61 RIGIDITY AND SCENE ANALYSIS

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INTRODUCTION

Rigidity and flexibility of frameworks (motions preserving lengths of bars) and scene analysis (liftings from plane polyhedral pictures to spatial polyhedra) are two core examples of a general class of geometric problems:

(a) Given a discrete configuration of points, lines, planes, ... in Euclidean space, and a set of geometric constraints (fixed lengths for rigidity, fixed incidences, and fixed projections of points for scene analysis), what is the set of solutions and what is its local form: discrete? k-dimensional?

(b) Given a structure satisfying the constraints, is it unique, or at least locally unique, up to trivial changes, such as congruences for rigidity, or vertical scale for liftings?

(c) How does this answer depend on the combinatorics of the structure and how does it depend on the specific geometry of the initial data or object?

The rigidity of frameworks examines points constrained by fixed distances between pairs of points, using vocabulary and linear techniques drawn from structural engineering: bars and joints, first-order rigidity and first-order flexes, and static rigidity and static self-stresses (Section 61.1). Section 61.2 describes some extended structures and their applications. Scene analysis and the dual concept of parallel drawings are described in Section 61.3. Finally, Section 61.4 describes reciprocal diagrams which form a fundamental geometric connection between liftings of polyhedral pictures and self-stresses in frameworks which continue to be used in structural engineering.

These core problems have a wide range of applications across many areas of applied geometry. The methods used and the results obtained for these problems serve as a model for what might be hoped for other sets of constraints (plane first-order results) and as a warning of the complexity that does arise (higher dimensions and broader forms of rigidity). The subject has a rich history, stretching back into at least the middle of the 19th century, in structural and mechanical engineering. Other independent rediscoveries and connections have arisen in crystallography and scene analysis. The range of constraints and applications has continued to expand over the last two decades. For more general geometric reconstruction problems, see Chapter 34.

61.1 RIGIDITY OF BAR FRAMEWORKS

Given a set of points in space, with certain distances to be preserved, what other configurations have the same distances? If we make small changes in the distances,

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will there be a small (linear scale) change in the position? What is the structure, locally and globally, of the algebraic variety of these "realizations"?

We begin with the simplest linear theory: first-order rigidity, and the equivalent dual static rigidity, which are the linearized (and therefore linear algebra) version of rigidity. Generic rigidity refers to first-order rigidity of "almost all" geometric positions of the underlying combinatorial structure. After the initial results presenting first-order rigidity (Section 61.1.1), the study divides into the combinatorics of generic rigidity, using graphs (Section 61.1.2); a few basic results on rigid and flexible frameworks (Section 61.1.3); and the geometry of special positions in first-order rigidity, using projective geometry (Section 61.1.4).

61.1.1 FIRST-ORDER RIGIDITY

GLOSSARY

Configuration of points in d-space: An assignment $p = (p_1, \ldots, p_v)$ of points $p_i \in \mathbb{R}^d$ to an index set V, where $v = |V|$.

Congruent configurations: Two configurations p and q in d-space, on the same set V, related by an isometry T of $\mathbb{R}^d$ (with $T(p_i) = q_i$ for all $i \in V$).

Bar framework in d-space $G(p)$ (or ***(bar-joint) framework***): A graph $G = (V, E)$ (no loops or multiple edges) and a configuration p in d-space for the vertices V, such that $p_i \neq p_j$ for each edge $\{i, j\} \in E$ (Figure 61.1.1(a)). An edge $\{i, j\} \in E$ is called a ***bar*** in $G(p)$,

First-order flex (or ***infinitesimal motion***): For a bar framework $G(p)$, an assignment of velocities $p' : V \to \mathbb{R}^d$, such that for each edge $\{i, j\} \in E$: $(p_i - p_j) \cdot (p'_i - p'_j) = 0$ (Figure 61.1.1(c,d), where the arrows represent nonzero velocities).

Trivial first-order flex: A first-order flex p' that is the derivative of a flex of congruent frameworks (Figure 61.1.1(c)). (There is a fixed skew-symmetric matrix S (a rotation) and a fixed vector t (a translation) such that, for all vertices $i \in V$, $p'_i = p_i S + t$.)

First-order flexible (or ***infinitesimally flexible) framework***: A framework $G(p)$ with a nontrivial first-order flex (Figure 61.1.1(d)).

First-order rigid (or ***infinitesimally rigid) framework***: A bar framework $G(p)$ for which every first-order flex is trivial (Figures 61.1.1(a), 61.1.2(a)).

Rigidity matrix: For a framework $G(p)$ in d-space, $R_G(p)$ is the $|E| \times d|V|$ matrix for the system of equations: $(p_i - p_j) \cdot (p'_i - p'_j) = 0$ in the unknown velocities ${p'}_i$. The first-order flex equations are expressed as

$$R_G(p){p'}^T = \begin{bmatrix} \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & (p_i - p_j) & \cdots & (p_j - p_i) & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \end{bmatrix} \times {p'}^T = 0^T.$$

Self-stress: For a framework $G(p)$, a row dependence ω for the rigidity matrix: $\omega R_G(p) = 0$. Equivalently, an assignment of scalars ω_{ij} to the edges such that at each vertex i, $\sum_{\{j|\ \{i,j\} \in E\ \}} \omega_{ij}(p_i - p_j) = 0$ (placing these "internal forces" $\omega_{ij}(p_i - p_j)$ in equilibrium at vertex i). $\omega_{ij} < 0$ is tension, $\omega_{ij} < 0$ is compression.

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Independent framework: A bar framework $G(p)$ for which the rigidity matrix has independent rows. Equivalently, there is only the zero self-stress.

Isostatic framework: A framework that is first-order rigid and independent.

Generically rigid graph in d-space: A graph G for which the frameworks $G(p)$ are first-order rigid on an open dense subset of configurations p in d-space (Figures 61.1.1(a) and 61.1.2(a)).

Special position of a graph G in d-space: Any configuration $p \in \mathbb{R}^{dv}$ such that the rigidity matrix $R_G(p)$ has rank smaller than the maximum rank (the rank at a configuration with algebraically independent coordinates); See Figures 61.1.1(d) and 61.1.2(b) for examples. The special positions form an algebraic variety. A configuration p is ***regular*** if it is not special (that is $R_G(p)$ is maximum rank over all configurations).

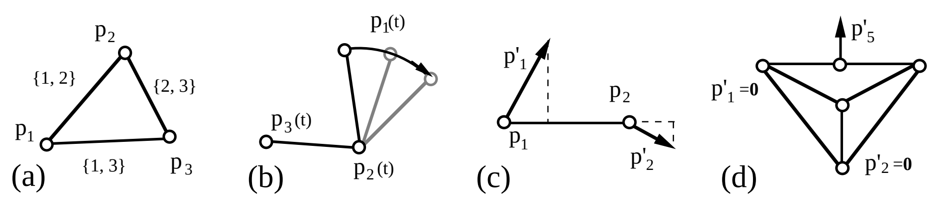

FIGURE 61.1.1

A rigid framework (a); a flexible framework (b); a first-order flex (c) and a first-order flexible, but rigid framework (d).

BASIC CONNECTIONS

Because the constraints $|p_i - p_j| = |q_i - q_j|$ are algebraic in the coordinates of the points (after squaring), we can work with the Jacobian matrix formed by the partial derivatives of these equations — the rigidity matrix of the framework.

The dimension of the space of trivial first-order motions of a framework in d-space is $\binom{d+1}{2}$ provided $|V| \geq d$ (the velocities generated by d translations and by $\binom{d}{2}$ rotations form a basis).

THEOREM 61.1.1 First-order Rank

A framework $G(p)$ with $|V| \geq d$ is first-order rigid if and only if the rigidity matrix $R_G(p)$ has rank $d|V| - \binom{d+1}{2}$.

A framework $G(p)$ with few vertices, $|V| \leq d$, is isostatic if and only if the rigidity matrix $R_G(p)$ has rank $\binom{v}{2}$ (if and only if G is the complete graph on V and the points p_i do not lie in an affine space of dimension $|V| - 2$).

First-order rigidity is linear algebra, with first-order rigid frameworks, self-stresses, and isostatic frameworks playing the roles of spanning sets, linear dependence, and bases of the row space for the rigidity matrix of the complete graph on the configuration p.

There is a dual theory of static rigidity for bar frameworks. Where first-order rigidity focuses on the kernel of the rigidity matrix (first-order flexes) and on the column space and column rank, static rigidity focuses on the cokernel of the rigidity matrix (the self-stresses) and on the row space of the rigidity matrix (the resolvable static loads). Methods from both approaches are widely used [CW82, Whi84a,

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Whi96], although in this chapter we present the results primarily in the vocabulary of first-order rigidity.

THEOREM 61.1.2 Isostatic Frameworks

For a framework $G(p)$ in d-space, with $|V| \geq d$, the following are equivalent:

(a) *$G(p)$ is isostatic (first-order rigid and independent);*

(b) *$G(p)$ is first-order rigid with $|E| = d|V| - \binom{d+1}{2}$;*

(c) *$G(p)$ is independent with $|E| = d|V| - \binom{d+1}{2}$;*

(d) *$G(p)$ is first-order rigid, and removing any one bar (but no vertices) leaves a first-order flexible framework.*

First-order rigidity of a framework $G(p)$ is a robust property: a small change in the configuration p preserves this rigidity. Independence implies that the lengths of bars are robust: any small change in these distances can be realized by a nearby configuration. On the other hand, self-stresses mean that one of the distances is algebraically dependent on the others: many small changes in the distances will have no realizations, or no nearby realizations.

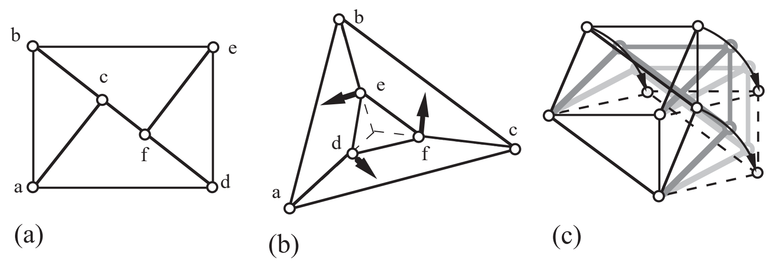

FIGURE 61.1.2
The graph G is realized as (a) a first-order rigid framework; as (b) a first-order flexible but rigid framework; as (c) a finitely flexible framework.

We note that the regular configurations of a graph G are an open dense subset.

THEOREM 61.1.3 Generic Rigidity Theorem

For a graph G and a fixed dimension d the following are equivalent:

(a) *G is generically rigid in d-space;*

(b) *for each configuration $p \in \mathbb{R}^{dv}$ using algebraically independent numbers over the rationals as coordinates, the framework $G(p)$ is first-order rigid;*

(c) *$G(p)$ is first-order rigid for all regular configurations;*

(d) *$G(p)$ is first-order rigid for some configuration.*

61.1.2 COMBINATORICS FOR GENERIC RIGIDITY

The major goal in generic rigidity is a combinatorial characterization of graphs that are generically rigid in d-space. The companion problem is to find efficient combinatorial algorithms to test graphs for generic rigidity. For the plane (and the line), this is solved. Beyond the plane the results are essentially incomplete, but some significant partial results are available.

GLOSSARY

Generically d-independent: A graph G for which some (equivalently, almost all) configurations p produce independent frameworks in d-space.

Generically d-isostatic graph: A graph G for which some (equivalently, almost all) configurations p produce isostatic frameworks in d-space.

Generic d-circuit: A graph G that is dependent for all configurations p in d-space but for all edges $\{i,j\} \in E$, $G - \{i,j\}$ is generically independent in d-space.

Complete bipartite graph: A graph $K_{m,n} = (A \cup B, A \times B)$, where A and B are disjoint sets of cardinality $|A| = m$ and $|B| = n$.

Triangulated d-pseudomanifold: A finite set of d-simplices (sets of $d+1$) with the property that each d subset of a simplex (facet) occurs in exactly two simplices, any two simplices are connected by a path of simplices and shared facets, and any two simplices sharing a vertex are connected through other simplices at this vertex. (For example, the triangles, edges, and vertices of a closed triangulated 2-surface without boundary form a 2-pseudomanifold.) See Section 16.3.

Henneberg d-construction for a graph G: A sequence $(V_d, E_d), \ldots, (V_n, E_n)$ of graphs, such that (Figure 61.1.6(a)):

(i) For each index $d < j \leq n$, (V_j, E_j) is obtained from (V_{j-1}, E_{j-1}) by

vertex addition: attaching a new vertex by d edges (Figure 61.1.3(a) for $d = 2$), or

edge splitting: replacing an edge from (V_{j-1}, E_{j-1}) with a new vertex joined to its ends and to $d-1$ other vertices (Figure 61.1.3(b) for $d = 2$); and

(ii) (V_d, E_d) is the complete graph on d vertices, and $(V_n, E_n) = G$.

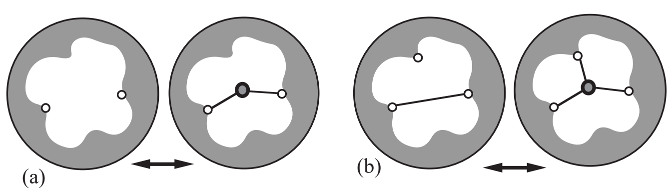

FIGURE 61.1.3

Both vertex addition (a) and edge split (b) preserve generic rigidity and generic independence.

Proper 3Tree2 partition: A partition of the edges of a graph into three trees, such that each vertex is attached to exactly two of these trees and no nontrivial subtrees of distinct trees T_i have the same support (i.e., the same vertices) (Figure 61.1.6(b)).

Proper 2Tree partition: A partition of the edges of a graph into two spanning trees, such that no nontrivial subtrees of distinct trees T_i have the same support (i.e., the same vertices) (Figure 61.1.6(c)).

d-connected graph: A graph G such that removing any $d-1$ vertices (and all incident edges) leaves a connected graph. (Equivalently, a graph such that any two vertices can be connected by at least d paths that are vertex-disjoint except for their endpoints.)

BASIC PROPERTIES IN ALL DIMENSIONS

THEOREM 61.1.4 Necessary Counts and Connectivity Theorem

If a graph G is generically d-isostatic, then, if $|V| \geq d$, $|E| = d|V| - \binom{d+1}{2}$, and for every $V' \subseteq V$ with $|V'| \geq d$, the number of edges in the subgraph of G induced by V' is at most $d|V'| - \binom{d+1}{2}$.

If G is a generically d-isostatic graph with $|V| > d$, then G is d-connected.

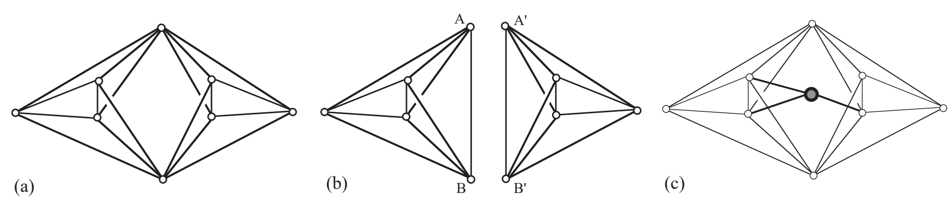

FIGURE 61.1.4

The double banana framework (a) is generically flexible, though it satisfies the basic counts of Theorem 61.1.4. It is generically dependent (b) and remains both flexible and dependent even when extended to be 3-connected (c).

For dimensions 1 and 2, the first count alone is sufficient for generic rigidity (see below). For dimensions $d > 2$, these two conditions are not enough to characterize the generically d-isostatic graphs (Table 61.2.1). Figure 61.1.4(a) shows a generically flexible counterexample for the sufficiency of the counts in dimension 3.

THEOREM 61.1.5 Bipartite Graphs [Whi84b]

A complete bipartite graph $K_{m,n}$, with $m > 1$, is generically rigid in dimension d if and only if $m + n \geq \binom{d+2}{2}$ and $m, n > d$.

INDUCTIVE CONSTRUCTIONS FOR ISOSTATIC GRAPHS

Inductive constructions for graphs that preserve generic rigidity are used both to prove theorems for general classes of frameworks and to analyze particular graphs [TW85, NR14].

THEOREM 61.1.6 Vertex Addition Theorem

The set of generically d-isostatic graphs is closed under vertex addition and under the deletion of vertices of degree d (Figure 61.1.3(a) for $d = 2$).

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THEOREM 61.1.7 Edge Split Theorem

The set of generically d-isostatic graphs is closed under edge splitting. Conversely, given a vertex v_0 of degree $d+1$ there is some pair j,k not in E and adjacent to v_0 such that adding that edge gives a d-isostatic graph. (Figure 61.1.3(b) for $d = 2$).

THEOREM 61.1.8 Construction Theorem

If a graph G is obtained by a Henneberg d-construction, then G is generically d-isostatic (Figure 61.1.6(a) for $d = 2$).

THEOREM 61.1.9 Gluing Theorem

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are generically d-rigid graphs sharing at least d vertices, then $G = (V_1 \cup V_2, E_1 \cup E_2)$ is generically d-rigid.

Vertex splitting is defined in Figure 61.1.5 [Whi91].

THEOREM 61.1.10 Vertex Splitting Theorem

If the graph G' is a vertex split of a generically d-isostatic graph G on d edges (Figure 61.1.5(a) for $d = 3$) *or a vertex split on $d-1$ edges* (Figure 61.1.5(b) for $d = 3$), *then G' is generically d-isostatic.*

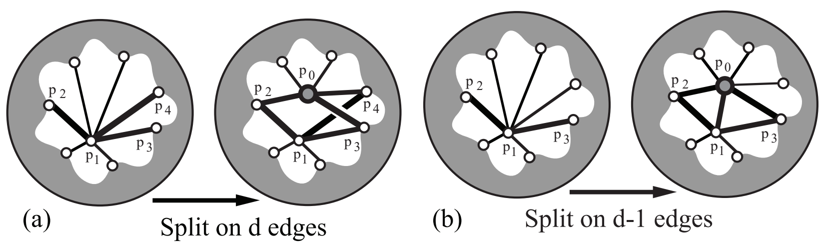

FIGURE 61.1.5
Two forms of vertex split (a) on d edges and (b) $d-1$ edges preserve generic rigidity and generic independence.

PLANE ISOSTATIC GRAPHS

Many plane results are expressed in terms of trees in the graph, building on a simpler correspondence between rigidity on the line and the connectivity of the graph. The earliest currently known complete proof of what is now known as Laman's Theorem (Theorem 61.1.11(b)) is [Poll27], though earlier statements of the result appear in the literature without complete proofs. The following is folklore.

THEOREM 61.1.11 Line Rigidity

For graph G and configuration p on the line with $p_i \neq p_j$ for all $\{i,j\} \in E$, the following are equivalent:

(a) *$G(p)$ is minimal among rigid frameworks on the line with these vertices;*

(b) *$G(p)$ is isostatic on the line;*

(c) *G is a spanning tree on the vertices;*

(d) *$|E| = |V| - 1$ and for every nonempty subset E' with vertices V', $|E'| \leq |V'| - 1$.*

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THEOREM 61.1.12 Plane Isostatic Graphs Theorem

For a graph G with $|V| \geq 2$, the following are equivalent:

(a) *G is generically isostatic in the plane;*

(b) *$|E| = 2|V| - 3$, and for every subgraph (V', E') with $|V'| \geq 2$ vertices, $|E'| \leq 2|V'| - 3$ (Laman's theorem; see also Table 61.2.1);*

(c) *there is a Henneberg 2-construction for G (Henneberg's theorem);*

(d) *E has a proper 3Tree2 partition (Crapo's theorem);*

(e) *for each $\{i, j\} \in E$, the multigraph obtained by doubling the edge $\{i, j\}$ is the union of two spanning trees (Recski's theorem).*

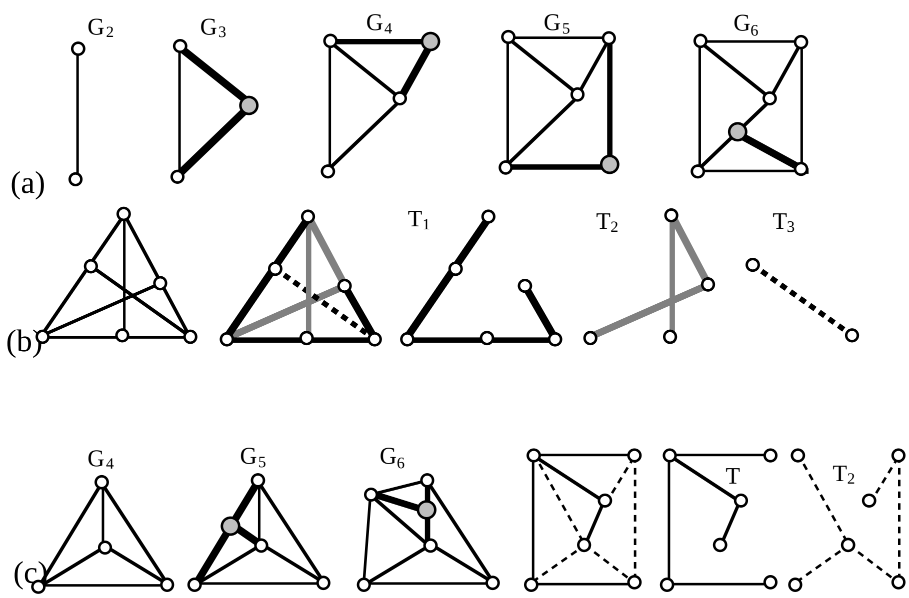


FIGURE 61.1.6

Figure (a) gives a Henneberg 2-construction for the graph in Figure 61.1.2. Figure (b) gives a decomposition into 3 trees, two at each vertex. Figure (c) gives inductive steps for a generic 2-circuit which decomposes into two spanning trees.

THEOREM 61.1.13 Plane 2-Circuits Theorem

For a graph G with $|V| \geq 2$, the following are equivalent:

(a) *G is a generic 2-circuit;*

(b) *$|E| = 2|V| - 2$, and for every proper subset E' on vertices V', $|E'| \leq 2|V'| - 3$;*

(c) *there is a construction for G from K_4, using only edge splitting and gluing (Berg and Jordán's theorem);*

(d) *E has a proper 2Tree partition.*

Figure 61.1.6(c) shows the construction for a 2-circuit, and an associated 2Tree partition. For 2-circuits with planar graphs, the planar dual is also a 2-circuit. The inductive techniques given above, and others, form dual pairs of constructions for these planar 2-circuits, pairing edge splits with vertex splits [BCW02].

THEOREM 61.1.14 Sufficient Connectivity [LY82]

If a graph G is 6-connected, then G is generically rigid in the plane.

There are 5-connected graphs that are not generically rigid in the plane. However, it was shown in [JJ09] that the 6-connectivity condition in Theorem 61.1.14 can be replaced by the weaker condition of "6-mixed connectivity."

ALGORITHMS FOR GENERIC 2-RIGIDITY

Each of the combinatorial characterizations has an associated polynomial time algorithm for verifying whether a graph is generically 2-isostatic:

(i) Counts: This can be checked by an $O(|V|^2)$ algorithm based on bipartite matchings or network flows on an associated graph [Sug86]. An efficient and widely used implementation is the "pebble game" [LS08].

(ii) 2-construction: Existence of a 2-construction can be checked by an $O(2^{|V|})$ algorithm, but a proposed 2-construction can be verified in $O(|V|)$ time.

(iii) 3Tree2 covering: Existence can be checked by an $O(|V|^2)$ matroid partition algorithm [Cra90].

(iv) 2Tree partition: All required 2Tree partitions can be found by a matroidal algorithm of order $O(|V|^3)$.

GENERICALLY RIGID GRAPHS IN HIGHER DIMENSIONS

Most of the results are covered by the initial summary for all dimensions d. Special results apply to the graphs of triangulated polytopes, as well as more general triangulated surfaces (see Section 16.3 for definitions).

THEOREM 61.1.15 Triangulated Pseudomanifolds Theorem

For $d \geq 2$, the graph of a triangulated d-pseudomanifold is generically $(d+1)$-rigid.

In particular, the graph of any closed triangulated 2-surface without boundary is generically rigid in 3-space (Fogelsanger's theorem), and the graph of any triangulated sphere is generically 3-isostatic (Gluck's theorem) [Fog88, Whi96]. Beyond the triangulated spheres in 3-space, most of these graphs are not isostatic, but are dependent. These results can all be proven by inductions using vertex splitting (and reduction techniques based on edge contractions).

These results have also been modified to take a triangulated sphere (and manifolds) and replace some discs by rigid blocks, and make other discs into holes. The resulting structures are *block and hole polyhedra* [FW13, CKP15a, CKP15b]). Here is a sample theorem, among a range of recent results.

THEOREM 61.1.16 Isostatic Towers [FW13]

A block and hole sphere, with one k-gon hole and one k-gon block, is generically isostatic if and only if there are k vertex disjoint paths connecting the hole and the block.

As a consequence of the recently proved Molecular Theorem 61.2.4, we describe a new class of generically 3-rigid graphs in Theorem 61.2.5. Further notable results regarding the generic independence of graphs in Euclidean d-space are the following.

THEOREM 61.1.17 Generic Independence in d-Space

Let $G = (V, E)$ be a graph with $|V| \geq d$ and let $G(p)$ be a generic framework in $\mathbb{R}^d$.

(a) *If $1 \leq d \leq 4$, and G is K_{d+2}-minor free, then $G(p)$ is independent [Nev07];*

(b) *if d is even, and for every $V' \subseteq V$ with $|V'| \geq 2$, the number of edges in the subgraph of G induced by V', $i(V')$, is at most $(\frac{d}{2}+1)|V'|-(d+1)$, then $G(p)$ is independent [JJ05a].*

If $d = 3$ and $i(V') \leq \frac{1}{2}(5|V'| - 7)$ for all $V' \subseteq V$ with $|V'| \geq 2$, then $G(p)$ is independent [JJ05a].

It is unknown whether the statement in (a) also holds for $d = 5$, but it is known to be false for $d \geq 6$. It is conjectured that the statement in (b) for even d also holds for all $d \geq 2$.

OPEN PROBLEMS

There is no combinatorial characterization of generically 3-isostatic graphs. There are a variety of conjectures towards such characterizations. Some older and recent work offers conjectures [JJ05b, JJ06]. We present one of the conjectures [TW85].

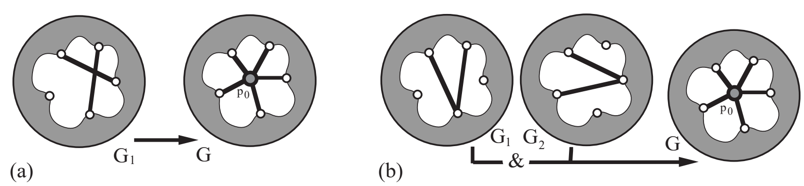


FIGURE 61.1.7
Two possible ways (a, b) to generate 5-valent vertices which are conjectured to preserve first-order rigidity.

CONJECTURE 61.1.18 *3-D Replacement Conjecture*

The X-replacement in Figure 61.1.7(a) *takes a graph G_1 that is generically rigid in 3-space to a graph G that is generically rigid in 3-space.*

The double V-replacement in Figure 61.1.7(b) *takes two graphs G_1, G_2 that are generically rigid in 3-space to a graph G that is generically rigid in 3-space.*

Every 3-isostatic graph is generated by an "extended Henneberg 3-construction," which uses these two moves along with edge splitting and vertex addition. What is unproven is that *only* 3-isostatic graphs are generated in this way.

The plane analogue of X-replacement is true for plane generic rigidity (without adding the fifth bar) [TW85, BCW02], and the 4-space analogue is false for some graphs (with two extra bars added in this analogue). If these conjectured steps prove correct in 3-space, then we would have inductive techniques to generate the graphs of all isostatic frameworks in 3-space, but the algorithm would be exponential.

For 4-space, there is no conjecture that has held up against the known counterexamples based on generically 4-flexible complete bipartite graphs such as $K_{7,7}$. We note that a related constraint matroid from multi-variate splines shares many of these inductive constructions, as well as gaps [Whi96]. However, X-replacement is known to work in the analogue of all dimensions there.

It is known that $d(d+1)$ connectivity is the minimum connectivity which is sufficient for body-bar frameworks. There is no confirmed polynomial bound on the minimum connectivity sufficient for generic rigidity in d-space for $d > 2$. We do have conjectures.

CONJECTURE 61.1.19 *Sufficient Connectivity Conjecture*

If a graph G is $d(d+1)$-connected, then G is generically rigid in d-space.

A graph can be checked for generic 3-rigidity by a "brute force" $O(2^{2^{|V|}})$ time algorithm. Assign the points independent variables as coordinates, form the rigidity matrix, then check the rank by symbolic computation. On the other hand, if numerical coordinates are chosen for the points "at random," then the rank of this numerical matrix ($O(|E|^3)$) will be the generic value, with probability 1. This gives a randomized polynomial-time algorithm, but there is no known deterministic algorithm that runs in polynomial, or even exponential, time.

61.1.3 RIGID AND FLEXIBLE FRAMEWORKS

GLOSSARY

Bar equivalence: Two frameworks $G(p)$ and $G(q)$ such that all bars have the same length in both configurations: $|p_i - p_j| = |q_i - q_j|$ for all bars $\{i,j\} \in E$.

Analytic flex: An analytic function $p(t) : [0,1) \to \mathbb{R}^{vd}$ such that $G(p(0))$ is bar-equivalent to $G(p(t))$ for all t (i.e., all bars have constant length).

Flexible framework: A bar framework $G(p)$ in $\mathbb{R}^d$ with an analytic flex $p(t)$ such that $p(0) = p$ but p is not congruent to $p(t)$ for all $t > 0$ (Figure 61.1.1(b).

Rigid framework: A bar framework $G(p)$ in d-space that is not flexible (Figure 61.1.1(a,d)).

BASIC CONNECTIONS

Because the constraints $|p_i - p_j| = |q_i - q_j|$ are algebraic in the coordinates of the points (after squaring), many alternate definitions of a "rigid framework" are equivalent. These connections depend on results such as the curve selection theorem of algebraic geometry or the inverse function theorem. A key early paper is [AR78].

THEOREM 61.1.20 Alternate Rigidity Definitions

For a bar framework $G(p)$ the following conditions are equivalent:

(a) *the framework is rigid;*

(b) *for every continuous path, or* **continuous flex** *of* $G(p)$, $p(t) \in \mathbb{R}^{vd}$, $0 \le t < 1$ *and* $p(0) = p$, *such that* $G(p(t))$ *is bar-equivalent to* $G(p)$ *for all* t, $p(t)$ *is congruent to* p *for all* t;

(c) *there is an* $\epsilon > 0$ *such that if* $G(p)$ *and* $G(q)$ *are bar-equivalent and* $|p-q| < \epsilon$, *then* p *is congruent to* q.

Essentially, the first derivative of a nontrivial analytic flex is a nontrivial first-order flex: $D_t\big((p_i(t) - p_j(t))^2 = c_{ij}\big)\big|_{t=0} \Rightarrow 2(p_i - p_j)\cdot(p'_i - p'_j) = 0$. (If this first derivative is trivial, then the earliest nontrivial derivative is a first-order motion.) This result is related to general forms of the inverse function theorem.

THEOREM 61.1.21 First-order Rigid to Rigid

If a bar framework $G(p)$ *is first-order rigid, then* $G(p)$ *is rigid.*

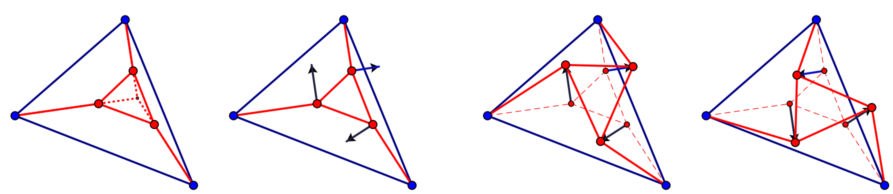
Rigidity and first-order rigidity are equivalent in most situations [AR78].

THEOREM 61.1.22 Rigid to First-order Rigid

If a bar framework $G(p)$ *is regular, then* $G(p)$ *is first-order rigid if and only if* $G(p)$ *is rigid.*

The other way to express this equivalence is that if a framework $G(p)$ has a first-order flex at a regular configuration p, then $G(p)$ is flexible.

Some first-order flexes are not the derivatives of analytic flexes (Figures 61.1.1(d) and 61.1.2(b)). However, a nontrivial first-order flex for a framework does guarantee a pair of nearby noncongruent, bar-equivalent frameworks. The averaging technique can also be used for alternative proofs that first-order rigidity implies rigidity.



(a) (b) (c) (d)

FIGURE 61.1.8

Given an infinitesimal motion of a framework (a), the velocity vectors (b) push forward (c) and pull backwards (d) to create two equivalent, non-congruent frameworks (de-averaging).

THEOREM 61.1.23 Averaging Theorem

If the points of a configuration p *affinely span* d*-space, then the assignment* p' *is a nontrivial first-order flex of* $G(p)$ *if and only if the frameworks* $G(p+p')$ *and* $G(p-p')$ *are bar-equivalent and not congruent (Figure 61.1.8).*

Whereas first-order rigidity is projectively invariant, rigidity itself is not projectively invariant—or even affinely invariant. It is a purely Euclidean property.

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61.1.4 GEOMETRY OF FIRST-ORDER RIGIDITY

GLOSSARY

Projective transform of a d-configuration p: A d-configuration q on the same vertices, such that there is an invertible matrix T of size $(d+1)\times(d+1)$ making $T(p_i, 1) = \lambda_i(q_i, 1)$ (where $(p_i, 1)$ is the vector p_i extended with an additional 1 — the affine coordinates of p_i).

Affine spanning set for d-space: A configuration p of points such that every point $q_0 \in \mathbb{R}^d$ can be expressed as an affine combination of the p_i: $q_0 = \sum_i \lambda_i p_i$, with $\sum_i \lambda_i = 1$. (Equivalently, the affine coordinates $(p_i, 1)$ span the vector space $\mathbb{R}^{d+1}$.)

Cone graph: The graph $G * u$ obtained from $G = (V, E)$ by adding a new vertex u and the $|V|$ edges (u, i) for all vertices $i \in V$.

Cone projection from p_0: For a $(d{+}1)$-configuration p on V, a configuration $q = \Pi_0(p)$ in d-space (placed as a hyperplane in $(d{+}1)$-space) on the vertices $V \setminus \{0\}$, such that $p_i \neq p_0$ is on the line $q_i p_0$ for all $i \neq 0$ (see also Chapter 63.2).

BASIC RESULTS

THEOREM 61.1.24 First-Order Flex Test

If the points of a configuration p on the vertices V affinely span d-space, then a first-order motion p' is nontrivial if and only if there is some pair h, k (not a bar) such that: $(p_h - p_k)\cdot(p'_h - p'_k) \neq 0$.

Projective invariance of static and infinitesimal rigidity has been known for more than 150 years, but is often forgotten [CW82].

THEOREM 61.1.25 Projective Invariance

If a framework $G(p)$ is first-order rigid (isostatic, independent) and $q = T(p)$ is a projective transform of p, then $G(q)$ is first-order rigid (isostatic, independent, respectively).

The following result provides an alternate proof of projective invariance as well as a corresponding generic result for cones.

THEOREM 61.1.26 Coning Theorem

A framework $G(\Pi_0 p)$ is first-order rigid (isostatic, independent) in d-space if and only if the cone $(G * u)(p)$ is first-order rigid (isostatic, independent, respectively) in $(d{+}1)$-space.

The special positions of a graph in d-space are rare, since they form a proper algebraic variety (essentially generated by minors of the rigidity matrix with variables for the coordinates of points). For a generically isostatic graph, this set of special positions can be described by the zeros of a single polynomial [WW83].

THEOREM 61.1.27 Pure Condition

For any graph G that is generically isostatic in d-space, there is a homogeneous polynomial $C_G(x_{1,1}, \dots, x_{1,d}, \dots, x_{|V|,1}, \dots, x_{|V|,d})$ such that $G(p)$ is first-order flexible if and only if $C_G(p_1, \dots, p_{|V|}) = 0$. C_G is of degree $(val_i + 1 - d)$ in the variables $(x_{i,1}, \dots, x_{i,d})$ for each vertex i of degree val_i in the graph.

Since Grassmann algebra (Chapter 60) is the appropriate language for these projective properties, these pure conditions C_G are polynomials in the Grassmann algebra. Section 60.4 contains several examples of these polynomial conditions.

THEOREM 61.1.28 Quadrics for Bipartite Graphs [Whi84b]

For a complete bipartite graph $K_{m,n}$ and $d > 1$, the framework $K_{m,n}(p)$, with $p(A)$ and $p(B)$ each affinely spanning d-space, is first-order flexible if and only if all the points $p(A \cup B)$ lie on a quadric surface of d-space (Figure 61.1.9).

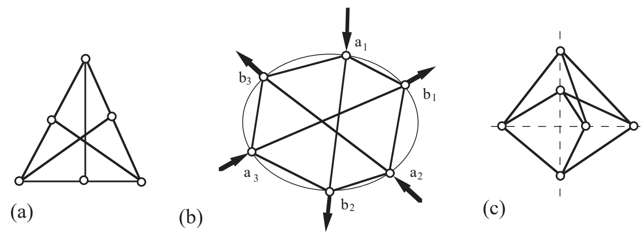

FIGURE 61.1.9
Three realizations of the complete bipartite graph $K_{3,3}$ which are (a) first-order rigid; (b) first-order flexible on a conic; and (c) flexible.

The following classical result describes an important open set of configurations that are regular for triangulated spheres [Whi84a].

THEOREM 61.1.29 Extended Cauchy Theorem

If $G(p)$ consists of the vertices and edges of a convex simplicial d-polytope, then $G(p)$ is first-order rigid in d-space.

If $G(p)$ consists of the vertices and edges of a strictly convex polyhedron in 3-space, then $G(p)$ is independent.

We recall that Steinitz's theorem guarantees that every 3-connected planar graph has a realization as the edges of a strictly convex polyhedron in 3-space, which gives Gluck's theorem. Connelly [Con78] gives a nonconvex (but not self-intersecting) triangulated sphere (with nine vertices) that is flexible. For many graphs, such as a triangulated torus (Theorem 61.1.15), we do not have even one specific configuration that gives a first-order rigid framework, only the guarantee that "almost all" configurations will work.

The Carpenter's Rule problem on straightening plane embedded polygonal paths and convexifying plane embedded polygons uses independence of appropriate bar frameworks (see Chapter 9).

There are further significant consequences of the underlying projective geometry of first-order rigidity [CW82]. The concepts of first-order rigidity and first-order flexibility, as well as the dual statics, can be expressed in any of the Cayley-Klein metrics that are extracted from the shared underlying projective space. This family includes the spherical metric, the hyperbolic metric, and others [SW12]. It is possible to express first-order rigidity in entirely projective terms that are essentially independent of the metric. In this way, the points "at infinity" in the Euclidean space can be fully integrated into first-order rigidity. For example, throughout Section 61.2.1 on Body-Bar and Body-Hinge structures, points at infinity can be included in the Cayley Algebra for bars, hinges, and centers of motion. However, in some metrics such as the hyperbolic metric, there is a singular set (the sphere at infinity, also known as the *absolute*) on which rigidity equations have distinct properties. This transfer goes back to Pogorelov and has been reworked in [SW07].

THEOREM 61.1.30 Transfer of Metrics

For a given graph G and a fixed point p in projective space of dimension d, the framework $G(p)$ is first-order rigid in Euclidean space if and only if $G(p)$ is first-order rigid in any alternate Cayley-Klein metric, with p not containing points on the absolute.

The most extreme projective transformation is polarity. The infinitesimal rigidity of specific body-bar and body-hinge frameworks are invariant under polarity (see Section 61.2.1).

In the plane, polarity switches points and lines of edges of a framework to lines and points of intersection [Whi89]. Surprisingly, this transformation turns plane infinitesimal motions into lifting motions of the lines into 3-space, which preserve the intersections — a result related to parallel drawings and liftings (Section 61.3). This transformation also takes tensegrity frameworks into weavings of lines in the plane. In both settings, the plane results correspond to interesting spatial results about when line configurations with specified intersections, or specific over-under patterns, can separate in 3-space or can only occur in flat (plane) formations.

61.2 RIGIDITY OF OTHER RELATED STRUCTURES

A number of related structures have also been investigated for first-order rigidity. In particular, body-bar, body-hinge, and molecular structures have important practical applications in engineering, robotics, and chemistry. For these structures, there exist basically complete combinatorial theories for all dimensions. Moreover, in mechanical and structural engineering it is common to analyze the rigidity and flexibility of pinned frameworks rather than "free-floating" frameworks. Finally, a new area of application of rigidity theory is the control of formations of autonomous agents. A notion of "directed rigidity" has recently been developed for this purpose.

61.2.1 BODY-BAR AND BODY-HINGE STRUCTURES

GLOSSARY

Body-bar framework in d-space $G(b)$: An undirected multigraph G (with no loops) and a bar-configuration $b : E \to \Lambda^2(\mathbb{R}^{d+1})$ that maps each edge $e = \{i, j\}$

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to the 2-extensor $\hat{p}_{e,i} \vee \hat{p}_{e,j}$ which indicates the Plücker coordinates of the bar connecting the point $p_{e,i}$ in the body i and the point $p_{e,j}$ in the body j. (For any $p \in \mathbb{R}^d$, $\hat{p}$ denotes the homogeneous coordinates of p, i.e., $\hat{p} = \begin{pmatrix} p \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1}$.) See also Chapter 60. Note that while an edge $\{i,j\}$ is an unordered pair, $\hat{p}_{e,i} \vee \hat{p}_{e,j}$ is ordered. For deciding whether $G(b)$ is first-order rigid, however, we just need the linear space spanned by $\hat{p}_{e,i} \vee \hat{p}_{e,j}$ and hence this ordering is irrelevant.

First-order flex of a body-bar framework $G(b)$***:*** A map $m : V \to \mathbb{R}^{\binom{d+1}{2}}$ satisfying $(m(i) - m(j)) \cdot b(e) = 0$ for all $e = \{i,j\} \in E$. A first-order flex m of $G(b)$ is ***trivial*** if $m(i) = m(j)$ for all $i, j \in V$ (i.e., if each vertex has the same "screw center" $m(i)$), and $G(b)$ is ***first-order rigid*** if every first-order flex of $G(b)$ is trivial.

Body-hinge framework in d-space $G(h)$: An undirected graph G and a hinge-configuration $b : E \to \Lambda^{d-1}(\mathbb{R}^{d+1})$ that maps each edge $e = \{i,j\}$ to the $(d-1)$-extensor $\hat{p}_{e,1} \vee \hat{p}_{e,2} \vee \cdots \vee \hat{p}_{e,d-1}$, which indicates the Plücker coordinates of a hinge, i.e., a $(d-1)$-dimensional simplex determined by the points $p_{e,1}, \ldots, p_{e,d-1}$ in the bodies of i and j.

First-order flex of a body-hinge framework $G(h)$***:*** A map $m : V \to \mathbb{R}^{\binom{d+1}{2}}$ satisfying $(m(i) - m(j)) \in \text{span}\{h(e)\}$ for all $e = \{i,j\} \in E$. A first-order flex m of $G(h)$ is ***trivial*** if $m(i) = m(j)$ for all $i, j \in V$, and $G(h)$ is ***first-order rigid*** if every first-order flex of $G(b)$ is trivial.

Panel-hinge framework: A body-hinge framework $G(h)$ with the property that for each $v \in V$, all of the $(d-2)$-dimensional affine subspaces $h(e)$ for the edges e incident to v are contained in a common $(d-1)$-dimensional affine subspace (i.e., a hyperplane).

Molecular framework: A body-hinge framework $G(h)$ with the property that for each $v \in V$, all of the hinge spaces $h(e)$ for the edges e incident to v contain a common point.

BODY-BAR FRAMEWORKS

Three-dimensional body-bar frameworks provide a useful model for many physical structures, linkages and robotic mechanisms. Tay established a combinatorial characterization of the multigraphs that are first-order rigid for almost all realizations as a body-bar framework in $\mathbb{R}^d$, for any dimension d [Tay84, WW87].

THEOREM 61.2.1 Tay's Theorem

For a multigraph G the following are equivalent:

(a) *for some line assignment of bars $b_{i,j}$ in d-space to the edges $\{i,j\}$ of G, the body-bar framework $G(b)$ is first-order rigid;*

(b) *for almost all bar configurations b, the body-bar framework $G(b)$ is first-order rigid;*

(c) *G contains $\binom{d+1}{2}$ edge-disjoint spanning trees.*

Condition (c) in Theorem 61.2.1 can be checked efficiently via a pebble game algorithm in $O(|V||E|)$ time. Note that a d-dimensional body-bar framework in

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which the attachment points of the bars affinely span d-space can be modeled as a bar-joint framework, with each body replaced by a first-order rigid bar-joint framework on these points. Therefore, Theorem 61.2.1 offers a fast combinatorial algorithm for the generically first-order rigid graphs in dimension $d \geq 3$ for the special class of body-bar frameworks. [WW87] offers some initial geometric criteria for the infinitesimal flexibility of generically isostatic multigraphs.

BODY-HINGE FRAMEWORKS

A body-hinge framework is a collection of rigid bodies, indexed by V, which are connected in pairs along hinges (lines in 3-space), indexed by edges of a graph. The bodies each move, preserving the contacts at the hinges. Of particular interest in applications are body-hinge frameworks in 3-space. Such structures could be modeled as bar-joint frameworks, with each hinge replaced by a pair of joints and each body replaced by a first-order rigid framework on the joints of its hinges (and other joints if desired). Since a hinge in 3-space removes 5 of the 6 relative degrees of freedom between a pair of rigid bodies, a 3-dimensional body-hinge framework can also be modeled as a body-bar framework by replacing each hinge with 5 independent bars, each intersecting the hinge line. It was shown by Tay and Whiteley in [TW84] that this special geometry of the bar configuration does not induce any additional first-order flexibility. The following result says that Tay's theorem regarding "generic" body-bar frameworks also applies to "generic" body-hinge frameworks [TW84].

THEOREM 61.2.2 Tay and Whiteley's Theorem

For a graph G the following are equivalent:

(a) *for some hinge assignment $h_{i,j}$ in d-space to the edges $\{i, j\}$ of G, the body-hinge framework $G(h)$ is first-order rigid;*

(b) *for almost all hinge assignments h, the body-hinge framework $G(h)$ is first-order rigid;*

(c) *if each edge of the graph is replaced by $\binom{d+1}{2}-1$ copies, the resulting multigraph contains $\binom{d+1}{2}$ edge-disjoint spanning trees.*

The following geometric result for body-hinge frameworks in 3-space was shown in [CW82] and extended to block and hole polyhedra in [FRW12].

THEOREM 61.2.3 Spherical Flexes and Stresses

Given an abstract spherical structure (see Section 61.3) $S = (V, F; \underline{E})$, *and an assignment of distinct points $p_i \in \mathbb{R}^3$ to the vertices, the following two conditions are equivalent:*

(a) *the bar framework $G(p)$ on $G = (V, E)$ has a nontrivial self-stress;*

(b) *the body-hinge framework on the dual graph $G^* = (F, E^*)$ with hinge lines $p_i p_j$ for each edge $\{i, j\}$ of G is first-order flexible.*

PANEL-HINGE AND MOLECULAR FRAMEWORKS

Panel-hinge and molecular frameworks are body-hinge frameworks with special types of hinge configurations. While 3-dimensional panel-hinge frameworks are common structures in engineering, the polar molecular frameworks provide a useful model for biomolecules, with the bodies and hinges representing the atoms and lines of the bond lines, respectively [Whi05]. Crapo and Whiteley showed in [CW82] that first-order rigidity is invariant under projective polarity.

Tay and Whiteley conjectured that the special geometry of the hinge configurations in panel-hinge frameworks (or in their projective duals) do not give rise to any added first-order flexibility [TW84]. This conjecture was confirmed by Katoh and Tanigawa [KT11]; the 2-dimensional case of this theorem was proven by Jackson and Jordán in [JJ08b].

THEOREM 61.2.4 Panel-Hinge and Molecular Theorem

For a graph G the following are equivalent:

(a) *G can be realized as a first-order rigid body-hinge framework in $\mathbb{R}^d$;*

(b) *G can be realized as a first-order rigid panel-hinge framework in $\mathbb{R}^d$;*

(c) *G can be realized as a first-order rigid molecular framework in $\mathbb{R}^d$;*

(d) *if each edge of G is replaced by $\binom{d+1}{2} - 1$ copies, the resulting multigraph contains $\binom{d+1}{2}$ edge-disjoint spanning trees.*

In the special case of dimension $d = 3$, this result says that if each body is realized with all hinges of each body concurrent in a single point (molecular structures), the "generic" rigidity is still measured by the existence of six spanning trees in the corresponding multigraph. The pebble game algorithm for checking this condition is implemented in several software packages (such as FIRST [JRKT01]) for analyzing protein flexibility [Whi05].

For $d = 3$ there is an equivalent result expressed in terms of bar-joint frameworks. The *square* of a graph G is obtained from G by joining each pair of vertices of distance two in G in the graph.

COROLLARY 61.2.5 Square Graphs

For a graph G, $5G$ contains 6 edge-disjoint spanning trees if and only if G^2 is first-order rigid in 3-space.

Recent work of [JJ07, JJ08a] derives some further corollaries. (1) G^2 is independent if and only if G^2 satisfies $|E'| \leq 3|V'| - 6$ for all vertex-induced subgraphs (V', E') of G^2 with $|V'| \geq 3$. This was conjectured by Jacobs, and one direction was previously known. (2) The Molecular Theorem implies that there exists an efficient algorithm to identify the maximal rigid subgraphs of a molecular bar and joint graph. Moreover, it was shown in [Jor12] that a 7-connected molecular graph (G^2) is generically rigid in 3-space.

61.2.2 FRAMEWORKS SUPPORTED ON SURFACES

Recent work has analyzed the rigidity of 3-dimensional bar frameworks whose joints are constrained to lie on a 2-dimensional irreducible algebraic variety embedded in $\mathbb{R}^3$, with a particular focus on classical surfaces such as spheres, cylinders and cones. Any first-order motion of such a framework must have the property that all of its velocity vectors are tangential to the surface. The following theorem summarizes the key combinatorial results for such structures. The proofs are based on Henneberg-type inductive constructions. We refer the reader to [NOP12, NOP14] for details.

THEOREM 61.2.6 Frameworks on Surfaces

Let $G = (V, E)$ be a graph, and let $\mathcal{M}$ be an irreducible algebraic surface in $\mathbb{R}^3$. Further, let $k \in \{1, 2\}$ be the dimension of the group of Euclidean isometries supported by $\mathcal{M}$. Then almost all realizations $G(p)$ of G with $p : V \to \mathcal{M}$ are isostatic on $\mathcal{M}$ if and only if G is a complete graph on $1, 2, 3$ or 4 vertices, or satisfies $|E| = 2|V| - k$ and for every non-empty subgraph (V', E'), $|E'| \le 2|V'| - k$.

Note that the small complete graphs are special cases, as they are considered isostatic in this context, even though their realizations may have tangential first-order flexes which are not tangential isometries of the surface. Theorem 61.2.6 takes care of the cases where $\mathcal{M}$ is a circular cylinder ($k = 2$) or a circular cone, torus, or surface of revolution ($k = 1$), for example. For $k = 3$, this is the transfer of Laman's Theorem to the sphere (or concentric spheres) [SW12]. Compare to Table 61.2.1. It is not yet known how to extend these results to surfaces with $k = 0$.

61.2.3 FRAMEWORKS IN NON-EUCLIDEAN NORMED SPACES

Another recent research strand is the development of a rigidity theory for a general non-Euclidean normed space, such as $\mathbb{R}^d$ equipped with an ℓ^q norm $\|\cdot\|_q$, or a polyhedral norm $\|\cdot\|_{\mathcal{P}}$, where the unit ball is a polyhedron $\mathcal{P}$. The mathematical foundation for this theory, as well as basic tools and methods, were established in [Kit15, KP14, KS15]. For the ℓ^q norms, where $1 \le q \le \infty$, $q \ne 2$ the following analogue of Laman's theorem was proved in [KP14].

THEOREM 61.2.7 Rigid graphs in $(\mathbb{R}^2, \|\cdot\|_q)$, where $1 < q < \infty$, $q \ne 2$

Let G be a graph and q be an integer with $1 < q < \infty$, $q \ne 2$. Almost all realizations $G(p)$ of G in $(\mathbb{R}^2, \|\cdot\|_q)$ are isostatic if and only if G satisfies $|E| = 2|V| - 2$ and for every non-empty subgraph (V', E'), $|E'| \le 2|V'| - 2$.

Since the polyhedral norms (such as the well-known ℓ^1 or ℓ^∞ norms) are not smooth, some new features emerge in the rigidity theory for these norms. For example, the set of regular realizations of a graph G is no longer dense. Moreover, there exist configurations p, where the 'rigidity map' which records the edge lengths of the framework $G(p)$ is not differentiable. For a space $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$, where d is an arbitrary dimension, the first-order rigidity of frameworks may be characterized in terms of edge-colourings of G which are induced by the placement of each bar relative to the facets of the unit ball (see [Kit15, KP14]).

For $d = 2$, there also exists an analogue of Laman's theorem for polyhedral norms [Kit15, KP14]. A framework $G(p)$ in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ is *well-positioned* if the rigidity map is differentiable at p. This is the case if and only if $p_i - p_j$ is contained in the interior of the conical hull of some facet of the unit ball, for each edge $\{i, j\}$.

THEOREM 61.2.8 Rigid graphs in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$

Let G be a graph and $\|\cdot\|_{\mathcal{P}}$ be a polyhedral norm in $\mathbb{R}^2$. There exists a well-positioned isostatic framework $G(p)$ in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$ if and only if G satisfies $|E| = 2|V| - 2$ and for every non-empty subgraph (V', E'), $|E'| \le 2|V'| - 2$.

Theorems 61.2.7 and 61.2.8 have not yet been extended to higher dimensions.

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61.2.4 PINNED GRAPHS AND DIRECTED GRAPHS

GLOSSARY

Pinned graph: A simple graph $G = (V_I \cup V_P, E)$ whose vertex set is partitioned into the sets V_I and V_P, and whose edge set E has the property that each edge in E is incident to at least one vertex in V_I. The vertices in V_I are called ***inner*** and the vertices in V_P are called ***pinned***.

Pinned framework in d-space: A triple (G, p, P), where $G = (V_I \cup V_P, E)$ is a pinned graph, and $p : V_I \to \mathbb{R}^d$ and $P : V_P \to \mathbb{R}^d$ are configurations of points in d-space.

First-order rigid pinned framework: A pinned framework (G, p, P) whose pinned rigidity matrix $R_G(p, P)$ has rank $d|V_I|$, where $R_G(p, P)$ is obtained from the standard rigidity matrix of $G(p, P)$ by removing the columns corresponding to the vertices in V_P. Further, (G, p, P) is ***independent*** if the rows of $R_G(p, P)$ are independent, and ***isostatic*** if it is first-order rigid and independent.

Pinned d-isostatic graph: A pinned graph $G = (V_I \cup V_P, E)$ for which some (equivalently, almost all) realizations (G, p, P) in $\mathbb{R}^d$ are isostatic.

Pinned graph composition: For two pinned graphs $G = (V_I \cup V_P, E)$ and $H = (W_I \cup W_P, F)$, and an injective map $c : V_P \to W_I \cup W_P$, the pinned graph $C(G, H)$ obtained from G and H by identifying the pinned vertices V_P of G with their images $c(V_P)$.

d-Assur graph: A d-isostatic pinned graph $G = (V_I \cup V_P, E)$ which is minimal in the sense that for all proper subsets of vertices $V_{I'} \cup V_{P'}$, where $V_{I'} \neq \emptyset$, $V_{I'} \cup V_{P'}$ induces a pinned subgraph $G' = (V_{I'} \cup V_{P'}, E')$ with $|E'| \leq d|V_{I'}| - 1$.

Orientation: For a simple graph $G = (V, E)$, an assignment of a direction (ordering) to each edge in E. The resulting structure is called a ***directed graph***.

d-directed orientation: For a pinned graph G, an orientation of G such that every inner vertex has out-degree exactly d and every pinned vertex has out-degree exactly 0.

Strongly connected: A directed graph G is called strongly connected if and only if for any two vertices i and j in G, there is a directed path from i to j and from j to i. The ***strongly connected components*** of G are the strongly connected subgraphs which are maximal in the sense that a subgraph cannot be enlarged to another strongly connected subgraph by including additional vertices and its incident edges.

Formation in d-space $G(p)$: A directed graph $G = (V, E)$ (no loops or multiple edges) and a configuration p in d-space for the vertices V.

Persistent formation: A formation $G(p)$ with the property that there exists an open neighborhood $N_\epsilon(p)$ of p so that every configuration q in $N_\epsilon(p)$ with the distance set defined by $d_{ij} = |p_i - p_j|$ is congruent to p.

Generically persistent graph in d-space: A directed graph G for which the formations $G(p)$ are persistent for almost all configurations p in d-space.

Minimally persistent graph: A directed graph which is generically persistent and has the property that the removal of any edge yields a directed graph which is not generically persistent.

Leader-follower graph: A directed graph $G = (V, E)$ which has a vertex v_0 of out-degree 0 (called the *leader*), a vertex v_1 of out-degree 1 (called the *follower*), and $(v_1, v_0) \in E$.

FIRST ORDER RIGIDITY OF PINNED GRAPHS

We have necessary conditions for a pinned graph to be d-isostatic, as well as necessary and sufficient conditions in the plane. These are presented in Table 61.2.1. For pinned frameworks in the plane with generic inner joints, the pins only need to be in general position for isostaticity.

TABLE 61.2.1

STRUCTURE	NECESSARY COUNTS IN d-SPACE	CHARACTERIZATION IN THE PLANE
bar-joint (61.1.2)	$\lvert E\rvert = d\lvert V\rvert - \binom{d+1}{2}$, $\lvert V\rvert \geq d$ $\lvert E'\rvert \leq d\lvert V'\rvert - \binom{d+1}{2}$, $\lvert V'\rvert \geq d$	$\lvert E\rvert = 2\lvert V\rvert - 3$, $\lvert V\rvert \geq 2$ $\lvert E'\rvert \leq 2\lvert V'\rvert - 3$, $\lvert V'\rvert \geq 2$
pinned bar-joint (61.2.4)	$\lvert E\rvert = d\lvert V_I\rvert$ $\lvert E'\rvert \leq d\lvert V_{I'}\rvert$, $\lvert V_{P'}\rvert \geq d$ $\lvert E'\rvert \leq d\lvert V_{I'}\rvert - \binom{d+1-k}{2}$, $\lvert V_{P'}\rvert = k$, $0 \leq k < d$	$\lvert E\rvert = 2\lvert V_I\rvert$ $\lvert E'\rvert \leq 2\lvert V_{I'}\rvert$, $\lvert V_{P'}\rvert \geq 2$ $\lvert E'\rvert \leq 2\lvert V_{I'}\rvert - \binom{3-k}{2}$, $\lvert V_{P'}\rvert = k$, $0 \leq k < 2$

For dimension $d \geq 3$, the conditions in Table 61.2.1 are not sufficient for G to be pinned d-isostatic. The bar-joint frameworks in 3-space (also referred to as "floating frameworks" in the engineering community) provide the core counterexamples (see Figure 61.1.4).

Where the counts give necessary and sufficient conditions ($d = 2$ in Table 61.2.1, and parallel drawing for $d \geq 2$ in Table 61.3.1), the counts alone define a matroid ([Whi96] Appendix A, [Whi88b]). Similarly, the counts for body-bar frameworks in Section 61.2.1 define a matroid. For $d \geq 3$, the counts are no longer enough to define the full independence structure of a matroid [Whi96].

ASSUR DECOMPOSITION OF PINNED GRAPHS

An important and widely used method in the analysis and synthesis of mechanisms is the decomposition of mechanical linkages into fundamental minimal components, each of which permits a simplified analysis. This layered approach for analysing mechanisms was originally developed by Assur and has now been reworked using rigidity theory. A mechanical linkage is converted into a pinned isostatic framework (G, p, P) in $\mathbb{R}^d$ by (i) transforming it into a flexible pinned framework with a designated driver (e.g., a piston changing the length between a pair of vertices) and (ii) replacing the driver by a rigid bar. The resulting d-isostatic pinned graph G decomposes uniquely into a partially ordered set of minimal pinned isostatic components (Assur graphs). G is reconstructed from these components by repeated pinned graph compositions (following the partial order) [SSW10a, SSW13].

This d-Assur decomposition of G may be algorithmically obtained as follows [SSW13]. Find a d-directed orientation of the d-isostatic pinned graph by applying the $d|V|$ pebble-game algorithm to G [LS08]. In this directed graph, we condense all pinned vertices into a single pinned vertex (making it into a sink), and then determine the strongly connected components of this graph (via the $O(|E|)$ time Tarjan's algorithm). When we extend the strongly connected components by the outgoing edges from the component, we obtain the d-Assur decomposition. The partial order is the acyclic directed graph obtained from contracting each strongly connected component to a single vertex, and discarding any multiple edges that may arise (Figure 61.2.1(c,d)).

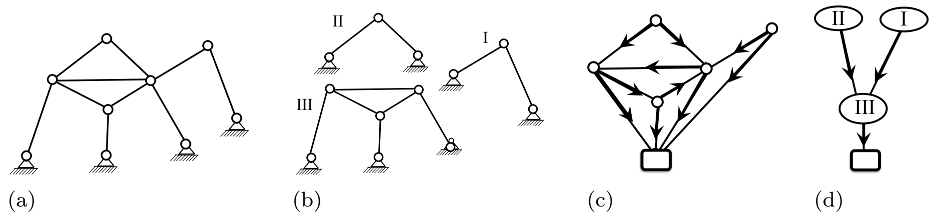

FIGURE 61.2.1
A pinned 2-isostatic graph G (a) and its 2-Assur decomposition (b). The directed graph in (c) is obtained from a 2-directed orientation on G by contracting all pinned vertices to a single vertex. (d) presents the partial order of the strongly connected components.

THEOREM 61.2.9 d-Assur Decomposition [SSW13]

Given a pinned d-isostatic graph G, there is a d-directed orientation of G. With any such d-directed orientation, the following decompositions are equivalent:

(a) *the d-Assur decomposition of G;*

(b) *the strongly connected decomposition into extended components associated with the d-directed orientation (with all pinned vertices identified);*

(c) *the block-triangular decomposition of the pinned rigidity matrix into a maximal number of components for some linear order extending the partial order of (i) or equivalently (ii).*

THEOREM 61.2.10 Plane Assur Graphs [SSW10a]

Let $G = (V_I \cup V_P, E)$ be a pinned isostatic graph. Then the following are equivalent:

(a) *G is 2-Assur;*

(b) *if the set V_P is contracted to a single vertex, inducing the unpinned graph (or multigraph) G^* with edge set E, then G^* is a generic 2-circuit;*

(c) *either G has a single inner vertex of degree 2 or each time we delete a vertex, the resulting pinned graph has a non-zero motion of all inner vertices (in generic position);*

(d) *deletion of any edge from G results in a pinned graph that has a non-zero motion of all inner vertices (in generic position).*

The purely combinatorial condition (b) can be checked by fast pebble-game algorithms. It also follows from (b) that all 2-Assur graphs on at least 5 vertices can be obtained from basic Assur graphs by a sequence of simple graph moves (such as edge splitting) and pin-rearrangements. This is a corollary of the inductive construction of plane circuits in [BJ02]. Such inductive constructions allow engineers to easily generate basic building blocks for synthesizing new linkages. Condition (c) allows a quick check of the 2-Assur property for smaller graphs, and (d) guarantees that a driver inserted for an arbitrary edge will (generically) put all inner vertices into motion, as will putting one pin into motion.

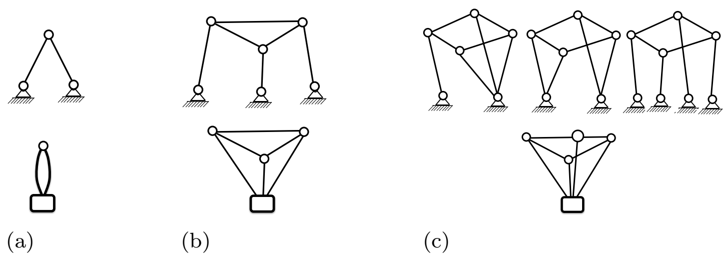

FIGURE 61.2.2
2-Assur graphs and their corresponding generic 2-circuits. One circuit can correspond to a family of Assur graphs (c).

Alternatively, 2-Assur graphs can be characterized geometrically, based on special singular realizations. See [SSW10b] for details.

For dimension $d \geq 3$, a good combinatorial characterization of d-Assur graphs is not known. The fundamental property of 2-Assur graphs given in Theorem 61.2.10 does not, in general, hold for d-Assur graphs where $d \geq 3$ (see [SSW13]). The decomposition of a structure into "Assur components" has been extended to bar frameworks with symmetries [NSSW14] and to tensegrity frameworks [STB$^+$09].

CONTROL OF FORMATIONS

Another application of rigidity theory is the distributed control of groups of mobile robots that enable them to "hold formation" in the absence of a centralised control or global coordinate system (e.g., systems moving indoors, underwater or in other GPS-denied environments). See [AYFH08, OPA15] for recent surveys on this topic. Rigidity-based methods are just made for this task, as they allow the robust control using only distance (or angle) measurements. The task of sensing and maintaining a prescribed distance between a pair of agents may either be shared by both agents or assigned to a single designated agent. The first case is modelled via an undirected graph, and techniques from rigidity theory can be applied directly. For the second case, we assign an orientation to each edge, where the edge (i, j) directed from i to j indicates that agent i is responsible for maintaining its distance to agent j. An adapted "directed rigidity" (or "persistence") theory is required to study the control of formations of this type (see [HADB07]).

Intuitively, a formation is persistent if, provided that each agent is trying to satisfy all the distance constraints for which it is responsible, all the agents can in fact accomplish this task, and, consequently, the shape of the entire formation is preserved as the formation is moving continuously. In the following, we restrict attention to formations in $\mathbb{R}^2$, and we say that a formation $G(p)$ (or directed graph G) is rigid if the bar framework (undirected graph) obtained from $G(p)$ (G) by

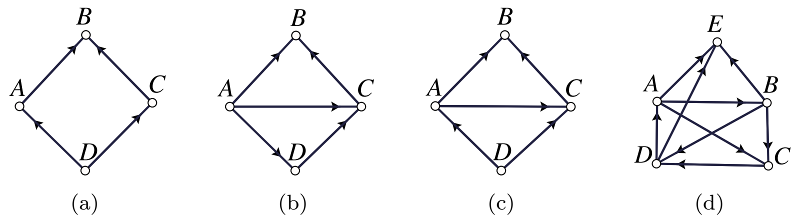

FIGURE 61.2.3
Formations in $\mathbb{R}^2$: (a) is not rigid and hence not persistent; (b) is rigid, but not persistent, since A may be unable to satisfy its distance constraints; (c) is minimally persistent; (d) is rigid but not persistent by [BJ08] due to the K_4 subgraph induced by A, B, C, D.

removing the orientation of the edges is (generically) 2-rigid. Persistence beyond minimal persistence has some advantages in situations where some directed links may be broken by obstacles or lost data.

THEOREM 61.2.11 Combin. Characterization of Persistence in 2D [HADB07]
A directed graph is generically persistent if and only if all those subgraphs, obtained by removing outgoing edges from vertices with out-degree larger than 2 *until all the vertices have out-degree smaller than or equal to 2, are rigid.*

Theorem 61.2.11 readily provides a (non-polynomial) algorithm to check the persistence of a directed graph. It remains open whether persistence can also be checked in polynomial time. For special classes, efficient algorithms are known [HADB07, BJ08].

Clearly, rigidity is a necessary, but not sufficient condition for persistence. It is an open problem to decide whether all generically rigid graphs have persistent orientations. However, affirmative answers exist for various classes of graphs, including wheels, power graphs of cycles, complete graphs, generically minimally rigid graphs, and generically rigid graphs with $|E| = 2|V| - 2$ [HADB07, BJ08].

The directed graphs that are both generically minimally rigid and persistent are precisely the minimally persistent graphs [HADB07]. Further, a directed graph is minimally persistent if and only if it is generically minimally rigid and no vertex has more than two outgoing edges. For leader follower formations, this orientation can be efficiently found by applying the pebble-game algorithm and drawing pebbles to the leader-follower edge, as the ground. However, an additional cycle-reversal operation is needed to sequentially build *all* minimally persistent directed graphs.

Real-life scenarios often require formations to be reconfigured due to changes in the environment, the presence of obstacles along the paths of agents, or unexpected losses of agents, for example. For this, various types of splitting, merging and closing ranks operations have been developed for formations. For formations modeled by *undirected* graphs, we refer the reader to [EAM$^+$04, OM02]. For *directed* graphs, initial results can be found in [HYFA08]. With leader-follower formations, Assur decompositions also provide an additional tool.

Recent work confirms that many of the results for generically persistent graphs extend to 3- and higher-dimensional spaces [YHF$^+$07]. While the concept of persistence is applicable in dimension $d \geq 3$, it is not always sufficient to guarantee the desired stability of the formation [YHF$^+$07]. A natural question is the extension of a persistence theory to formations based on other types of geometric constraint

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systems, such as 2- or higher-dimensional formations consisting of points, lines or other rigid building blocks, linked by (possibly mixed) angle, direction (bearings) and distance constraints. See the references in [OPA15]) and Section 61.3.

61.2.5 TENSEGRITY FRAMEWORKS

In a tensegrity framework, we replace some (or all) of the equalities for bars with inequalities for the distances—corresponding to *cables* (the distance can shrink but not expand) and *struts* (the distance can expand but not shrink). The study of these inequalities introduces techniques from linear programming. Further results for tensegrity frameworks appear in Chapter 63.

GLOSSARY

Signed graph: A graph with a partition of the edges into three classes, written $G_{\pm} = (V; E_-, E_0, E_+)$.

Tensegrity framework $G_{\pm}(p)$ in $\mathbb{R}^d$: A signed graph $G_{\pm} = (V; E_-, E_0, E_+)$ and a configuration p on V.

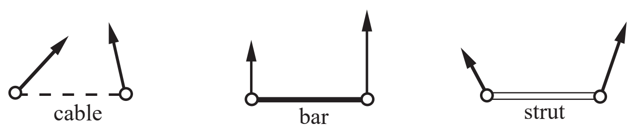

FIGURE 61.2.4
A tensegrity framework has (a) cables that can get shorter; (b) bars that cannot change length, and (c) struts that can get longer.

Cables, bars, struts: For a tensegrity framework $G_{\pm}(p)$, the members of E_-, of E_0, and of E_+, respectively. In figures, cables are indicated by dashed lines, struts by double thin lines, and bars by single thick lines (see Figure 61.2.4).

$G_{\pm}(p)$ ***dominates*** $G_{\pm}(q)$***:*** For each edge, the appropriate condition holds:

$$\begin{aligned} |p_i - p_j| &\geq |q_i - q_j| \quad \text{when} \quad \{i,j\} \in E_- \\ |p_i - p_j| &= |q_i - q_j| \quad \text{when} \quad \{i,j\} \in E_0 \\ |p_i - p_j| &\leq |q_i - q_j| \quad \text{when} \quad \{i,j\} \in E_+. \end{aligned}$$

Rigid tensegrity framework $G_{\pm}(p)$***:*** For every analytic path $p(t)$ in $\mathbb{R}^{vd}$, $0 \leq t < 1$, if $p(0) = p$ and $G(p)$ dominates $G(p(t))$ for all t, then p is congruent to $p(t)$ for all t.

First-order flex of a tensegrity framework $G_{\pm}$: An assignment $p' : V \to \mathbb{R}^d$ of velocities to the vertices such that, for each edge $\{i,j\} \in E$ (Figure 61.2.4),

$$\begin{aligned} (p_j - p_i) \cdot (p'_j - p'_i) &\leq 0 \quad \text{for cables} \quad \{i,j\} \in E_- \\ (p_j - p_i) \cdot (p'_j - p'_i) &= 0 \quad \text{for bars} \quad \{i,j\} \in E_0 \\ (p_j - p_i) \cdot (p'_j - p'_i) &\geq 0 \quad \text{for struts} \quad \{i,j\} \in E_+. \end{aligned}$$

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Trivial first-order flex: A first-order flex p' of a tensegrity framework $G_{\pm}(p)$ such that $p'_i = Sp_i + t$ for all vertices i, with a fixed skew-symmetric matrix S and vector t.

First-order rigid: A tensegrity framework $G_{\pm}(p)$ is first-order rigid if every first-order flex is trivial, and ***first-order flexible*** otherwise.

Proper self-stress on a tensegrity framework $G_{\pm}(p)$: An assignment ω of scalars to the edges of G such that:

(a) $\omega_{ij} \geq 0$ for cables $\{i,j\} \in E_-$;

(b) $\omega_{ij} \leq 0$ for struts $\{i,j\} \in E_+$; and

(c) for each vertex i, $\sum_{\{j \mid \{i,j\} \in E\}} \omega_{ij}(p_j - p_i) = 0$.

Strict self-stress: A proper self-stress ω with strict inequalities in (a) and (b).

Underlying bar framework: For a tensegrity framework $G_{\pm}(p)$, the bar framework $G(p)$ on the unsigned graph $G = (V,E)$, where $E = E_- \cup E_0 \cup E_-$ (Figure 61.2.5(a,b)).

BASIC RESULTS

The equivalent definitions of "rigidity" and the basic connections between rigidity and first-order rigidity all transfer directly to tensegrity frameworks [RW81].

THEOREM 61.2.12 First-Order Stress Test

A tensegrity framework $G_{\pm}(p)$ is first-order rigid if and only if the underlying bar framework $G(p)$ is first-order rigid and there is a strict self-stress on $G_{\pm}(p)$ (Figure 61.2.5 (a,b)).

This connection to self-stresses means that any first-order rigid tensegrity framework with at least one cable or strut has $|E| > d|V| - \binom{d+1}{2}$ edges.

THEOREM 61.2.13 Reversal Theorem

A tensegrity framework $G_{\pm}(p)$ is first-order rigid if and only if the reversed framework $G^r_{\pm}(p)$ is first-order rigid, where the graph $G^r_{\pm}$ interchanges cables and struts (Figure 61.2.5 (a,c)).

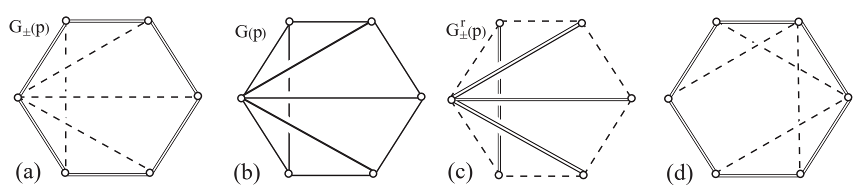

FIGURE 61.2.5

The tensegrity framework (a) is built with a proper stress on the first-order rigid bar framework (b). Reversing the stress (c) is also first-order rigid. (d) is another tensegrity polygon which is also first-order rigid.

There is no single "generic" behavior for a signed graph $G_{\pm}$. If some configuration produces a first-order rigid framework for a graph $G_{\pm}$, then the set of all such configurations is open but not dense. The algebraic variety of "special positions" of the underlying unsigned graph divides the configuration space into open subsets, in some of which all configurations are rigid, and in others, none are. The required sign pattern for a self-stress can change as you cross such a boundary [WW83].

The first-order rigidity of a tensegrity framework is projectively invariant, with the proviso that a cable (strut) $\{i, j\}$ is switched to a strut (cable) whenever $\lambda_i \lambda_j < 0$ for the projective transformation.

THEOREM 61.2.14 Stress Existence

If a tensegrity framework $G_{\pm}(p)$ with at least one cable or strut is rigid, then there is a nonzero proper self-stress.

A number of results relate minima of quadratic energy functions to the rigidity of tensegrity frameworks. These energy results are not invariant under projective transformations, but such rigidity is preserved under "small" affine transformations. This is one result, drawn from results on second-order rigidity [CW96].

THEOREM 61.2.15 Rigidity Stress Test

A tensegrity framework $G_{\pm}(p)$ is rigid if, for each nontrivial first-order motion p' of $G_{\pm}(p)$, there is a proper self-stress $\omega^{p'}$ making $\sum_{ij} \omega_{ij}^{p'}(p'_i - p'_j) \cdot (p'_i - p'_j) > 0$.

61.3 SCENE ANALYSIS

The problem of reconstructing spatial objects (polyhedra or polyhedral surfaces) from a single plane picture is basic to several applications. This section summarizes the combinatorial results for "generic pictures" (Section 61.3.1). Section 61.3.2 presents a polar "parallel configurations" interpretation of the same abstract mathematics and Section 61.3.3 presents connections to other fields of discrete geometry. In addition, liftings and scene analysis have strong connections to the recent work on 'affine rigidity' [GGLT13].

Within approximation theory, the study of multivariate splines considers surfaces which are piecewise polynomial of bounded degree, extending the polyhedral scenes which are piecewise linear (see Chapter 56). There are many analogies between the theory of rigidity and the theory of multivariate splines at the level of matroidal combinatorics and geometry which are described in [Whi96, Whi98].

61.3.1 COMBINATORICS OF PLANE POLYHEDRAL PICTURES

GLOSSARY

Polyhedral incidence structure S: An abstract set of ***vertices*** V, an abstract set of ***faces*** F and a set of ***incidences*** $I \subset V \times F$.

d-scene for an incidence structure $S = (V, F; I)$: A pair of location maps, $p : V \to \mathbb{R}^d$, $p_i = (x_i, \ldots, z_i, w_i)$ and $P : F \to \mathbb{R}^d$, $P^j = (A^j, \ldots, C^j, D^j)$, such that, for each $(i, j) \in I$: $A^j x_i + \ldots + C^j z_i + w_i + D^j = 0$. (We assume that no hyperplane is vertical, i.e., is parallel to the vector $(0, 0, \ldots, 0, 1)$.)

(d−1)-picture of an incidence structure S: A location map $r : V \to \mathbb{R}^{d-1}$, $r_i = (x_i, \ldots, z_i)$ (Figure 61.3.1(a)).

Lifting of a $(d-1)$-picture $S(r)$: A d-scene $S(p, P)$ with vertical projection $\Pi(p) = r$ (Figure 61.3.1(b)). (I.e., if $p_i = (x_i, \ldots, z_i, w_i)$, then $r_i = (x_i, \ldots, z_i) = \Pi(p_i)$).

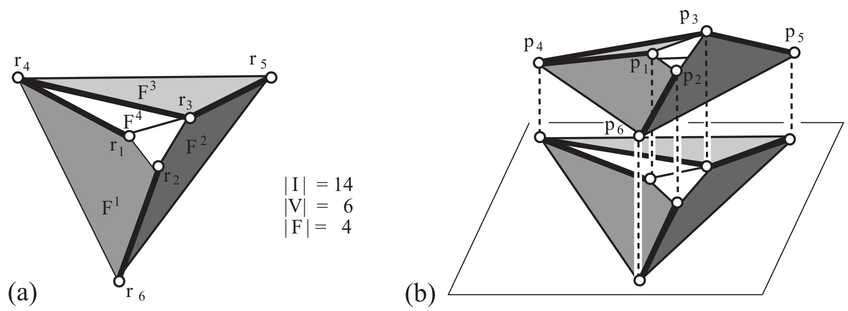

FIGURE 61.3.1
A generic polyhedral picture and an associated sharp scene.

Lifting matrix for a picture $S(r)$: The $|I| \times (|V| + d|F|)$ coefficient matrix $M_S(r)$ of the system of equations for liftings of a picture $S(r)$: for each $(i, j) \in I$, $A^j x_i + \ldots + C^j z_i + w_i + D^j = 0$, where the variables are ordered:

$$\ldots, w_i, \ldots \; ; \; \ldots, A^j, \ldots, C^j, D^j, \ldots.$$

Sharp picture: A $(d-1)$-picture $S(r)$ that has a lifting $S(p, P)$ with a distinct hyperplane for each face (Figure 61.3.1(a,b)).

BASIC RESULTS

Since the incidence equations are linear, there is no distinction between "continuous liftings" and "first-order liftings." Since the rank of the lifting matrix is determined by a polynomial process on the entries, "generic properties" of pictures have several characterizations. These were conjectured by Sugihara and proven in [Whi88a]. The larger overview of the problems can be found in [Sug86].

We find the usual correspondence of one generic sharp picture and all generic realizations being sharp.

The generic properties of a structure are robust: all small changes in such a sharp picture are also sharp pictures and small changes in the points of a sharp picture require only small changes in the sharp lifting. Even special positions of such structures will always have nontrivial liftings, although these may not be sharp. However, up to numerical round-off, all pictures "are generic." Other structures that are not generically sharp (Figure 61.3.2(a)) may have sharp pictures in special positions (Figure 61.3.2(b)), but a small change in the position of even one point can destroy this sharpness.

The incidence equations allow certain "trivial" changes to a lifted scene that will preserve the picture—generated by adding a single plane H^0 to all existing planes: $P^j_* = H^0 + P^j$; and by changes in vertical scale in the scene: $w^*_i = \lambda w_i$. This space of ***lifting equivalences*** has dimension $d+1$, provided the points of the

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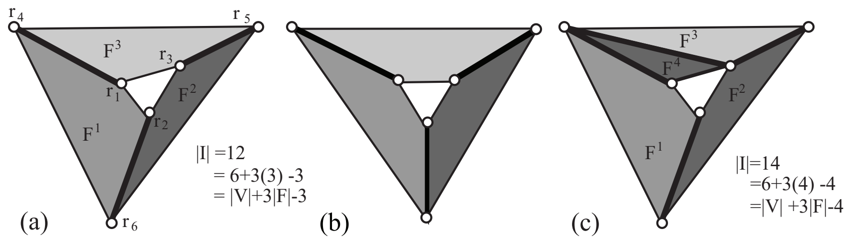

FIGURE 61.3.2
The generic picture (a) has only trivial liftings, while the special position (b) has sharp liftings. The modified picture (c) has sharp liftings (see Figure 61.3.1(b)).

scene do not lie in a single hyperplane. These results are connected to entries in Table 61.2.1.

THEOREM 61.3.1 Picture Theorem

A generic picture of an incidence structure $S = (V, F; I)$ with at least two faces has a sharp lifting, unique up to lifting equivalence, if and only if $|I| = |V| + d|F| - (d+1)$ and, for all subsets I' of incidences on at least two faces, $|I'| \leq |V'| + d|F'| - (d+1)$ (Figure 61.3.2(a,c)).

A generic picture of an incidence structure $S = (V, F; I)$ has independent rows in the lifting matrix if and only if for all nonempty subsets I' of incidences, $|I'| \leq |V'| + d|F'| - d$ (Figure 61.3.2(a)).

ALGORITHMS

Any part of a structure with $|I'| = |V'| + d|F'| - d$ independent incidences will be forced to be coplanar over a picture with algebraically independent coordinates for the points. If the structure is not generically sharp, then an effective, robust lifting algorithm consists of selecting a subset of vertices for which the incidences are sharp, then "correcting" the position of the other vertices based on calculations in the resulting scene. This requires effective algorithms for selecting such a set of incidences. Sugihara and Imai have implemented $O(|I|^2)$ time algorithms for finding maximal generically sharp (independent) structures using modified bipartite matching on the incidence structure [Sug86].

61.3.2 PARALLEL DRAWINGS

The mathematical structure defined for polyhedral pictures has another, dual interpretation: the polar of a "point constrained by one projection" is a "hyperplane constrained by an assigned normal." Two configurations sharing the prescribed normals are "parallel drawings" of one another [Whi88a]. These geometric patterns, used by engineering draftsmen in the nineteenth century, have reappeared in a number of branches of discrete geometry. This dual interpretation also establishes a basic connection between the geometry and combinatorics of scene analysis and the geometry and combinatorics of first-order rigidity of frameworks.

GLOSSARY

Parallel d-scenes for an incidence structure: Two d-scenes $S(p,P)$, $S(q,Q)$ such that for each face j, $P^j||Q^j$ (that is, the first $d-1$ coordinates are equal) (Figure 61.3.3). (For convenience, not necessity, we stick with the "nonvertical" scenes of the previous section.)

Nontrivially parallel d-scene for a d-scene $S(p,P)$: A parallel d-scene $S(q,Q)$, such that the configuration q is not a translation or dilatation of the configuration p (Figure 61.3.3 for $d=2$).

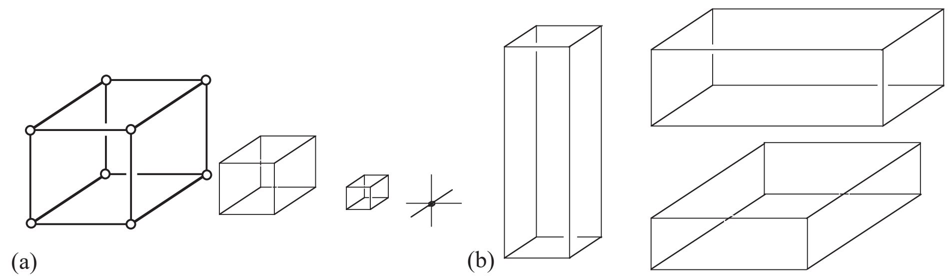

FIGURE 61.3.3
The cube graph has trivial parallel drawings (a), and a space of non-trivial drawings (b).

Directions for the faces: An assignment of d-vectors $D^j = (A^j, \dots, C^j)$ to $j \in F$.

d-scene realizing directions D: A d-scene $S(p,P)$ such that for each face $j \in F$, the first $d-1$ coordinates of P^j and D^j coincide.

Parallel drawing matrix for directions D in d-space: The $|I| \times (|V| + d|F|)$ matrix $M_S(D)$ for the system of equations for each incidence $(i,j) \in I$: $A^j x_i + B^j y_i + \dots + C^j z_i + w_i + D^j = 0$, where the variables are ordered:

$$\dots, D^j, \dots \; ; \; \dots, x_i, y_i, \dots, z_i, w_i, \dots .$$

DL***-framework:*** A *mixed graph* $G_{DL} = (V; D, L)$ *with two classes of edges (not necessarily disjoint)* D *(for directions) and* L *(for lengths) and a configuration* $p : V \to \mathbb{R}^2$. *A first-order flex is a map* $p' : V \to \mathbb{R}^2$ *such that (i) for an edge in* D: $(p_i - p_j)^\perp \cdot (p'_i - p'_j) = 0$ *and (ii) for an edge in* L: $(p_i - p_j) \cdot (p'_i - p'_j) = 0$. *The trivial first-order flexes are translations, and a framework* $G_{DL}(p)$ *is* ***tight*** *if all first-order flexes are trivial (Figure 61.3.5). In* $\mathbb{R}^d$ *we have a multi-graph where* D *can contain up to* $d-1$ *copies of a pair and* $(p_i - p_j)^\perp$ *represents a vector selected from the* $(d-1)$*-dimensional space of normals to the vector* $(p_i - p_j)$.

BASIC RESULTS

All results for polyhedral pictures dualize to parallel drawings. Again, for parallel drawings there is no distinction between continuous changes and first-order changes. The trivially parallel drawings, generated by d translations and one dilation towards a point, form a vector space of dimension $d+1$, provided there are at least two distinct points (Figure 61.3.3(a)). (A trivially parallel drawing may even have all points coincident, though the faces will still have assigned directions (Figure 61.3.3(a)).)

THEOREM 61.3.2 Parallel Drawing Theorem for Scenes

For generic selections of the directions D in d-space for the faces, a structure $S = (V, F; I)$ has a realization $S(p, P)$ with all points p distinct if and only if, for every nonempty set I' of incidences involving at least two points $V(I')$ and faces $F(I')$, $|I'| \le d|V(I')| + |F(I')| - (d+1)$ (Figure 61.3.3(a)).

In particular, a configuration p, P with distinct points realizing generic directions for the incidence structure is unique, up to translation and dilatation, if and only if $|I| = d|V| + |F| - (d+1)$ and $|I'| \le d|V'| + |F'| - (d+1)$.

Of course other nontrivially parallel drawings will also occur if the rank is smaller than $d|V'| + |F'| - (d+1)$ (Figure 61.3.3(b), with a generic rank 1 less than required for $d = 2$, and a geometric rank, as drawn, 2 less than required).

Figure 61.3.3 may also be interpreted as the parallel drawings of a "cube in 3-space." For spherical polyhedra, there is an isomorphism between the nontrivially parallel drawings in 3-space (the parallel drawings modulo the trivial drawings) and the nontrivially parallel drawings in a plane projection [CW94]. Only the dimension (4 vs. 3) of the trivially parallel drawings will change with the projection.

61.3.3 CONNECTIONS TO OTHER FIELDS

FIRST-ORDER MOTIONS AND PARALLEL DRAWINGS

For any plane framework, if we turn the vectors of a first-order motion 90° (say clockwise), they become the vectors joining p to a parallel drawing q of the framework (Figure 61.3.4(a,b)). The converse is also true: a result that is folklore in the structural engineering community.

THEOREM 61.3.3 Parallel Drawing Test for Plane First-order Flexes

A plane framework $G(p)$ has a nontrivial first-order flex if and only if the configuration $G(p)$ has a nontrivially parallel drawing $G(q)$ (Figure 61.3.4(b,c)).

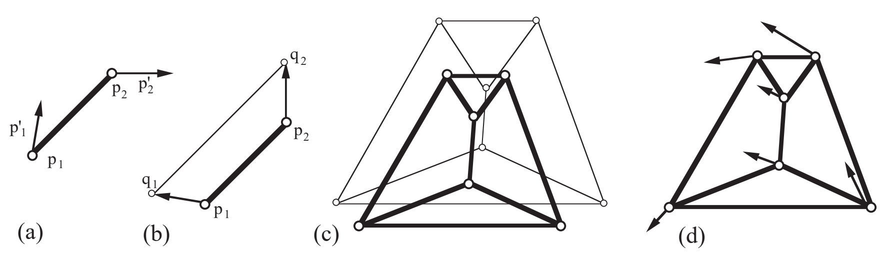

FIGURE 61.3.4

In the plane, first-order flexes (a,d) correspond to parallel drawings (c,d), by taking the vectors of one and turning them all 90^o.

In this transfer, the translations go to translations, and the rotations become dilations, or scalings.

Because of this connection, combinatorial and geometric results for plane first-order rigidity and for plane parallel drawings have numerous deep connections. For example, Laman's theorem (Theorem 61.1.12(b)) is a corollary of the parallel drawing theorem, for $d = 2$. In higher dimensions, the connection is one-way: a nontrivially parallel drawing of a "framework" (the "direction of an edge" is represented by $d-1$ facets through the two points) induces one (or more) nontrivial first-order motions of the corresponding bar framework. The theory of parallel drawing in higher dimensions is more complete and has simpler algorithms than the theory of first-order rigidity in higher dimensions, generalizing almost all results for plane first-order rigidity and plane parallel drawings [Whi96]. In Table 61.3.1, we give the counts that are necessary and sufficient for independence of parallel drawings of graphs and multi-graphs over all dimensions.

TABLE 61.3.1

STRUCTURE	NECESSARY COUNTS IN d-SPACE	CHARACTERIZATION IN THE PLANE
parallel drawing (61.3.3)	$\lvert D\rvert = d\lvert V\rvert - (d+1)$, $\lvert V\rvert \geq 2$ $\lvert D'\rvert \leq d\lvert V'\rvert - (d+1)$, $\lvert V'\rvert \geq 2$ (Also sufficient in d-space!)	$\lvert D\rvert = 2\lvert V\rvert - 3$, $\lvert V\rvert \geq 2$ $\lvert D'\rvert \leq 2\lvert V'\rvert - 3$, $\lvert V'\rvert \geq 2$
mixed (DL) (61.3.3)	$\lvert D\rvert + \lvert L\rvert = d\lvert V\rvert - d$, $\lvert V\rvert \geq 1$ $\lvert D'\rvert + \lvert L'\rvert \leq d\lvert V'\rvert - d$, $\lvert V'\rvert \geq 1$ $\lvert D'\rvert \leq d\lvert V'\rvert - (d+1)$, $\lvert V'\rvert \geq 2$ $\lvert L'\rvert \leq d\lvert V'\rvert - \binom{d+1}{2}$, $\lvert V'\rvert \geq d$	$\lvert D\rvert + \lvert L\rvert = 2\lvert V\rvert - 2$, $\lvert V\rvert \geq 1$ $\lvert D'\rvert + \lvert L'\rvert \leq 2\lvert V'\rvert - 2$, $\lvert V'\rvert \geq 1$ $\lvert D'\rvert \leq 2\lvert V'\rvert - 3$, $\lvert V'\rvert \geq 2$ $\lvert L'\rvert \leq 2\lvert V'\rvert - 3$, $\lvert V'\rvert \geq 2$

DIRECTION-LENGTH FRAMEWORKS IN THE PLANE

We can combine the information from parallel drawing and first-order rigidity through the direction-length or DL-frameworks (Figure 61.3.5). Notice that the same pair of vertices can be both a direction and a length.

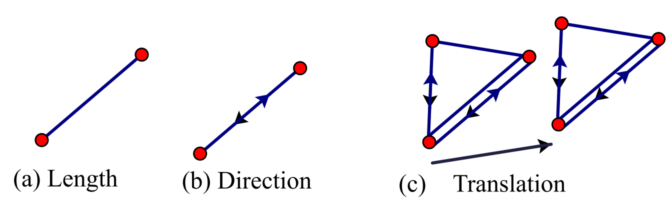

FIGURE 61.3.5
DL-frameworks have both length constraints (a) and direction constraints (b). The only trivial motions are translations (c).

Since the only trivial first-order flexes are translations, a *tight* framework in $d = 2$ will have the minimal count of $|D| + |L| = 2|V| - 2$. Note, however, that the maximum size of a pure graph with only lengths (or with only directions) is still $|D| \leq 2|V| - 3$ and $|L| \leq 2|V| - 3$ for $|V| \geq 2$. These are both necessary and sufficient (Table 61.3.1). The extended necessary counts for $d \geq 2$ also appear in Table 61.3.1.

An additional observation is that swapping directions and lengths for $d = 2$ preserves the counts and generic tightness. Working with the corresponding rigidity matrix, this swapping actually holds for any specific geometric realization [SW99]. These conditions can be checked by running the pebble game three times (once for D, once for L, and once overall).

ANGLES IN CAD

In plane computer-aided design, many different patterns of constraints (lengths, angles, incidences of points and lines, etc.) are used to design or describe configurations of points and lines, up to congruence or local congruence. With distances between points, the geometry becomes that of first-order rigidity. If angles and incidences are added, even the problems of "generic rigidity" of constraints are unsolved (and perhaps not solvable in polynomial time). However, special designs, mixing lengths, distances of points to lines, and trees of angles have been solved, using direct extensions of the techniques and results for plane frameworks, and plane parallel drawings, and the DL-frameworks ([SW99]).

For the specific constraints of distances between points, angles between lines, and distances between points and lines, some new combinatorial characterizations have recently been published [JO16]. Recent work has developed combinatorial characterizations for point-line frameworks, via projections of frameworks from the sphere, with points at infinity becoming lines which have angle constraints, and point-line distances as added constraints [EJN$^+$17].

MINKOWSKI DECOMPOSABILITY

By a theorem of Shephard, a polytope is decomposable as the Minkowski sum of two simpler polyhedra if and only if the faces and vertices of the polytope (or the edges and vertices of the polytope) have a nontrivially parallel drawing. Many characterizations of Minkowski indecomposable polytopes can be deduced directly from results for parallel d-scenes (or equivalently, for polyhedral pictures of the polar polytope). This includes the projective invariance of Minkowski decomposability.

61.4 RECIPROCAL DIAGRAMS

The reciprocal diagram is a geometric construction that has appeared, independently over a 140-year span [Max64, Cre72], in areas such as "graphical statics" (drafting techniques for resolving forces), scene analysis, and computational geometry.

Continuing work in structural engineering is developing reciprocal diagrams as a tool for design and as a tool for analysis [BMMM16, MBMK16, BO07, Tac12]. Related papers extend reciprocal diagrams to 3-space and higher dimensions [Mic08, Ryb99].

GLOSSARY

Abstract spherical polyhedron $S = (V, F; \underline{E})$: For a 2-connected planar graph $G_S = (V, E_S)$, embedded on a sphere (or in the plane), we record the vertices as V and the regions as faces F, and rewrite the directed edges $\underline{E}$ as ordered 4-tuples $\underline{e} = \langle h, i; j, k\rangle$, where the edge from vertex h to vertex i has face j on the right and face k on the left. (The reversed edge $-\underline{e} = \langle i, h; k, j\rangle$ runs from i to h, with k on the right.)

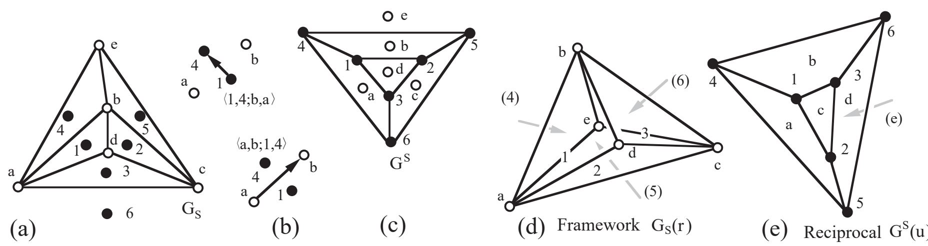

FIGURE 61.4.1
A planar graph, with its dual (a,c), (b,d) will have reciprocal drawings, with dual edges mutually perpendicular, if and only if there is a self-stress.

Dual abstract spherical polyhedron: The abstract spherical polyhedron S^* formed by switching the roles of V and F, and switching the pairs of indices in each ordered edge $\underline{e} = \langle h, i; j, k\rangle$ into $\underline{e}^* = \langle j, k; i, h\rangle$. (Also the abstract spherical polyhedron formed by the dual planar graph $G^S = (F, E^S)$ of the original planar graph (Figure 61.4.1(a,c)).)

Proper spatial spherical polyhedron: An assignment of points $p_i = (x_i, y_i, z_i)$ to the vertices and planes $P^j = (A^j, B^j, D^j)$ to the faces of an abstract spherical polyhedron $(V, F; \underline{E})$, such that if vertex i and face j share an edge, then the point lies on the plane: $A^j x_i + B^j y_i + z_i + D^j = 0$; and at each edge the two vertices are distinct points and the two faces have distinct planes.

Projection of a proper spatial polyhedron $S(p, P)$: The plane framework $G_S(r)$, where r is the vertical projection of the points p (i.e., $r_i = \Pi p_i = (x_i, y_i)$) (Figure 61.4.2).

Gradient diagram of a proper spatial polyhedron $S(p, P)$: The plane framework $G^S(s)$, where $s_j = (A^j, B^j)$ is (minus) the gradient of the plane P^j (Figure 61.4.2).

Reciprocal diagrams: For an abstract spherical polyhedron S, two frameworks $G_S(r)$ and $G^S(s)$ on the graph and the dual graph of the polyhedron, such that for each directed edge $\langle h, i; j, k\rangle \in \underline{E}$, $(r_h - r_i) \cdot (s_j - s_k) = 0$ (Figure 61.4.1(d,e)).

BASIC RESULTS

Reciprocal diagrams have deep connections to both of our previous topics:

(a) Given a spatial scene on a spherical structure, with no faces vertical, the vertical projection and the gradient diagram are reciprocal diagrams. (This follows because the difference of the gradients at an edge is a vector perpendicular to the vertical plane through the edge.)

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(b) Given a pair of reciprocal diagrams on $S = (V, F; \underline{E})$, then for each edge $\underline{e} = \langle h, i; j, k\rangle$ the scalars ω_{ij} defined by $\omega(r_h - r_i) = (s_j - s_k)^{\perp}$ (where $^{\perp}$ means rotate by 90° clockwise) form a self-stress on the framework $G_S(r)$. (This follows because the closed polygon of a face in $G^S(s)$ is, after $^{\perp}$, the vector sum for the "vertex equilibrium" in the self-stress condition.)

These facts can be extended to other oriented polyhedra and their projections. The real surprise is that, for spherical polyhedra, the converses hold and all these concepts are equivalent (an observation dating back to Clerk Maxwell and the drafting techniques of graphical statics). The first complete proof appears to be in [KW04].

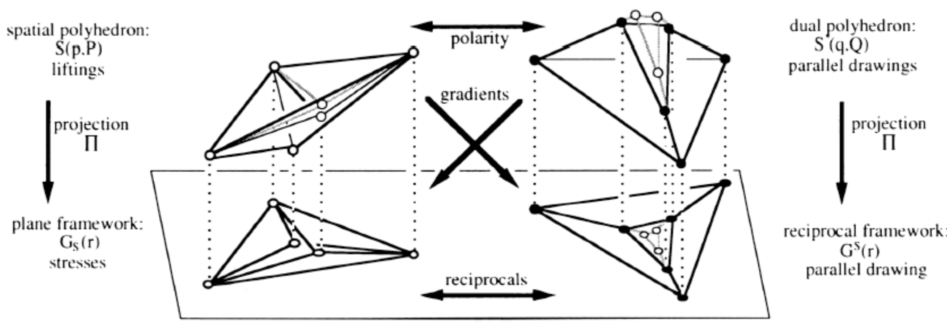

FIGURE 61.4.2
For planar graphs, the reciprocal pairs are entwined with a pair of polar polyhedra by projection and lifting.

THEOREM 61.4.1 Maxwell's Theorem

For an an abstract spherical polyhedron $(V, F; \underline{E})$, the following are equivalent:

(a) *The framework $G_S(r)$, with the vertices of each edge distinct, has a self-stress nonzero on all edges;*

(b) *$G_S(r)$ has a reciprocal framework $G^S(s)$ with the vertices of each edge distinct;*

(c) *$G_S(r)$ is the vertical projection of a proper spatial polyhedron $S(p, P)$;*

(d) *$G_S(r)$ is the gradient diagram of a proper spatial polyhedron $S^*(q, Q)$.*

There are other refinements of this theorem, that connect the space of self-stresses of $G_S(r)$ with the space of parallel drawings (and first-order flexes) of the reciprocal $G^S(s)$, the space of polyhedra $S(p, P)$ with the same projection, and the space of parallel drawings of $S^*(q, Q)$ [CW94] (Figure 61.4.2).

A second refinement connects the local convexity of the edge of the polyhedron with the sign of the self-stress.

THEOREM 61.4.2 Convex Self-stress

The vertical projection of a strictly convex polyhedron, with no faces vertical, produces a plane framework with a self-stress that is < 0 on the boundary edges (the edges bounding the infinite region of the plane) and > 0 on all edges interior to this boundary polygon.

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A plane Delaunay triangulation also has a basic "reciprocal" relationship to the plane Voronoi diagram: the edges joining vertices at the centers of the regions are perpendicular to edges of the polygon of the Voronoi regions surrounding the vertex. This pair of reciprocals is directly related to the projection of a spatial convex polyhedral cap, as are generalized Voronoi diagrams [WABC13]. See Section 27.1.

This pattern of "reciprocal constructions" and the connection to liftings to polytopes in the next dimension generalizes to higher dimensions [CW94]. For example, for Voronoi diagrams and Delaunay simplicial complexes, the edges of one are perpendicular to facets of the other, in all dimensions. Moreover, for appropriate sphere-like homology, the existence of a reciprocal corresponds to the existence of nontrivial liftings [CW94, Ryb99]. Such geometric structures are also related to k-rigidity and to combinatorial proofs of the g-theorem in polyhedral combinatorics [TW00].

61.5 SOURCES AND RELATED MATERIALS

SURVEYS AND BASIC SOURCES

All results not given an explicit reference can be traced through these surveys:

[CW96]: A presentation of basic results for concepts of rigidity between first-order rigidity and rigidity for tensegrity frameworks.

[CWW14]: A recent book with a range of contributions on rigidity.

[Wiki]: A range of older preprints, including chapters from a draft book on rigidity.

[DT99]: Conference papers related to applications of rigidity theory.

[GSS93]: An older monograph devoted to combinatorial results for the graphs of generically rigid frameworks, with an extensive bibliography on many aspects of rigidity.

[Ros00]: An older thesis that explores in depth multiple topics of this chapter and their connections.

[SSS17]: A new handbook on constraint theory — to appear shortly.

[Sug86]: A monograph on the reconstruction of spatial polyhedral objects from plane pictures.

[Whi96]: An expository article presenting matroidal aspects of first-order rigidity, scene analysis, and multivariate splines.

RELATED CHAPTERS

Chapter 9: Geometry and topology of polygonal linkages
Chapter 27: Voronoi diagrams and Delaunay triangulations
Chapter 34: Geometric reconstruction problems
Chapter 51: Robotics
Chapter 55: Graph drawing
Chapter 60: Geometric applications of the Grassmann-Cayley algebra

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Chapter 62: Rigidity of symmetric frameworks
Chapter 63: Global rigidity

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