The Unit Graph on Z, Q, R: An Exposition
By William Gasarch

1 Introduction

Most of this writuep is from Chilakamarri [1].

We consider the following three graphs.

Definition 1. Let $n \ge 1z$. We use x and y for n tuples.

1. G_{Z^n} is the graph with

$$\begin{aligned} V &= \mathsf{Z}^n \\ E &= \{(x,y) \in \mathsf{Z}^n \times \mathsf{Z}^n : d(x,y) = 1\} \end{aligned}$$

2. G_{Q^n} is the graph with

$$\begin{aligned} V &= \mathsf{Q}^n \\ E &= \{(x,y) \in \mathsf{Q}^n \times \mathsf{Q}^n : d(x,y) = 1\} \end{aligned}$$

3. G_{R^n} is the graph with

$$\begin{aligned} V &= \mathsf{R}^n \\ E &= \{(x,y) \in \mathsf{R}^n \times \mathsf{R}^n : d(x,y) = 1\} \end{aligned}$$

The chromatic number of G_{R^2}, $\chi(G_{\mathsf{R}^2})$, is a well known open problem. It is called the *Hadwiger-Nelson* problem. See HERE for the Wikipedia site. It is fairly easy to prove that $4 \le \chi(G_{\mathsf{R}^2}) \le 7$. Audrey de Grey [2] used a computer proof to show that $5 \le \chi(G_{\mathsf{R}^2})$.

In this paper we explore graph properties of G_{Z^n}, G_{Q^n}, and G_{R^n}.

2 Connectivity: The $n = 1$ Case

This is trivial but we include it for completeness.

Theorem 2. *Let $n = 1$. Then G_{Z^n}, G_{Q^n}, and G_{R^n} are all disconnected.*

Proof. Let $x = 0$ and $y = \frac{1}{2}$. If x and y are connected then the $d(x,y) \in \mathsf{N}$. Hence x and y are not connected. $\square$

For the rest of this section we assume $n \ge 2$.

3 Connectivity for G_{Z^n}

This is trivial but we include it for completeness.

Theorem 3. *Let $n \ge 2$. Then G_{Z^n} is connected.*

Proof. Let $x, y \in \mathsf{Z}^n$.

Let
$x = (x_1, \ldots, x_n)$
$y = (y_1, \ldots, y_n)$.

Assume $x_1 < y_1$ (the proof for $x_1 > y_1$ is similar).
The following is a path in G_{Z^n}.
$(x_1, x_2, \ldots, x_n)$, $(x_1+1, x_2, \ldots, x_n)$, $(x_1+2, x_2, \ldots, x_n)$, …, $(y_1, x_2, \ldots, x_n)$.
Repeat this procedure on each coordinate to get $(y_1, \ldots, y_n)$. □

4 Connectivity for G_{Q^n}

This is the most interesting case for connectivity. We will prove that

1. For $n = 2, 3, 4$ G_{Q^n} is disconnected.
2. For $n \geq 5$, G_{Q^n} is connected.

We will need the following lemma for both parts. We omit the proof which is just simple calculation

Lemma 4. *Let* $a, b, c \in \mathsf{Z}$.

1. $a^2 \equiv 0, 1, 4 \pmod 8$.

2. $a^2 + b^2 \equiv 0, 1, 2, 4, 5 \pmod 8$.

3. $a^2 + b^2 + c^2 \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 8$.

4.1 For $n \in \{2, 3, 4\}$, G_{Q^n} is not Connected

With the benefit of hindsight, we note a difference between $n = 2, 3, 4$ and $n \geq 5$.

Lemma 5.

1. If $a_1^2 + a_2^2 = b^2$ *and* $\gcd(a_1, a_2, b) = 1$ *then* $b \not\equiv 0 \pmod 4$.

2. If $a_1^2 + a_2^2 + a_3^2 = b^2$ *and* $\gcd(a_1, a_2, a_3, b) = 1$ *then* $b \not\equiv 0 \pmod 4$.

3. If $a_1^2 + a_2^2 + a_3^2 + a_4^2 = b^2$ *and* $\gcd(a_1, a_2, a_3, a_4, b) = 1$ *then* $b \not\equiv 0 \pmod 4$.

4. There exists $a_1, a_2, a_3, a_4, , a_5, b$ *such that* $\gcd(a_1, a_2, a_3, a_4, a_5, b) = 1$ *and* $b \equiv 0 \pmod 4$.

Proof.

The proofs of parts 1,2, and 3 are by contradiction. We use the fact that if for all $a \in \mathsf{Z}$, $a^2 \equiv 0, 1, 4 \pmod 8$.

1. Assume $a_1^2 + a_2^2 = b^2$. and $b \equiv 0 \pmod 4$. Then $b^2 \equiv 0 \pmod 8$. Hence

$$a_1^2 + a_2^2 = b^2 \equiv 0 \pmod 8.$$

Since $\gcd(a_1, a_2, b) = 1$, at least one of a_1, a_2 is odd. Assume its a_1. Then

$$a_1^2 \equiv 1 \pmod 8.$$

By Lemma 4.a

$$a_2^2 \equiv 0, 1, 4 \pmod 8.$$

Hence

$$a_1^2 + a_2^2 \equiv 1, 2, 5 \not\equiv 0 \pmod 8$$

which is a contradiction.

2. Assume $a_1^2 + a_2^2 + a_3^2 = b^2$ and $b \equiv 0 \pmod 4$. Then $b^2 \equiv 0 \pmod 8$. Hence

$$a_1^2 + a_2^2 a_3^2 = b^2 \equiv 0 \pmod 8.$$

Since $\gcd(a_1, a_2, a_3) = 1$, at least one of a_1, a_2, a_3 is odd. Assume its a_1. Then

$$a_1^2 \equiv 1 \pmod 8.$$

By Lemma 4.b

$$a_2^2 + a_3^2 \equiv 0, 1, 2, 4, 5 \pmod 8$$

Hence

$$a_1^2 + a_2^2 + a_3^2 \equiv 1, 2, 3, 4, 6 \not\equiv 0 \pmod 8$$

which is a contradiction.

3. Assume $a_1^2 + a_2^2 + a_3^2 + a_4^2 = b^2$. and $b \equiv 0 \pmod 4$. Then $b^2 \equiv 0 \pmod 8$. Hence

$$a_1^2 + a_2^2 a_3^2 + a_4^2 = b^2 \equiv 0 \pmod 8.$$

Since $\gcd(a_1, a_2, a_3, a_4, b) = 1$, at least one of a_1, a_2, a_3, a_4 is odd. Assume its a_1. Then

$$a_1^2 \equiv 1 \pmod 8.$$

By Lemma 4.c

$$a_2 + a_3^2 + a_4^2 \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 8.$$

Hence

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 \equiv 1, 2, 3, 4, 5, 6, 7 \not\equiv 0 \pmod 8.$$

which is a contradiction.

4. The following values satisfy the conditions: $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 19$. $b = 20$.

□

Lemma 6. *If a sum of rationals equals $\frac{1}{4}$ then at least one of them has a denominator divisible by 4.*

Proof. Assume, by way of contradiction, that there exists rationals $\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}$ such that

$$\sum_{i=1}^{k} \frac{a_i}{b_i} = \frac{1}{4}$$

and, for all i, $b_i \not\equiv 0 \pmod 4$.

Partition $\{1, \ldots, k\}$ as follows.

1. Let X be the set of all i such that

$$b_i \not\equiv 0 \pmod 2.$$

Note that

$$\sum_{i \in X} \frac{a_i}{b_i} = \frac{c_X}{d_X}$$

where $c_X \not\equiv 0 \pmod 2$.

2. Let Y be the set of all i such that

$$b_i \equiv 2 \pmod 4.$$

For $i \in Y$ let c_i be such that $b_i = 4c_i + 2$. Note that

$$\sum_{i \in Y} \frac{a_i}{b_i} = \sum_{i \in Y} \frac{a_i}{4c_i + 2} = \frac{1}{2} \sum_{i \in Y} \frac{a_i}{22c_i + 1} = \frac{c_Y}{2d_Y}$$

where $c_Y \not\equiv 0 \pmod 2$.

Since $(\forall i)[b_i \not\equiv 0 \pmod 4]$, $X \cup Y$ is a partition of $\{1, \ldots, k\}$. Hence using the comments made when defining the partition we have

$$\sum_{i=1}^{k} \frac{a_i}{b_i} = \sum_{i \in X} \frac{a_i}{b_i} + \sum_{i \in Y} \frac{a_i}{b_i} = \frac{c_X}{d_X} + \frac{c_Y}{2d_Y} = \frac{1}{4}.$$

Multiply both sides by $4d_X d_Y$ to get

$$4c_X d_Y + 2c_Y d_X = 1.$$

The left hand side is even and he right hand side is odd, which is a contradiction. □

We state but do not prove a generalization of Lemma 6. We will not be needing it.

Lemma 7. *Let p be a prime and $e \geq 1$. If a sum of rationals equals $\frac{1}{p^e}$ then at least one of them has a denominator divisible by p^e.*

Lemma 8. *Let $n \geq 1$. Let $p \in \mathsf{Q}^2$. Then there exists a_1, a_2, b such that the following hold.*

1. *$p = (\frac{a_1}{b}, \frac{a_2}{b})$ (note that both fractions have the same denominator).*
2. *$\gcd(a_1, a_2, b) = 1$.*

Proof. Let p be given as

$$(\frac{c_1}{d_1}, \frac{c_2}{d_2}).$$

Then p is also

$$(\frac{c_1d_2}{d_1d_2}, \frac{c_2d_1}{d_1d_2}).$$

If $\gcd(c_1d_2, c_2, d_1, d_1d_2) = 1$ then we set $a_1 = c_1d_2$, $a_2 = c_2d_1$, $b = d_1d_2$. If $\gcd(c_1d_2, c_2, d_1, d_1d_2) = e \geq 2$ then we set $a_1 = c_1d_2/e$, $a_2 = c_2d_1/e$, $b = d_1d_2/e$. □

Theorem 9. *The graphs G_{Q^2}, G_{Q^3}, and G_{Q^4} are not connected.*

Proof. We do the proof for G_{Q^2}. The proof is almost identical for G_{Q^3} and G_{Q^4}. We will note the one place we use $n = 2$ and say how to modify for G_{Q^3} and G_{Q^4}.

Assume, by way of contradiction, that G_{Q^2} is connected. Let $x = (0,0)$ and $y = (\frac{1}{4}, 0)$. Let the path between them be

$$x, x_1, x_2, \ldots, x_k, y.$$

$d(x, x_1) = 1$. So $x - x_1$ is on the unit sphere. $d(x_1, x_2) = 1$. So $x_2 - x_1$ is on the unit sphere.

$\vdots$

$d(x_{k-1}, x_k) = 1$. So $x_k - x_{k-1}$ is on the unit sphere.

$d(x_k, y) = 1$. So $y - x_k$ is on the unit sphere.

Add up all of those points on the unit sphere. You get

$$(x - x_1) + (x_1 - x_2) + \cdots + (x_k - x_{k-1}) + y - x_k = x + y = y.$$

UPSHOT: $(\frac{1}{4}, 0)$ is the sum of points on the unit sphere.

Let $z_1, \ldots, z_k$ be the points on the unit sphere that add up to $(\frac{1}{4}, 0)$. For $1 \leq i \leq k$ let $z_i = (\frac{a_{i1}}{b_i}, \frac{a_{i2}}{b_i}$ with $\gcd(a_{i1}, a_{i2}, b_i) = 1$ (we are using Lemma 8).

Since z_i is on the unit sphere

$$a_{i1}^2 + a_{i2}^2 = b_i^2.$$

By Lemma 5.a, $b_i \not\equiv 0 \pmod 4$. (We use Lemma 5.a since we are dealing with G_{Q^2}. For G_{Q^3} we use Lemma 5.b. For G_{Q^4} we use Lemma 5.c.) More to the point,

$$(\forall 1 \leq i \leq k)[b_i \not\equiv 0 \pmod 4].$$

Since $\sum_{i=1}^{k} z_i = (\frac{1}{4}, 0)$.

$$\sum_{i=1}^{k} \frac{a_{i1}}{b_i} = \frac{1}{4}.$$

By Lemma 6

$$(\exists 1 \leq i \leq k)[b_i \equiv 0 \pmod 4).$$

This is a contradiction.

□

4.2 For $n \geq 5$ G_{Q^n} is Connected

With the benefit of hindsight, we note a difference between $n = 2, 3, 4$ and $n \geq 5$.

Lemma 10.

1. *Let $n \geq 5$. For all $N \in \mathsf{N}$, $4N^2$ can be written as the sum of n squares, one of which is 1.*
2. *Let $n \leq 4$. For an infinite number of $N \in \mathsf{Z}$, $4N^2$ cannot be written as the sum of n squares, one of which is 1.*

Proof.

1. Recall that every number is the sum of 4 squares. Hence there exists a, b, c, d such that

$$4N^2 - 1 = a^2 + b^2 + c^2 + d^2$$

$$4N^2 = a^2 + b^2 + c^2 + d^2 + 1$$

2. Let $N \equiv 0 \pmod 2$. Assume, by way of contradiction, that there exists a, b, c such that

$$4N^2 = a^2 + b^2 + c^2 + 1$$

$$4N^2 - 1 = a^2 + b^2 + c^2$$

$$4N^2 - 1 \equiv a^2 + b^2 + c^2 \pmod 8.$$

Since $N \equiv 0 \pmod 2$ the left hand side is $\equiv 7 \pmod 8$. By Lemma 4.c the right hand side is $\equiv 0, 1, 2, 3, 4, 5, 6$. Hence they are not equal mod 8. That is a contradiction.

□

Lemma 11. *Let $N \in \mathsf{Z} - \{0\}$.*

1. *In G_{Q^5} there is a path between $(0, 0, 0, 0, 0)$ and $(\frac{1}{N}, 0, 0, 0, 0)$.*
2. *Let $n \geq 5$. Let $1 \leq i \leq n$. In G_{Q^n} there is a path between $(0, \ldots, 0)$ and $(0, 0, \ldots, 0\frac{1}{N}, 0, \ldots, 0)$ (the $\frac{1}{N}$ is in the ith place).*

Proof. We prove part 1. The proof of part 2 is similar.

By Lemma 10 there exists a, b, c, d such that

$$4N^2 = 1 + a^2 + b^2 + c^2 + d^2$$

Divide by $4N^2$ to get:

$$1 = \left(\frac{1}{2N}\right)^2 + \left(\frac{a}{2N}\right)^2 + \left(\frac{b}{2N}\right)^2 + \left(\frac{c}{2N}\right)^2 + \left(\frac{d}{2N}\right)^2$$

Hence the following 2^5 vectors are all on the Q^5-unit sphere:

$$\left(\pm\frac{1}{2N}, \pm\frac{a}{2N}, \pm\frac{b}{2N}, \pm\frac{c}{2N}, \pm\frac{d}{2N},\right)$$

We now describe the path from $(0,0,0,0,0)$ to $(\frac{1}{N},0,0,0,0)$ by adding just two Q^5-unit sphere vectors to $(0,0,0,0,0)$ to get $(\frac{1}{N},0,0,0,0)$

$$(0,0,0,0,0)+\left(\frac{1}{2N},\frac{a}{2N},\frac{b}{2N},\frac{c}{2N},\frac{d}{2N},\right)+\left(\frac{1}{2N},-\frac{a}{2N},-\frac{b}{2N},-\frac{c}{2N},-\frac{d}{2N},\right)=\left(\frac{1}{N},0,0,0,0\right)$$

□

Lemma 12. *Let $n \ge 1$. Let $x, y \in G_{\mathsf{Q}^n}$. If there is a path from 0^n to x and from 0^n to y then there is a path from 0^n to $x+y$.*

Proof. □

Theorem 13. *Let $n \ge 5$. Then G_{Q^n} is connected.*

Proof. We show that, for every vertex x of G_{Q^n}, there is a path from $(0,\ldots,0)$ to x. Let

$$x = (\frac{a_1}{N_1},\ldots,\frac{b_n}{N_n}).$$

By Lemma 11 there is a path from 0^n to $(\frac{1}{N},0,\ldots,0)$. From this and Lemma 12 there is a path from 0^n to.

CONTINUE LATER

□

References

[1] K. B. Chilakamarri. Unit distance graphs in rational n-space. *Discrete Mathematics*, 69:213–218, 1988. link.

[2] A. de Grey. The chromatic number of the plane is at least 5. *Geocombinatorics*, 28:5–18, 2018. arxiv link.