The Muffin Problem

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Who is Not Here

1. Rishi on zoom
2. Dylan on zoom
3. Faye hopefully on zoom
4. Ilya maybe on zoom
5. Fikur could not make it.

How it Begann

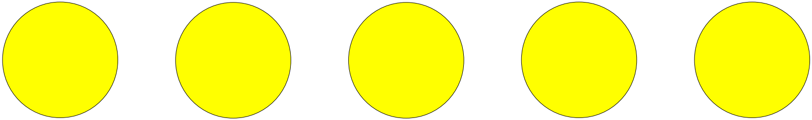
A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by **Alan Frank**:

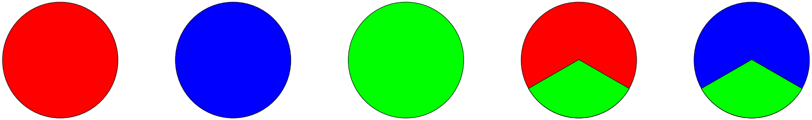
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ *where nobody gets a tiny sliver?*



5 Muffins, 3 Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

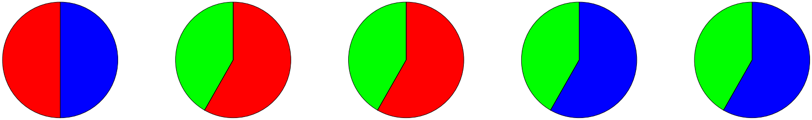
Work on it with your neighbor

5 Muffins, 3 People–Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

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5 Muffins, 3 People–Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. (**Henceforth:** All muffins cut into $\geq$ **2** pieces.)

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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$. (**Henceforth:** All muffins cut into 2 pieces.)

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Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

What Else Was in the Pamphlet?

The pamphlet also had asked about

1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
3. a few others

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- Find a technique that solves many problems (e.g., Floor-Ceiling).
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- Find a new technique **which was interesting**.
- Lather, Rinse, Repeat.

General Problem

***f*(*m*, *s*)** be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5,3) = \frac{5}{12}$ here.

We have shown $f(m,s)$ exists, is rational, and is computable using a **Mixed Int Program**.

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We have shown $f(m,s)$ exists, is rational, and is computable using a **Mixed Int Program**.
This was a case of a Theorem in **Applied Math** being used to prove a Theorem in **Pure Math**.

Amazing Results!/Amazing Theorems!

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2. $f(52, 11) = \frac{83}{176}$.
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Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

Conventions

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4. If assuming $f(m,s) > \alpha > \frac{1}{3}$, assume all muffin in ≤ 2 pcs.
5. $f(m,s) > \alpha > \frac{1}{3}$, so exactly 2 pcs, is common case.

FC Thm Generalizes $f(5,3) \le \frac{5}{12}$

$$f(m,s) \le \mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

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Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s\lceil 2m/s\rceil}$.

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The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

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Note: A Mod 3 Pattern.
Theorem: For all $m \geq 3$, $f(m,3) = \mathrm{FC}(m,3)$.

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$f(4k+2,4) = \frac{1}{2}$.

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$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m, 4) = \text{FC}(m, 4)$.

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Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m, 4) = \mathrm{FC}(m, 4)$.
FC-Conjecture: For all m, s with $m \geq s$, $f(m, s) = \mathrm{FC}(m, s)$.

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For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k+1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

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Note: A Mod 5 Pattern.
Theorem: For all $m \geq 5$ **except m=11**, $f(m, 5) = \mathrm{FC}(m, 5)$.

What About FIVE students, ELEVEN muffins?

$$f(11,5) \leq \max\left\{\frac{1}{3}, \min\left\{\frac{11}{5\lceil 22/5\rceil}, 1-\frac{11}{5\lfloor 22/5\rfloor}\right\}\right\} = \frac{11}{25}.$$

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We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.

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We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.
We found a protocol with smallest piece $\frac{13}{30} = 0.4333\ldots$.

1. Divide 1 muffin $(\frac{15}{30}, \frac{15}{30})$.
2. Divide 2 muffins $(\frac{14}{30}, \frac{16}{30})$.
3. Divide 8 muffins $(\frac{13}{30}, \frac{17}{30})$.
4. Give 2 students $[\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30}]$
5. Give 1 students $[\frac{16}{30}, \frac{16}{30}, \frac{17}{30}, \frac{17}{30}]$
6. Give 2 students $[\frac{15}{30}, \frac{17}{30}, \frac{17}{30}, \frac{17}{30}]$

So Now What?

We have:

$$\frac{13}{30} \leq f(11,5) \leq \frac{11}{25} \qquad \text{Diff}= 0.006666\ldots$$

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Options:

1. $f(11,5) = \frac{11}{25}$. Need to find procedure.
2. $f(11,5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11,5)$ in between. Need to find both.
4. $f(11,5)$ unknown to science!

Vote

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Vote WE SHOW: $\mathbf{f(11,5) = \frac{13}{30}}$. **Exciting** new technique!

Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let x be a piece from muffin M.
The *other piece* from muffin M is the *buddy of* x.

Note that the buddy of x is of size

$$1 - x.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \le \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \le \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\ge \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

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Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

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(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
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$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets $w \le x \le y \le z$ and $w \le \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \le \frac{1}{2}$

$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets $w \le x \le y \le z$ and $w \le \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \le \frac{1}{2}$

$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \ge \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets $w \le x \le y \le z$ and $w \le \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \le \frac{1}{2}$

$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \ge \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of z.

$$B(z) \le 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$

$f(11,5)=\frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets $w \le x \le y \le z$ and $w \le \frac{1}{2}$.
Since $w+x+y+z=\frac{11}{5}$ and $w \le \frac{1}{2}$

$$x+y+z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \ge \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of z.

$$B(z) \le 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$

GREAT! This is where $\frac{13}{30}$ comes from!

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.2: All 4-shares are $> \frac{1}{2}$. There are $4s_4 = 12$ 4-shares. There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

INT Method

Proof that $f(11,5) \le \frac{13}{30}$ was an example of the HALF method.

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INT Method

Proof that $f(11,5) \leq \frac{13}{30}$ was an example of the HALF method.

FC or HALF worked on everything with $s = 3, 4, 5, \ldots, 23$.

Then we found a case where neither FC nor HALF worked.

We found a new method: INT.

More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\leq \frac{19}{44}$.

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Case 1: A student gets ≥ 6 shares. Some piece $\le \frac{24}{11 \times 6} < \frac{19}{44}$.

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Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\leq \frac{19}{44}$.

Case 1: A student gets ≥ 6 shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

Case 2: A student gets ≤ 3 shares. Some piece $\geq \frac{24}{11 \times 3} = \frac{8}{11}$.
Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

More Sophisticated INT: $f(24,11) \le \frac{19}{44}$

Assume $(24,11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
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Case 1: A student gets ≥ 6 shares. Some piece $\le \frac{24}{11\times 6} < \frac{19}{44}$.

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Buddy of that piece $\le 1 - \frac{8}{11} \le \frac{3}{11} < \frac{19}{44}$.

Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.
5-students: a student who gets 5 shares. s_5 is the number of them.

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$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \end{aligned}$$

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4-share: a share that a 4-student who gets.
5-share: a share that a 5-student who gets.

$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \end{aligned}$$

$s_4 = 7$. Hence there are $4s_4 = 4 \times 7 = 28$ 4-shares.
$s_5 = 4$. Hence there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.1 and 3.2: Too Big or Too Small

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Case 3.1: There is a share $\geq \frac{25}{44}$. Then its buddy is

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Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: There is a share $\ge \frac{25}{44}$. Then its buddy is

$$\le 1 - \frac{25}{44} = \frac{19}{44}$$

Case 3.2: There is a share $\le \frac{19}{44}$. Duh.
Henceforth assume that all shares are in

$$\left(\frac{19}{44}, \frac{25}{44}\right)$$

$$\underset{\frac{19}{44}}{(} \qquad\qquad\qquad \underset{\frac{25}{44}}{)}$$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.

Since $v + w + x + y + z = \frac{24}{11}$ and $z \geq \frac{20}{44}$

Case 3.3: Some 5-shares $\ge \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\ge \frac{20}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume Alice has $v \le w \le x \le y \le z$ and $z \ge \frac{20}{44}$.

Since $v + w + x + y + z = \frac{24}{11}$ and $z \ge \frac{20}{44}$

$$v + w + x + y \le \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

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5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

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Since $v + w + x + y + z = \frac{24}{11}$ and $z \geq \frac{20}{44}$

$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

$$v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

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$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

$$v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

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$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

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Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \qquad\qquad \underset{\frac{25}{44}}{)}$$

Case 3.3: Some 5-shares $\ge \frac{20}{44}$

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Claim: If some 5-shares is $\ge \frac{20}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume Alice has $v \le w \le x \le y \le z$ and $z \ge \frac{20}{44}$.

Since $v + w + x + y + z = \frac{24}{11}$ and $z \ge \frac{20}{44}$

$$v + w + x + y \le \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

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Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \qquad\qquad \underset{\frac{25}{44}}{)}$$

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume Alice has $w \le x \le y \le z \le$ and $w \le \frac{21}{44}$.

Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume Alice has $w \le x \le y \le z \le$ and $w \le \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \le \frac{21}{44}$

Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume Alice has $w \le x \le y \le z \le$ and $w \le \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \le \frac{21}{44}$

$$x + y + z \ge \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume Alice has $w \le x \le y \le z \le$ and $w \le \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \le \frac{21}{44}$

$$x + y + z \ge \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

$$z \ge \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

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4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume Alice has $w \le x \le y \le z \le$ and $w \le \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \le \frac{21}{44}$

$$x + y + z \ge \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

$$z \ge \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of z is of size

$$\le 1 - \frac{25}{44} = \frac{19}{44}$$

Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume Alice has $w \le x \le y \le z \le$ and $w \le \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \le \frac{21}{44}$

$$x + y + z \ge \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

$$z \ge \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of z is of size

$$\le 1 - \frac{25}{44} = \frac{19}{44}$$

Henceforth we assume all 4-shares are in

$$\left(\frac{21}{44}, \frac{25}{44}\right).$$

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

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Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.
Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.
Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.
Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.
Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad \text{0 shs} \quad \underset{\frac{21}{44}}{](} \quad \text{28 4-shs} \quad \underset{\frac{25}{44}}{)}$$

More Refined Picture of What is Going On

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad \text{0 shs} \quad \underset{\frac{21}{44}}{](} \quad \text{28 4-shs} \quad \underset{\frac{25}{44}}{)}$$

More Refined Picture of What is Going On

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Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

More Refined Picture of What is Going On

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad \text{0 shs} \quad \underset{\frac{21}{44}}{](} \quad \text{28 4-shs} \quad \underset{\frac{25}{44}}{)}$$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$. QED.

More Refined Picture of What is Going On

(	20 5-shs	)[	0 shs	](	28 4-shs	)
$\frac{19}{44}$		$\frac{20}{44}$		$\frac{21}{44}$		$\frac{25}{44}$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$. QED.
The following picture captures what we know so far.

(	20 5-shs	)[	0	](	8 S4-shs	)[	0	](	20 L4-shs	)
$\frac{19}{44}$		$\frac{20}{44}$		$\frac{21}{44}$		$\frac{23}{44}$		$\frac{24}{44}$		$\frac{25}{44}$

More Refined Picture of What is Going On

(	20 5-shs	)[	0 shs	](	28 4-shs	)
$\frac{19}{44}$		$\frac{20}{44}$		$\frac{21}{44}$		$\frac{25}{44}$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$. QED.
The following picture captures what we know so far.

(	20 5-shs	)[	0	](	8 S4-shs	)[	0	](	20 L4-shs	)
$\frac{19}{44}$		$\frac{20}{44}$		$\frac{21}{44}$		$\frac{23}{44}$		$\frac{24}{44}$		$\frac{25}{44}$

S4= Small 4-shares
L4= Large 4-shares. L4 shares, 5-share: **buddies**, so $|L4|=20$.

Diagram

(	20 5-shs	)[	0	](	8 S4-shs	)[	0	](	20 L4-shs	)
$\frac{19}{44}$		$\frac{20}{44}$		$\frac{21}{44}$		$\frac{23}{44}$		$\frac{24}{44}$		$\frac{25}{44}$

Diagram

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0 \quad \underset{\frac{21}{44}}{](} \quad \text{8 S4-shs} \quad \underset{\frac{23}{44}}{)[} \quad 0 \quad \underset{\frac{24}{44}}{](} \quad \text{20 L4-shs} \quad \underset{\frac{25}{44}}{)}$$

Claim 2: Every 4-student has at least 3 L4 shares.

Diagram

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0 \quad \underset{\frac{21}{44}}{](} \quad \text{8 S4-shs} \quad \underset{\frac{23}{44}}{)[} \quad 0 \quad \underset{\frac{24}{44}}{](} \quad \text{20 L4-shs} \quad \underset{\frac{25}{44}}{)}$$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

Diagram

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Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

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Hence there are ≥ 21 L4-shares.

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Hence there are ≥ 21 L4-shares. But there are only 20.

GAPS Method

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Example of GAPS Technique: $f(31,19) \le \frac{54}{133}$

We show $f(31,19) \le \frac{54}{133}$.
Assume $(31,19)$-procedure with smallest piece $> \frac{54}{133}$.

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By INT-technique methods obtain:
$s_3 = 14$, $s_4 = 5$.

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The following are the only students who are allowed.
$e(1,5,5)$.
$e(2,4,5)$,
$e(3,4,5)$.
$e(4,4,4)$.

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$e(1,5,5)$. Let the number of such students be x
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$e(4,4,4)$. Let the number of such students be z.

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1) $|J_2| = |J_3|$,
only students using J_2 are $e(2,4,5)$ – they use one share each,
only students using J_3 are $e(3,4,5)$ – they use one share each.
Hence $y_1 = y_2$. We call them both y.

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3) Since $s_3 = 14$, $x + 2y + z = 14$.

$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}$.
Contradiction.

MATRIX Technique: $f(5,3) \geq \frac{5}{12}$

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$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0,2,0)$, m_1 muffins of this type.
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Natural Number Solution: $m_1 = 1$, $m_2 = 4$, $s_1 = 2$, $s_2 = 1$

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5) **Look for Nat Numb sol.** If find can translate into procedure.

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5. One corollary of the work: $f(m, s)$ only depends on m/s.

TV Show Leverage and Our Book

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2. Jacob and Daniel: Programmers (codes up techniques)
3. Bill: The Mastermind (guides the work and writes it up)

How it worked

We kept increasing s.

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Also a chapter that sketched out Scott H's method.

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  Second Year Royalties: \$50.00. I'm up by \$10.00. Wow!

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  Third Year Royalties: The royalties did not cover the cost of the muffins you are enjoying.