Funky Dice: An Exposition

William Gasarch - University of MD

Attendence

Should be here

Olivia Guo
Yifei Lin
Fikur Mikuria
Kate Patrabansh
Ricky Sun
Amie Zeng
Ilya Hag (might be on zoom)
Rishi Cherukuri (will be on zoom)

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Won't be here but will watch the recording

Jonah Chopra-Khan
Jessica Hsieh
Chaewoon Kyoung
Sujan Poudel
Dylan Schenker

If You Roll Two Standard 6-Sided Dice Then

1. 2: (1,1). ONE way. Prob $\frac{1}{36}$.
2. 3: (1,2), (2,1). TWO ways. Prob $\frac{1}{18}$.
3. 4: (1,3), (2,2), (3,1). THREE ways. Prob $\frac{1}{12}$.
4. 5: (1,4), (2,3), (3,2), (4,1). FOUR ways. Prob $\frac{1}{9}$.
5. 6: (1,5), (2,4), (3,3), (4,2), (5,1) FIVE ways. Prob $\frac{5}{36}$.

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6. 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) SIX ways. Prob $\frac{1}{6}$.

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8. 9: (3,6), (4,5), (5,4), (6,3) FOUR ways. Prob $\frac{1}{9}$.
9. 10: (4,6), (5,5), (6,4) THREE ways. Prob $\frac{1}{12}$.
10. 11: (5,6), (6,5) TWO ways. Prob $\frac{1}{18}$.
11. 12: (6,6) ONE way. Prob $\frac{1}{36}$.

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1. Can we load two 6-sided dice so that every number from 2 to 12 has the **same** probability. Called **fair sums**.
2. Can you label the dice something other than $\{1, \ldots, 6\}$ and $\{1, \ldots, 6\}$ and get the same probabilities you get with standard dice?

Loaded Dice

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Fair Dice Yield Unfair Sums

Fair Die:

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How Unfair?: $1/6 - 1/36 \sim 0.139$ unfair.

What Are Loaded Dice?

Def: A **Die** is a 6-tuple $(p_1, p_2, p_3, p_4, p_5, p_6)$ such that $0 \leq p_i \leq 1$ and $\sum_{i=1}^{6} p_i = 1$.

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3. Answer on next slide.

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The coefficient of x^i is $\mathrm{Prob}(\mathrm{sum} = \mathrm{i})$

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Continued on Next Slide.

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Does $x^{10} + x^9 + \cdots + x + 1$ have any real roots?

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1. r root of $x^{10} + \cdots + x + 1 \implies r$ root of $x^{11} - 1$ & $r \neq 1$.
2. r root of $x^{11} - 1$ & $r \neq 1 \implies r$ root of $x^{10} + \cdots + x + 1$.

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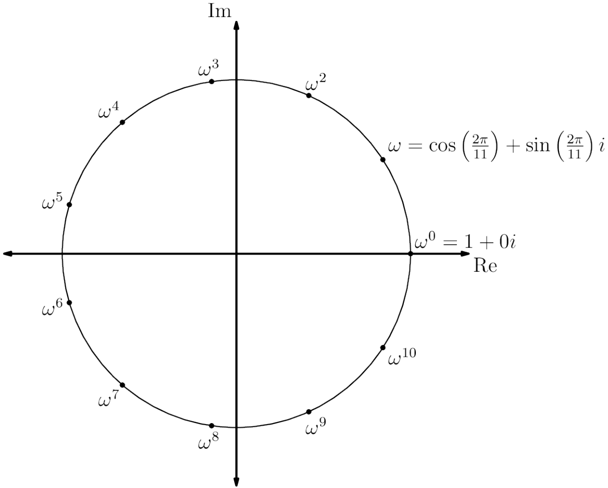
The roots of $x^{11} - 1$ are on the complex unit circle. See Next Slide.

The 11th Roots of Unity: Only Real one is 1



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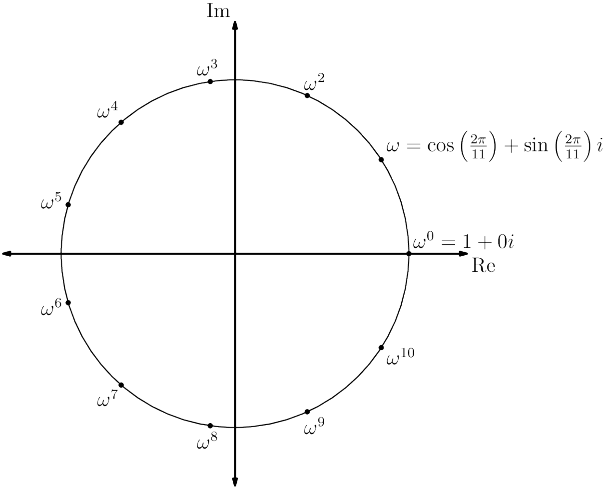

1 is only real 11th root of unity. $x^{10} + \cdots + 1 = 0$: **no** real roots.

No Dice (cont)

Recap

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Contradiction

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Answer on next slide

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1. The proof that for even d you **cannot** load two d-sided dice to get fair sums is similar to what we did for two 6-sided dice.
2. The proof that for odd d you **cannot** load two d-sided dice to get fair sums requires new techniques.

Can You Ever Load Dice to Get Fair Sums?

Is there a $d_1, d_2 \geq 2$ such that there are d_1-sided and d_2-sided dice that give fair sums?

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Fame! One paper refers to **The Gasarch-Kruskal Thm**.

How Close To Uniform Can You Get?

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How far are normal dice from uniform?

$$2(1/11-1/36)^2+2(1/11-1/18)^2+2(1/11-1/12)^2+2(1/9-1/11)^2+$$

$$2(5/36-1/11)^2)+(1/6-1/11)^2 \sim 0.0217$$

How Close To Uniform Can You Get? (cont)

Thm The optimal pair of 6-sided dice is $(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$ and $(\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})$.

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Distance from Uniform is $\frac{1}{352} \sim 0.0028$.

Measuring Unfairness

One measure is distance from uniform which is what Asgarli, Hartclass, Ostrov, Walden used.

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The optimal pair of n-sided dice is
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The distance from uniform is $\frac{1}{2(2n-1)(3n-2)}$.

Different Labels on Dice

William Gasarch - University of MD

Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So $(1, 2, 2, 3, 5, 8)$ would be allowed.

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Answer on next slide.

Yes We Kam!

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YES. There a non-standard labeling of a pair of 6-sided dice so that the dice yield the SAME probabilities as the standard dice.

We prove this on the next slide.

Let Polynomials Do The Work For You!

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)$$

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Coefficient of x^n is number of ways to get n.

Example of Non-Standard Labelings

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1. 12: TWO ways. Prob $\frac{1}{18}$.
2. 9: THREE ways. Prob $\frac{1}{12}$.
3. 8: NINE ways. Prob $\frac{1}{4}$.
4. 6: TWO ways. Prob $\frac{1}{18}$.

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4. 6: TWO ways. Prob $\frac{1}{18}$.
5. 5: TWELVE ways. Prob $\frac{1}{3}$.
6. 4: FOUR ways. Prob $\frac{1}{9}$.
7. 3: THREE ways. Prob $\frac{1}{12}$.
8. 2: ONE ways. Prob $\frac{1}{36}$.

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Question Is there a nonstandard labeling of two 6-sided dice that gives the same probabilities as the standard dice?

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$$(x^6+x^5+x^4+x^3+x^2+x)^2.$$

Is there a Non-Standard Labeling That... Cont.

$$(x^{a_1}+x^{a_2}+x^{a_3}+x^{a_4}+x^{a_5}+x^{a_6})(x^{b_1}+x^{b_2}+x^{b_3}+x^{b_4}+x^{b_5}+x^{b_6}) =$$

$$(x^6+x^5+x^4+x^3+x^2+x)^2 = x^2(x^5+x^4+x^3+x^2+x+1)^2 =$$

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

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$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8+x^6+x^5+x^4+x^3+x.$
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So desired dice are $(1,2,2,3,3,4)$ and $(1,3,4,5,6,8)$.

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What About Two d-Sided Dice?

For which $d \geq 2$ are there two non-standard d-sided dice that have the same prob as standard dice? **VOTE:**

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Answer on Next Slide

What About Two d-Sided Dice?

Answer There are two non-standard d-sided dice iff d is non-prime.

What About Two d-Sided Dice?

Answer There are two non-standard d-sided dice iff d is non-prime.

The proof is similar to what we did, though requires some thought.

What About d_1, d_2-Sided Dice?

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?
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3. Gallian and Rusin's paper exactly characterizes when this is possible:
   `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/nonstandarddice.pdf`
   The paper only looked at n d-sided dice and I do not know of a later paper. Thats why the question of d_1, d_2 is **Unknown to Science**.

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1. George Sicherman first posed the problem and solved it in 1978.
   The dice produced are sometimes called **Sicherman Dice**.
   You can buy these dice on the web!
2. Gasarch has an exposition on this material:
   `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/billdice.pdf`
3. Gallian and Rusin's paper exactly characterizes when this is possible:
   `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/nonstandarddice.pdf`
   The paper only looked at n d-sided dice and I do not know of a later paper. Thats why the question of d_1, d_2 is **Unknown to Science**.
   Or maybe just **Unknown to Bill**.

Parting Thoughts

William Gasarch - University of MD

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2. Polynomials are useful for problems with dice since multiplication gives information.
3. It is remarkable that a problem about dice lead to looking at complex roots of polynomials!