1 \otimes (\Delta_2 \bullet \bot) \xrightarrow{\delta_L} (\Delta_1 \otimes \Delta_2) \bullet \bot$$

whereas (2) yields

$$\Gamma_1 \otimes \Gamma_2 \xrightarrow{f \otimes g} \Delta_1 \otimes \Delta_2 \xrightarrow{\lambda^{\bullet}_{\Delta_1 \otimes \Delta_2}} (\Delta_1 \otimes \Delta_2) \bullet \bot$$

These morphisms are equal according to the diagram in Figure 2.10, where the two smaller rectangles commute by naturality, and the outer "triangle" commutes by coherence of linear distribution. We have made use of the equality $f \otimes (\rho^{\bullet -1}_{\Delta_2} \circ g) = (f \otimes (g \bullet \bot)) \circ (\Gamma_1 \otimes \rho^{\bullet -1}_{\Gamma_2})$ which is a consequence of functoriality of $\otimes$ and naturality of ρ. A similar diagram shows that (3) constructs the same morphism as (2), and hence also (1).

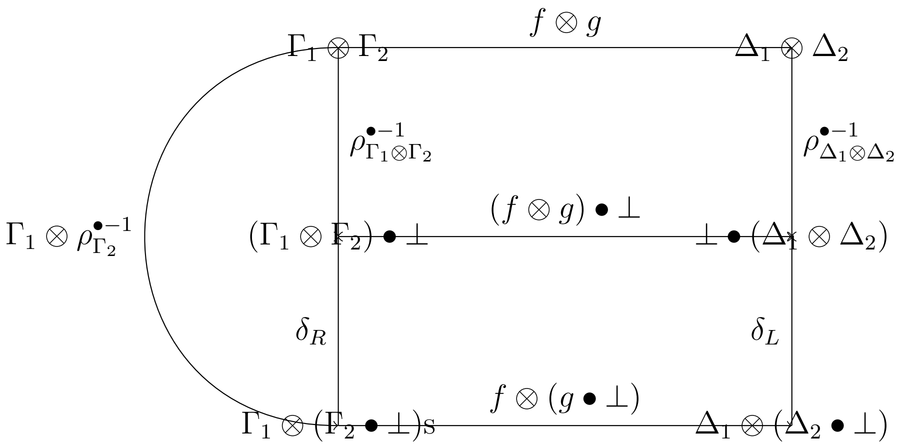


Figure 2.10: Diagrams like these force the identification of many proofs

□

Now we return to proof nets, this time with units. The rules of the logic are now given in Table 2.10. A reasonable way to represent links for units is shown in Figure 2.11. For the $\bot$ introduction rule, the “anchor point” A remains free – the link exists merely to indicate the stage at which the unit is introduced.

$\dfrac{}{\vdash A, A^{\bot}}$ (id)	$\dfrac{\vdash \Gamma, A \qquad \vdash A^{\bot}, \Delta}{\vdash \Gamma, \Delta}$ (cut)	$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \otimes \mathbf{1}}$ $\mathbf{1}$
$\dfrac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}$ $\otimes$	$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A ⅋ B}$ ⅋	$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A, \bot}$ $\bot$

Table 2.10: Rules for one-sided MLL, treating sequents as multi-sets

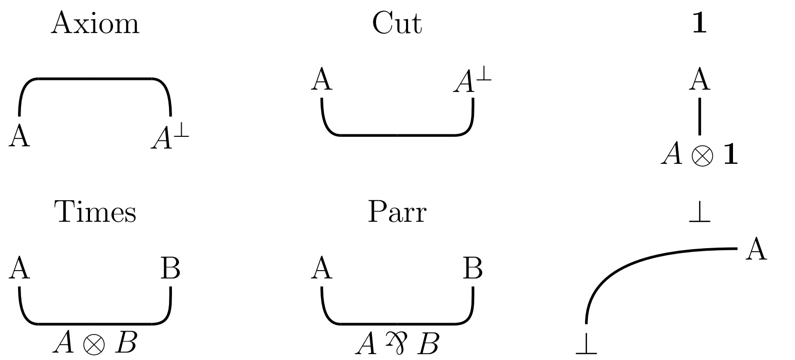


Figure 2.11: The links for multiplicative proof nets, including units

We begin to notice problems immediately. Consider the innocuous proofs in Figure 2.12 and the corresponding proof nets in Figure 2.13. The result is three proofs which are identified in $*$-autonomous categories, but three syntactically distinct (non-isomorphic) proof nets.

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,\text{id}}{\vdash A, \bot, A^\perp}\,\bot \qquad \dfrac{}{\vdash B, B^\perp}\,\text{id}}{\vdash A \otimes B, \bot, A^\perp, B^\perp}\,\otimes$$

$$\dfrac{\dfrac{}{\vdash A, A^\perp}\,\text{id} \qquad \dfrac{\dfrac{}{\vdash B, B^\perp}\,\text{id}}{\vdash B, \bot, B^\perp}\,\bot}{\vdash A \otimes B, \bot, A^\perp B^\perp}\,\otimes$$

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,\text{id} \qquad \dfrac{}{\vdash B, B^\perp}\,\text{id}}{\vdash A \otimes B, A^\perp, B^\perp}\,\otimes}{\vdash A \otimes B, \bot, A^\perp, B^\perp}\,\bot$$

Figure 2.12: Simple proofs involving bottom introduction

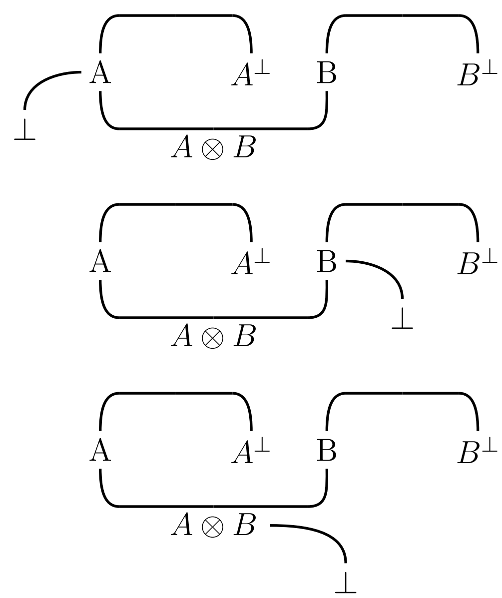


Figure 2.13: The proof nets corresponding to the proofs in Figure 2.12

How "bad" is the situation? Our attempt to build the unit rules into proof nets was somewhat ad-hoc. Conceivably there could exist *some* method of extending proof nets for which equality of proofs is syntactical, or at least relatively easy to characterize. In fact there is apparently no good (read: computationally tractable) way to do this – the 2014 paper [8] proves that proof equivalence for MLL (i.e., the word problem for $*$-autonomous categories) is PSPACE-complete. Therefore any scheme for presenting proofs nets involving units is, in general, either intractable when converting from a deduction to a net, or when determining whether two nets represent the same morphism. Put another way, there is no simple notion of string diagrams for describing general morphisms in $*$-autonomous categories which yields easily-calculated normal forms.

The conjecture being developed in our work is that morphisms for $*$-autonomous categories are best thought of as surfaces, and equality of morphisms is some topological notion of equality between surfaces (which, in general, one would not expect to be a tractable problem). We turn to these ideas in the next chapter.

Remark 2.8. If there are no proof nets for the full system MLL, then what is the proper categorical notion for the fragment MLL$^-$, for which proof nets are canonical? A reasonable criterion for answering this question is that such proof nets ought to generate a free model of such a category. Two different approaches to this line of research are explored in the papers [4] and [9].

Chapter 3

$*$-Autonomy and Frobenius Pseudoalgebras

3.1 Surfaces in PROF

The theorem that makes our work possible states that $*$-autonomous categories have a characterization as objects of a monoidal bicategory **PROF**, which is useful in light of the fact that monoidal bicategories have a convenient notation, the surface calculus, described in Chapter 8 of [10]. This section introduces **PROF** and this calculus simultaneously. Our reference for profunctor theory is [2] and the notes [14]. We assume familiarity with bicategories – Appendix A contains background material on the subject.

A note on the surface calculus: There are at least two ways of thinking about it. First, we can depict 2-cells as actual three-dimensional surfaces. This is the perspective we will use when considering equivalence of surfaces in Chapter 4. Another presentation is to depict 1-cells as string diagrams, and depict 2-cells as arrows between them which describe "local rewrites." The rewrite perspective will prove more useful in Section 3.2. Here, we will present both perspectives side-by-side. We do assume familiarity with string diagrams for monoidal categories, described in [10].

> *Remark* 3.1. From the rewrite perspective, the structure of monoidal bicategories ensures that the rewrites are local operations, in the sense that rewrites which can be "spatially separated" commute with each other. We use this fact to build commutative diagrams of rewrites like in Figure 3.35.

The bicategory **PROF** is essentially an expansion of the monoidal bicategory **CAT** of categories, functors, and natural transformations. A profunctor $F\colon A \nrightarrow B$ from A to B is simply a functor $B^{op} \times A \to \mathrm{Set}$ (thus, the type of an arrow differs depending on whether it is considered in **PROF** or **CAT** – notice we use a slash to indicate when an arrow is being thought of as a profunctor). This means profunctors are particular functors. However, they are also generalizations of functors.

In the surface calculus, a profunctor $F\colon A \nrightarrow B$ from A to B is drawn as in Figure

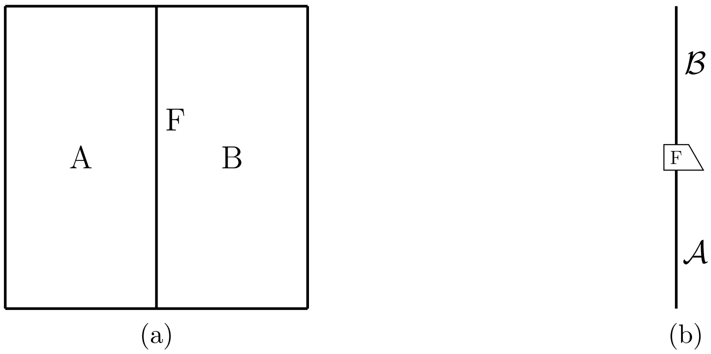


Figure 3.1: A Profunctor F from A to B, as a surface and as a string diagram

3.1 as either a surface (3.1a) or a string diagram (3.1b).

Any functor $A \xrightarrow{F} B$ between categories induces two profunctors $\mathcal{A} \overset{F_*}{\nrightarrow} \mathcal{B}$ and $\mathcal{B} \overset{F^*}{\nrightarrow} \mathcal{A}$ in **PROF**, namely

$$\mathcal{A} \overset{F_*}{\nrightarrow} \mathcal{B} \text{ defined on objects by } (b, a) \mapsto \mathcal{B}(b, Fa)$$

and

$$\mathcal{B} \overset{F^*}{\nrightarrow} \mathcal{A} \text{ defined on objects by } (a, b) \mapsto \mathcal{B}(Fa, b)$$

Not every profunctor is induced in this fashion however. Profunctors naturally isomorphic (see below) to ones of the form F_* as above are called *representable*. We shall also show that natural transformations induce 2-cells in **PROF**, and moreover these embeddings preserve the structure of composition in **CAT**. This is what we mean by the fact that **CAT** embeds into **PROF**. In fact there are two such embeddings: the covariant ($F \mapsto F_*$) and the contravariant ($F \mapsto F^*$)

Since profunctors are particular functors, we can consider natural transformations between them in the usual sense. These are the 2-cells in **PROF**. That is, given two profunctors $A \overset{F}{\nrightarrow} B$ and $A \overset{G}{\nrightarrow} B$ of the same type, a 2-cell $\mu \colon F \Rightarrow G$ between them in **PROF** is a natural transformation between F and G as functors $B^{op} \times A \to \text{Set}$. A natural transformation $\mu \colon F \Rightarrow G$ is drawn as in Figure 3.2. Vertical composition is drawn as in Figure 3.3.

Composition of profunctors is less straightforward to define than that of functors. Notice that a profunctor $\mathcal{A} \overset{F}{\nrightarrow} \mathcal{B}$ is essentially a functor (which we call $\hat{F}$, the "transpose" of F) $\mathcal{A} \xrightarrow{\hat{F}} \hat{\mathcal{B}}$, where by $\hat{\mathcal{B}}$ we mean the category whose objects are functors $\mathcal{B}^{op} \to \text{Set}$. Now, given two profunctors $A \overset{F}{\nrightarrow} B$ and $B \overset{G}{\nrightarrow} C$, we can "officially" define their composition as the profunctor $G \circ F$ whose transpose is $L_{Y_B}(\hat{G}) \circ \hat{F}$, where by $L_{Y_B}(\hat{G})$ we mean the left Kan extension of $\hat{G}$ along the Yoneda functor $B \to \hat{B}$ which maps $b \mapsto \mathcal{B}(-, b)$. A discussion of Kan extensions would take us outside the scope of this paper – fortunately there is an easier way. One shows that this definition is equivalent to

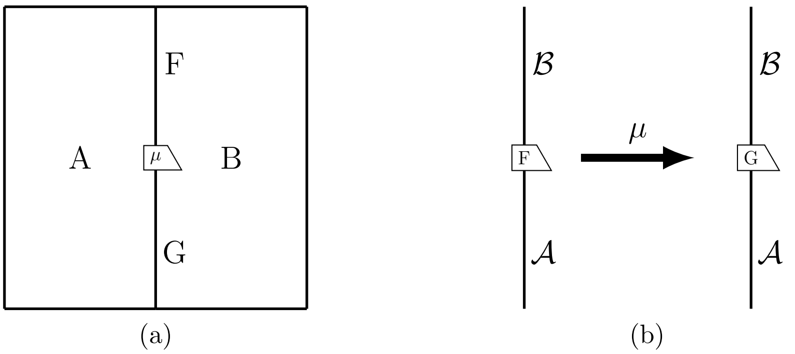


Figure 3.2: A natural transformation μ from F to G, as a surface or a rewrite

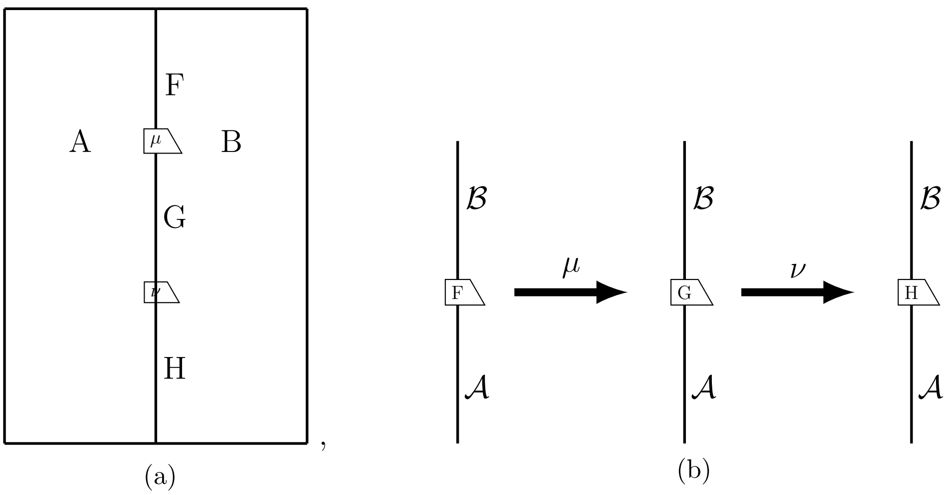


Figure 3.3: The vertical composite of a 2-cell μ from F to G and another ν from G to H

.

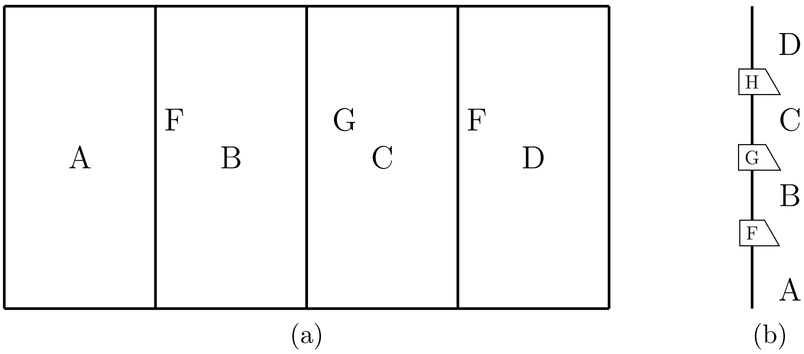


Figure 3.4: The composition of several profunctors

$$(G \circ F)(c, a) = \int^{x \in B} G(c, x) \times F(x, a)$$

Here the intergral notation means the coend of the "integrand," the functor of type $\mathcal{B}^{op} \times \mathcal{B} \to \text{Set}$ given by $(b_1, b_2) \mapsto G(c, b_2) \times F(b_1, a)$. While a discussion of coends would also take us beyond our current scope, this object has a straightforward characterization: Let Φ be the integrand, the functor defined above. We obtain two actions (in a sort of categorified, "many-object" sense) of $\mathcal{B}$ on Φ of type,

$$\coprod_{b_1, b_2 \in \mathcal{B}} \Phi(b_2, b_1) \times \mathcal{B}(b_1, b_2) \longrightarrow \coprod_{b \in \mathcal{B}} \Phi(b, b).$$

Notice the the coproduct on the left hand side is the set of all $((y, x), f)$ where $c \xrightarrow{y} b_1$, $b_2 \xrightarrow{x} a$, and $b_1 \xrightarrow{f} b_2$. A left action is given by mapping this object to $(f \cdot y, x)$ where $f \cdot y := f \circ y$. Similarly, a right action maps it to $(y, x \cdot f)$ where $x \cdot f := x \circ f$. The coend above is the (object part of the) coequalizer of these two actions. This implies that up to isomorphism, $(G \circ F)(c, a)$ is given on objects by

$$(c, a) \mapsto \big(\coprod_{b \in \mathcal{B}} G(c, b) \times F(b, a)\big) / \sim$$

where $\sim$ is the equivalence relation generated by the relation $(f \cdot y, x) \sim (y, x \cdot f)$ for $b \xrightarrow{f} b'$. As an exercise, one can verify that the composition of representable profunctors works as one might expect: $(G_* \circ F_*) \simeq (G \circ F)_*$.

Composition of profunctors is only associative up to a natural isomorphism, which is allowed in a bicategory. The composition of profunctors, for instance of $A \overset{F}{\nrightarrow} B$, $B \overset{G}{\nrightarrow} C$, and $C \overset{H}{\nrightarrow} D$, is drawn as in Figure 3.4. (However the surface by itself only depicts the composite up to isomorphism.) Up to isomorphism, the identity profunctor on a category $\mathcal{A}$ is given by the hom-functor $(a', a) \mapsto \mathcal{A}(a', a)$.

Natural transformations of functors lift to ones between representable profunctors. Given $\mu \colon F \Rightarrow G$, we obtain a 2-cell $\mathcal{B}(-, F-) \Rightarrow \mathcal{B}(-, G-)$ defined by $(b \xrightarrow{f} Fa) \mapsto (b \to Fa \xrightarrow{\mu_a} Ga)$. Similarly we obtain a 2-cell $\mathcal{B}(G-, -) \to \mathcal{B}(F-, -)$ by pre-composition.

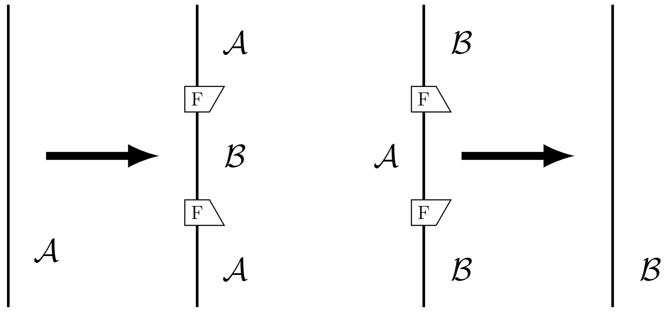


Figure 3.5: Unit and counit of an adjunction

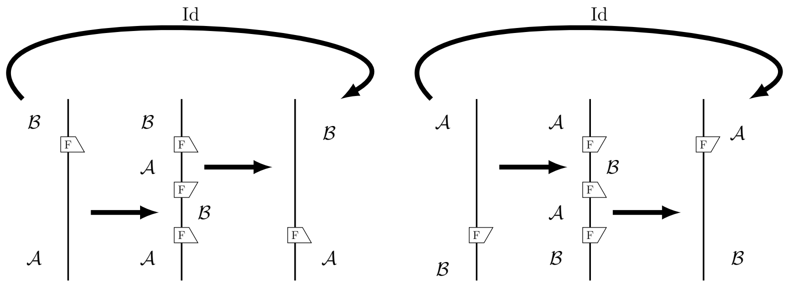


Figure 3.6: Commutativity satisfied by the unit and counit of an adjunction. Here, we are using the position of the arrows to indicate where the rewrites are taking place.

Given two functors there is an evident natural transformation $\mathcal{A}(a', a) \to \int^{b\in\mathcal{B}} \mathcal{B}(Fa', b) \times \mathcal{B}(b, Fa)$ (i.e. $1_{\mathcal{A}} \to F^* \circ F_*$). There is also a transformation $\int^{a\in\mathcal{A}} \mathcal{B}(b', Fa) \times \mathcal{B}(Fa, b) \to \mathcal{B}(b', b)$ (i.e. $F_* \circ F^* \to 1_{\mathcal{B}}$). In fact we can state more than this, in the form of a theorem we will state without proof

Theorem 3.1. *For a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$, F_* is left adjoint to F^*. Moreover this is a characterization – a profunctor with a right adjoint is representable.* □

We indicate the (unique up to isomorphism) right adjoint to a profunctor F in the string diagrams by vertically flipping the slanted box, but keeping the same label F (This is clearer by looking at the pictures). Then the adjointness property means there are two 2-cells as in Figure 3.5, subject to two commutative diagrams (Figure 3.6).

Finally, note that by taking Cartesian products, **PROF** can be made into a monoidal bicategory just as **CAT**, where the unit $\mathbf{1_{PROF}}$ is given by the single-object, single-arrow category. We write $\boxtimes$ for this product. The product is symmetric by taking the maps which "swap" the two factors. Notice also that the category $\mathcal{A}^{op}$ gives a left and right dual object for each category $\mathcal{A}$ in **PROF**, making it autonomous (see Appendix A). This gives us several "cup" and "cap" diagrams (Figure 3.7), where for instance the cup $\mathcal{A} \times \mathcal{A}^{op} \to \mathbf{1_{PROF}}$ is just the hom-functor $(a, a') \mapsto \mathcal{A}(a', a)$ and the rest are defined similarly. The cups and caps feature invertible 2-cells, shown in Figure 3.8, a bicategorical generalization of the "snake equations" for objects in monoidal categories.

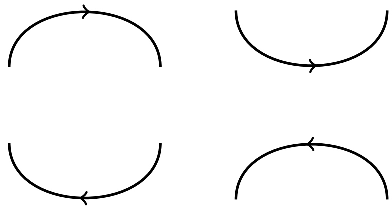


Figure 3.7: Cups and caps witnessing duality

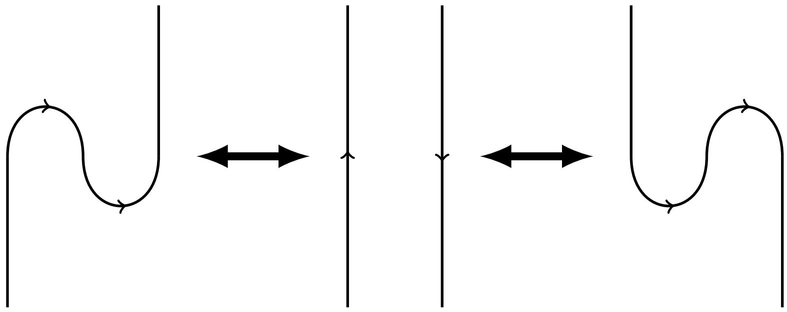


Figure 3.8: 2-cells like these allow us to "pull strings tight," or alternatively to bend them.

3.2 Frobenius Pseudoalgebras

Throughout the following, let $\mathcal{B}$ be an arbitrary symmetric monoidal bicategory.

Definition 3.1. A *pseudomonoid* $(A, \oplus, 1_\oplus, \alpha_\oplus, l_\oplus, r_\oplus)$ in $\mathcal{B}$ is given from

- An object $A \in \mathcal{B}_0$
- Two 1-cells $A \boxtimes A \xrightarrow{\oplus} A$ and $\mathbf{I}_\mathcal{B} \xrightarrow{1_\oplus} A$ (Figure 3.9)
- A 2-cell $\alpha_\oplus$ witnessing the associativity of $\oplus$ and two $(l_\oplus, r_\oplus)$ witnessing the unitality of $1_\oplus$ with $\oplus$ (Figure 3.10)
- Subject to certain coherence conditions: that diagrams of Figures 3.11 and 3.12 must commute.

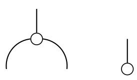


Figure 3.9: Monoid 1-cells

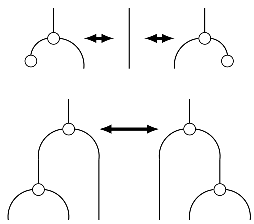


Figure 3.10: Monoid 2-cells

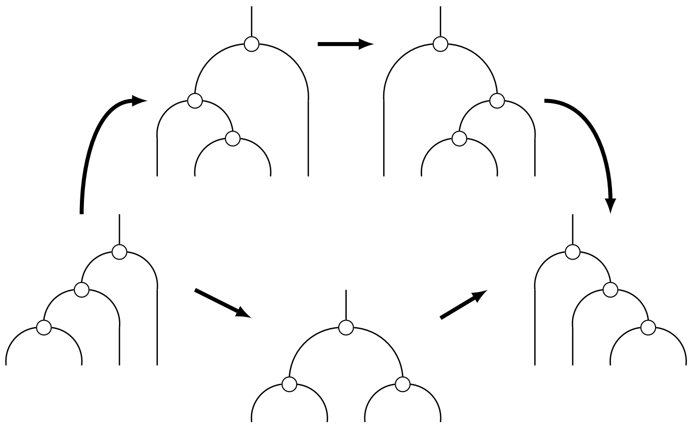


Figure 3.11

Remark 3.2. Similarly to our conventions with monoidal categories (which the reader can verify are precisely pseudomonoids in **CAT**), we often abbreviate the data of a pseudomonoid to simply $(A, \oplus, 1_\oplus)$. We will never conflate the pseudomonoid for the object, however, since we will consider several such structures on one object simultaneously.
As a convention, in our diagrams a white dot corresponds to what is $\oplus$ in the text. A black dot always corresponds to $\otimes$.

The definition of a pseudocomonoid is obtained by vertically flipping the direction of the 1-cells above. Hereafter we may simply drop the unwieldy prefix "pseudo-," but we do not mean to imply that the structures under consideration are strict in any way.

If the pseudomonoid is equipped with an additional coherent invertible 2-cell as in Figure 3.13, then it is *commutative* (and likewise for pseudocomonoids). One of the coherence conditions is shown in Figure 3.14, and another is generated by horizontally flipping the 1-cells pictured there.

Definition 3.2. Suppose $(A, \oplus, 1_\oplus)$ is a pseudomonoid in an autonomous $\mathcal{B}$. There is an induced comonoid on A° defined from the data in Figure 3.15. We call this the *dual comonoid*. Likewise for pseudocomonoids, which induce dual pseudomonoids.

Definition 3.3. A Frobenius pseudoalgebra $(A, \oplus, 1_\oplus, \otimes, 1_\otimes, f)$ in $\mathcal{B}$ is given by

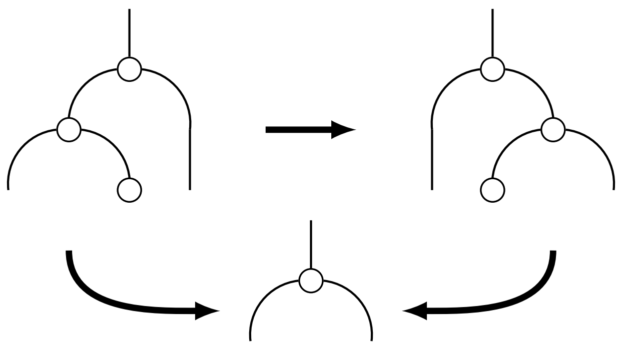


Figure 3.12

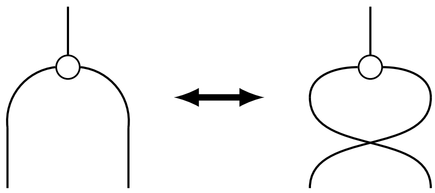


Figure 3.13: Commutativity of a pseudomonoid

- The data $(A, \oplus, 1_\oplus)$ of a pseudomonoid on A
- The data $(A, \otimes, 1_\otimes)$ of a pseudocomonoid on the same object
- An invertible 2-cell f as in Figure 3.16

There is an additional 2-cell for these structures which comes for free.

Theorem 3.2. *Suppose we have the data $(A, \oplus, 1_\oplus, \otimes, 1_\otimes, f)$ of a Frobenius pseudoalgebra. Then we can also equip it with two invertible 2-cells shown in Figure 3.17.*

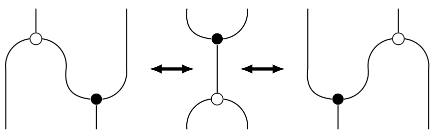


Figure 3.17: 2-cells of these types are a fundamental to the study of Frobenius pseudoalgebras. We shall call these the "Frobenius rewrites."

Proof. We shall construct just one of the cells (Figure 3.18), and the other follows from composing with the Frobenius 2-cell in Figure 3.16.

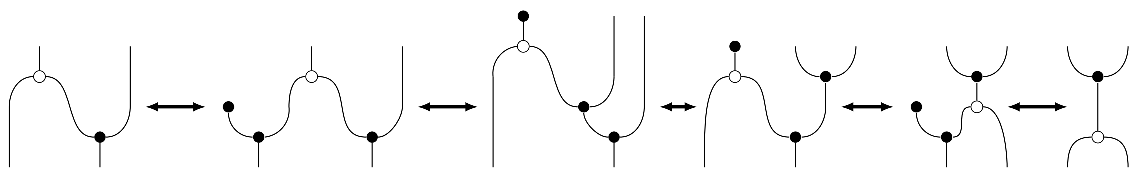


Figure 3.18

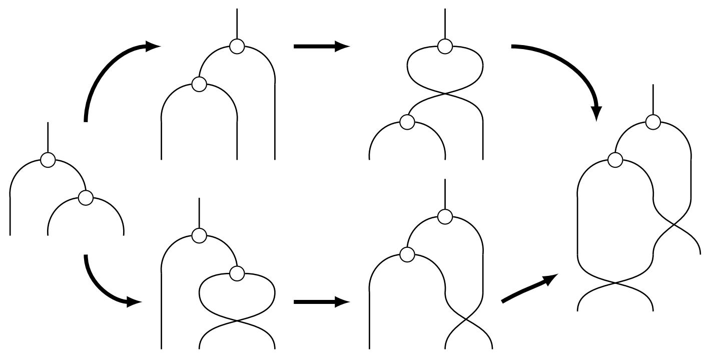


Figure 3.14: Coherence required of commutativity

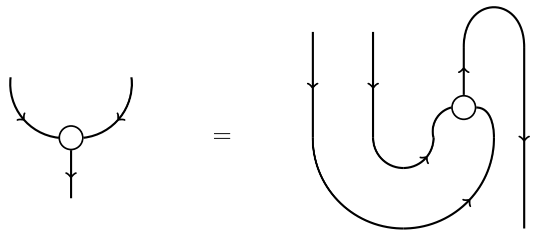


Figure 3.15: The dual comonoid

□

It is not hard to show that in the commutative case, we have a converse to the above theorem – either one of the of the 2-cells given is sufficient to derive a 2-cell making the structure Frobenius.

To study $*$-autonomous categories, we impose further conditions on the algebra.

Definition 3.4. If the monoid and comonoid (and their units) of a Frobenius pseudoalgebra have adjoints, the resulting structure is called a †-Frobenius pseudoalgebra.

This terminology is also used to describe situations where the comonoid itself is adjoint to the monoid. We are considering a more general situation. By specializing to the case where $\mathcal{B}$ is the autonomous monoidal bicategory **PROF**, we shall demonstrate below that we have arrived at a definition of $*$-autonomous categories. To see this, note that the axioms of a $*$-autonomous category guarantee a few things:

- The two monoidal structures on the category $\mathcal{A}$ are precisely two pseudomonoidal structures on $\mathcal{A}$ as an object of **CAT**, which become pseudomonoidal structures on $\mathcal{A}$ in **PROF**. The adjoints (which are given, for instance, by functors like $X, Y, Z \mapsto \mathcal{A}(X \otimes Y, Z)$) induce pseudocomonoidal structures as well. Furthermore we can take the duals of all of these objects to give (co)monoidal structures on $\mathcal{A}^{op}$.
- The covariant embedding of S, a profunctor $\mathcal{A} \nrightarrow \mathcal{A}^{op}$ defined by $(a', a) \mapsto \mathcal{A}^{op}(Sa', a)(= \mathcal{A}(a, Sa'))$, is a weak inverse to the covariant embedding of S^{op}.

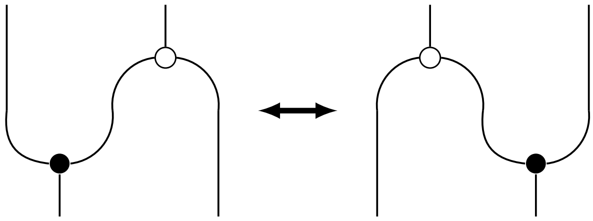


Figure 3.16: The "Frobeniusator"

We also have several duality isomorphisms like the one shown in Figure 3.19. They are expressing the various isomorphisms like $\mathcal{A}(x, Sy) \simeq \mathcal{A}(y, Sx)$ which are a consequence of the self-adjointness of S.

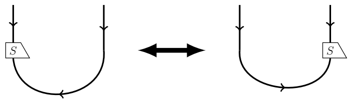


Figure 3.19: We often use rewrites like these, a consequence of S being left and right adjoint to S^{op}.

- The isomorphism

$$\mathcal{A}(A \otimes B, SC) \simeq \mathcal{A}(A, S(B \otimes C))$$

  gives us a local rewrite of either of the equivalent forms depicted in Figures 3.20 3.21 (we will use both interchangeably – we could always achieve one rewrite from the other by appealing to the "snake" 2-cells relating the cups and caps between A and A^{op}).

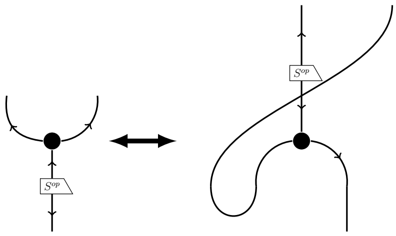


Figure 3.20: One way of picturing the hom-set isomorphism of $*$-autonomy.

Theorems 3.3 and 3.4 show that $*$-autonomous categories can be characterized as certain objects in **PROF**. These theorems are based on a set of observations by Street in [18].

> *Remark* 3.3. In some of the diagrams below we do not depict excessive fiddling with wires. Implicitly we are making heavy use of the snake 2-cells and interchange law for monoidal bicategories.

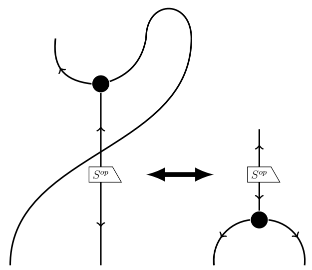


Figure 3.21: An equivalent way of picturing the hom-set isomorphism of $*$-autonomy.

Theorem 3.3. *$*$-autonomous categories induce commutative $\dagger$-Frobenius pseudoalgebras in* **PROF**, *for which the adjoint of $\otimes$ is the comonoid and $\bullet$ is the monoid.*

Proof. We already know that $*$-autonomous categories induce the commutative monoid and comonoid. What we must verify is the existence of an invertible 2-cell between the left (Figure 3.22) and right (Figure 3.23) configurations. Figure 3.24 shows that this follows from the commutativity of the (co)monoids, the fact that $\mathbf{Id}_{\mathcal{A}} \simeq S^{op} \circ S$, and the $*$-autonomy rewrite described above. $\square$

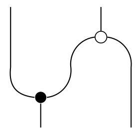


Figure 3.22: $(A, B, C, D) \mapsto \int^{x \in A} \mathcal{A}(A \otimes X, C) \times \mathcal{A}(B, S(SX \otimes SD))$

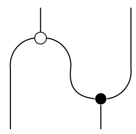


Figure 3.23: $(A, B, C, D) \mapsto \int^{x \in A} \mathcal{A}(A, S(SC \otimes SX)) \times \mathcal{A}(X \otimes B, D)$

Theorem 3.4. *The previous theorem is essentially a characterization. Commutative $\dagger$-Frobenius pseudoalgebras in* **PROF** *are $*$-autonomous categories.*

Proof. Suppose $\mathcal{A}$ is a Frobenius pseudoalgebra. First, we appeal to Theorem 3.1 to see that the monoid and its unit are representable, say by a functor $\mathcal{A} \times \mathcal{A} \xrightarrow{\bullet} \mathcal{A}$ and another of type $\mathbf{1}_{\mathbf{PROF}} \to \mathcal{A}$ (which is equivalent to giving an object in $\mathcal{A}$). From the 2-cells witnessing the associativity, unitality, and commutativity of the pseudomonoid, and the coherence of the pseudomonoid, we can reconstruct appropriately coherent natural transformations to complete the monoidal structure on $\mathcal{A}$. (For instance, the 2-cells allows us to construct a natural isomorphism $\mathcal{A}(A, A) \times \mathcal{A}(B, B) \times \mathcal{A}(C, C) \to \mathcal{A}(A \otimes (B \otimes C), (A \otimes B) \otimes C)$, from which it is easy to obtain the component $\alpha_{a,b,c}$ of the associator transformation.) Similarly, the dual pseudomonoid induced by the pseudocomonoid has a right adjoint, giving a monoidal structure on $\mathcal{A}^{op}$, which is

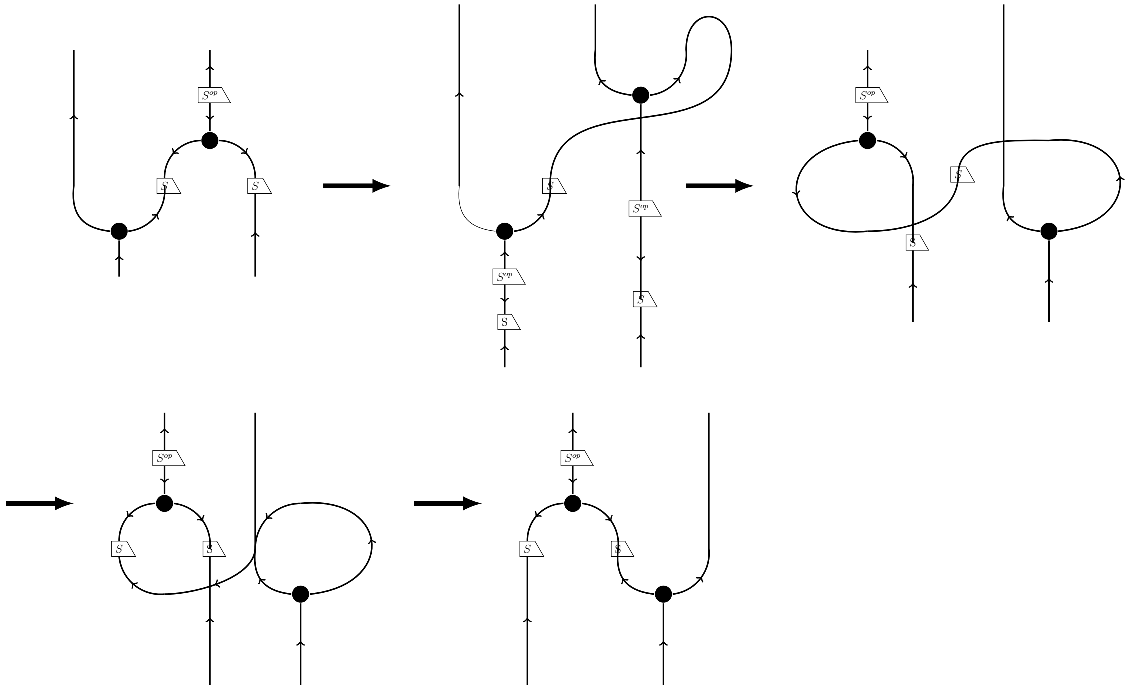


Figure 3.24: The Frobenius law is a direct consequence of the axioms of $*$-autonomous categories. The last 2-cell applied to the braiding of the $\bullet$ monoidal structure, which can be defined from that of the $\otimes$ monoid

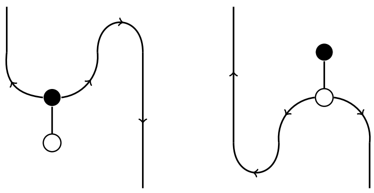


Figure 3.25: These 1-cells are weak inverses to each other, allowing us to define a contravariant self-adjoint equivalence S on $\mathcal{A}$

equivalent to giving one on $\mathcal{A}$. The "snake" 2-cells and the Frobenius law 2-cells (particularly the one depicted in Theorem 3.2), shows that the two 2-cells in Figure 3.25 are left and right inverse to each other. It is well known that equivalences can be refined to be adjoint equivalences – hence the 1-cells are adjoint to each other and are representable. This observation also gives us the isomorphism $\mathbf{Id}_{\mathcal{A}} \simeq S^{op} \circ S$. We must verify the natural isomorphism

$$\mathcal{A}(A \otimes B, SC) \simeq \mathcal{A}(A, S(B \otimes C)).$$

It is established from the invertible 2-cell in Figure 3.26. □

Definition 3.5. For a given Frobenius pseudoalgebra, if the diagram of Figure 3.27 (and the one built from taking mirror images of 1-cells) commutes, the structure is considered *coherent* (with units).

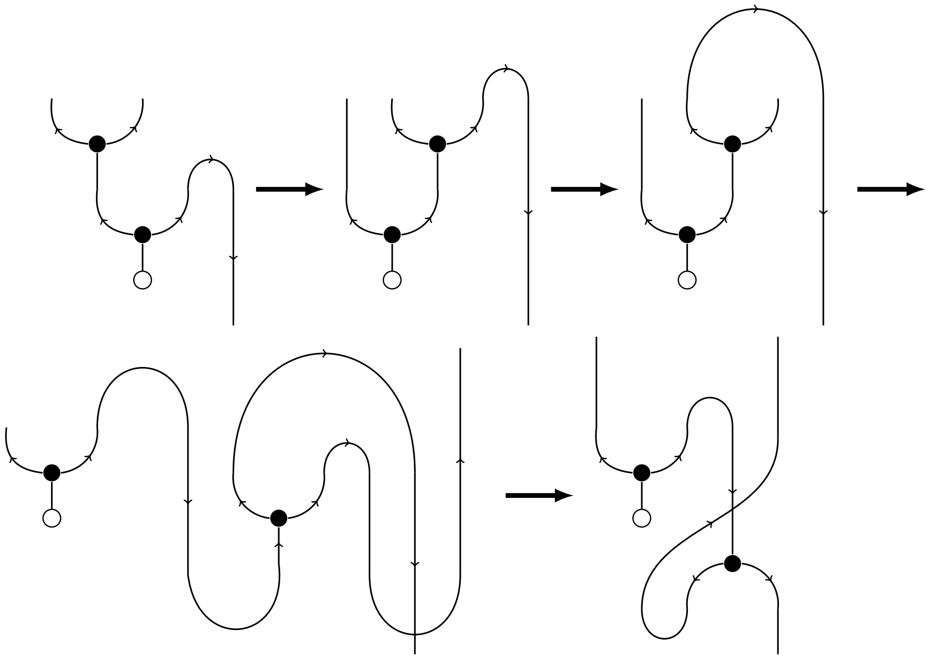


Figure 3.26: The hom-set isomorphism of $*$-autonomous categories. We have used associativity, the definition of the dual pseudomonoid, commutativity, and several (implied) instances of the snake 2-cells and interchange law.

Currently it is thought every Frobenius pseudoalgebra is coherent in this manner. Some motivation is given in Chapter 4, where visualizing the resulting surfaces yields two figures which are homotopy equivalent. For now, we shall take this coherence as an axiom and consider its consequences.

What is the advantage of considering such structures? One example is the coherence of linear distribution. We return to the derivation of linear distribution in $*$-autonomous categories as described in [15]. We can visualize this construction in surfaces diagrams. A close examination shows the first morphsim of type $A \otimes S(A \otimes B) \to B$ is derived by a transformation which looks like that shown in Figure 3.28. The second, of type $SB \to (SB \multimap C) \multimap C$, is effectively what is shown in Figure 3.29. The isomorphism from Figure 2.9 is given in Figure 3.30. The full derivation of linear distribution is then shown in 3.31.

Using the adjoint properties of the (co)monoids, giving a 2-cell of the type in Figure 3.31 is equivalent to giving one as in 3.32. Instead of deriving a rewrite like 3.32 through this approach, however, we can employ the Frobenius rewrites, whose use greatly simplifies matters.

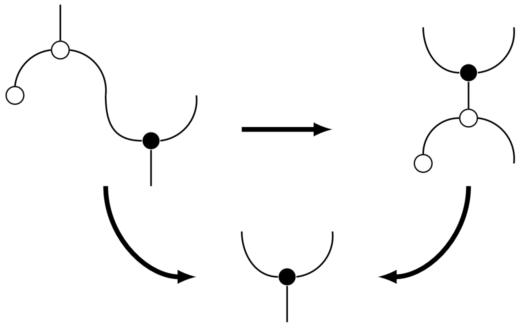


Figure 3.27: One of the unit coherence properties for Frobenius pseudoalgebras. Ongoing work seeks to characterize pseudoalgebras properties like these.

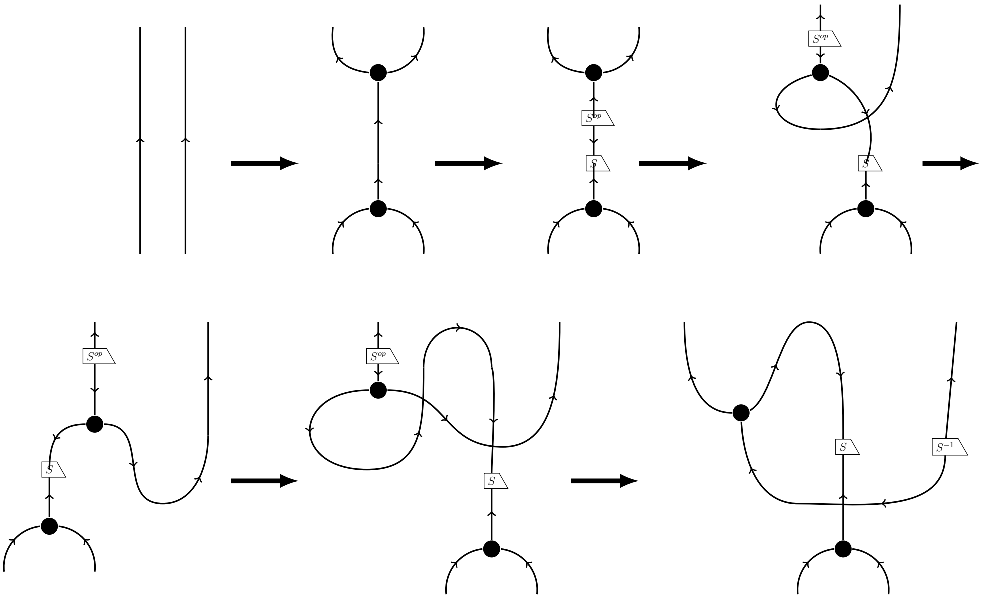


Figure 3.28: A natural transformation $\mathcal{A}(A, A) \times \mathcal{A}(B, B) \to \mathcal{A}(A \otimes S(A \otimes B), SB)$

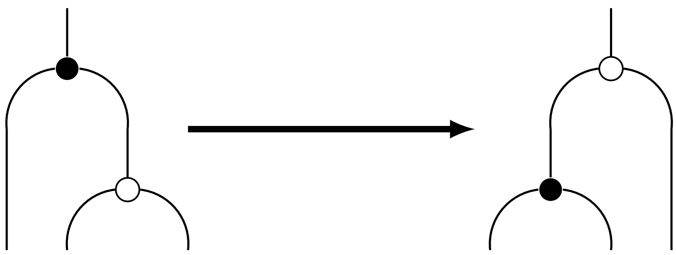


Figure 3.32: The type of linear distribution.

Theorem 3.5. *A †-Frobenius pseudoalgebra has a linear distribution, a 2-cell of the form Figure 3.32.*

Proof. See Figure 3.33

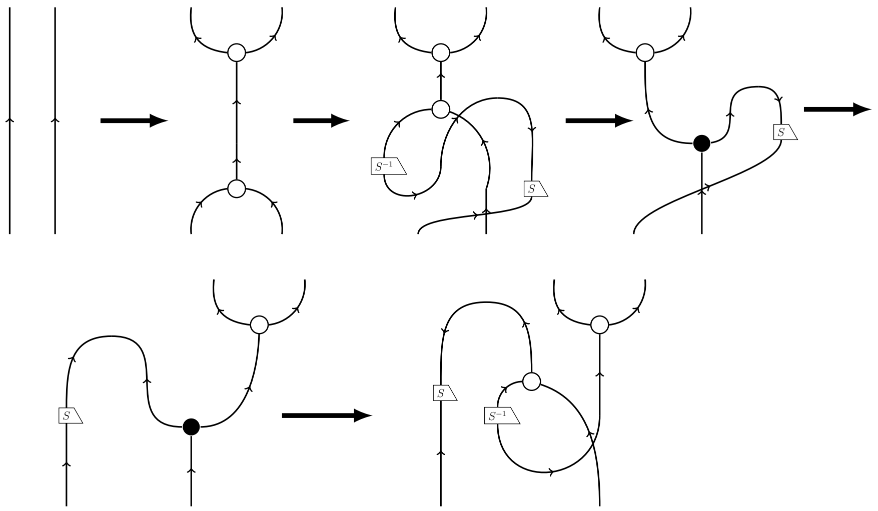


Figure 3.29: A natural transformation $\mathcal{A}(B, B) \times \mathcal{A}(C, C) \to \mathcal{A}(SB, (SB \multimap C) \multimap C)$

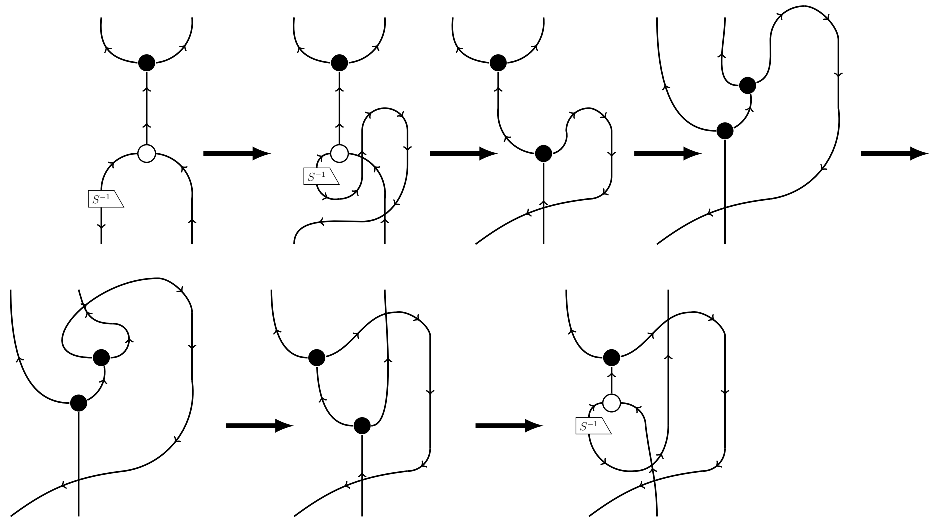


Figure 3.30: A natural transformation $\mathcal{A}(A \otimes X, Y \multimap C) \times \mathcal{A}(A \otimes Y, X \multimap C)$

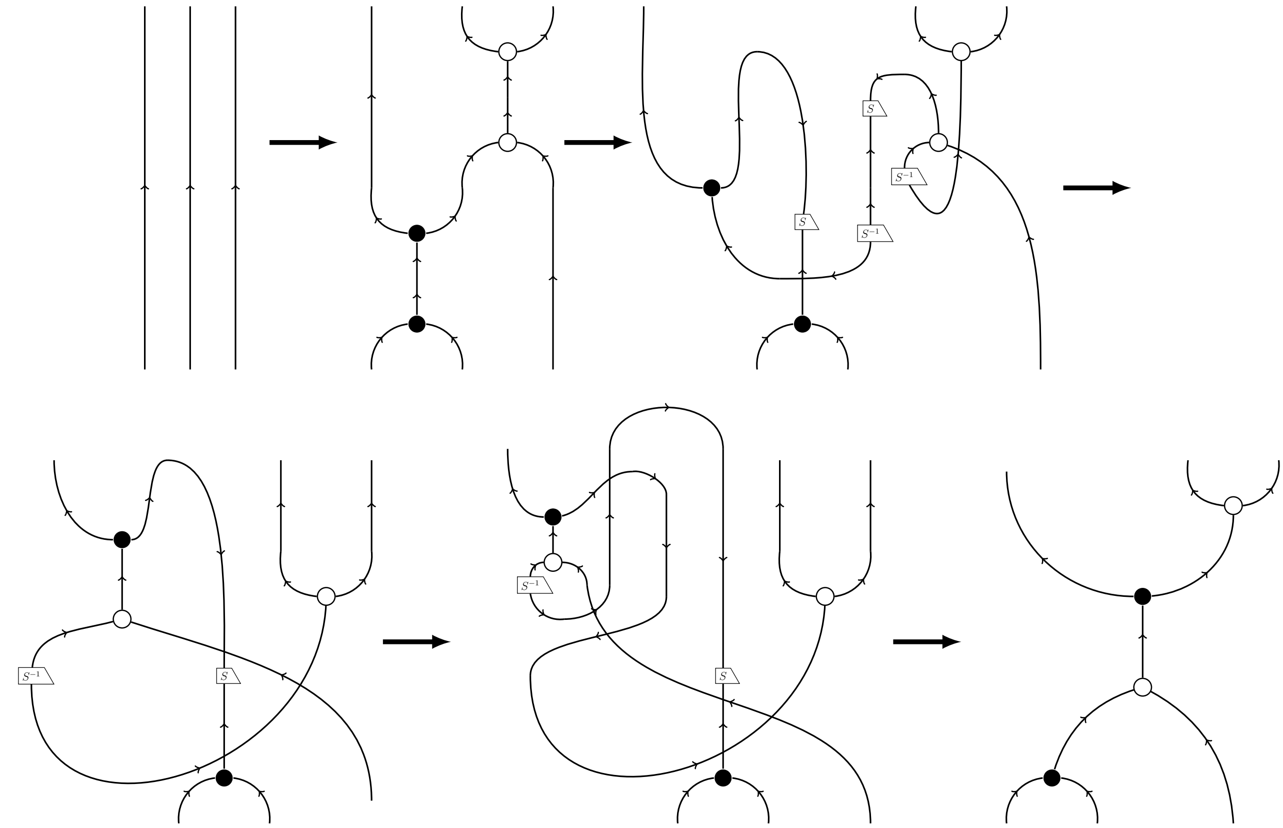


Figure 3.31: A natural transformation $\mathcal{A}(A,A) \times \mathcal{A}(B,B) \times \mathcal{A}(C,C) \to \mathcal{A}(A \otimes (B \bullet C), (A \otimes B) \bullet C)$, built from the rewrites in Figures 3.28, 3.29, and 3.30.

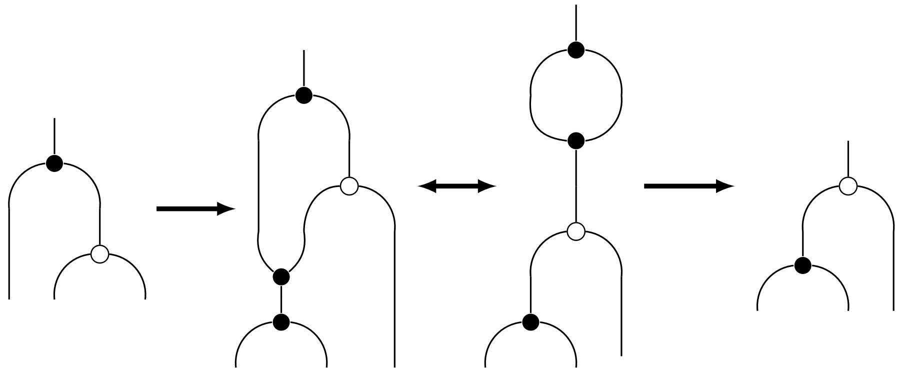


Figure 3.33: Linear distribution derived from Frobenius rewrites

□

It is not currently known whether the resulting transformation of hom-sets through this approach is identical to using the linear distribution of Figure 3.31. However, the Frobenius approach seems to simplify the coherence properties of linear distribution, if we are correct in speculating that all Frobenius pseudoalgebras are coherent with units.

Theorem 3.6. *For a given †-Frobenius pseudoalgebra, if the Frobenius law is coherent with units as in Figure 3.27, then the linear distribution of Theorem 3.5 is coherent with units in the sense that Figure 3.34 commutes (These are prototypical examples – We also require diagrams built from mirror images of these 1-cells to commute, for instance).*

Proof. See Figure 3.35.

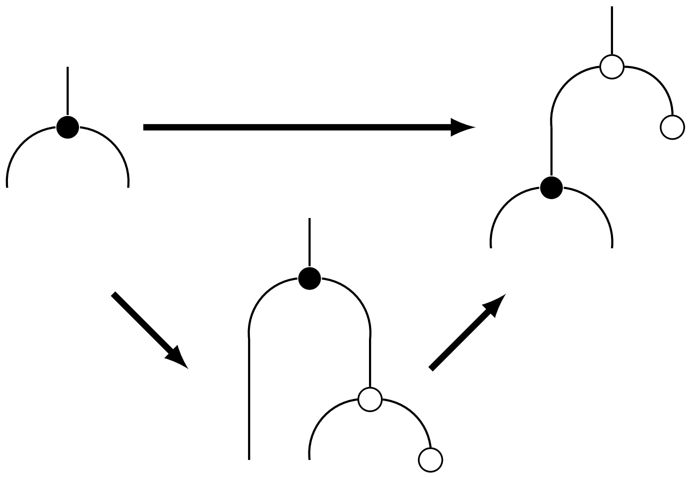


Figure 3.34: Prototypical example of the coherence required of linear distribution

□

We will conclude this chapter with a some conjectures which form the grounds for future work.

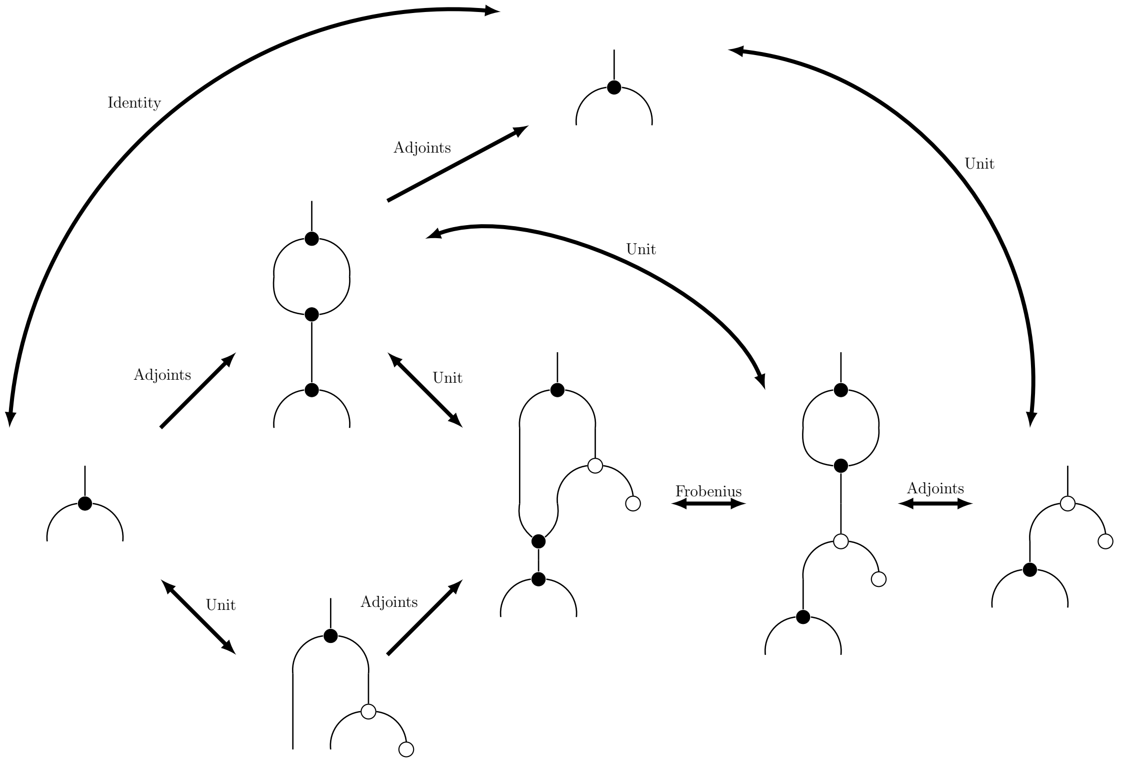


Figure 3.35: Coherence for the linear distribution rewrite of Figure 3.33, assuming a coherent Frobenius law. Following diagram along the bottom gives the linear distribution defined in Figure 3.33. Along the top, we get a simple unit introduction.

Conjecture 3.1. *All Frobenius pseudoalgebras are coherent with units in the sense we have defined.*

The next conjecture is somewhat imprecise and largely dependent on the previous one.

Conjecture 3.2. *The coherence of linear distribution in $*$-autonomous categories can be inferred from the coherence properties of Frobenius pseudoalgebras.*

In the next chapter we shall examine the correlation between identifications of proofs and equivalence of surfaces in the surface calculus.

Chapter 4

Proofs as Surfaces

In this chapter we shall look at a few examples of how linear logic fits into this idea of surface deformations (/string diagram rewrites). We shall also visualize a few examples of three-dimensional surfaces to motivate the ongoing work in this area. In general, we have some choice on exactly how to draw the surface corresponding to a rule or proof in linear logic. What we demonstrate here are some possibilities.

First notice for a $*$-autonomous category $\mathcal{A}$ that there are isomorphisms $\mathcal{A}(A, A) \simeq \mathcal{A}(\mathbf{I}, S(A \otimes SA)) \simeq \mathcal{A}(A \otimes SA, \perp)$, corresponding to the 2-cells in Figure 4.1. This is essentially a general property of Frobenius pseudoalgebras: cups and caps are determined up to an invertible 2-cell, and the Frobenius 2-cells, combined with the fact that S and S^{op} are weak inverses, guarantee that the 1-cells pictured give a duality between $\mathcal{A}$ and $\mathcal{A}^{op}$.

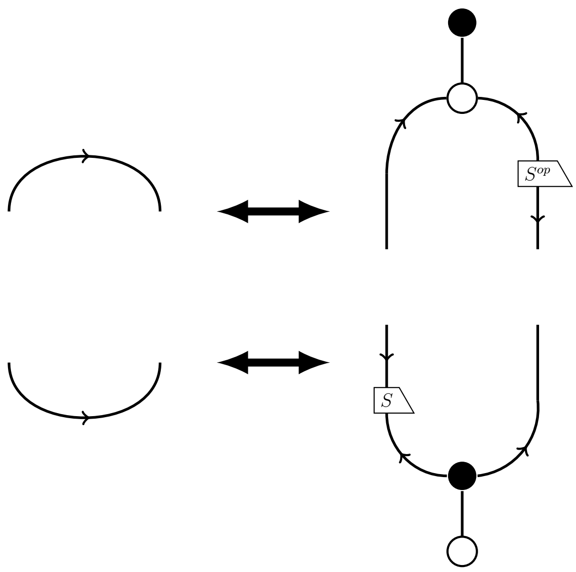


Figure 4.1

The two-sided introduction rule

$$\frac{}{A \vdash A} \text{ id}$$

has a simple interpretation: The natural transformation $\mathbf{1}_{\mathbf{PROF}}(\star, \star) \to \mathcal{A}(A, A)$ which sends $id_\star \mapsto id_A$. It is shown in Figure 4.2.

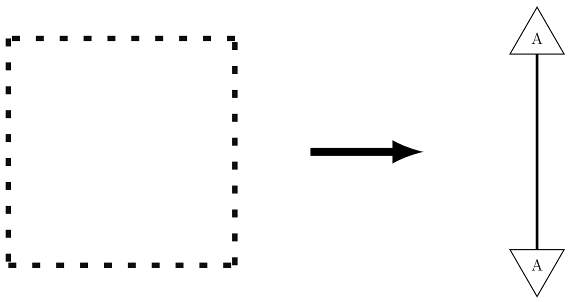


Figure 4.2

The one-sided introduction rule can be seen through a regular introduction combined with the isomorphism between caps above.

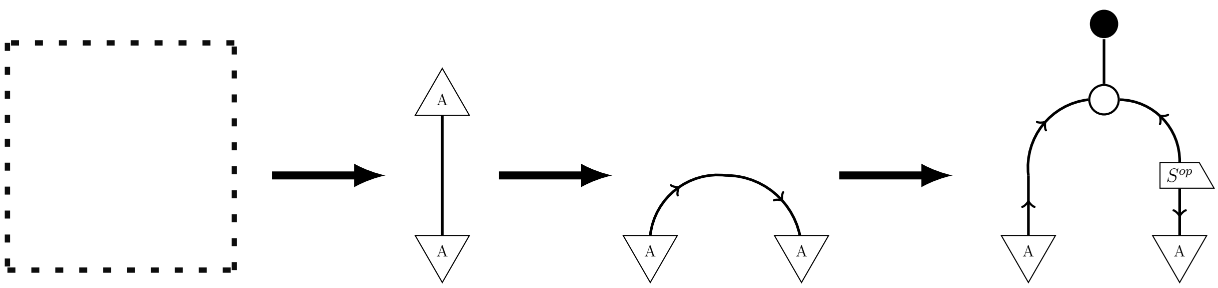


Figure 4.3

The negation laws are essentially the same as the $*$-autonomy isomorphism (see Figure 4.4).

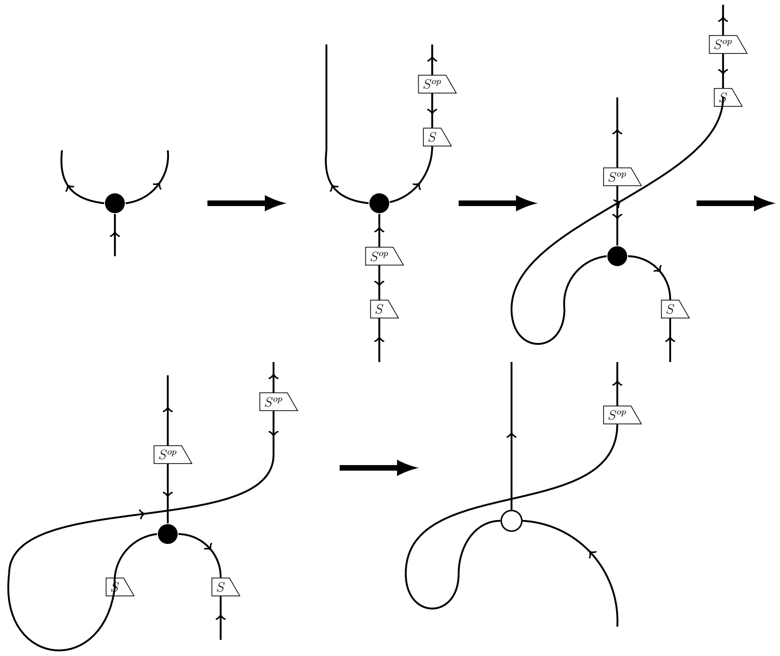


Figure 4.4

Now consider (cut).

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{(cut)}$$

Suppose we fix a linear distribution 2-cell based on the approach in [15] (recall that this is defined though Figures 3.28, 3.29, 3.30, and 3.31, and then using the adjoint properties of the monoids to derive a 2-cell of the form 3.32). Then the interpretation of (cut), as given in Chapter 2, has the visualization shown in Figure 4.5. It is suspected that an equally good (except much simpler) approach is to use a Frobenius rewrite like in Figure 4.6.

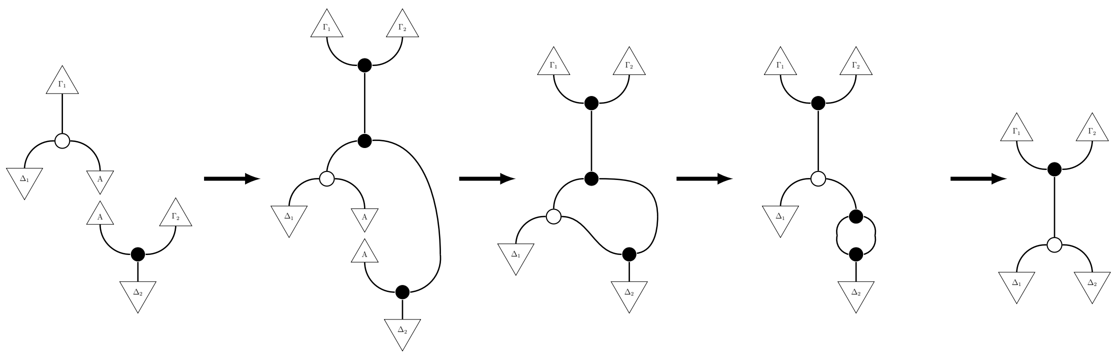


Figure 4.5: The interpretation of the (cut) rule given in Chapter 2

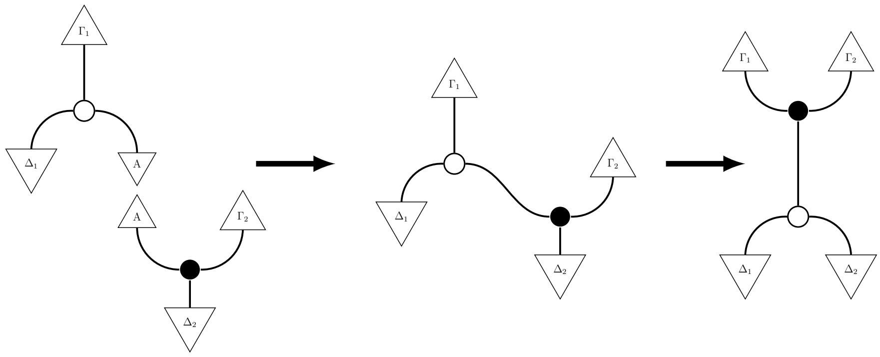


Figure 4.6: A candidate for interpreting the (cut) rule based on Frobenius rewrites

Essentially the same discussion applies to the interpretation of the $(\otimes R)$ rule – the standard approach is through linear distribution, but Frobenius rewrites seem to offer a simpler approach. We shall look at some examples to highlight this. In the simplest case, the $\otimes$ rule is simply given by the natural transformation $\mathcal{A}(A, A) \times \mathcal{A}(B, B) \to \mathcal{A}(A \otimes B, A \otimes B)$ which forms part of the adjunction in **PROF** between the $\otimes$ monoid and its adjoint, a comonoid. For instance, the proof in Figure 4.7 has the surface of Figure 4.8.

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{id} \qquad \dfrac{\dfrac{}{B \vdash B}\,\mathrm{id} \qquad \dfrac{}{C \vdash C}\,\mathrm{id}}{B, C \vdash B \otimes C}\,\otimes\mathrm{R}}{A, B, C \vdash A \otimes (B \otimes C)}\,\otimes\mathrm{R}}{A, B \otimes C \vdash A \otimes (B \otimes C)}\,\otimes\mathrm{L}$$

Figure 4.7: A particularly simple use of the $(\otimes R)$ rule

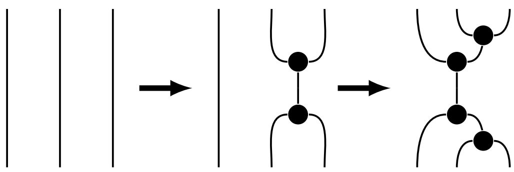


Figure 4.8

The real interest has to do with way the $\otimes$ monoid interacts with the $\oplus$ monoid. Consider the one-sided proof in Figure 4.9. The proof net is shown in Figure 4.10. We give two surfaces for this deduction. Figure 4.11 constructs a morphism of the corresponding type in $*$-autonomous categories based on the linear distribution. Figure 4.12 accomplishes the same thing using Frobenius rewrites. We believe that reading the proof net from top to bottom roughly corresponds with reading the rewrites from left to right as shown.

$$\dfrac{\dfrac{}{\vdash A^{\perp} A}\,\text{id} \qquad \dfrac{}{\vdash B^{\perp}, B}\,\text{id}}{\vdash A^{\perp}, A \otimes B, B^{\perp}}\,\otimes\text{R}$$

Figure 4.9

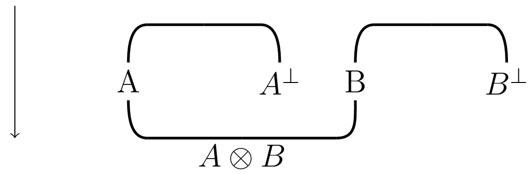


Figure 4.10: The proof net for Figure 4.9. The downward arrow loosely corresponds to the arrows running from left to right in the rewrite (surface) diagrams.

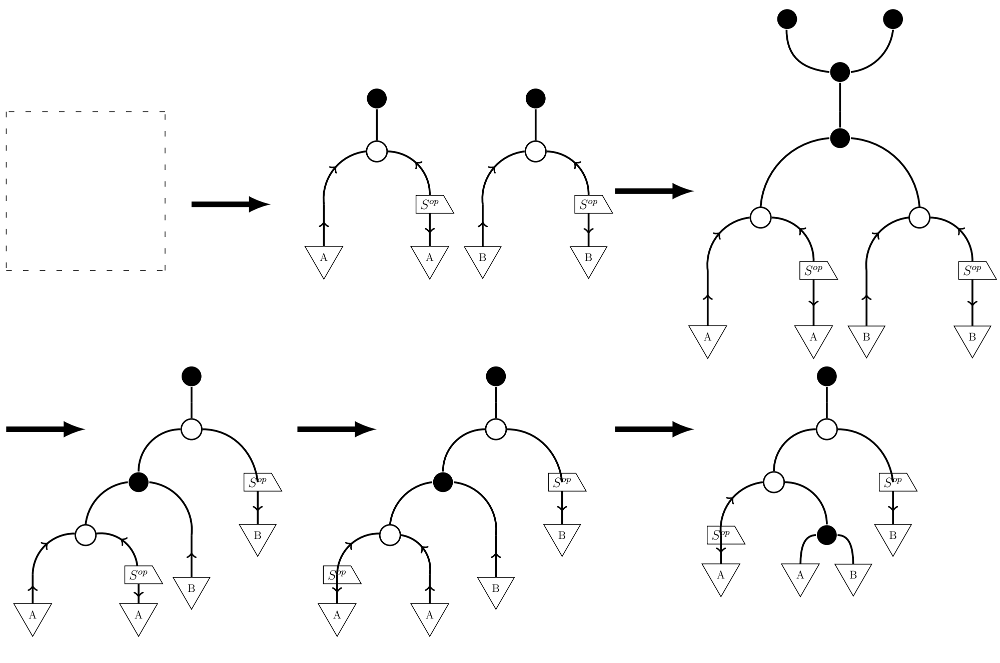


Figure 4.11: A surface diagram using a linear distribution 2-cell to construct a morphism $\mathbf{I} \to (SA \bullet (A \otimes B)) \bullet SB$ in a $*$-autonomous category

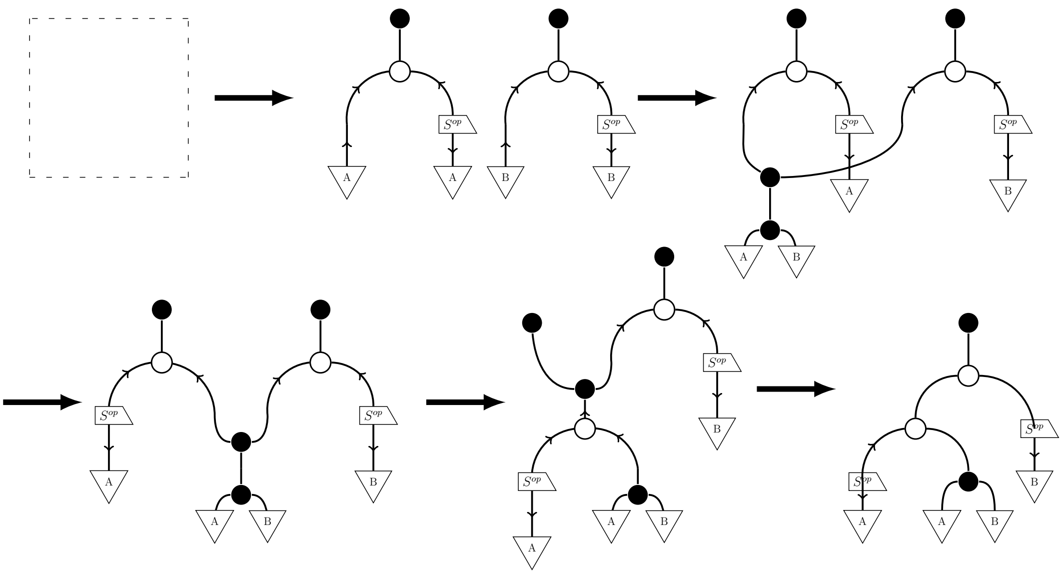


Figure 4.12: A surface diagram using Frobenius rewrites to construct a morphism $\mathbf{I} \to (SA \bullet (A \otimes B)) \bullet SB$ in a $*$-autonomous category

Compare the Frobenius-based surface in Figure 4.12 to the proof net in Figure 4.10. The first stages of rewrites appear to correspond with the geometry of the proof net. In fact it seems that the surface depicts the proof net being drawn – beginning from an empty page, we introduce two identity axioms, then "wire together" two objects whose conjunction we want to take. We also see something new in the surface diagram: a series of rewrites which appear to depict the proof net being rewritten into a normal form. The final 1-cell depicted is the hom-set $\mathcal{A}(\mathbf{I}, (SA \bullet (A \otimes B)) \bullet SB)$, which directly corresponds with the sequent $\vdash A^{\perp}, A \otimes B, B^{\perp}$ in the deduction shown in Figure 4.10. It seems likely that the existing criteria for proof net correctness have to do with characterizing when configurations of 1-cells like the ones shown can be rewritten into a form like the final 1-cell pictured using Frobenius rewrites.

Now suppose the underlying psuedoalgebra is coherent with the units. Returning to the examples from Figures 2.12 and 2.13 in Chapter 2, we find that the diagram in Figure 4.13 commutes. Whereas the proof nets of Figure 2.13 had to be identified for reasons removed from proof net theory *per se* (that is, because of complicated coherence conditions satisfied by the linear distribution natural transformations defined in [15]), here we see grounds for identification of proof nets which are more immediate.

Finally, we shall indicate the potential to use topological arguments in surface identification. First, a simple example: unit introduction and elimination. Recall the unitality 2-cell from Figure 3.10 in Chapter 3. The units have the property indicated in Figure 4.14 (where we have slightly distorted the geometry to make the visualization as a surface easier). Visualized as a surface, this invertibility property states that the surfaces of Figure 4.15 are equal. The two surfaces are seen to be homotopy equivalent.

Now consider adjoint profunctors. The diagram in Figure 3.6, specialized to the

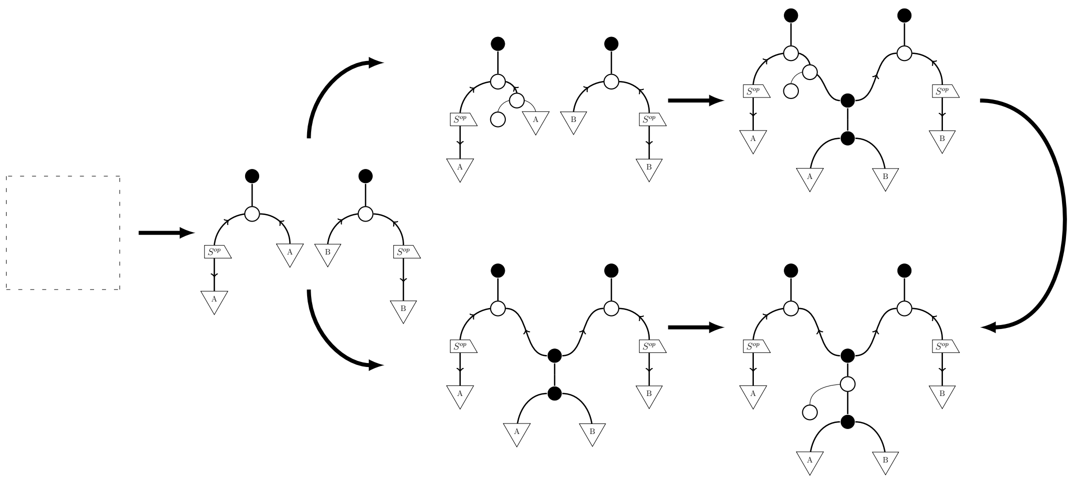


Figure 4.13: The coherence axiom of Frobenius pseudoalgebras implies this diagram commutes.

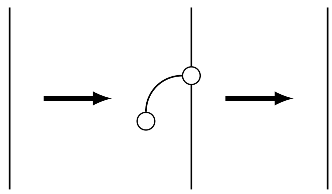


Figure 4.14: The unitors are isomorphisms.

case where the representable profunctor is induced by the $\otimes$ functor, becomes what is shown in Figure 4.16.

The 2-cell of type $1_{A \boxtimes A} \to \otimes^* \circ \otimes_*$, has a surface visualization given by Figure 4.17. The other 2-cell, which has type $\otimes_* \circ \otimes^* \to 1_A$, has the visualization given in Figure 4.18. Putting them together, one of the equations of the adjunction states that the 2-cell in Figure 4.19 is the same as that in Figure 4.20.

Now we look at linear distribution. The type of linear distribution (from Figure 3.32) has the three-dimensional visualization shown in 4.21. We have discussed the fact that [15] defines a particular 2-cell of this type using the axioms of $*$-autonomous categories, and this cell satisfies many coherence conditions, including for example the fact that Figure 4.22 is the same as Figure 4.23.

The supposed coherence property of Frobenius pseudoalgebras states that Figures 4.24 and 4.25 define the same 2-cell. Again we see that the surfaces are related by some notion of deformation.

We conjecture that this idea can be formalized and applied to all derivations in linear logic.

Conjecture 4.1. *There is a precise notion of deformation of surfaces, and a choice of 2-cells necessary to interpret the rule of linear logic, such that two proofs in linear logic are identified in $*$-autonomous categories if and only if the corresponding surface diagrams can be deformed into each other in that sense.*

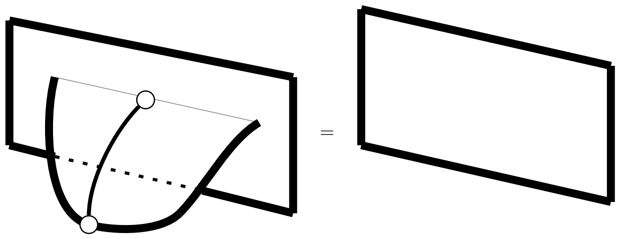


Figure 4.15: Unit introduction followed by elimination is equal to the identity

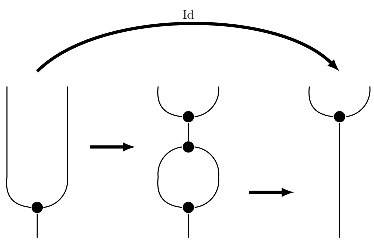


Figure 4.16: This diagram commutes as part of the adjunction between $\otimes^*$ and $\otimes_*$

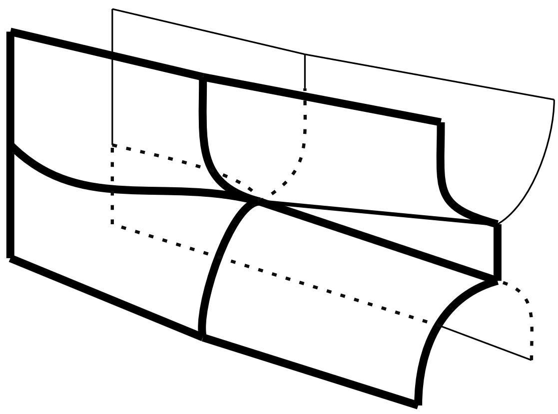


Figure 4.17: One of the 2-cells of the adjunction between the covariant and contravariant embeddings of $\otimes$ into **PROF**.

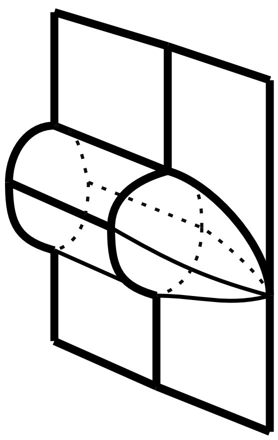


Figure 4.18: Another one of the 2-cells of the adjunction between the covariant and contravariant embeddings of $\otimes$ into **PROF**.

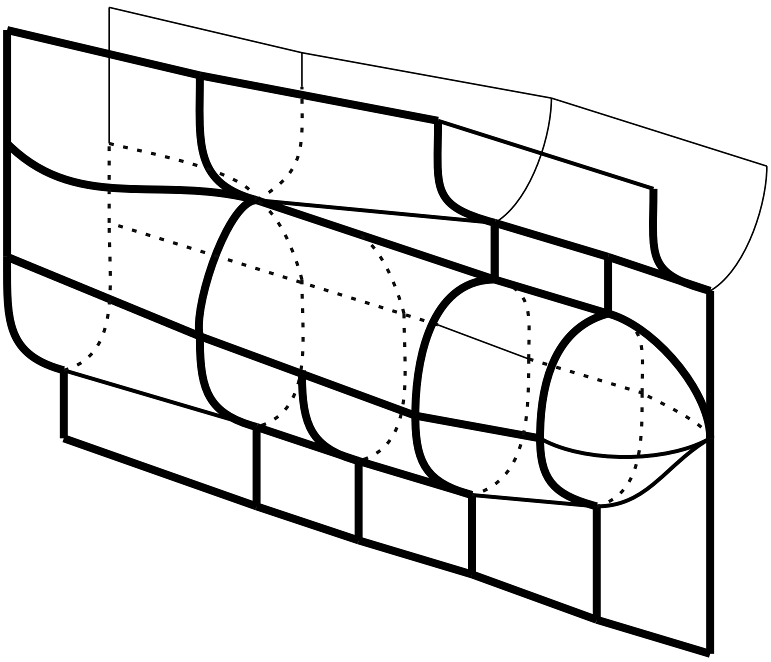


Figure 4.19: This surface is the same 2-cell as Figure 4.20.

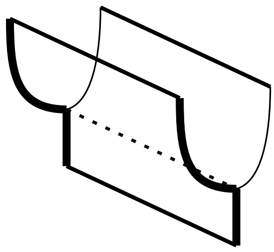


Figure 4.20: This 2-cell is merely the identity on the hom-functor $(A, B, C) \mapsto \mathcal{A}(A \otimes B, C)$

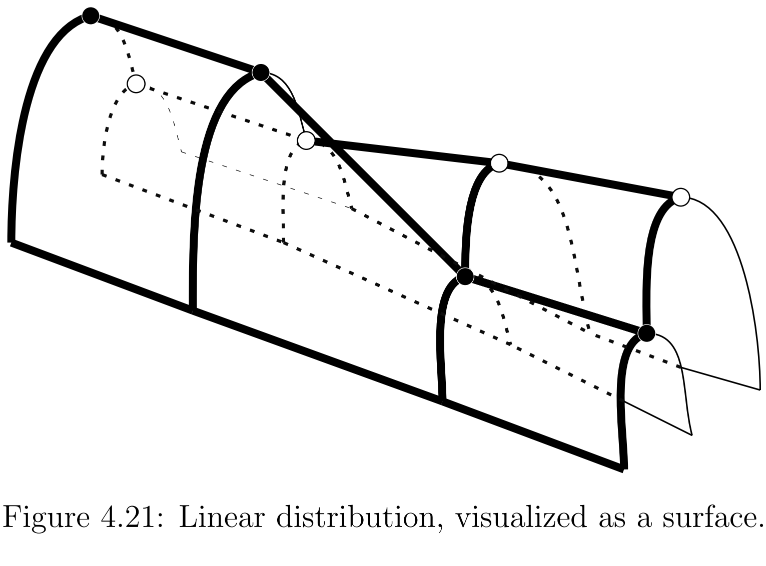


Figure 4.21: Linear distribution, visualized as a surface.

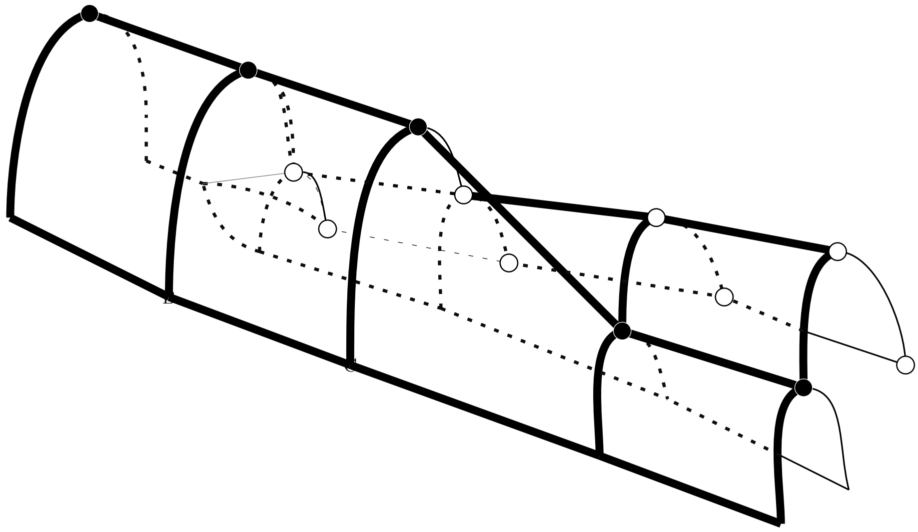


Figure 4.22: When linear distribution is given according to [15], this surface defines the same 2-cell as 4.23.

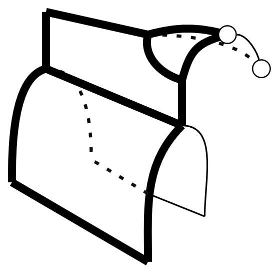


Figure 4.23: A simple unit introduction, shown as a surface.

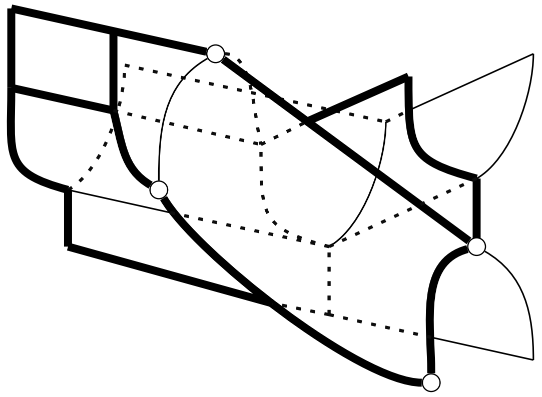


Figure 4.24: A unit introduction followed by a Frobenius rewrite.

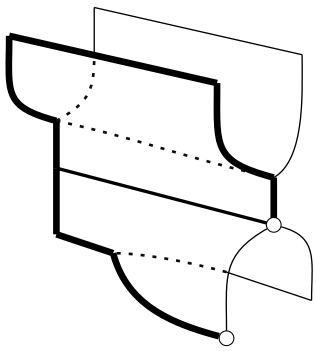


Figure 4.25: A simple unit introduction

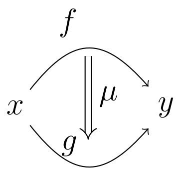
Appendix A

Symmetric Monoidal Bicategories

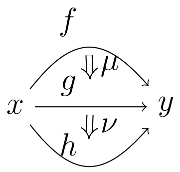
This section is a quick guide to defining symmetric monoidal bicategories. Our primary reference for this section is [16], where even more full definitions are given.

Definition A.1. A *bicategory* $\mathcal{B}$ consists of several pieces of data:

- A class $\mathcal{B}_0$ of objects (0-cells) x, y, z ...
- For each ordered pair (x, y) of objects, a category $\mathcal{B}(x, y)$ whose objects are *1-cells* from x to y, and whose morphisms are *2-cells*. A 2-cell $f \overset{\mu}{\Rightarrow} g$, where f and g are 1-cells from x to y, is written

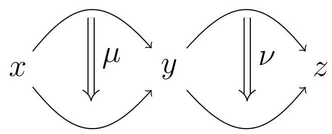
  Composition in $\mathcal{B}(x, y)$ is known as vertical composition of 2-cells in $\mathcal{B}$, due to the diagrammatic notation.

- For each object $x \in \mathcal{C}$, a distinguished 1-cell $1_x \in \mathcal{C}(x, x)$, the *identity 1-cell* at x.

$$x \xrightarrow{1_x} x$$

- For each ordered triple (x, y, z) of objects, a functor $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \xrightarrow{\circ_{x,y,z}} \mathcal{C}(x, z)$, the *horizontal composition* functor. For two 1-cells, say $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$, we write their horizontal composite as $g \circ f$, eliding subscripts.



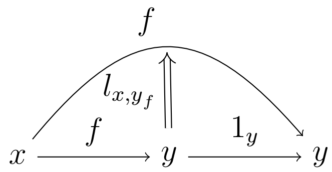
- For each pair (x, y) of objects, two natural isomorphisms known respectively as the left and right *unitors*,

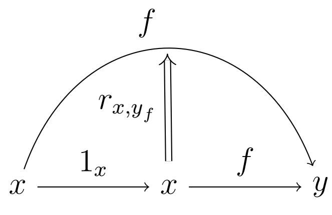
$$l_{x,y} \colon \circ_{x,y,y} \circ \big(\mathbf{I_y} \times \mathbf{Id}_{\mathcal{C}(x,y)}\big) \to \mathbf{Id}_{\mathcal{C}(x,y)}$$

$$r_{x,y} \colon \circ_{x,x,y} \circ \big(\mathbf{Id}_{\mathcal{C}(x,y)} \times \mathbf{I_x}\big) \to \mathbf{Id}_{\mathcal{C}(x,y)}$$

  where, for instance, $\big(\mathbf{Id}_{\mathcal{C}(x,y)} \times \mathbf{I_x}\big)$ is the functor sending the 1-cell $x \xrightarrow{f} y$ to $f \circ_{x,x,y} 1_x$, with a straightforward definition on 2-cells.

  Thus, such transformations consist of invertible 2-cells which look like the following:





  natural in the variable f.

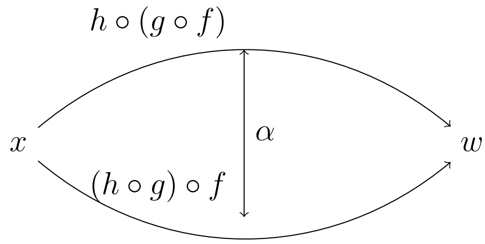
- For each quadruple $x, y, z, w \in \mathcal{C}$, an *associator* natural isomorphism

$$a_{x,y,z,w} \colon \circ_{x,y,w} \circ \big(\circ_{y,z,w} \times \mathbf{Id}_{\mathcal{C}(x,y)}\big) \to \circ_{y,z,w} \circ \big(\mathbf{Id}_{\mathcal{C}(z,w)} \times \circ_{x,y,z}\big)$$

  Thus we can read diagrams like

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$$

uniquely up to canonical isomorphism, because of 2-cells which look like



- These data are required to satisfy two sets of coherence conditions, namely the pentagon identity and the triangle identity, given by commutative diagrams (in the hom-categories) of the form

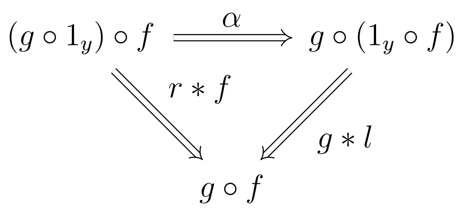


Figure A.1: The Triangle Identity

and

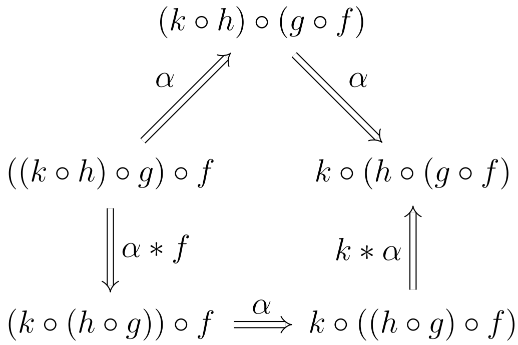
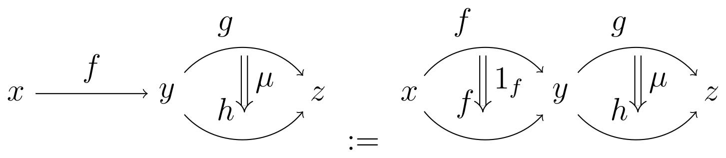


Figure A.2: The Pentagon Identity

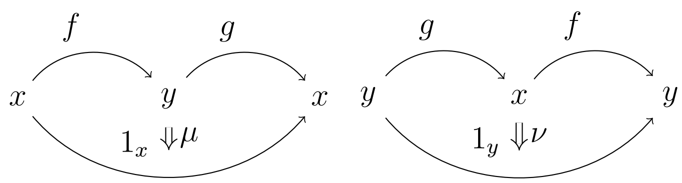
Throughout the following, assume $\mathcal{B}$ and $\mathcal{C}$ are bicategories.

Definition A.2 (Whiskering)**.** Given a 1-morphism $x \xrightarrow{f} y$ and a 2-cell $g \overset{\mu}{\Rightarrow} h$ between two 1-morphisms of type $y \to z$, we can "compose" f with μ to define a new 2-cell, $\mu \circ 1_f$, from $f \circ_{x,y,z} g$ to $f \circ_{x,y,z} h$. This 2-cell is written μf, and the operation is known as *whiskering* (μ from the left by f). Similarly we can define whiskering from the right.



Whiskering is “associative” in the sense that writing $\mu f g$ is unambiguous.

Definition A.3. In $\mathcal{B}$, an *equivalence* (f, g, μ, ν) is a pair of 1-morphisms $x \xrightarrow{f} y$, $y \xrightarrow{g} x$, an invertible 2-cell $\mu\colon (g \circ f) \simeq 1_x$, and invertible $\nu\colon (f \circ g) \simeq 1_y$.



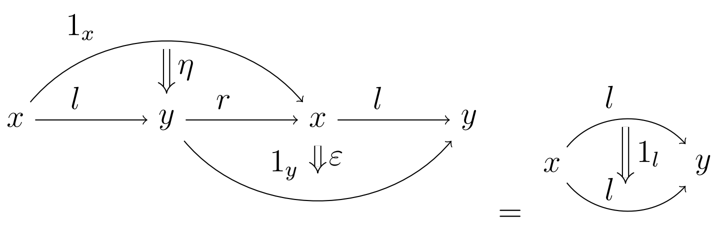
Definition A.4. In $\mathcal{B}$, an *adjunction* $(l, r, \eta, \varepsilon)$ is a pair of 1-morphisms $x \xrightarrow{l} y$, $y \xrightarrow{r} x$, a 2-cell $1_x \overset{\eta}{\Rightarrow} r \circ l$, and a 2-cell $l \circ r \overset{\varepsilon}{\Rightarrow} 1_y$, such that:

$$(r\varepsilon) \circ (\eta r) = 1_r$$

and

$$(\varepsilon l) \circ (l\eta) = 1_l$$

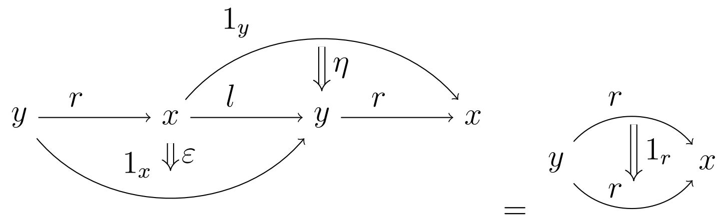
That is,



We write $l \dashv r$, or more explicitly $l \dashv_{\nu}^{\varepsilon} r$

> *Remark* A.1. Alternate terminology says that l is left dual to r, and r a right dual to l.

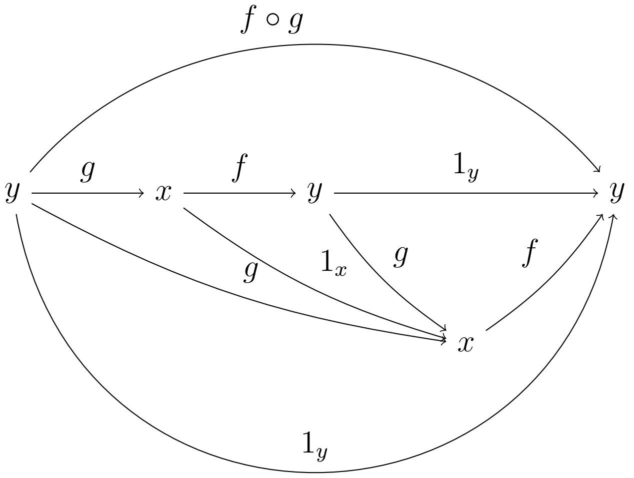
Definition A.5. In $\mathcal{B}$, an adjunction $(l, r, \eta, \varepsilon)$ is an *adjoint equivalence* when η and ε are invertible.



It is clear that an adjoint equivalence is a kind of equivalence, especially a more structured or coherent one. However the property of being an adjoint equivalence is not more general than the less structured one, since we can always rechoose our 2-cells to form an adjoint equivalence.

Theorem A.1. *Suppose (f, g, μ, ν) is an equivalence in $\mathcal{B}$. Then we can find an η, ε such that $(f, g, \eta, \varepsilon)$ is an adjoint equivalence.*

Proof. We will just give the construction here. In fact we can simply take $\eta = \mu$ and choose $\varepsilon = f \circ g \xrightarrow{fg\nu^{-1}} fgfg \xrightarrow{f\eta^{-1}} fg \xrightarrow{\nu} 1$.



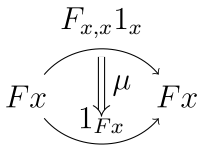
□

The proper notion of a "morphism between bicategories" is given by a *pseudofunctor*.

Definition A.6. A pseudofunctor F from $\mathcal{B}$ to a bicategory $\mathcal{C}$ is given by the following data:

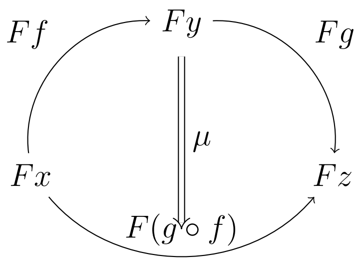
- For each $x \in \mathcal{B}_0$, an object $Fx \in \mathcal{C}_0$.
- For each (x, y), a functor $F_{x,y} \colon \mathcal{B}(x, y) \to \mathcal{C}(Fx, Fy)$

- For each $x \in \mathcal{B}_0$, an invertible 2-cell between the identity $1_{Fx} \in \mathcal{C}(Fx, Fx)$ and



$F_{x,x}1_x$.

- For each x, y, z in $\mathcal{B}_0$, a natural isomorphism between the functors $\circ^{\mathcal{C}}_{Fx,Fy,Fz} \circ (F_{b,c} \times F_{a,b})$ and $F_{a,c} \circ (\circ_{a,b,c})$. Thus a class of 2-cells



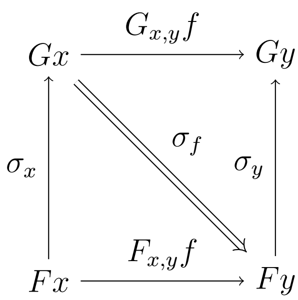
- These data are required to satisfy some coherence conditions discussed in [16].

Before defining a monoidal structure on a bicategory, we need to define the notion of a transformation between pseudofunctors and a modification between transformations.

Definition A.7. Given two pseudofunctors $F, G\colon \mathcal{B} \to \mathcal{C}$ between bicategories, a transformation $\sigma\colon F \to G$ is given by

- 1-cells $Fx \xrightarrow{\sigma_x} Gx$ for each $x \in \mathcal{B}_0$.

- Natural isomorphisms $\sigma_{x,y}\colon \sigma_x \circ F_{x,y} \to G_{x,y} \circ \sigma_y$, thus consisting of invertible 2-cells σ_f of the form



  natural in the variable f.

- These data are required to satisfy coherence conditions that are further discussed in [16].

Definition A.8. Given two transformations $\sigma, \tau \colon F \to G$ between pseudofunctors, a modification $\sigma \overset{\Sigma}{\to} G$ consists of 2-cells $\sigma_x \overset{\Sigma_x}{\longrightarrow} \tau_x$ such that all diagrams like the following commute.

$$
\begin{array}{ccc}
Gx, yf \circ \sigma_y & \overset{\Sigma_x * id}{\Longrightarrow} & G_{x,y}f \circ \tau_y \\
\sigma_f \Big\Uparrow & & \Big\Uparrow \tau_f \\
\sigma_x \circ F_{x,y} & \overset{id * \Sigma_y}{\Longrightarrow} & \tau_x \circ Fx, y
\end{array}
$$

We are now in a position to define a symmetric monoidal structure on a bicategory. Our definition is terse – see [16] for more details.

Definition A.9. A symmetric monoidal bicategory $(\mathcal{B}, \boxtimes, \mathbf{I}, \alpha, l, r, \pi, \lambda, \varrho)$.

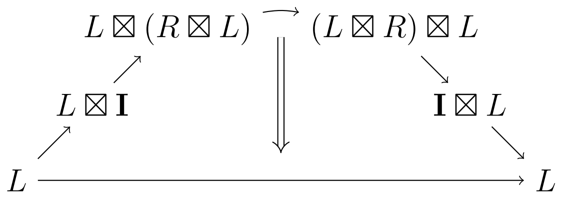
- A pseudofunctor $\boxtimes \colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$
- An associator transformation and two unitor transformations, consisting of adjoint equivalences
- Invertible modifications expressing coherence of the associator and unitors
- A braiding transformation β, also an adjoint equivalence with chosen data
- Invertible modifications expressing the coherence of β with the associators and unitors
- An invertible modification expressing the symmetry of β
- A wealth of coherence equations between all of these data

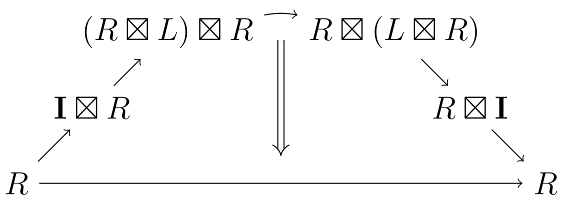
The coherence and structure of a symmetric monoidal bicategory allow us to examine 2-cells through the surface diagrammatic notation and the 1-cell "local rewrite" notation used in Chapter 3.

Definition A.10 (Biduality)**.** In a symmetric monoidal bicategory $\mathcal{B}$, a bidual pairing $(L, R, e, n, \alpha, \beta)$ is composed of

- Two objects $L, R \in \mathcal{B}_0$
- 1-morphisms $\mathbf{I} \overset{n}{\to} R \boxtimes L$, $L \boxtimes R \overset{e}{\to} \mathbf{I}$

- Two invertible 2-cells α and β





If there exists a bidual pairing $(L, R, e, n, \alpha, \beta)$, we say that L is left bidual to R, and R right bidual to L.

Definition A.11. If every object in $\mathcal{B}$ has a (left/right) bidual, then $\mathcal{B}$ is (left/right) *autonomous*.

Bibliography

[1] Michael Barr. “*-Autonomous Categories”. In: *Lecture Notes in Mathematics*. Vol. 752. Springer-Verlag, 1979, pp. 1–100.

[2] Jean Bénabou. “Distributors at Work”. June 2000. URL: `http://www.mathematik.tu-darmstadt.de/~streicher/FIBR/DiWo.pdf`.

[3] Vincent Danos and Laurent Regnier. “The Structure of Multiplicatives”. In: *Arch. Math. Logic* 28 (1989), pp. 181–203.

[4] K. Dosen and Z. Petri. *Proof-Net Categories*. 2005.

[5] Jean-Yves Girard. “A New Constructive Logic: Classical Logic”. In: *Mathematical Structures in Computer Science* 01 (1991), pp. 255–296.

[6] Jean-Yves Girard. “Linear Logic”. In: *Theoretical Computer Science* (1996).

[7] Jean-Yves Girard. “Proof-nets: The parallel syntax for proof-theory”. In: *Logic and Algebra*. Marcel Dekker, 1996, pp. 97–124.

[8] Willem Heijltjes and Lutz Straburger. “No Proof Nets for MLL: Proof equivalence in MLL with units is PSPACE-complete”. In: *Proc. Login in Computer Science* 2(4:3) (2014), pp. 1–44.

[9] Willem Heijltjes and Lutz Straburger. “Proof nets and semi-star-autonomous categories”. In: *Mathematical Structures in Computer Science* FirstView (Aug. 2015), pp. 1–40. ISSN: 1469-8072. URL: `http://journals.cambridge.org/article_S0960129514000395`.

[10] Chris Heunen and Jamie Vicary. “Categorical Quantum Mechanics: An Introduction”. Course Notes. Hilary Term 2015. URL: `https://www.cs.ox.ac.uk/files/7051/notes.pdf`.

[11] Dominic Hughes. “Simple multiplicative proof nets with units”. 2005.

[12] Yves Lafont Jean-Yves Girard and Paul Taylor. *Proofs and Types*. Cambridge University Press, 1989.

[13] Francois Lamarche and Lutz Straburger. “From proof nets to the free *-autonomous category.” In: *Logical Methods in Computer Science* 2(4:3) (2006), pp. 1–44.

[14] Fosco Loregian. “This is the co/end, my only co/friend”. 2015. URL: `http://arxiv.org/pdf/1501.02503v2`.

[15] J.R.B. Cocket R. A. G. Seely. “Weakly Distributive Categories”. In: *Journal of Pure and Applied Algebra* 114 (1997).

[16] Christopher John Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Field Theories*. 2009.

[17] R.A.G. Seely. “Linear Logic, $*$-autonomous Categories and Cofree Coalgebras”. In: *In Categories in Computer Science and Logic*. American Mathematical Society, 1989, pp. 371–382.

[18] Ross Street. “Frobenius monads and pseudomonoids”. In: *2-categories Companion 73*. 2004, pp. 3930–3948.