Quantum Bits, Entanglement, and the CHSH Game

Exposition by
William Gasarch and Evan Golub

September 6, 2024

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3. We describe the CHSH game.
4. We give a strategy for the CHSH game where (1) the 2 players are classical, and (2) **the prob of winning is 0.75.** We note that one can prove this is the best two players can do.
5. We give a strategy for the CHSH game where (1) the 2 players have qubits that are entangled, and (2) **the prob of winning is larger than 0.75.**

Quantum Bits I: Measure Once

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On next slide we show that M_θ is unitary.

Proof that M_θ is Unitary

Let $v = (\alpha, \beta)$ be a vector. We show $N(M_\theta(v)) = N(v)$.

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos(\theta)\alpha - \sin(\theta)\beta \\ \sin(\theta)\alpha + \cos(\theta)\beta \end{pmatrix}$$

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Def A **qubit** is something in physics that has a state. The state is an ordered pair (α, β) such that $\alpha^2 + \beta^2 = 1$. If a qubit is in state (α, β) then, when the qubit is measured, the prob that the bit is 0 is α^2 and the prob the bit is 1 is β^2.

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We will elaborate on this on the next slide.

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So she changes the state of the qubit before measuring it.
This is referred to as **measuring the qubit in a different basis** or in a **different frame.**

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Next two slides have the first and second coordinate of $M_{\frac{\pi}{6}}(v)$

Example (cont)

First coordinate of $M_{\frac{\pi}{6}}(v)$ is
$\cos(\theta)\alpha - \sin(\theta)\beta = \cos(\frac{\pi}{6})\frac{1}{\sqrt{2}} - \sin(\frac{\pi}{6})\frac{1}{\sqrt{2}}$
$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} - \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

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Second coordinate of $M_{\frac{\pi}{6}}(v)$ is
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The next few slides investigate this issue further.

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3. For $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$, Pr(0) goes from $\frac{1}{2}$ to 1.
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She is going to measure the qubit in bases θ. Let $w = M_\theta(v)$
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$\theta = \pi/60$: $\Pr(0) = 0.448$, close to $\frac{1}{2}$.

As θ gets bigger what happens?

1. For $0 \le \theta \le \frac{\pi}{4}$, $\Pr(0)$ goes from $\frac{1}{2}$ to 0.
2. For $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$, $\Pr(0)$ goes from 0 to $\frac{1}{2}$.
3. For $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$, $\Pr(0)$ goes from $\frac{1}{2}$ to 1.
4. For $\frac{3\pi}{4} \le \theta \le \pi$, $\Pr(0)$ goes from 1 to $\frac{1}{2}$.

The next few slides give actual numbers.

$0 \le \theta \le \frac{\pi}{4}$

θ	α	β	$\Pr(0) = \alpha^2$	$\Pr(1) = \beta^2$
0	+0.707	+0.707	0.5	0.5
$\pi/60$	+0.669	+0.743	0.448	0.552
$2\pi/60$	+0.629	+0.777	0.396	0.604
$3\pi/60$	+0.588	+0.809	0.345	0.655
$4\pi/60$	+0.545	+0.839	0.297	0.703
$5\pi/60$	+0.500	+0.866	0.250	0.750
$6\pi/60$	+0.454	+0.891	0.206	0.794
$7\pi/60$	+0.407	+0.914	0.165	0.835
$8\pi/60$	+0.358	+0.934	0.128	0.872
$9\pi/60$	+0.309	+0.951	0.095	0.905
$10\pi/60$	+0.259	+0.966	0.067	0.933
$11\pi/60$	+0.208	+0.978	0.043	0.957
$12\pi/60$	+0.156	+0.988	0.024	0.976
$13\pi/60$	+0.105	+0.995	0.011	0.989
$14\pi/60$	+0.052	+0.999	0.003	0.997
$15\pi/60$	+0.000	+1.000	0.000	1.000

$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

θ	α	β	$\Pr(0) = \alpha^2$	$\Pr(1) = \beta^2$
$15\pi/60$	$+0.000$	$+1.000$	0.000	1.000
$16\pi/60$	-0.052	$+0.999$	0.003	0.997
$17\pi/60$	-0.105	$+0.995$	0.011	0.989
$18\pi/60$	-0.156	$+0.988$	0.024	0.976
$19\pi/60$	-0.208	$+0.978$	0.043	0.957
$20\pi/60$	-0.259	$+0.966$	0.067	0.933
$21\pi/60$	-0.309	$+0.951$	0.095	0.905
$22\pi/60$	-0.358	$+0.934$	0.128	0.872
$23\pi/60$	-0.407	$+0.914$	0.165	0.835
$24\pi/60$	-0.454	$+0.891$	0.206	0.794
$25\pi/60$	-0.500	$+0.866$	0.250	0.750
$26\pi/60$	-0.545	$+0.839$	0.297	0.703
$27\pi/60$	-0.588	$+0.809$	0.345	0.655
$28\pi/60$	-0.629	$+0.777$	0.396	0.604
$29\pi/60$	-0.669	$+0.743$	0.448	0.552
$30\pi/60$	-0.707	$+0.707$	0.500	0.500

$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$

θ	α	β	$\Pr(0) = \alpha^2$	$\Pr(1) = \beta^2$
$30\pi/60$	-0.707	$+0.707$	0.500	0.500
$31\pi/60$	-0.743	$+0.669$	0.552	0.448
$32\pi/60$	-0.777	$+0.629$	0.604	0.396
$33\pi/60$	-0.809	$+0.588$	0.655	0.345
$34\pi/60$	-0.839	$+0.545$	0.703	0.297
$35\pi/60$	-0.866	$+0.500$	0.750	0.250
$36\pi/60$	-0.891	$+0.454$	0.794	0.206
$37\pi/60$	-0.914	$+0.407$	0.835	0.165
$38\pi/60$	-0.934	$+0.358$	0.872	0.128
$39\pi/60$	-0.951	$+0.309$	0.905	0.095
$40\pi/60$	-0.966	$+0.259$	0.933	0.067
$41\pi/60$	-0.978	$+0.208$	0.957	0.043
$42\pi/60$	-0.988	$+0.156$	0.976	0.024
$43\pi/60$	-0.995	$+0.105$	0.989	0.011
$44\pi/60$	-0.999	$+0.052$	0.997	0.003
$45\pi/60$	-1.000	$+0.000$	1.000	0.000

$\frac{3\pi}{4} \leq \theta \leq \pi$

θ	α	β	$\Pr(0) = \alpha^2$	$\Pr(1) = \beta^2$
$45\pi/60$	-1.000	$+0.000$	1.000	0.000
$46\pi/60$	-0.999	-0.052	0.997	0.003
$47\pi/60$	-0.995	-0.105	0.989	0.011
$48\pi/60$	-0.988	-0.156	0.976	0.024
$49\pi/60$	-0.978	-0.208	0.957	0.043
$50\pi/60$	-0.966	-0.259	0.933	0.067
$51\pi/60$	-0.951	-0.309	0.905	0.095
$52\pi/60$	-0.934	-0.358	0.872	0.128
$53\pi/60$	-0.914	-0.407	0.835	0.165
$54\pi/60$	-0.891	-0.454	0.794	0.206
$55\pi/60$	-0.866	-0.500	0.750	0.250
$56\pi/60$	-0.839	-0.545	0.703	0.297
$57\pi/60$	-0.809	-0.588	0.655	0.345
$58\pi/60$	-0.777	-0.629	0.604	0.396
$59\pi/60$	-0.743	-0.669	0.552	0.448
$60\pi/60$	-0.707	-0.707	0.500	0.500

Quantum Bits II: Measure Twice

Exposition by
William Gasarch and Evan Golub

September 6, 2024

Measuring a Qubit Twice in the Standard Basis

Alice has a qubit in state $v = (\alpha, \beta)$.

Measuring a Qubit Twice in the Standard Basis

Alice has a qubit in state $v = (\alpha, \beta)$.
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Measuring a Qubit Twice in the Standard Basis

Alice has a qubit in state $v = (\alpha, \beta)$.
1) Alice measures it in the st. basis and gets bit b.
2) Alice **then** measures the qubit in the st. basis again.

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Alice has a qubit in state $v = (\alpha, \beta)$.
1) Alice measures it in the st. basis and gets bit b.
2) Alice **then** measures the qubit in the st. basis again.
She will get b.

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Alice has a qubit in state $v = (\alpha, \beta)$.
1) Alice measures it in the st. basis and gets bit b.
2) Alice **then** measures the qubit in the st. basis again.
She will get b. She cannot get anything else.

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This is **not** weird. Here is a classical analog:

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Alice has a box that has a coin in it with sides labelled 0 and 1.

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She will get b. She cannot get anything else.

This is **not** weird. Here is a classical analog:

Alice has a box that has a coin in it with sides labelled 0 and 1.
She opens it and sees a b face up.
She closes it. She opens it again. She still sees a b.

Measuring a Qubit in Standard Basis and Non-Standard Basis

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1) Alice measures a qubit in st. basis, gets bit b. Use v.

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1) Alice measures a qubit in st. basis, gets bit b. Use v.
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The prob that she gets that same b again is $\cos^2(\theta)$.
Why $\cos^2(\theta)$? We will explain that on two later slide titled:

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3) Next slide generalizes this.

Measuring a Qubit in Two Different Basis

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Alice has a qubit in state $v = (\alpha, \beta)$.
1) Alice measures the qubit in basis θ_1, so in state $w = M_{\theta_1}(v)$, and gets bit b.

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1) Alice measures the qubit in basis θ_1, so in state $w = M_{\theta_1}(v)$, and gets bit b.
2) Alice then measures the qubit in basis θ_2, so in state $w' = M_{\theta_2}(w)$.
The prob that she gets that same b again is $\cos^2(\theta_1 - \theta_2)$.
Why $\cos^2(\theta_1 - \theta_2)$? We will explain that on two slide titled

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EVAN AND BILL- MIGHT PUT PICTURE HERE.

2 People Measure a Qubit in Two Different Basis

Alice has a qubit in state $v = (\alpha, \beta)$.

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The prob that he gets that b is $\cos^2(\theta_1 - \theta_2)$.

Example

Alice has a qubit in state $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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$\Pr(0) =$ Prob that Bob and Alice agree $= \cos^2(0 - \frac{\pi}{6}) = 0.75$.

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$\Pr(1) = 1 - \Pr(0) = 0.25$.

Why $\cos^2(\theta)$: Collapsing

Measuring qubits is not passive.

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That is why when Bob then measures v in the st. basis he will get 0.

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We will discuss what happens if Bob then measures in basis θ.

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That is why when Bob then measures v in the st. basis he will get 0.
We will discuss what happens if Bob then measures in basis θ.

Similarly if Alice measures v in the st. basis and gets 1 then v collapses to $(0, 1)$.

Why $\cos^2(\theta)$: The Math

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Alice has a qubit in state $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
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Bob then measures in basis θ. But note that state is now $(1, 0)$.

Why $\cos^2(\theta)$: The Math

Alice has a qubit in state $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
Case 1 Alice measures the qubit and gets 0. State collapses to $(1, 0)$.
Bob then measures in basis θ. But note that state is now $(1, 0)$.
$M_\theta(v) = (\cos(\theta), \sin(\theta))$.

Why $\cos^2(\theta)$: The Math

Alice has a qubit in state $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
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Case 2 Alice measures v in st. basis and gets 1. state collapses to $(0, 1)$.

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Alice has a qubit in state $v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
Case 1 Alice measures the qubit and gets 0. State collapses to $(1, 0)$.
Bob then measures in basis θ. But note that state is now $(1, 0)$.
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Case 2 Alice measures v in st. basis and gets 1. state collapses to $(0, 1)$.
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$M_\theta(v) = (-\sin(\theta), \cos(\theta))$.
Prob that Bob and Alice will agree is $\Pr(1) = \cos^2(\theta)$.

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$M_\theta(v) = (\cos(\theta), \sin(\theta))$.

Prob that Bob and Alice will agree is $\Pr(0) = \cos^2(\theta)$.

Case 2 Alice measures v in st. basis and gets 1. state collapses to $(0, 1)$.

Bob then measures in θ basis. But note that state is now $(0, 1)$.

$M_\theta(v) = (-\sin(\theta), \cos(\theta))$.

Prob that Bob and Alice will agree is $\Pr(1) = \cos^2(\theta)$.

In Both Cases The Prob that Alice and Bob Agree is $\cos^2(\theta)$

Entanglement

Exposition by
William Gasarch and Evan Golub

September 6, 2024

Alice and Bob Like to Share

We say what Alice and Bob can do if they have qubits that are entangled in a certain way.

Alice and Bob Like to Share

We say what Alice and Bob can do if they have qubits that are entangled in a certain way.

We first describe four scenarios without quantum entanglement to later contrast the case of qubits that are entangled to other cases, both classical and quantum.

Two Scenarios That are Not Weird

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1) Charles picks $b \in \{0, 1\}$ uniformly at random. Charles gives Alice a box with b in it, and Bob a box with the b in it.

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If Alice opens the box and sees b, she knows that Bob's box also has b.

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2) Charles gives Alice and Bob each a qubit in state (α, β). Alice & Bob both know α & β. Alice measures in the st. basis and gets a 1.

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If Alice opens the box and sees b, she knows that Bob's box also has b.

2) Charles gives Alice and Bob each a qubit in state (α, β). Alice & Bob both know α & β. Alice measures in the st. basis and gets a 1.

Alice knows the prob that Bob gets a 1 is α^2, but she knew this before she measured.

Two More Scenarios That are Not Weird

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3) Alice has a qubit. Alice measures it in the st. basis and gets 0. Hence the state is now $(1, 0)$. Alice gives the qubit to Bob. He measures it in the st. basis.

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Two More Scenarios That are Not Weird

3) Alice has a qubit. Alice measures it in the st. basis and gets 0. Hence the state is now $(1, 0)$. Alice gives the qubit to Bob. He measures it in the st. basis.

Alice **knows** that he will get a 0.

4) Alice has a qubit. Alice measures it in the st. basis and gets 0. The state is now $(1, 0)$. Alice gives the qubit to Bob. He rotates the qubit state by $\frac{\pi}{6}$ (so multiplies its state by $M_{\frac{\pi}{6}}$). The qubit is now in state $(\cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}))$. He measures the qubit.
$\Pr(0) = \cos^2(0 - \frac{\pi}{6})^2 = 0.75$, $\Pr(1) = 0.25$.

Two More Scenarios That are Not Weird

3) Alice has a qubit. Alice measures it in the st. basis and gets 0. Hence the state is now $(1, 0)$. Alice gives the qubit to Bob. He measures it in the st. basis.

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In the four scenarios above the qubits were not connected. We will now discuss **Quantum Entanglement** where the qubits are connected.

Alice and Bob Have Entangled Qubits

Alice has a qubit in state $v_A = (\alpha, \beta)$.

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We will only deal with the case where v_A and v_B are an **EPR pair** (EPR stands for Einstein, Podolsky, Rosen) which is the simplest case of Entanglement. EPR pairs are also called **Bell Pairs.**

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We will define properties of EPR pairs on the next slide.

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3) If Alice measures $M_{\theta_A}(v_A)$ and Bob measures $M_{\theta_B}(v_B)$ then the probability that they get the same answer is $\cos^2(\theta_A - \theta_B)$. For example, if $\theta_A - \theta_B$ is close to 0 then the probability that they get the same answer is close to 1.

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Note This is weird. The two entangled qubits are **different** and may be **far apart** yet they are instantaneously linked together.

Contrast Independent Pairs and EPR pairs

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$$\Pr(\text{Bob gets } 0) = 0.5 \qquad \Pr(\text{Bob gets } 1) = 0.5.$$

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$$\Pr(\text{Bob gets } 1) = \Pr(\text{Alice \& Bob disagree}) = 1 - \cos^2\left(\frac{\pi}{6} - 0\right) = 0.25.$$

The CHSH Game

Exposition by
William Gasarch and Evan Golub

September 6, 2024

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1. Charles sends Alice a bit x and Bob a bit y. Both x and y were chosen uniformly at random.
2. Alice sends Charles a bit a. Bob sends Charles a bit b.
3. If $x \wedge y = a \oplus b$ then Alice and Bob win. Else they lose.

Classic Strategies

On the next few slides we discuss strategies with an eye towards asking how often they win.

All 0 Strategy

Since $x \wedge y$ is mostly 0, make $a \oplus b$ always 0, so a strong strategy is for Alice and Bob to both send 0.

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0	0	0	0	0	0	Y
0	1	0	0	0	0	Y
1	0	0	0	0	0	Y
1	1	0	0	1	0	N

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Alice and Bob win with probability 0.75.

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Next slide analyzes the prob that they win.

Analyzing the Mostly 0 Strategy

x	y	coin	a	b	$x \wedge y$	$a \oplus b$	Wins?
0	0	0	0	0	0	0	Y
0	0	1	0	0	0	0	Y
0	1	0	0	0	0	0	Y
0	1	1	0	0	0	0	Y
1	0	0	0	0	0	0	Y
1	0	1	1	0	0	1	N
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1	0	0	0	0	0	0	Y
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1	0	0	0	0	0	0	Y
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1	1	0	0	0	1	0	N
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In the first four rows the coin flip is irrelevant.
If $(x, y) = (1, 0)$ then they win if the coin is 0, so prob $1 - p$.

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0	1	0	0	0	0	0	Y
0	1	1	0	0	0	0	Y
1	0	0	0	0	0	0	Y
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1) $(x, y) \in \{(0, 0), (0, 1)\}$. Thats prob $\frac{1}{2}$.

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Hence they win when any of the following happen:

1) $(x, y) \in \{(0, 0), (0, 1)\}$. Thats prob $\frac{1}{2}$.

2) $(x, y) = (1, 0)$ and the coin is 0. Thats prob $\frac{1}{4} \times (1 - p)$.

Analyzing the Mostly 0 Strategy

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Analyzing the Mostly 0 Strategy

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So the prob of winning is $\frac{1}{2} + \frac{1-p}{4} + \frac{p}{4} = \frac{3}{4} = 0.75$.

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So the prob of winning is $\frac{1}{2} + \frac{1-p}{4} + \frac{p}{4} = \frac{3}{4} = 0.75$. No better.

Is There a Better Strategy?

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1. There is no deterministic strategy that can win with probability more than 0.75.
2. There is no randomized strategy that can win with probability more than 0.75.

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then Alice and Bob have a strategy that wins the CHSH game with probability $\frac{13}{16} = 0.8125 > 0.75$.

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3. $y = 0$: Bob measures $M_{\frac{\pi}{6}}(v_B)$. b is result.

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3. $y = 0$: Bob measures $M_{\frac{\pi}{6}}(v_B)$. b is result.
4. $y = 1$: Bob measures $M_{\frac{\pi}{2}}(v_B)$. b is result.

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Hence the prob of a win is
$\frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{3}{4} + \frac{1}{4} \times 1 = \frac{13}{16} = 0.8125$.

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2. This is evidence that quantum mechanics is correct.
3. There are things we can do **better** in the quantum world than in the classical world.

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If Alice and Bob share an EPR pair then Alice and Bob have a strategy that wins the CHSH game with Prob 0.8125

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Answer on the next slide.

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The exact prob of winning is $\cos^2(\frac{\pi}{8})$.

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Answer to a particular part of this problem on the Next Page

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Can Alice and Bob obtain a higher prob of winning with a different choice of angles?

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No.

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No.
This can be proven solving maximizing

$$\cos^2(x_0 - y_0) + \cos^2(x_0 - y_1) + \cos^2(x_1 - y_0) + \cos^2(x_1 - y_1).$$

EVAN AND BILL- BILL ESP- CHECK ON THIS- VERIFY THIS IS WHAT YOU NEED TO MAXIMIZE AND FIND THE MAX.

Can Alice and Bob Do Better With a Diff Approach? More EPR-Pairs?

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In 1980 Tsirelson proved the following:

Can Alice and Bob Do Better With a Diff Approach? More EPR-Pairs?

In 1980 Tsirelson proved the following:

Even allowing Alice and Bob to share many EPR pairs, there is no strategy that gives a prob of winning $> \cos^2(\frac{\pi}{8})$**.**

Final Thoughts

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4. I am amazed that with a shared EPR pair Alice and Bob can do **so much better**. I would have have thought something like $0.75 + \epsilon$.

Final Thoughts

1. Classically: There is a strategy for CHSH that has prob of winning 0.75 and it is known you cannot do better than that.
2. If Alice and Bob share an EPR pair then there is a strategy that has prob of winning $\cos^2(\frac{\pi}{8}) \sim 0.853$.
3. I am amazed that with a shared EPR pair Alice and Bob can do better.
4. I am amazed that with a shared EPR pair Alice and Bob can do **so much better**. I would have have thought something like $0.75 + \epsilon$.
5. Even with many EPR pairs and any kind of strategy Alice and Bob cannot do better than $\cos^2(\frac{\pi}{8})$. I am not amazed this is true, but I am amazed its been proven. (Proof is hard.)