Putting Different Numbers on Dice: An Exposition
By William Gasarch

1 Introduction

All of the ideas in this paper are from Gallian and Rusin [1].

In 1978 Martin Gardner [2] discussed the following problem, which was first posed by George Sicherman (paraphrased):

How many ways can you label two six sided dice with positive naturals such that you get the same probabilities as the standard labeling?

The answer was two. The standard way and one more way. Gallian and Rusin [1] generalized this to many more sets of dice.

In our paper we discuss the original case of 2 six-sided dice and then look at a few more pairs of dice.

2 Two 6-Sided Dice

Identify a 6-sided dice with sides $a_1 > \cdots > a_6$ with the polynomial

$$x^{a_1} + \cdots + x^{a_6}.$$

Assume there are two dice: $a_1 > \cdots > a_6$ and $b_1 > \cdots > b_6$ that yield the same probabilities as the standard dice. Look at

$$(x^{a_1} + \cdots + x^{a_6})(x^{b_1} + \cdots + x^{b_6}).$$

The coefficient of x^c in the above is the number of ways to roll a c.

To make the pair of dice yield the same probabilities as standard dice we need:

$$(x^{a_1} + \cdots + x^{a_6})(x^{b_1} + \cdots + x^{b_6}) = (x^6 + \cdots + x)(x^6 + \cdots + x).$$

$$= x^2(x^5 + \cdots + 1)^2 = x^2(x+1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2.$$

SO we need to find all ways to factor

$$x^2(x+1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2$$

into two factors such that the following holds.

1. Each factor has no constant term so it has exactly one of the x's.

2. Each factor has six terms in it ($2x$ is 2 terms). So if $P(x)$ is a factor then $P(1) = 6$.

A factor will be of the form

$$P(x) = x(x+1)^a(x^2 - x + 1)^b(x^2 + x + 1)^c$$

where $a, b, c \in \{0, 1, 2\}$. Since $P(1) = 6$ we have $6 = 2^a 1^b 3^c$. So we have $a = 1$ and $c = 1$ and $b \in \{0, 1, 2\}$. We look at the three polynomials from $b \in \{0, 1, 2\}$ and then later see which combinations can be multiplied to obtain

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2$$

Case $b=0$: Then the polynomials for the die is
$x(x+1)(x^2+x+1) = x^4+2x^3+2x^2+x$.
So the die has numbers $(1,2,2,3,3,4)$.
Case $b=1$: Then the polynomial for the die is
$x(x+1)(x^2-x+1)(x^2+x+1) = (x^6+x^5+x^4+x^3+x^2+x)$.
So the die has numbers $(1,2,3,4,5,6)$
Case $b=2$: Then the polynomial for the die is
$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8+x^6+x^5+x^4+x^3+x$.
So the die has numbers $(1,3,4,5,6,8)$.
There are two ways to put together these polynomials to get

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2$$

1. The $b=0$ and $b=2$ case, if multiplied together give

   $$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2$$

   Hence we have a pair of *non standard dice*

   $(1,2,2,3,3,4)$ and $(1,3,4,5,6,8)$

   that yield the same probabilities as the standard dice.

2. The $b=1$ and $b=1$ case, if multiplied together give

   $$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2$$

   This leads to the *standard dice*

   $(1,2,3,4,5,6)$ and $(1,2,3,4,5,6)$.

UPSHOT: There is exactly one pair of non-standard dice that gives the same probabilities as the the standard dice.

3 Other Pairs of Dice

Now that we know how these proofs go, we skip straight to the polynomial factorization. We denote a factor as $P(x)$.

3.1 Two 2-Sided Dice

$$(x^2+x)^2 = x^2(x+1)^2.$$

$P(x) = x(x+1)^a$ where $a \in \{0,1,2\}$. Since $P(1) = 2$ we have $2^a = 2$, so $a = 1$. There is only one case and it is the standard one.

3.2 Two 3-Sided Dice

$$(x^3+x^2+x)(x^3+x^2+x) = x^2(x^2+x+1)^2$$

$P(x) = x(x^2+x+1)^a$ where $a \in \{0,1,2\}$. Since $P(1) = 3$ we have $3^a = 3$, so $a = 1$. There is only one case and it is the standard one.

3.3 Two 4-Sided Dice

$$(x^4 + x^3 + x^2 + x)^2 = x^2(x+1)^2(x^2+1)^2.$$

$P(x) = x(x+1)^a(x^2+1)^b$ where $a, b \in \{0,1,2\}$. We want $P(1) = 4$, so $2^a 2^b = 4$.
Case 1 $(a,b) = (0,2)$ and $(a,b) = (2,0)$.

1. $x(x^2+1)^2 = x(x^4+2x^2+1) = x^5 + 2x^3 + x$. Die is
   $(5,3,3,1)$.
2. $x(x+1)^2 = x(x^2+2x+1) = x^3 + 2x^2 + x$. Die is
   $(3,2,2,1)$.

Case 2 $(a,b) = (1,1)$ and $(a,b) = (1,1)$. This yields the standard dice where both are $(4,3,2,1)$. We omit details.

3.4 Two 5-Sided Dice

$$(x^5 + x^4 + x^3 + x^2 + x)^2 = x^2(x^4 + x^3 + x^2 + x)^2.$$

$P(x) = x(x^4 + x^3 + x^2 + x)^a$ where $a \in \{0,1,2\}$. We want $P(1) = 5$, so $a = 1$. This only yields the standard dice. We omit details.

3.5 Two 6-Sided Dice

See Section 2.

3.6 Two p-Sided Dice for p Prime

Note that if the number of sides is 2,3, or 5 then the only solution is the standard one. Hmmm. Also note that 2,3 and 5 are primes. It turns out that this is not a coincidence.

Theorem 1. *Let p be a prime. The only way that two p-sided dice can be labeled with positive naturals so that the probability is the same as standard dice is if they are labeled to be standard dice.*

Proof. As in the examples throughout this paper it comes down to factoring

$$(x^p + \cdots + x)^2 = x^2(x^{p-1} + \cdots + 1)^2.$$

Recall *Eisenstein's Irreducibility Criterion:* Let $f(x) = a_n x^n + \cdots + a_0 \in \mathbf{Z}[x]$. If the following hold then $f(x)$ is irreducible.

1. p does not divide a_n.
2. p does divide $a_{n-1}, \ldots, a_0$.
3. p^2 does not divide a_0.

We want to show that $f(x) = x^{p-1} + \cdots + 1$ is irreducible. Alas, $f(x)$ does not satisfy the Eisenstein Criterion. But $f(x+1)$ does. Since $f(x+1)$ is irreducible, $f(x)$ is irreducible.

Back to the proof of the theorem. Since $x^{p-1} + \cdots + 1$ is irreducible, the only possible factorizations in $\mathbf{Z}[x]$ of $x^2(x^{p-1} + \cdots + 1)^2$ where x appears is of the form

$P(x) = x(x^{p-1} + \cdots + 1)^a$ where $a \in \{0,1,2\}$. Since we need $P(x) = p$ we get $a = 1$. Since there is only one case, it is the standard dice case.

□

Note Gallian and Rusin [1] showed that there is a nonstandard solution iff n is not prime.
For more on this problem see the paper of Gallian and Rusin [1].

References

[1] J. Gallian and D. Rusin. Cycltomic polynomials and nonstandard dice. *Discrete Mathematics*, 27, 1979.

[2] M. Gardner. Mathematical games. *Scientific American*, 238:19–32, 1978.