1 Limits On How Well We Can Approximate $a + \sqrt{b}$ With Rationals

We want to find see how well we can approximation $a+\sqrt{b}$ where $a, b \in \mathsf{Q}-\mathsf{Z}$. We will determine a, b, Δ to satisfy the following:

$$(\exists^\infty q \in \mathsf{N})(\exists p \in \mathsf{N})\left[\left|\frac{p}{q} - (a+\sqrt{b})\right| < \frac{\Delta}{q^2}\right] \implies \text{ A CONTRADICTION.}$$

If b is NOT squarefree then we would just factor out the squares. So we will also want b is square free.

Assume p, q, Δ are such that $0 < \frac{p}{q} - (a+\sqrt{b}) < \frac{\Delta}{q^2}$. (The case where the diff is negative is similar.)

We will find (a, b, Δ) such that if q is large we get a contradiction.

There exists $\delta < \Delta$ such that

$$\frac{p}{q} - (a+\sqrt{b}) = \frac{\delta}{q^2}.$$

$$p - q(a+\sqrt{b}) = \frac{\delta}{q}$$

$$\frac{\delta}{q} = p - aq - \sqrt{b}q$$

$$\frac{\delta}{q} + \sqrt{b}q = p - aq$$

$$\left(\frac{\delta}{q} + \sqrt{b}q\right)^2 = (p-aq)^2$$

$$\frac{\delta^2}{q^2} + 2\frac{\delta}{q}\sqrt{b}q + q^2 b = p^2 - 2apq + q^2a^2$$

$$\frac{\delta^2}{q^2} + 2\delta\sqrt{b} = p^2 - 2apq + q^2a^2 - q^2b = p^2 - 2apq + q^2(a^2 - b)$$

Want that as $q \to \infty$ LHS $\notin \mathsf{Z}$ and RHS $\in \mathsf{Z}$.

1.1 Making LHS $\notin \mathsf{Z}$

The LHS is

$$\frac{\delta^2}{q^2} + 2\delta\sqrt{b} = p^2 - 2apq$$

If q is large then $\frac{\delta^2}{q^2}$ is very small. We will also make $2\delta\sqrt{b}$ small so that the sum is < 1.

We first see how big q has to be and then how to set Δ which bounds δ.

Need

$$\frac{\delta^2}{q^2} + 2\delta\sqrt{b} < 1$$

$$\delta^2 + 2\delta\sqrt{b}q^2 < q^2$$

$$\delta^2 + 2\delta\sqrt{b}q^2 < q^2$$

$$\delta^2 < q^2(1 - 2\delta\sqrt{b})$$

$$q^2 > \frac{\delta^2}{1 - 2\delta\sqrt{b}}$$

$$q > \sqrt{\frac{\delta^2}{1 - 2\delta\sqrt{b}}}$$

We need to pick δ so that
$1 - 2\delta\sqrt{b} > 0$.
$1 > 2\delta\sqrt{b}$
$\delta < \frac{1}{\sqrt{b}}$.
Since $\delta < \Delta$ we can take $\Delta = \frac{1}{\sqrt{b}}$.

Upshot To get LHS $\notin \mathsf{Z}$

1. $q > \sqrt{\frac{\delta^2}{1-2\delta\sqrt{b}}}$
2. $\Delta = \frac{1}{\sqrt{b}}$.

1.2 Making RHS $\in \mathsf{Z}$

The RHS is

$$p^2 - 2apq + q^2(a^2 - b)$$

Note that $p, q \in \mathsf{N}$. Hence we need to make have $2apq \in \mathsf{N}$ and $a^2 - b \in \mathsf{Z}$. Recall that $a, b \in \mathsf{Q} - \mathsf{N}$.

To make $2apq \in \mathsf{Z}$ we take $a \in \{-\frac{1}{2}, \frac{1}{2}\}$.

1. $a = \frac{1}{2}$: To make $q^2(a^2 - b) \in \mathsf{Z}$ we need $q^2(\frac{1}{4} - b) \in \mathsf{Z}$. Hence

$$b \in X = \left\{\frac{4c+1}{4} : c \in \mathsf{N}\right\}.$$

2. $a = -\frac{1}{2}$: To make $q^2(a^2 - b) \in \mathsf{Z}$ we need $q^2(\frac{1}{4} - b) \in \mathsf{Z}$. Hence

$$b \in Y = \left\{\frac{4c+1}{4} : c \in \mathsf{N}\right\}.$$

Upshot To get RHS $\in \mathsf{Z}$ we do one of the following:

1. $a = \frac{1}{2}$, $b = \frac{4c+1}{4}$, $4c + 1$ NOT a square.
2. $a = -\frac{1}{2}$, $b = \frac{4c+1}{4}$, $4c + 1$ NOT a square.

1.3 Actual Numbers

Since Δ only depends on b we will take $a = \frac{1}{2}$ and not consider the $a = -\frac{1}{2}$ case.

a	b	$\Delta = \frac{1}{2\sqrt{b}}$
$\frac{1}{2}$	$\frac{5}{4}$	$\frac{1}{\sqrt{5}}$
$\frac{1}{2}$	$\frac{9}{4}$	NO GOOD- $9 \in \mathsf{SQ}$
$\frac{1}{2}$	$\frac{13}{4}$	$\frac{1}{\sqrt{13}}$
$\frac{1}{2}$	$\frac{17}{4}$	$\frac{1}{\sqrt{17}}$

Clearly the winner is $\frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$.