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UNIT-DISTANCE GRAPHS IN RATIONAL n-SPACES

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Let U_n be the infinite graph with n-dimensional rational space Q^n as vertex set and two vertices joined by an edge if and only if the distance between them is exactly 1. The connectedness and clique numbers of the graphs U_n are discussed.

1. Introduction and definitions

Let R^n and Q^n denote real and rational n-space, equipped with the usual Euclidean metric. Let G_n denote the infinite graph whose vertices are the points of R^n, two vertices adjacent if and only if the distance between them is exactly 1. It is easy to see that G_n is connected for $n \geq 2$ and the maximum number of points in R^n that are pairwise unit distance apart (the clique number of G_n) is $n+1$ for $n \geq 1$. However, the chromatic number of G_n is so far unknown for $n \geq 2$ [1].

Let U_n be the subgraph of G_n induced by those vertices that are in Q^n. In Section 2 we shall prove that U_n is connected if and only if $n \geq 5$. In Section 3 we shall determine the clique number $\omega(n)$ of U_n. For even n, $\omega(n)$ is $n+1$ or n according as $n+1$ is or is not a perfect square. For odd n, if the diophantine equation $nx^2 - 2(n-1)y^2 = z^2$ has an integer solution (x, y, z) with $x \neq 0$, then $\omega(n) = n+1$ or n according as $\frac{1}{2}(n+1)$ is or is not a perfect square; otherwise, $\omega(n) = n-1$.

2. The connectedness of U_n

In this section we shall first prove that U_1, U_2, U_3, and U_4 are all disconnected and prove that U_n is connected for $n \geq 5$.

Lemma 1. *There is no path in U_4 connecting the origin* $(0, 0, 0, 0)$ *to* $(\frac{1}{4}, 0, 0, 0)$.

Proof. Suppose there is. Then, equivalently, there are finitely many points on the unit sphere in Q^4 whose sum is $(\frac{1}{4}, 0, 0, 0)$. Let $(a_1/b,\ a_2/b,\ a_3/b,\ a_4/b)$ be such a point, where a_1, a_2, a_3, a_4, and b have no common factor and

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = b^2. \tag{1}$$

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If b is divisible by 4, then at least one of a_1, a_2, a_3, a_4 is odd, and so the left-hand side of (1) is not divisible by 8 whereas the right-hand side is. (Recall that the only squares modulo 8 are 0, 1, and 4.) Thus b is either odd or twice an odd integer. But the sum of a finite number of fractions with denominators of this form cannot be equal to $\frac{1}{4}$. This completes the proof of the lemma. □

Theorem 2. *The graphs U_1, U_2, U_3, and U_4 are all disconnected.*

Proof. This follows immediately from Lemma 1, since there are obvious subgraphs of U_4 that contain the points $(0,0,0,0)$ and $(\frac{1}{4},0,0,0)$ and are isomorphic to U_1, U_2, and U_3, respectively. □

Theorem 3. *The graph U_n is connected for $n \geq 5$.*

Proof. First note that if there exist two paths in U_n, one connecting $\mathbf{0}$ to x and the other connecting $\mathbf{0}$ to y, then there exists a path from $\mathbf{0}$ to $x+y$ in U_n. With this observation, it suffices to show that there is a path from $\mathbf{0}$ to $(0,0,\ldots,0,1/N,0,\ldots,0)$ in U_n for every non-zero integer N with $1/N$ in the ith coordinate for $i=1,2,\ldots,n$. Consider the integer $4N^2-1$. Since it is positive it can be written as a sum of four squares by Lagrange's Four Square Theorem. Hence, $4N^2-1=a^2+b^2+c^2+d^2$ for some integers a, b, c, and d, or, equivalently,

$$1=\left(\frac{1}{2N}\right)^2+\left(\frac{a}{2N}\right)^2+\left(\frac{b}{2N}\right)^2+\left(\frac{c}{2N}\right)^2+\left(\frac{d}{2N}\right)^2. \tag{2}$$

So, there are edges in U_n joining $\mathbf{0}$ and

$$\left(\frac{1}{2N}, \pm\frac{a}{2N}, \pm\frac{b}{2N}, \pm\frac{c}{2N}, \pm\frac{d}{2N}, 0, 0, \ldots, 0\right).$$

This shows that there is a path of length 2 in U_n connecting $\mathbf{0}$ to $(1/N,0,0,\ldots,0)$. By repeating the above with $1/2N$ in the ith coordinate, the desired path is obtained. This completes the proof of the theorem. □

3. The clique number of U_n

A set of points will be called *unidistant* if they are pairwise unit distance apart. Let $\omega(n)$ denote the maximum number of unidistant points in Q^n (the clique number of U_n). We may remark that any unidistant set can be translated so that the translated unidistant set contains $\mathbf{0}$. In this section, we first find bounds for $\omega(n)$ and then evaluate $\omega(n)$.

Lemma 4. $\omega(n) \leq n+1$.

Proof. Let $\{\mathbf{0}, y_1, y_2, \ldots, y_r\}$ be a unidistant set in Q^n. Let A be the $r \times n$ matrix whose rows are $y_1, y_2, \ldots, y_r$. Now the $r \times r$ matrix AA^T has 1's on the principal diagonal and $\frac{1}{2}$ everywhere else. AA^T is a non-singular matrix and so,

$$r = \text{rank}(AA^T) \leqslant \text{rank}(A) \leqslant n.$$

From this it follows immediately that $\omega(n) \leqslant n + 1$. This completes the proof of the lemma. □

Lemma 5. *If $n \geqslant 4$, then $\omega(n) \geqslant n$ if n is even and $\omega(n) \geqslant n - 1$ if n is odd.*

Proof. If n is even, define a set S_n of n unidistant points as follows:

$$\begin{aligned}
x_1 &= \mathbf{0} \\
x_2 &= (1, 0, 0, \ldots, 0) \\
x_3 &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0) \\
x_4 &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \ldots, 0) \\
x_5 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0) \\
x_6 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \tfrac{1}{2}, -\tfrac{1}{2}, 0, \ldots, 0) \\
&\vdots \\
x_{n-1} &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0, \tfrac{1}{2}, \tfrac{1}{2}) \\
x_n &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0, \tfrac{1}{2}, -\tfrac{1}{2})
\end{aligned}$$

If n is odd, define a set T_n of $n - 1$ unidistant points by adding an extra coordinate zero to the end of each vector in S_{n-1}. □

Theorem 6. $\omega(n) = n + 1$ *if and only if a set of n unidistant points exist in Q^n and $(n + 1)/2^n$ is a rational square.*

Proof. If $\omega(n) = n + 1$, then with no loss of generality let $\{\mathbf{0}, x_1, \ldots, x_n\}$ be a set of the $n+1$ unidistant points in Q^n. Let A be the $n \times n$ matrix having $x_1, x_2, \ldots, x_n$ as its rows. It is clear that $\det(A)$ (the determinant of A) is a rational number. Now $\det(AA^T) = (n + 1)/2^n =$ square of $\det(A)$, thus showing that $(n + 1)/2^n$ is a rational square.

Suppose $(n + 1)/2^n$ is a rational square and $\{\mathbf{0}, x_1, \ldots, x_{n-1}\}$ is a unidistant set of n points. We will construct a point x_n so that $\{\mathbf{0}, x_1, \ldots, x_n\}$ is a unidistant set in Q^n. Consider the $(n - 1) \times n$ matrix B having $x_1, \ldots, x_{n-1}$ as its rows. Let B_i be the $(n - 1) \times (n - 1)$ matrix obtained from B by deleting its ith column, and let $a_i = (-1)^{i+1} \det(B_i)$, for $i = 1, 2, \ldots, n$. Defining a vector $x = (a_1, a_2, \ldots, a_n)$, we observe that it has the following properties;

(1) x is in Q^n,

(2) x is orthogonal to $x_1, x_2, \ldots, x_{n-1}$ (follows from construction),

(3) $\|x\|^2 = \det(BB^T) = n/2^{n-1}$ (easily verified and also a consequence of the Cauchy–Binet Theorem).

Define a vector $x_n = kx + c$, where

$$c = \frac{1}{n}(x_1 + x_2 + \cdots + x_{n-1})$$

and

$$k = \frac{2^{n-1}}{n}\sqrt{\frac{n+1}{2^n}}.$$

The vector x_n is in Q^n since k is a rational number. From properties (2) and (3) above, it follows that

$$\begin{aligned} \|x_n\|^2 &= k^2\|x\|^2 + 2kx \cdot c + \|c\|^2 \\ &= \frac{2^{2n-2}}{n^2}\frac{n+1}{2^n}\frac{n}{2^{n-1}} + 0 + \frac{1}{n^2}\left(n - 1 + \frac{(n-1)(n-2)}{2}\right) \\ &= \frac{n+1}{2n} + \frac{n-1}{2n} = 1 \end{aligned}$$

and

$$\begin{aligned} \|x_n - x_i\|^2 &= \|x_n\|^2 - 2x_n \cdot x_i + \|x_i\|^2 \\ &= 1 - \frac{2}{n}\left(1 + \frac{n-2}{2}\right) + 1 = 1, \quad \text{for } i = 1, 2, \ldots, n-1. \end{aligned} \tag{4}$$

This completes the proof. □

Theorem 7. *If n is even, then $\omega(n) = n+1$ if $n+1$ is a perfect square and $\omega(n) = n$ otherwise.*

Proof. If $n \geqslant 4$, this follows immediately from Lemma 5 and Theorem 6. If $n = 2$, the result is a simple exercise. In fact, Woodall [4] shows that U_2 is two-colorable (bipartite). □

In what follows, we shall need the following theorem:

Theorem (Hall and Ryser [2]). *Let A be a non-singular $n \times n$ matrix with entries from a field of characteristic $\neq 2$, and suppose that $AA^{\mathrm{T}} = D_1 \oplus D_2$, the direct sum of two square matrices D_1 and D_2 of orders r and s respectively ($r + s = n$). Let M be an arbitrary $r \times n$ matrix such that $MM^{\mathrm{T}} = D_1$. Then there exists an $n \times n$ matrix Z having M as its first r rows such that $ZZ^{\mathrm{T}} = D_1 \oplus D_2$.*

Lemma 8. *Let U and V be two unidistant sets of $n-1$ points in Q^n. Then there is a rational orthogonal transformation (preserving distances and inner products) that maps U onto V. In particular, there is a point u in Q^n that is unidistant from all points in U if and only if there is a point v in Q^n that is unidistant from all points in V.*

Proof. There is no loss of generality in supposing that $\mathbf{0}$ is in both U and V, so that we can write

$$U=\{\mathbf{0}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{n-2}\} \quad \text{and} \quad V=\{\mathbf{0}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-2}\}.$$

Let $\boldsymbol{u}_{n-1}$ and $\boldsymbol{u}_n$ be independent vectors in Q^n that are orthogonal to all the vectors in U. Let A be the $n \times n$ matrix with rows $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ and let M be the $(n-2) \times n$ matrix with rows $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-2}$. Then A is non-singular, $AA^{\mathrm{T}}=D_1 \oplus D_2$ and $MM^{\mathrm{T}}=D_1$, where D_1 is a square matrix of order $n-2$ with 1's on the principal diagonal and $\frac{1}{2}$ everywhere else, and D_2 is a non-singular 2×2 matrix. By Hall and Ryser's theorem, there exists an $n \times n$ matrix Z having M as its first $n-2$ rows such that $ZZ^{\mathrm{T}}=D_1 \oplus D_2$. Let $L=Z^{-1}A$. Then L is a rational matrix such that $\boldsymbol{v}_i L=\boldsymbol{u}_i$, for $i=1, 2, \ldots, n-2$. Moreover, L is an orthogonal matrix, because $(Z^{\mathrm{T}})^{-1}Z^{-1}AA^{\mathrm{T}}=I$ and so $LL^{\mathrm{T}}=Z^{-1}AA^{\mathrm{T}}(Z^{-1})^{\mathrm{T}}=I$. This completes the proof of Lemma 8. □

Theorem 9. *Let n be an odd integer $\geqslant 5$. If the diophantine equation*

$$nx^2-2(n-1)y^2=z^2 \tag{5}$$

has an integer solution (x, y, z) with $x \neq 0$, then $\omega(n)=n+1$ or n according as $\frac{1}{2}(n+1)$ is or is not a perfect square; otherwise $\omega(n)=n-1$.

Proof. In view of Theorem 6, it suffices to prove that $\omega(n) \geqslant n$ if and only if (5) has an integer solution with $x \neq 0$. By Lemma 8, $\omega(n) \geqslant n$ if and only if there is a point $\boldsymbol{x}$ in Q^n that is unidistant from all the $n-1$ points in the set T_n of Lemma 5. Let

$$\boldsymbol{x}=(t_1, s_1, t_2, s_2, \ldots, t_m, s_m, r)$$

be such a point, where $m=\frac{1}{2}(n-1)$. It follows immediately that $t_1=\frac{1}{2}$, $s_2=s_3=\cdots=s_m=0$, $t_2=t_3=\cdots=t_m=\frac{1}{2}-s_1$ and $s_1^2+(m-1)(\frac{1}{2}-s_1)^2+r^2=\frac{3}{4}$. Solving for s_1 in terms of r,

$$s_1=\frac{m-1 \pm \sqrt{n-4mr^2}}{2m}. \tag{6}$$

Thus there exists a point $\boldsymbol{x}$ in Q^n as required if and only if there exists a rational number $r=y/x$ such that $n-4mr^2$ is a rational square, say $(z/x)^2$; that is, if and only if eq. (5) has an integer solution with $x \neq 0$. This completes the proof of Theorem 9. □

The above theorem is also true for $n=1$ and $n=3$. For $n=3$, the result is a simple exercise. The chromatic number of U_3 is 2. Robertson [3] has shown that the chromatic number of U_4 is 4. These results will be reported in a separate paper dealing mainly with the coloring of graphs U_n.

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